Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

On Laplace Equation Solution in Orthogonal Similar Oblate Spheroidal Coordinates

Pavel Strunz

Nuclear Physics Institute of the CAS, 25068, Řež, Czech Republic
E-mail: strunz@ujf.cas.cz

Received: 19 May 2024; **Revised:** 5 July 2024; **Accepted:** 16 July 2024

Abstract: Orthogonal coordinate systems enable expressing the boundary conditions of differential equations in accord with the physical boundaries of the problem. It can significantly simplify calculations. The orthogonal similar oblate spheroidal (SOS) coordinate system can be particularly useful for a physical processes description inside or in the vicinity of the bodies or particles with the geometry of an oblate spheroid. The interior solution of the Laplace equation in the SOS coordinates was recently found; however, the exterior solution was missing. The exterior solution of the azimuthally symmetric Laplace equation in the SOS coordinates is derived. In the steps leading to this solution, important formulas of the SOS algebra are found. Various forms of the Laplace operator in the SOS coordinates in azimuthally symmetric case are shown. General transformation between two different SOS coordinate systems is derived. It is determined that the SOS harmonics are physically the same as the solid harmonics. Further, a formula expressing any generalized Legendre polynomial as a finite sum of monomials is found. The reported relations have potential application in geophysics, astrophysics, electrostatics, electromagnetism, fluid flow and solid state physics (e.g., ferroic inclusions).

Keywords: Laplace equation, similar oblate spheroidal coordinates, harmonic function, Legendre function, potential theory, ferroic nanoparticles

MSC: 35A08, 31B05, 35J05, 35A25

1. Introduction

Curvilinear coordinate systems are valuable tools particularly for differential equations solutions, especially in the potential theory. They enable expressing the boundary conditions of differential equations in a way greatly simplifying the calculations when the coordinate surfaces fit the physical boundaries of the problem [1]. A range of field problems that can be handled effectively depends on the number of well developed coordinate systems. Amongst them, the orthogonal curvilinear coordinate systems [2] are the most useful.

Recently, the earlier suggested [3] similar oblate spheroidal (SOS) orthogonal coordinate system was finalized [4, 5]. The similar oblate spheroidal coordinates are distinct from all the well known standard orthogonal coordinate systems [2, 6], the confocal oblate spheroidal (COS) system not excluding. The SOS system can be a powerful tool for a description of field or physical processes inside or in the vicinity of the bodies or particles with an oblate spheroidal geometry. Such objects range (but are not limited to) from ferroic nanoparticles through planets up to galaxies.

DOI: https://doi.org/10.37256/cm.6220254965

In electrostatics and solid state physics, the SOS coordinates could find application in description of electric field potential in ferroelectric materials (e.g., ferroelectric nanocomposites) containing dielectric inclusions (or vice versa-ferroelectric inclusions in dielectric matrix) [7], particularly the inclusions of spheroidal shape [8].

Other areas where the novel coordinates could facilitate solution of physical problems are fluid flow and electromagnetic wave scattering in an environment exhibiting special geometry. The use of specific orthogonal coordinate systems is of a large advantage in such cases [9–11]. It was also shown that using non-Cartesian coordinates (particularly spheroidal) can bring advantages for solution of Stokes flow [12]. Solution of differential equations (continuity, momentum, motion), including Laplace and Poisson equations, connected to such problems can be obviously achieved more simply within a proper coordinate system.

In the field of atmospheric physics, the SOS coordinates could be of help for better modeling of geopotential surfaces, allowing for a better description of the spatial variation of the apparent gravity [3]. In astrophysics, similar oblate spheroids are frequently used for modeling of iso-density levels inside galaxies [13, 14].

Although the derived basic relations for the SOS system are analytical, they cannot be expressed in a closed form. They employ convergent infinite power series with generalized binomial coefficients, which can be-nevertheless-handled with the use of combinatorial identities. This fact makes from calculations in the SOS system an interesting mathematical topic.

The analytical coordinate transformation from the SOS coordinates to the Cartesian system and the metric scale factors were already determined [4, 5], as well as the formulas necessary for the transformation of a vector field between the SOS system and the Cartesian coordinates [5].

Recently, the Laplace equation solution for the interior space was derived [15] in the SOS coordinates. Nevertheless, the found harmonic functions cannot be used or easily extended to the exterior space (i.e., for the space ranging up to infinity). Therefore, a separate solution valid in the exterior has to be searched for.

This article deals with solution of Laplace differential equation in the SOS coordinates in the exterior space for azimuthally symmetric case. A full solution of the Laplace equation would represent an important step in the development of the SOS coordinate system and its applicability in physics and other fields. The complete determination of harmonic functions would also help to find solutions of more complex differential equations in SOS coordinates.

The organization of the text is following. First, a summary of the SOS coordinate system is provided together with the already found relations relevant for the present derivation. Then, preparatory considerations which enable achieving the aim of the article are carried out. Finally, the exterior solution is found, and the complete solution of the Laplace equation in the azimuthally symmetric case in the SOS coordinates is reported.

As some derivations within the article are very lengthy when carried out in a full detail, a large part of them is moved to Supplements to this article.

2. Summary of the previous results

2.1 *SOS coordinates*

The similar oblate spheroidal coordinates, introduced by White et al. [3], are qualitatively different from the standard orthogonal coordinate systems [2, 6] thanks to the fact that the second coordinate surfaces family is not of the second degree or of the fourth degree but it is formed by general power functions $z \sim x^{1+\mu}$ (i.e., with a real-number exponent $1 + \mu$) rotated around the z-axis [3, 4]. These are orthogonal to the similar oblate spheroids representing the first set of the coordinate surfaces (For terminological clarity, a spheroid means in this text an ellipsoid of revolution or rotational ellipsoid. An oblate spheroid is a quadric surface obtained by rotating an ellipse about the shorter principal axis).

For the SOS coordinate system $(R,\ \nu,\ \lambda)$ [4], the basic coordinate surfaces of the R coordinate are similar oblate spheroids given in 3D Cartesian coordinates $(x_{3D},\ y_{3D},\ z_{3D})$ by the formula

$$x_{3D}^2 + y_{3D}^2 + (1+\mu) z_{3D}^2 = R^2 \qquad (1)$$

The R coordinate value is equal to the equatorial radius of the particular spheroid from the family. The parameter μ characterizes the oblateness of the whole family of the similar oblate spheroids (the larger the parameter μ, the flatter the spheroid). The parameter $\mu > 0$ for oblate spheroids. The minor and the major semi-axes of each member of the spheroid family have the ratio $(1+\mu)^{1/2}$. As a limit (when $\mu = 0$), a sphere (and spherical coordinate surfaces) is obtained by (1). A special reference spheroid is introduced with the equatorial radius R_0, usually coinciding with the reference surface of the object for which the SOS coordinate system is to be applied.

The second set of the coordinate surfaces, orthogonal to the similar oblate spheroids defined above, are power functions in 3D of the form [3, 4]

$$z_{3D} = \frac{1}{\sqrt{1+\mu}} \frac{1}{R_0^{\mu}} \frac{\sin \nu}{\cos^{1+\mu} \nu} \left(\sqrt{x_{3D}^2 + y_{3D}^2} \right)^{1+\mu} \tag{2}$$

The labeling, i.e., the coordinate ν corresponding to these surfaces, is equivalent to the so called parametric latitude [4]. The coordinate ν is also equivalent to the parameter used for the standard parametric equation of the ellipse, representing the meridional section of the reference spheroid mentioned above, having a special equatorial radius (major semi-axis) equal to R_0, i.e.,

$$\sqrt{x_{3D}^2 + y_{3D}^2} = R_0 \cos \nu \quad \text{and} \quad z_{3D} = \frac{R_0}{\sqrt{1+\mu}} \sin \nu \tag{3}$$

Finally, the third set of the coordinate surfaces, orthogonal to the previous two, are semi-infinite planes containing the rotation axis. The associated coordinate is the longitude angle λ, which is the same as its equivalent coordinate in the spherical coordinate system.

