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Research Article

Infinite-Horizon Probability of Ruin for a Variable-Memory Counting Process (Hawkes Process)

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Abstract: The reserve $R(t)$ of an insurance company denotes the accumulated capital up to time ttt throughout its operational span. While this reserve can be modeled using stochastic processes, the endeavor remains notably intricate. The mathematical complexity presents a considerable obstacle for researchers within this domain. Prior research has predominantly focused on reserve models constructed from Markovian processes. In this manuscript, our attention shifts to risk models derived from non-Markovian processes, particularly those with claim arrivals governed by Hawkes processes. Before delving into the core subject matter, we provide an extensive review of the literature on counting processes with variable memory. This includes a thorough exploration of Hawkes processes, Gerber-Shiu functions, integro-differential equations, and Laplace transforms. These elements are crucial for computing the probability of ruin over an infinite time horizon, especially in scenarios where interdependence exists between inter-claim times. Additionally, we present the requisite mathematical tools essential for comprehending the contents of this article, specifically tailored for individuals with a foundational understanding of actuarial science. Furthermore, we have successfully determined the probability of ruin, marking a significant milestone in our investigation.

Keywords: risk models, hawkes processes, counting processes, probability of ruin

MSC: 91B05, 62H05

1. Introduction

The mathematical theory of insurance provides a rigorous framework for understanding the risks inherent in insurance activities and for developing effective management strategies. Indeed, a better understanding of the underlying mechanisms of claim occurrence enables insurers to more accurately assess their risks and optimize their reserves. Hawkes processes, a class of self-exciting counting processes, have proven particularly relevant for modeling claim arrivals. Unlike Poisson processes, which assume independence between events, Hawkes processes capture the temporal dependence between claims, reflecting the tendency of events to cluster together in time. This characteristic makes them particularly suitable for modeling phenomena such as natural disasters or industrial accidents, where one event can increase the probability of occurrence of other similar events in the near future. The probability of ruin, which corresponds to the

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probability that an insurance company is no longer able to meet its obligations, is a central concept in actuarial science. It depends closely on the reserve model used to represent the evolution of the company's capital. In this paper, we propose a new reserve model based on a variable-memory counting process, inspired by the work of [1, 2].

Our objective is to study the impact of self-excitation on the probability of ruin. To do so, we build upon the work of [3, 4] and [5] who have demonstrated the interest of Hawkes processes in various fields of finance. In particular [5], studied the asymptotic behavior of ruin probabilities in a model where claim arrivals are modeled by a Hawkes process. This study delves into the impact of self-excitation on the probability of ruin for insurance companies. We begin by providing a comprehensive overview of classical methods used to calculate the probability of ruin (subsection 2.1). Next, we introduce our novel reserve model, detailing its underlying assumptions (subsection 2.2). Subsequently, we conduct an in-depth analysis of the law governing the inter-arrival times of claims within our model framework (subsection 2.3). Building on this analysis, we derive the integro-differential equation that describes the probability of ruin over an infinite time horizon. This equation is then tackled using the Laplace transform technique (subsection 2.3). Finally, in section 3, we propose a numerical simulation to illustrate and validate our theoretical findings Our results contribute to a better understanding of the impact of self-excitation on the solvency of insurance companies and open up new perspectives for risk management.

2. Preliminaries

Consider a filtered probability space $(\Omega,\ \mathscr{H},\ \{\mathscr{H}_t\},\ P)$, where Ω represents the sample space of all possible outcomes, $\mathscr{H}$ is a σ-algebra of subsets of Ω, $\{