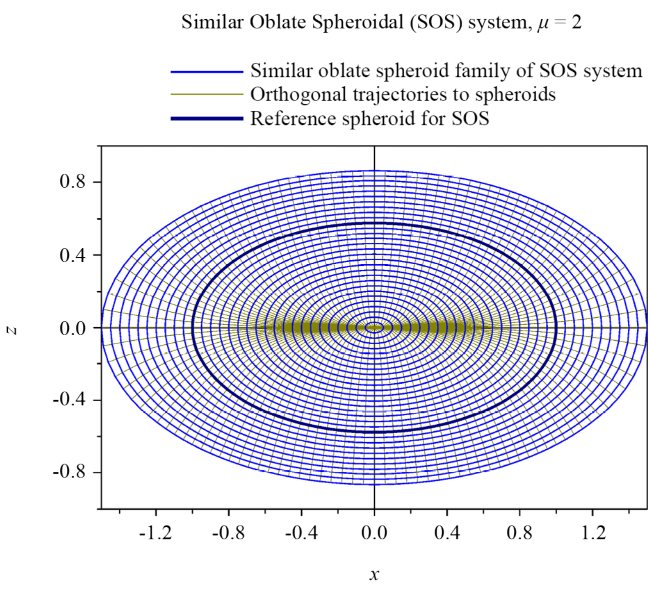

Figure 1. The cross section (x-z) of the SOS coordinate system along the meridian. The sets of the orthogonal SOS coordinate-system lines for the coordinates R and ν are displayed. The section of the reference spheroid with the equatorial radius $R_0 = 1$ is shown as well

Figure 1 displays the x-z cross section of the SOS coordinate system along the meridional plane for the oblateness parameter $\mu = 2$. The full description of the analytical solution of the SOS coordinates can be found in [4]. A shortened summary is reported in [5]. A limited description (to the extent needed for the tasks of this article) is given also further in this section.

2.2 *Basic relations of the SOS coordinate system*

A key role in the derivation of the SOS coordinate algebra (e.g., the coordinates transformation between the SOS and the Cartesian system [4]) plays the dimensionless real-number parameter W defined as

$$W = \left(\frac{R}{R_0}\right)^{\mu} \frac{\sin\nu}{\cos^{1+\mu}\,\nu} \tag{4}$$

The constant-W surfaces are straight half-lines starting at the origin and rotated around the system axis, i.e., cones [4]. In what follows, the calculation is restricted to the parametric latitude $\nu \in \left[0, \frac{\pi}{2}\right]$, i.e., to the first quadrant in Figure 1. With this limitation, the parameter W, see (4), is always non-negative which simplifies further derivations. Due to the symmetry (reflection with respect to the equator), expressions and solutions relevant for the SOS system in the complementary range $\nu \in \left[-\frac{\pi}{2}, 0\right)$ can be easily obtained.

The SOS coordinates can be transformed to the Cartesian coordinates using analytical expressions including infinite power series with generalized binomial coefficients [4, 5]. Relations enabling such an approach were first reported by Pólya and Szegö [16]. Due to the convergence limits of the involved power series, the expressions have to be derived separately in two regions: the so called "small-ν region" and separately in the "large-ν region" [4]. The boundary between the two regions is defined by the parameter W (see (5)) value fulfilling the following relation:

$$W_b(R,\,\nu) = \sqrt{\frac{\mu^{\mu}}{(1+\mu)^{1+\mu}}} = \text{constant} \tag{5}$$

This boundary surface is a straight half-line starting at the origin and rotated around the symmetry axis (and forming thus the surface of a cone) [4, 5]. Although W_b is a constant for a fixed μ, the coordinates R and ν vary along this half-line.

The metric scale factors-separately in the small-ν region and in the large-ν region-for R coordinate,

$$h_R = \sqrt{\sum_{k=0}^{\infty} \binom{-\mu k}{k} \left(W^2\right)^k} \quad \text{and} \quad h_R = \frac{1}{\sqrt{1+\mu}} \sqrt{\sum_{k=0}^{\infty} \binom{\frac{\mu}{1+\mu}k}{k} \left(W^{-\frac{2}{1+\mu}}\right)^k} \tag{6}$$

as well as for the ν coordinate,

$$h_\nu = \frac{R}{\sqrt{1+\mu}} \frac{\partial W}{\partial \nu} \sqrt{\sum_{k=0}^{\infty} \binom{-(\mu+2)-\mu k}{k} \left(W^2\right)^k} \quad \text{and}$$
$$h_\nu = \frac{R}{1+\mu} W^{-\frac{2+\mu}{1+\mu}} \frac{\partial W}{\partial \nu} \sqrt{\sum_{k=0}^{\infty} \binom{-\frac{2+\mu}{1+\mu} + \frac{\mu}{1+\mu}k}{k} \left(W^{-\frac{2}{1+\mu}}\right)^k} \tag{7}$$

were determined [4]. The left side relations show the formulae that hold for the small-ν region while the right side relations show the formulae that hold for the large-ν region.

The functions for the generalized cosine (f_C) and for the generalized sine (f_S) in the frame of the SOS coordinate system were found as well [5]:

$$f_C = \sqrt{\sum_{k=0}^{\infty} \binom{-1-\mu k}{k} (W^2)^k} \quad \text{and} \quad f_C = \frac{W^{-\frac{1}{1+\mu}}}{\sqrt{1+\mu}} \sqrt{\sum_{k=0}^{\infty} \binom{-\frac{1}{1+\mu}+\frac{\mu}{1+\mu}k}{k} \left(W^{-\frac{2}{1+\mu}}\right)^k} \tag{8}$$

$$f_S = W\sqrt{1+\mu}\sqrt{\sum_{k=0}^{\infty} \binom{-(1+\mu)-\mu k}{k} (W^2)^k} \quad \text{and} \quad f_S = \sqrt{\sum_{k=0}^{\infty} \binom{-1+\frac{\mu}{1+\mu}k}{k} \left(W^{-\frac{2}{1+\mu}}\right)^k} \tag{9}$$

The left sides show the formulae that hold for the small-ν region while the right sides show the formulae that hold for the large-ν region. Note that f_S, f_C depend solely on W (not separately on R and ν).

For the solution of the Laplace equation, it is of advantage to define a new function $s(W)$ as

$$s \equiv \frac{f_S}{h_R} \tag{10}$$

Using (9) and (6) and the known binomial identities [5], the power-series expressions of s are as follows:

$$s = W\sqrt{1+\mu}\sqrt{\sum_{k=0}^{\infty} \frac{-(1+\mu)}{-(1+\mu)-\mu k} \binom{-(1+\mu)-\mu k}{k} (W^2)^k} \tag{11}$$

in the small-ν region, and

$$s = \sqrt{1+\mu}\sqrt{\sum_{k=0}^{\infty} \frac{-1}{-1+\frac{\mu}{1+\mu}k} \binom{-1+\frac{\mu}{1+\mu}k}{k} \left(W^{-\frac{2}{1+\mu}}\right)^k} \tag{12}$$

in the large-ν region [5].

The functions listed in this sub-section contain infinite power series with generalized binomial coefficients. It is possible to deal with relations involving them with a help of known combinatorial identities [5, 16–20].

2.3 *Laplacian in the SOS coordinates*

The Laplace operator of a general scalar function $V(R, \nu)$, for example a scalar potential, in the SOS coordinates in the azimuthally symmetric case with the spheroid-coordinate surfaces of the R coordinate described by the oblateness parameter μ is

$$\Delta V = \frac{1}{\Im}\left[\frac{\partial}{\partial R}\left(\frac{\Im}{h_R^2}\frac{\partial V(R, \nu)}{\partial R}\right) + \frac{\partial}{\partial \nu}\left(\frac{\Im}{h_\nu^2}\frac{\partial V(R, \nu)}{\partial \nu}\right)\right] \tag{13}$$

In the case when V can be separated to an R-dependent part and a W-dependent part, i.e., $V = r(R)F(W)$, the Laplacian in the SOS coordinates in the azimuthally symmetric case can be rewritten with the help of product and chain rules for derivatives to the form

$$\Delta V = \frac{1}{h_R^2}\frac{d^2r(R)}{dR^2}F(W) + \frac{1}{R}\frac{dr(R)}{dR}\left[(\mu+3) - \frac{1}{h_R^2}\right]F(W) + \frac{1}{R^2}r(R)\left[\frac{(\mu-2)\mu}{h_R^2} + (\mu+3)\mu + \frac{h_R^2(1+\mu)^2}{f_C^2}\right] W\frac{dF(W)}{dW} + 2\frac{1}{R}\frac{dr(R)}{dR}\frac{\mu W}{h_R^2}\frac{dF(W)}{dW} + \frac{1}{R^2}r(R)\left(\frac{\mu^2}{h_R^2} + \frac{h_R^2(1+\mu)^2}{f_C^2 f_S^2}\right)W^2\frac{d^2F(W)}{dW^2} \tag{14}$$

(deduce it from (B20) in the Supplement B of the interior-solution article [15] and from the steps leading to it). The Laplacian (14) is a sum of terms which are always a product of the W-dependent expression and the R-dependent expression.

The Laplacian can be also expressed in s-terms:

$$\Delta V = \frac{1}{1+\mu}\left((1+\mu)+\mu s^2\right)F(s)\frac{d^2r(R)}{dR^2} + \left[(\mu+2) - \frac{\mu}{1+\mu}s^2\right]F(s)\frac{1}{R}\frac{dr(R)}{dR} + 2\frac{\mu}{1+\mu}\left[(1+\mu)-s^2\right] s\frac{dF(s)}{ds}\frac{1}{R}\frac{dr(R)}{dR} + \frac{3\mu s^2 - \left(3\mu^2+5\mu+2\right)}{1+\mu}s\frac{dF(s)}{ds}\frac{r(R)}{R^2} + \frac{\left[(1+\mu)-s^2\right]\left[(1+\mu)^2-\mu s^2\right]}{1+\mu}\frac{d^2F(s)}{ds^2}\frac{r(R)}{R^2} \tag{15}$$

This relation is derived in Appendix A.

When using the relations (14) or (15) for the Laplacian, the Laplace equation cannot be separated by a standard separation procedure (see e.g., [2]). Nevertheless, a special separation procedure described in [15] can be successfully employed.

For completeness, the Laplacian in the SOS coordinates is derived for the axially symmetric case for the non-separable function V_N in Supplement 1 as well (see (S1.23) in the Supplement). The steps leading to the result are similar to the ones described in Appendix A for the separable function V. The result for the non-separable function V_N is

$$\Delta V_N = \left[\frac{\mu s^2 + (1+\mu)}{1+\mu}\frac{\partial^2 V_N}{\partial R^2} + (1+\mu)^2\frac{(1+\mu) - s^2}{\mu s^2 + (1+\mu)}\frac{1}{R^2}\frac{\partial^2 V_N}{\partial s^2}\right] + \left[\frac{(1+\mu)(\mu+2) - \mu s^2}{1+\mu}\frac{1}{R}\frac{\partial V_N}{\partial R} - 2(1+\mu)^4\frac{s}{\left[\mu s^2 + (1+\mu)\right]^2}\frac{1}{R^2}\frac{\partial V_N}{\partial s}\right] \tag{16}$$

This represents the Laplacian expressed in R and s terms. Going still further, one can rewrite the derivatives in (16) from the form "with respect to s" to the form "with respect to v" (see Supplement 1, relation (S1.39)):

$$\Delta V_N = \frac{\mu s^2+(1+\mu)}{1+\mu}\frac{\partial^2 V_N}{\partial R^2} + \frac{(1+\mu)(\mu+2)-\mu s^2}{1+\mu}\frac{1}{R}\frac{\partial V_N}{\partial R} + \frac{(1+\mu)^2}{s^2}\frac{\mu s^2+(1+\mu)}{(1+\mu)-\ s^2}\frac{\sin^2\nu\ \cos^2\nu}{\left(1+\mu\ \sin^2\nu\right)^2}\frac{1}{R^2}\frac{\partial^2 V_N}{\partial \nu^2}$$

$$+\frac{(1+\mu)^2}{[(1+\mu)-\ s^2]}\frac{\sin\nu\ \cos\nu}{\left(1+\mu\ \sin^2\nu\right)}\left((1+\mu)-\frac{(1+\mu)+\mu s^2}{s^2}\frac{\left[2+3\mu+\mu^2\sin^2\nu\right]\sin^2\nu}{\left(1+\mu\ \sin^2\nu\ \right)^2}\right)\frac{1}{R^2}\frac{\partial V_N}{\partial \nu} \tag{17}$$

2.4 *The solution of Laplace equation in the interior space*

The Laplace equation in the orthogonal similar oblate spheroidal coordinates in the azimuthally symmetric case can be written as

$$\Delta V = \frac{\partial}{\partial R}\left(\frac{\Im}{h_R^2}\frac{\partial V(R,\ \nu)}{\partial R}\right)+\frac{\partial}{\partial \nu}\left(\frac{\Im}{h_\nu^2}\frac{\partial V(R,\ \nu)}{\partial \nu}\right)=0 \tag{18}$$

where the Laplacian is to be expressed in the form suitable for the special variable separation, i.e., (14) or (15).

The radial part of the separated Laplace equation has the solution (see [15], Equation (33))

$$r(R)\sim R^{K_d} \tag{19}$$

where the separation constant K_d of the Laplace equation is in principle not restricted to a limited range for this, radial, part of the Laplace equation solution. Therefore, it can be used both for the interior as well as for the exterior solution solid harmonics.

The angular part of the Laplace equation expressed in s terms in the case of variables separation is (see [15], Equation (48))

$$\left[(1+\mu)-s^2\right]\left[(1+\mu)^2-\mu s^2\right]\frac{d^2F(s)}{ds^2}\ +\ s\left[-(3\mu+2)(1+\mu)+2\mu(1+\mu)K_d+\mu\,(3-2K_d)\,s^2\right]\frac{dF(s)}{ds}$$

$$+\ K_d\left[(K_d-2)\,\mu s^2+(1+\mu)K_d+(1+\mu)^2\right]F(s)=0 \tag{20}$$

In [15], this equation was solved for positive values of the separation constant K_d (i.e., for the interior space) with a help of newly defined generalized Legendre functions. It was shown that a solution of (20) certainly exists when the separation constant K_d is a non-negative integer n. Then, the function $F(s)$ is proportional to the generalized Legendre function of the first kind, $P_n^{SI}(s)$, or of the second kind, $Q_n^{SI}(s)$.

In [15], Equation (102), the complete interior solution of the Laplace equation

$$V(R,\ \nu)=V(R,\ s)=\sum_{n=0}^{\infty}a_nR^nP_n^{SI}(s)+\sum_{n=0}^{\infty}b_nR^nQ_n^{SI}(s) \tag{21}$$

is presented. a_n and b_n are arbitrary real coefficients, which can be determined for the particular boundary conditions. The relation (21) thus represents the interior harmonic functions in the SOS coordinates. Several low-index generalized

Legendre functions can be seen in Tables 1 and 2 of [15]. For a better understanding, the first five generalized Legendre functions of the first kind (i.e., the generalized Legendre polynomials) are listed here as well:

$$P_0^{SI}(s) = 1, \quad P_1^{SI}(s) = \frac{1}{1+\mu}s, \quad P_2^{SI}(s) = \frac{1}{(1+\mu)^2}\frac{1}{2}\left[(\mu+3)s^2 - (1+\mu)^2\right],$$

$$P_3^{SI}(s) = \frac{1}{(1+\mu)^3}\frac{1}{2}\left[(3\mu+5)s^3 - 3(1+\mu)^2 s\right], \tag{22}$$

$$P_4^{SI}(s) = \frac{1}{(1+\mu)^4}\cdot\frac{1}{2^3}\cdot\left[\left(3\mu^2+30\mu+35\right)s^4 - 6(\mu+5)(1+\mu)^2 s^2 + 3(1+\mu)^4\right]$$

The higher degree polynomials can be retrieved by the Bonnet-like recursion formula [15]

$$P_{n+1}^{SI}(s) = \frac{2n+1}{n+1}\frac{1}{1+\mu}sP_n^{SI}(s) - \frac{n}{n+1}\left(1-\frac{\mu}{(1+\mu)^2}s^2\right)P_{n-1}^{SI}(s) \tag{23}$$

The same formula holds also for the generalized Legendre functions of the second kind $Q_n^{SI}(s)$.

However, the solution of (20) for negative integer values of the separation constant $K_d = -1-n$, $n \geq 0$, i.e., in fact for the exterior space, was not yet found, and it is the topic of the following sections.

3. Preparatory considerations for the exterior solution

3.1 *Transformation between two different SOS systems*

It will be of advantage to find-in addition to the relation for the position vector magnitude v_r reported in Equation (53) of [15]-also a relation for the angle χ between the position vector of a point (R, ν) and the equatorial plane. The angle χ is equivalent to the latitude, and (v_r, χ, λ) are thus in fact spherical coordinates (although not the polar ones). The sine of the angle χ satisfies clearly the relation

$$\sin\chi = \frac{z_{3D}}{v_r} \tag{24}$$

where, according to Equations (49) and (53) of the article [15],

$$z_{3D} = \frac{1}{1+\mu}Rs \quad \text{and} \quad v_r = R\sqrt{1-\frac{\mu}{(1+\mu)^2}s^2} \tag{25}$$

Combination of (24) and (25) results in

$$\sin\chi = \frac{sR}{(1+\mu)v_r} \quad \Rightarrow \quad v_r\sin\chi = \frac{1}{1+\mu}Rs \tag{26}$$

Then also (using (25) and (26))

$$\sin\chi = \frac{\frac{1}{1+\mu}Rs}{R\sqrt{1-\frac{\mu}{(1+\mu)^2}s^2}} = \frac{s}{\sqrt{(1+\mu)^2-\mu s^2}} \tag{27}$$

These relations between the SOS and the spherical coordinates also follow from the transformation between two different SOS systems, part of which is reported in [4]. As such transformation relations are significant, the complete formulae derivation of such general transformation between two different SOS systems is shown in the following paragraphs.

Assume two SOS systems: The first SOS system, (R, ν, λ), has the parameter of oblateness μ. The second SOS coordinate system, (R_V, ν_V, λ_V), has the parameter of oblateness $\mu_V \neq \mu$. Both systems are assumed to have the same equatorial radius R_0 of their reference spheroids. According to Equation (123) in [4] for the transformation of radial coordinate we have that

$$R_V = R\sqrt{\frac{1+\mu_V}{1+\mu} - \frac{\mu_V-\mu}{1+\mu}\sum_{M=0}^{\infty}\frac{-1}{-1-\mu M}\binom{-1-\mu M}{M}\left(W^2\right)^M} \tag{28}$$

in the small-ν region. As (see [4], Equations (10) and (47), or [5], Equations (6), (7) and (43))

$$\begin{aligned} x_{3D}^2 + y_{3D}^2 &= \left(\cos\lambda\ R\sum_{k=0}^{\infty}\binom{-\frac{1}{2}-\mu k}{k}\frac{-\frac{1}{2}}{-\frac{1}{2}-\mu k}\left(W^2\right)^k\right)^2 \\ &\quad + \left(\sin\lambda\ R\sum_{k=0}^{\infty}\binom{-\frac{1}{2}-\mu k}{k}\frac{-\frac{1}{2}}{-\frac{1}{2}-\mu k}\left(W^2\right)^k\right)^2 \\ &= R^2\left(\sum_{k=0}^{\infty}\binom{-\frac{1}{2}-\mu k}{k}\frac{-\frac{1}{2}}{-\frac{1}{2}-\mu k}\left(W^2\right)^k\right)^2 = R^2\left(p^{-\frac{1}{2}}\right)^2 = R^2p^{-1} \\ &= R^2\sum_{M=0}^{\infty}\frac{-1}{-1-\mu M}\binom{-1-\mu M}{M}\left(W^2\right)^M \end{aligned} \tag{29}$$

and as (see [15], Equation (52), and consider that $s = f_S/h_R$) the distance from the axis is also

$$x_{3D}^2 + y_{3D}^2 = R^2\left(1-\frac{1}{1+\mu}s^2\right) \tag{30}$$

the following identity holds

$$\sum_{M=0}^{\infty} \frac{-1}{-1-\mu M} \binom{-1-\mu M}{M} \left(W^2\right)^M = 1 - \frac{1}{1+\mu} s^2 \tag{31}$$

in the small-ν region. Then, (28) becomes

$$R_V = R\sqrt{\frac{1+\mu_V}{1+\mu} - \frac{\mu_V - \mu}{1+\mu}\left(1 - \frac{1}{1+\mu}s^2\right)} = R\sqrt{1 - \frac{\mu-\mu_V}{(1+\mu)^2}s^2} \tag{32}$$

Further (see [15], Equation (49), and considering that $s = f_S/h_R$)

$$z_{3D} = \frac{1}{1+\mu} Rs \tag{33}$$

in the first SOS system (R, ν, λ), whereas it is

$$z_{3D} = \frac{1}{1+\mu_V} R_V s_V \tag{34}$$

in the second SOS coordinate system (R_V, ν_V, λ_V). Equating (33) and (34) gives

$$\frac{1}{1+\mu_V} R_V s_V = \frac{1}{1+\mu} Rs \tag{35}$$

With the help of (32), we obtain

$$s_V = \frac{1+\mu_V}{1+\mu}\frac{R}{R_V}s = \frac{1+\mu_V}{1+\mu}\frac{1}{\sqrt{1-\frac{\mu-\mu_V}{(1+\mu)^2}s^2}}s = \frac{1+\mu_V}{\sqrt{(1+\mu)^2-(\mu-\mu_V)s^2}}s \tag{36}$$

The Equations (32) and (36) are sumarized as follows,

$$R_V = \frac{R}{1+\mu}\sqrt{(1+\mu)^2-(\mu-\mu_V)s^2} \qquad \text{and} \qquad s_V = \frac{1+\mu_V}{\sqrt{(1+\mu)^2-(\mu-\mu_V)s^2}}s \tag{37}$$

and they represent the transformation between any two SOS coordinate systems having the same equatorial radius R_0 of their reference spheroids. The same relation can be derived also in the large-ν region.

In the special case when the parameter of oblateness μ_V of the second SOS system is equal to zero, i.e., it is equivalent to the spherical coordinate system, the transformation (37) reads

$$v_r = R_V|_{\mu_V=0} = R\sqrt{1-\frac{\mu}{(1+\mu)^2}s^2} \qquad \text{and} \qquad \sin\chi = s_V|_{\mu_V=0} = \frac{s}{\sqrt{(1+\mu)^2-\mu s^2}} \tag{38}$$

These relations are equivalent to the ones in (25) and (27) derived previously for this special case $\mu_V = 0$. From (27) and (38), the following relations can be easily found:

$$s^2 = \frac{(1+\mu)^2}{\frac{1}{\sin^2\chi} + \mu} = \frac{(1+\mu)^2 \sin^2\chi}{1+\mu\sin^2\chi} \quad \Rightarrow \quad s = \frac{(1+\mu)\sin\chi}{\sqrt{1+\mu\sin^2\chi}} \quad \text{and} \quad \frac{1}{(1+\mu)^2 - \mu s^2} = \frac{1+\mu\sin^2\chi}{(1+\mu)^2} \tag{39}$$

$$R = \frac{(1+\mu)v_r}{\sqrt{(1+\mu)^2 - \mu s^2}} = v_r\sqrt{1+\mu\sin^2\chi} \tag{40}$$

Then, the following relation holds between the coordinates of the SOS and of the spherical systems:

$$R^n s^m = \left(v_r\sqrt{1+\mu\sin^2\chi}\right)^n \left(\frac{(1+\mu)\sin\chi}{\sqrt{1+\mu\sin^2\chi}}\right)^m = v_r^n\left(\sqrt{1+\mu\sin^2\chi}\right)^{n-m}(1+\mu)^m\sin^m\chi \tag{41}$$

The Laplace equation solution of the first kind in the interior space, $R^n P_n^{SI}(s)$, is in fact a series of terms involving $R^n s^m$, see (21) and (22). As we know (see Table 1 of [15]) that $n-m$ is always an even number, we can rewrite (41) in the form

$$\begin{aligned} R^n s^m &= (1+\mu)^m v_r^n\left(1+\mu\sin^2\chi\right)^{\frac{n-m}{2}}\sin^m\chi = (1+\mu)^m v_r^n\left[\sum_{k=0}^{\frac{(n-m)}{2}}\begin{pmatrix}\frac{n-m}{2}\\ k\end{pmatrix}\left(\mu\sin^2\chi\right)^k\right] \\ \sin^m\chi &= (1+\mu)^m v_r^n\sum_{k=0}^{\frac{(n-m)}{2}}\begin{pmatrix}\frac{n-m}{2}\\ k\end{pmatrix}(\sin\chi)^{2k+m} \end{aligned} \tag{42}$$

where the binomial expansion was used.

3.2 *Equivalence between the spherical solid harmonics and the SOS harmonics*

To proceed with the Laplace equation solution for the exterior space, it is worth to use the already found [15] interior solution (21) involving the generalized Legendre polynomials $P_n^{SI}(s)$. We can write them in the following form:

$$P_n^{SI}(s) = \sum_{N=0}^{\left[\frac{n}{2}\right]} s^{n-2N}A_{n,\,n-2N} \tag{43}$$

Here, $[n/2]$ denotes the floor function, i.e., the integer part of $n/2$. Note, that all $A_{n,\,k}$ coefficients are zero for $n-k$ being odd (such A's are, moreover, not included in the sum in the above formula). This corresponds to the shape of the already determined interior solutions $n=0$ to 6, see Table 1 of [15]. Further note, that a general formula for the coefficients $A_{n,\,n-2N}$ is not yet known and, presently, a Bonet-like recursion described by (23) is to be used to find a particular shape of a generalized Legendre function of degree n from the lower-degree functions. The formula (43) is, nevertheless, similar to the expresion for the standard Legendre polynomials $P_n(x)$ derived from the Rodriguez formula, i.e.,

$$P_n(x) = \sum_{L=0}^{\left[\frac{n}{2}\right]} x^{n-2L} \frac{1}{2^n} (-1)^L \binom{n}{L} \binom{2n-2L}{n} = \sum_{L=0}^{\left[\frac{n}{2}\right]} x^{n-2L} a_{n,\, n-2L} \tag{44}$$

(according to [17], Equation 3.133). The formula for the coefficients $a_{n,\, n-2L}$ is thus known for this case:

$$a_{n,\, n-2L} \equiv \frac{1}{2^n} (-1)^L \binom{n}{L} \binom{2n-2L}{n} \tag{45}$$

The interior harmonic function in the spherical coordinates is then

$$v_r^n P_n(x) = v_r^n \sum_{L=0}^{\left[\frac{n}{2}\right]} x^{n-2L} a_{n,\, n-2L} \tag{46}$$

The individual interior solution of n-th degree of the Laplace equation in the SOS coordinates is, on the other hand, proportional to

$$R^n P_n^{SI}(s) = R^n \sum_{N=0}^{[n/2]} s^{n-2N} A_{n,\, n-2N} = \sum_{N=0}^{[n/2]} R^n s^{n-2N} A_{n,\, n-2N} \tag{47}$$

We now insert, for $R^n s^{n-2N}$, the relation according to (42) and get

$$\begin{aligned} R^n P_n^{SI}(s) &= \sum_{N=0}^{[n/2]} A_{n,\, n-2N} (1+\mu)^{n-2N} v_r^n \sum_{k=0}^{N} \binom{N}{k} \mu^k (\sin\chi)^{2k+n-2N} \\ &= v_r^n \sum_{N=0}^{[n/2]} A_{n,\, n-2N} (1+\mu)^{n-2N} \sum_{k=0}^{N} \binom{N}{k} \mu^k (\sin\chi)^{2k+n-2N} \end{aligned} \tag{48}$$

Now, substitute k according to the relation $L = N - k$:

$$R^n P_n^{SI}(s) = v_r^n \sum_{N=0}^{[n/2]} A_{n,\, n-2N} (1+\mu)^{n-2N} \sum_{L=N}^{0} \binom{N}{N-L} \mu^{N-L} (\sin\chi)^{n-2L} \tag{49}$$

Further, change the direction of summation in the inner sum and take into account that $\binom{N}{N-L} = \binom{N}{L}$. Then

$$R^n P_n^{SI}(s) = v_r^n \sum_{N=0}^{[n/2]} \sum_{L=0}^{N} A_{n,\, n-2N} (1+\mu)^{n-2N} \binom{N}{L} \mu^{N-L} (\sin\chi)^{n-2L} \tag{50}$$

As the binomial coefficient would be zero for $L > N$ (calling negative number in binomial coefficient results in zero), we can expand counting in the inner sum up from N to $[n/2]$ without a change of the result:

$$R^n P_n^{SI}(s) = v_r^n \sum_{N=0}^{[n/2]} \sum_{L=0}^{[n/2]} A_{n,\, n-2N} (1+\mu)^{n-2N} \binom{N}{L} \mu^{N-L} (\sin\chi)^{n-2L} \tag{51}$$

Then, we can exchange the order of the two sums,

$$R^n P_n^{SI}(s) = v_r^n \sum_{L=0}^{[n/2]} \sum_{N=0}^{[n/2]} A_{n,\, n-2N} (1+\mu)^{n-2N} \binom{N}{L} \mu^{N-L} (\sin\chi)^{n-2L} \tag{52}$$

and put the sine power (depending on L, not N) out of the inner sum:

$$R^n P_n^{SI}(s) = v_r^n \sum_{L=0}^{[n/2]} (\sin\chi)^{n-2L} \sum_{N=0}^{[n/2]} A_{n,\, n-2N} \mu^{N-L} (1+\mu)^{n-2N} \binom{N}{L} \tag{53}$$

Equipped with the above derived relations (46) and (53), we can declare a significant statement which is to be confirmed later as output of the following analysis. Particularly, that the form of the individual Laplace equation interior solution of the first kind in the SOS coordinates is physically the same as the form of the individual Laplace equation solution of the first kind in the spherical coordinates. If this assumption is valid, then

$$R^n P_n^{SI}(s) = v_r^n P_n(\sin\chi), \quad n \geq 0 \tag{54}$$

i.e., that the solid harmonics of the first kind (the right side of (54)) and the SOS harmonics (the left side of (54)) are, each individually, equivalent for the interior space. That means that the form of the individual Laplace equation solutions of the first kind in the SOS coordinates, and in the spherical coordinates are physically the same (although expressed in different coordinates). They are not a linear combination of the others, but they are equal to each other one by one (Note, that such rule is not the case when the solid harmonics are compared for the solutions in the spherical coordinates and in the well known confocal oblate spheroidal coordinates).

The right-hand side of (54) is given by (46), whereas the left-hand side by (53). By comparison, we can see that the inner sum in (53) would be thus equal to the coefficients of the standard Legendre polynomials $a_{n,\, n-2L}$, see (45). Then, we get a relation between the coefficients of the generalized Legendre polynomials and the Legendre polynomials in the form

$$\sum_{N=0}^{[n/2]} A_{n,\, n-2N} \mu^{N-L} (1+\mu)^{n-2N} \binom{N}{L} = \frac{1}{2^n} (-1)^L \binom{n}{L} \binom{2n-2L}{n} = a_{n,\, n-2L} \tag{55}$$

As the binomial coefficient on the left side is non-zero only when $N \geq L$, we restrict the sum range:

$$\sum_{N=L}^{[n/2]} A_{n,\, n-2N} \mu^{N-L} (1+\mu)^{n-2N} \binom{N}{L} = \frac{1}{2^n} (-1)^L \binom{n}{L} \binom{2n-2L}{n} = a_{n,\, n-2L} \tag{56}$$

Note, that the right-hand side is independent of μ. Then, if our assumption holds, the terms containing μ need to cancel mutually on the left side. Using (56), it can be checked with the already found coefficients $A_{n,\,n-2N}$ for $n = 0$ to 6 whether we can obtain the coefficients of the standard Legendre polynomials $a_{n,\,n-2L}$ from the coefficients of the generalized Legendre polynomials $A_{n,\,n-2N}$. In the positive case, our assumption that the form of the individual Laplace equation solution of the first kind in SOS is physically the same as the form of the individual Laplace equation solution of the first kind in spherical coordinates would have a strong support.

The detailed check of the validity of (56) for n up to 6 is given in Supplement 2 with a positive output: its validity is confirmed. It indicates (although it is not proven for all cases) that the relation (53) could hold generally for any $n \geq 0$, and the solid harmonics are thus generally the same as the SOS harmonics of the first kind. Note, that such an equivalence is not the case for the confocal oblate spheroidal (COS) coordinates harmonics, which differ physically from the solid harmonics in the spherical coordinates: the COS harmonics can be obtained only as a linear combination of several spherical harmonics [21].

Note also that, although the solid harmonics are solutions of the Laplace equation in the spherical as well as in the SOS coordinates, the spherical harmonics (i.e., the solutions of the angular part of the separated Laplace equation in spherical coordinates) are not the solutions of the angular part of the separated Laplace equation in the SOS coordinates (20). The generalized Legendre functions have to be used instead.

We will use the assumption (54) in what follows for the determination of the exterior solution of the Laplace equation in the SOS coordinates. If the assumption holds, the final check with the model function assembled in this way and inserted into the equation has to lead to the fulfillment of the equation regardless of the value of n.

3.3 *Expressing generalized legendre polynomials as sum of monomials*

For testing of a model solution of the Laplace equation, it would be of large advantage to have generalized Legendre polynomials expressed as a sum of monomials with known coefficients. As a continuation of the analysis from the previous sub-section, the following formula for the coefficients of the generalized Legendre polynomials is derived (see Supplement 3, relation (S3.66)):

$$A_{n,\,n-2N} = \frac{1}{(1+\mu)^{n-2N}} \frac{1}{2^n} (-1)^N \binom{n}{N} \sum_{j=0}^{\left[\frac{n}{2}\right]-N} \binom{n-N}{j} \binom{2(n-N-j)}{n} \mu^j \tag{57}$$

The formula is obtained as an estimated generalization of several low-index coefficients. If this formula is later successfully used in the Laplace equation solution, it would confirm its general validity.

An arbitrary generalized Legendre polynomial of degree n can be, therefore, written as a sum of monomials without a need of any reccurence relation, similarly as the Rodriguez formula is used for the expression of the standard Legendre polynomials as a sum of monomials. Such a formula is (using (43) and (57)):

$$P_n^{SI}(s) = \frac{1}{2^n} \sum_{N=0}^{\left[\frac{n}{2}\right]} s^{n-2N} \frac{(-1)^N}{(1+\mu)^{n-2N}} \binom{n}{N} \sum_{j=0}^{\left[\frac{n}{2}\right]-N} \binom{n-N}{j} \binom{2(n-N-j)}{n} \mu^j \tag{58}$$

For $\mu = 0$, the relation (58) reduces to (44).

4. The exterior solution

4.1 *Model for the exterior solution*

In the previous section, it was strongly indicated that the solid harmonics and the SOS harmonics of the first kind in the interior are physically the same (see (54)). The present model for the exterior solution supposes that the same scheme is valid for the exterior solution, i.e.,

$$R^{-1-n} F_n^{SE}(s) = v_r^{-1-n} P_n(\sin\chi), \qquad n \geq 0 \tag{59}$$

where $n \geq 0$ clearly means negative exponents of R (and thus non-diverging functions on the left side for R approaching infinity), and $F_n^{SE}(s)$ is the foreseen exterior solution of the angular part of the Laplace Equation (20). The right side of (59) represents the classical solution of the Laplace equation in the exterior space in the spherical coordinates (the non-polar ones), whereas we use the still unknown function F_n^{SE} on the left side as a prospective solution of the angular part of the Laplace equation in the SOS coordinates, which is to be found. This function is not necessarily a polynomial. This approach also means that we expect separability of the exterior space solutions, similarly as was previously expected for the interior solution.

By multiplying (59) by v_r^{2n+1}, we get

$$v_r^{2n+1} R^{-1-n} F_n^{SE}(s) = v_r^n P_n(x) \tag{60}$$

We know (see (54)) that the right side of (60) is equal to $R^n P_n^{SI}(s)$. Thus

$$v_r^{2n+1} R^{-1-n} F_n^{SE}(s) = R^n P_n^{SI}(s) \tag{61}$$

Then

$$F_n^{SE}(s) = \frac{R^n}{v_r^{2n+1} R^{-1-n}} P_n^{SI}(s) = \frac{R^{2n+1}}{v_r^{2n+1}} P_n^{SI}(s) \tag{62}$$

When (25) is inserted into (62), we obtain for the angular part of the exterior solution in the SOS case

$$F_n^{SE}(s) = \frac{R^{2n+1}}{\left(\frac{R}{1+\mu}\sqrt{(1+\mu)^2 - \mu s^2}\right)^{2n+1}} P_n^{SI}(s) = \frac{(1+\mu)^{2n+1}}{\left(\sqrt{(1+\mu)^2 - \mu s^2}\right)^{2n+1}} P_n^{SI}(s) \tag{63}$$

It can be seen that the exterior solution model is based on the generalized Legendre polynomial but itself it is not a polynomial in s (except the case $\mu = 0$). It is a more general function.

4.2 *Proof for the exterior solution*

Further, we have to test if our assumptions are correct, i.e., whether (63) is in reality the solution of the angular part of the Laplace Equation (20) where the separation constant K_d is negative, particularly $K_d = -1-n$, $n \geq 0$. It would also confirm that all the assumptions leading to this solution model were correct. The proof is to be done generally, i.e., for any non-negative n, leading thus to the exterior space solution, see (59), with the negative exponents of the radial part (19) in such a case.

The proof strategy is simple: insertion of (63) into the angular part of the Laplace Equation (20) with plugged-in the polynomial $P_n^{SI}(s)$ according to (58). Nevertheless, the detailed performance of the proof is rather lengthy, and it is thus given in Supplement 4. It is finally proved in the Supplement that the model function (63) is an exterior space solution of the angular part of the Laplace equation in the SOS coordinates for any non-negative n. Therefore, according to (59),

$$R^{-1-n}\frac{(1+\mu)^{2n+1}}{\left(\sqrt{(1+\mu)^2-\mu s^2}\right)^{2n+1}}P_n^{SI}(s)=v_r^{-1-n}P_n(\sin\chi) \tag{64}$$

Then, also the assumption that the physical form of the SOS solution of the Laplace equation is the same as the solution in the spherical coordinates is confirmed.

5. The complete solution of the azimuthally symmetric Laplace equation and simple example of its use

The full Laplace equation solution of the first kind (both in the exterior as well as in the interior space) is thus

$$\begin{aligned} V(R,\,v)=V(R,\,s)&=\sum_{n=0}^{\infty}\left[a_{nI}R^nP_n^{SI}(s)+a_{nE}R^{-1-n}\frac{(1+\mu)^{2n+1}}{\left(\sqrt{(1+\mu)^2-\mu s^2}\right)^{2n+1}}P_n^{SI}(s)\right]\\ &=\sum_{n=0}^{\infty}\left[a_{nI}R^n+a_{nE}R^{-n-1}\frac{(1+\mu)^{2n+1}}{\left(\sqrt{(1+\mu)^2-\mu s^2}\right)^{2n+1}}\right]P_n^{SI}(s) \end{aligned} \tag{65}$$

For the exterior space, all the a_{nI} coefficients are zero (except perhaps a_{0I}, depending on the level to which the function V is set in infinity), while-for the interior space-all a_{nE} are equal to zero. In this relation, the generalized Legendre functions of the first kind are given by (58).

To demonstrate a simple application of the found solution, consider an example from the field of electrostatics. We intend to find the electrostatic potential outside an oblate spheroid with the equatorial radius b whose surface is held at a potential given in the SOS coordinates (the reference surface equatorial radius $R_0=b$) by

$$V(R_0,\,v)=V(b,\,v)=V_0+C\frac{1}{\sqrt{1-\dfrac{\mu}{1+\mu}\sin^2 v}} \tag{66}$$

where V_0 and C are constants and v is the parametric latitude. The relation (66) is the boundary condition of the problem.

The full Laplace equation solution of the first kind (65) applied in the exterior is to be used to find the exterior potential. As the solution cannot diverge in the infinity, all coefficients a_{nI} except a_{0I} have to be zero:

$$V(R,\,\nu)=V(R,\,s)=a_{0I}+\sum_{n=0}^{\infty}\left[a_{nE}R^{-1-n}\frac{(1+\mu)^{2n+1}}{\left(\sqrt{(1+\mu)^2-\mu s^2}\right)^{2n+1}}P_n^{SI}(s)\right] \tag{67}$$

It was previously found (see [15], Equation (63)) that the function s on the reference surface is

$$s|_{R=R_0}=\sqrt{1+\mu}\sin\nu \tag{68}$$

Considering this relation and the form of the generalized Legendre polynomials (see (22) or (58)), it can be concluded that the solution derived from (67) can have no term in the numerator contain any power of s. Then, the only non-zero term in the sum in (67) can be the one containing the zero-degree generalized Legendre polynomial, i.e. the one with $n=0$. The solution (67) in such a case simplifies to

$$V(R,\,\nu)=V(R,\,s)=a_{0I}+a_{0E}\frac{1}{R\sqrt{1-\dfrac{\mu}{(1+\mu)^2}s^2}} \tag{69}$$

This solution, used at the boundary and compared with the boundary condition (66), provides the values for the parameters a_{0I} and a_{0E}:

$$a_{0I}=V_0,\;\; a_{0E}=CR_0=Cb \tag{70}$$

Therefore, the particular solution is finally

$$V(R,\,\nu)=V(R,\,s)=V_0+C\frac{b}{R}\frac{1}{\sqrt{1-\dfrac{\mu}{(1+\mu)^2}s^2}} \tag{71}$$

Note, that the solution in this case conains only one term plus a constant. Other boundary conditions would lead to a more complex solutions derived from the general one given by (65), possibly also with infinite number of terms.

6. Conclusions

The exterior solution of the azimuthally symmetric Laplace equation in the SOS coordinates was found. The approach leading to this result was of different nature than the one used previously for the interior solution determination [15] and proved the exterior solutions of the first kind for any degree of generalized Legendre polynomial.

In the steps leading to the solution, important formulae were derived which hold for the SOS algebra. Various forms of the Laplace operator in the SOS coordinates in the azimuthally symmetric case were shown. General transformation between two different SOS coordinate systems was derived (see (37)). Its special case-the transformation between the SOS coordinate system and the spherical coordinate system (see (38))-was used to prove that the SOS harmonics are physically the same as the solid harmonics in the system of spherical coordinates, only expressed in a different form. Further, the formula expressing any generalized Legendre polynomial of degree n as a finite sum of monomials (i.e., without a need of a recurrence relation) is derived (see (58) with coefficients given by (57)).

The final proof of the validity of the solution justifies the assumptions made during the derivation. All the abovementioned results are novel and were not-to the author's knowledge-yet published. Finally, the general solution of the azimuthally symmetric Laplace equation in the SOS coordinates, both in the interior as well as in the exterior space, is presented.

Note however, that while the proof of the equivalence of solid harmonics (see (59) and (64)) was given for the exterior space harmonics generally, the equivalence for the interior solid harmonics (see (54)) was proven only up to the degree $n = 6$. To carry it out generally also in the interior, the angular part of the Laplace Equation (20) needs to be tested with generalized Legendre functions $P_n^{SI}(s)$ expressed as a sum of monomials (58) in a similar way as done in Supplement 4 for the exterior space. Such a general proof is, nevertheless, left for future derivations. Also, the second kind exterior solution of the Laplace equation is not derived in the present text. It can be hypothesized that it is of the same form as (65) which is valid with the generalized Legendre functions of the first kind, only with $P_n^{SI}(s)$ substituted by $Q_n^{SI}(s)$, i.e. the second kind generalized Legendre functions. Nevertheless, the functions $Q_n^{SI}(s)$ still need to be derived generally, similarly as done for $P_n^{SI}(s)$ in (58), and tested in the Laplace equation.

The found relations for the SOS coordinate system can have application in several fields of physics where fields inside or around objects of oblate spheroidal shape are investigated, including geophysics, astrophysics, electromagnetism, fluid flow, electrostatics and solid state physics (e.g., ferroic inclusions).

Data availability

This article deals with the derivation of theoretical relations within the similar oblate spheroidal coordinate system. There are thus no data sets to be disclosed.

Acknowledgments

The author acknowledges support from the long-term conceptual development project RVO 61389005 of the Nuclear Physics Institute of the Czech Academy of Sciences, and from the OP JAK project of MEYS Nr. CZ.02.01.01/00/22_008/0004591. The author thanks the teacher Ivana Malá form "ZŠ Strossmayerovo náměstí" who attracted his interest to mathematics.

Conflict of interest

The author declares no conflict of interests.

References

[1] Moon P, Spencer DE. *Foundations of Electrodynamics*. Princeton, NJ, USA: D. Van Nostrand Co.; 1960.

[2] Moon P, Spencer DE. *Field Theory Handbook (Including Coordinate Systems, Differential Equations and Their Solutions)*. Berlin, Germany: Springer-Verlag; 1961. p.31.

[3] White AA, Staniforth A, Wood N. Spheroidal coordinate systems for modelling global atmospheres. *Quarterly Journal of the Royal Meteorological Society*. 2008; 134: 261-270. Available from: https://doi.org/10.1002/qj.208.

[4] Strunz P. Analytical solution of similar oblate spheroidal coordinate system. *Celestial Mechanics and Dynamical Astronomy*. 2022; 134: 51. Available from: https://doi.org/10.1007/s10569-022-10099-z.

[5] Strunz P. Unit vectors for similar oblate spheroidal coordinates and vector transformation. *Zeitschrift Für Angewandte Mathematik und Physik*. 2023; 74: 172. Available from: https://doi.org/10.1007/s00033-023-02058-x.

[6] Komarov IV, Ponomarev LI, Slavjanov SJ. *Sferoidalnye u Kulonovskie Sferoidalnye Funkcii [Spheroidal and Kulonovsky Spheroidal Functions]*. Moskva, Russia: Nauka; 1976.

[7] Darinskii B, Sidorkin A, Sigov A, Popravko N. Influence of depolarizing fields and screening effects on phase transitions in ferroelectric composites. *Materials*. 2018; 11: 85.

[8] Huang C, Wang X, Zhao J. Large electrocaloric effects induced by multidomain-to-monodomain transition in ferroelectrics with electrical inclusions. *Frontiers in Energy Research*. 2023; 11: 1257567. Available from: https://doi.org/10.3389/fenrg.2023.1257567.

[9] Vafeas P, Papadopoulos PK, Lesselier D. Electromagnetic low-frequency dipolar excitation of two metal spheres in a conductive medium. *Journal of Applied Mathematics*. 2012; 2012: 628261. Available from: https://doi.org/10.1155/2012/628261.

[10] Vafeas P, Bakalis P, Papadopoulos PK. Effect of the magnetic field on the ferrofluid flow in a curved cylindrical annular duct. *Physics of Fluids*. 2019; 31: 117105. Available from: https://doi.org/10.1063/1.5122708.

[11] Vafeas P. Low-frequency dipolar electromagnetic scattering by a solid ellipsoid in lossless environment. *Studies in Applied Mathematics*. 2020; 2020: 12318. Available from: https://doi.org/10.1111/sapm.12318.

[12] Dassios G, Vafeas P. On the spheroidal semiseparation for Stokes flow. *Research Letters in Physics*. 2008; 2008: 135289. Available from: https://doi.org/10.1155/2008/135289.

[13] Schmidt M. A model of the distribution of mass in the galactic system. *Bulletin of the Astronomical Institute of the Netherlands*. 1956; 13: 15-41.

[14] Criss RE, Hofmeister AM. Density profiles of 51 galaxies from parameter-free inverse models of their measured rotation curves. *Galaxies*. 2020; 8(1): 19. Available from: https://doi.org/10.3390/galaxies8010019.

[15] Strunz P. Interior solution of azimuthally symmetric case of Laplace equation in orthogonal similar oblate spheroidal coordinates. *The European Physical Journal Plus*. 2024; 139: 409. Available from: https://doi.org/10.1140/epjp/s13360-024-05181-4.

[16] Pólya G, Szegö G. *Problems and Theorems in Analysis: Series, Integral Calculus, Theory of Functions*. New York, NY, USA: Springer-Verlag; 1977.

[17] Gould HW. Combinatorial identities: A standardized set of tables listing 500 binomial coefficient summations. *The Fibonacci Quarterly*. 1972; 10(6): 662-662.

[18] Chu W. Reciprocal formulae on binomial convolutions of Hagen-Rothe type. *Bollettino dell'Unione Matematica Italiana, Serie 9*. 2013; 6(3): 591-605.

[19] Seiden S. *Theoretical Computer Science Cheat Sheet v2.02*. 1994. Available from: http://www.tug.org/texshowcase/cheat.pdf [Accessed 19th May 2024].

[20] Gould HW. Some generalizations of Vandermonde's convolution. *The American Mathematical Monthly*. 1956; 63(2): 84-91. Available from: https://doi.org/10.1080/00029890.1956.11988763.

[21] Arnaoudov I, Venkov G. Relations between spheroidal and spherical harmonics. *Doklady Bolgarskoi Akademiya Nauk*. 2010; 63: 971-978.

Appendix A
The Laplacian in s-terms

When we want to express the Laplacian in the SOS coordinates in terms of R and s, we need several relations, first of all derivatives. With the help of the Supplement C of the interior-solution paper ([15], Equations (C25) and (C28)), we know the derivatives of the function $F(W)$ with respect to W expressed in terms of F derivatives with respect to s:

$$\frac{dF(W)}{dW} = \frac{dF(s)}{ds} \frac{\left[(1+\mu)-s^2\right]^{\frac{\mu+3}{2}}}{(1+\mu)^{\frac{\mu}{2}}\left[(1+\mu)+\mu s^2\right]} \tag{A1}$$

and

$$\frac{d^2F(W)}{dW^{\,2}} = \frac{d^2F(s)}{ds^2} \frac{\left[(1+\mu)-s^2\right]^{(\mu+3)}}{(1+\mu)^{\mu}\left[(1+\mu)+\mu s^2\right]^2} - \frac{dF(s)}{ds} s \frac{(1+\mu)^{(1-\mu)}\left[(1+\mu)-s^2\right]^{(\mu+2)}\left[3(1+\mu)+\mu s^2\right]}{\left[(1+\mu)+\mu s^2\right]^3} \tag{A2}$$

Also (see Equation (C21) of the Supplement C of [15]), W expressed in s terms is

$$W = \sqrt{\frac{\dfrac{1}{1+\mu}\ s^2}{\left(1\ -\ \dfrac{1}{1+\mu}\ s^2\right)^{1+\mu}}} = \sqrt{\frac{(1+\mu)^{\mu}s^2}{\left[(1+\mu)\ -\ s^2\right]^{1+\mu}}} \tag{A3}$$

Further, $\frac{1}{h_R^2}$ can be expressed according to Equation (A19) from the Appendix A of [15] with the result

$$\frac{1}{h_R^2} = \left(\frac{\mu}{1+\mu}s^2+1\right) \tag{A4}$$

Since further (see [15])

$$f_C^2 = 1 - f_S^2, \quad f_S^2 = \frac{1+\mu}{\mu}\left(1-h_R^2\right), \quad f_C^2 = \frac{(1+\mu)h_R^2-1}{\mu} \tag{A5}$$

then (14) can be written as

$$\Delta V = \left(\frac{\mu}{1+\mu}s^2+1\right)\frac{d^2r(R)}{dR^2}F(s)+\left[\frac{1}{R}\frac{dr(R)}{dR}(\mu+3)F(s)-\frac{1}{R}\frac{dr(R)}{dR}\left(\frac{\mu}{1+\mu}s^2+1\right)F(s)\right]$$

$$+\frac{1}{R^2}r(R)\left[(\mu-2)\mu\left(\frac{\mu}{1+\mu}s^2+1\right)+(\mu+3)\mu+\frac{(1+\mu)^2}{\frac{1-f_S^2}{h_R^2}}\right]\sqrt{\frac{(1+\mu)^{\mu}s^2}{[(1+\mu)\ -\ s^2]^{1+\mu}}}\frac{dF(W)}{dW}$$

$$+2\frac{1}{R}\frac{dr(R)}{dR}\mu\sqrt{\frac{(1+\mu)^{\mu}s^2}{[(1+\mu)\ -\ s^2]^{1+\mu}}}\left(\frac{\mu}{1+\mu}s^2+1\right)\frac{dF(W)}{dW} \tag{A6}$$

$$+\frac{1}{R^2}r(R)\left(\mu^2\left(\frac{\mu}{1+\mu}s^2+1\right)+\frac{(1+\mu)^2}{\frac{(1+\mu)h_R^2-1}{\mu}s^2}\right)\frac{(1+\mu)^{\mu}s^2}{[(1+\mu)\ -\ s^2]^{1+\mu}}\frac{d^2F(W)}{dW^2}$$

When inserting the derivatives (C25) and (C28), and after simplification, we arrive at

$$\Delta V = \left(\frac{\mu}{1+\mu}s^2+1\right)\frac{d^2r(R)}{dR^2}F(s)+\left[(\mu+2)-\frac{\mu}{1+\mu}s^2\right]\frac{1}{R}\frac{dr(R)}{dR}F(s)$$

$$+\frac{1}{R^2}r(R)\left[(\mu-2)\mu\left(\frac{\mu}{1+\mu}s^2+1\right)+(\mu+3)\mu+\frac{(1+\mu)^2}{\frac{\mu s^2+(1+\mu)\left(1-s^2\right)}{1+\mu}}\right]$$

$$s\frac{\left[(1+\mu)-s^2\right]}{\left[(1+\mu)+\mu s^2\right]}\frac{dF(s)}{ds}+2\frac{1}{R}\frac{dr(R)}{dR}s\frac{\mu}{1+\mu}\left[(1+\mu)-s^2\right]\frac{dF(s)}{ds} \tag{A7}$$

$$+\mu\left(\mu\frac{1}{1+\mu}\left((1+\mu)+\mu s^2\right)+\frac{(1+\mu)^2}{\frac{(1+\mu)s^2}{\frac{1}{1+\mu}\left(\mu s^2+(1+\mu)\right)}-s^2}\right)$$

$$\left[\frac{\left[(1+\mu)-s^2\right]^2}{\left[(1+\mu)+\mu s^2\right]^2}s^2\frac{d^2F(s)}{ds^2}-\frac{(1+\mu)\left[(1+\mu)-s^2\right]\left[3(1+\mu)+\mu s^2\right]}{\left[(1+\mu)+\mu s^2\right]^3}s^3\frac{dF(s)}{ds}\right]\frac{1}{R^2}r(R)$$

Simplification of some factors containing μ and s leads to

$$\Delta V = \left(\frac{\mu}{1+\mu}s^2+1\right)\frac{d^2r(R)}{dR^2}F(s)+\left[(\mu+2)-\frac{\mu}{1+\mu}s^2\right]\frac{1}{R}\frac{dr(R)}{dR}F(s)$$

$$+\frac{1}{R^2}r(R)\left[\frac{\left(\frac{\mu\mu\mu}{1+\mu}s^2-2\frac{\mu\mu}{1+\mu}s^2+2\mu\mu+\mu\right)\left[(1+\mu)-s^2\right]+(1+\mu)^3}{\left[(1+\mu)+\mu s^2\right]}\right]s\frac{dF(s)}{ds}$$

$$+2\frac{\mu}{1+\mu}\left[(1+\mu)-s^2\right]s\frac{1}{R}\frac{dr(R)}{dR}\frac{dF(s)}{ds} \tag{A8}$$

$$+\left(\frac{\mu^2s^2\left((1+\mu)-s^2\right)+(1+\mu)^3}{(1+\mu)}\right)$$

$$\left[\frac{(1+\mu)-s^2}{(1+\mu)+\mu s^2}\frac{d^2F(s)}{ds^2}-\frac{(1+\mu)\left[3(1+\mu)+\mu s^2\right]}{\left[(1+\mu)+\mu s^2\right]^2}s\frac{dF(s)}{ds}\right]\frac{1}{R^2}r(R)$$

Then, the identical derivatives are put together,

$$\Delta V = \left(\frac{\mu}{1+\mu}s^2+1\right)\frac{d^2r(R)}{dR^2}F(s)+\left[(\mu+2)-\frac{\mu}{1+\mu}s^2\right]\frac{1}{R}\frac{dr(R)}{dR}F(s)$$

$$+2\frac{\mu}{1+\mu}\left[(1+\mu)-s^2\right]s\frac{1}{R}\frac{dr(R)}{dR}\frac{dF(s)}{ds}-\left(\frac{\mu^2s^2\left((1+\mu)-s^2\right)+(1+\mu)^3}{(1+\mu)}\right)$$

$$\frac{(1+\mu)\left[3(1+\mu)+\mu s^2\right]}{\left[(1+\mu)+\mu s^2\right]^2}s\frac{1}{R^2}r(R)\frac{dF(s)}{ds} \tag{A9}$$

$$+\left[\frac{\left(\frac{\mu\mu\mu}{1+\mu}s^2-2\frac{\mu\mu}{1+\mu}s^2+2\mu\mu+\mu\right)\left[(1+\mu)-s^2\right]+(1+\mu)^3}{\left[(1+\mu)+\mu s^2\right]}\right]s\frac{1}{R^2}r(R)\frac{dF(s)}{ds}$$

$$+\frac{\mu^2s^2\left((1+\mu)-s^2\right)+(1+\mu)^3}{(1+\mu)}\frac{(1+\mu)-s^2}{(1+\mu)+\mu s^2}\frac{1}{R^2}r(R)\frac{d^2F(s)}{ds^2}$$

their pre-factors are joined, and the relation is simplified:

$$
\begin{aligned}
\Delta V = {} & \frac{1}{1+\mu}\left((1+\mu)+\mu s^2\right) F(s) \frac{d^2 r(R)}{dR^2} + \left[(\mu+2) - \frac{\mu}{1+\mu} s^2\right] F(s) \frac{1}{R} \frac{dr(R)}{dR} \\
& + 2\frac{\mu}{1+\mu}\left[(1+\mu) - s^2\right] s \frac{dF(s)}{ds} \frac{1}{R} \frac{dr(R)}{dR} + \frac{3\mu s^2 - \left(3\mu^2 + 5\mu + 2\right)}{1+\mu} s \frac{dF(s)}{ds} \frac{r(R)}{R^2} \\
& + \frac{\left[(1+\mu) - s^2\right]\left[(1+\mu)^2 - \mu s^2\right]}{1+\mu} \frac{d^2 F(s)}{ds^2} \frac{r(R)}{R^2}
\end{aligned}
\tag{A10}
$$

The above relation represents the Laplacian in the SOS coordinates expressed in s-terms instead of W-terms.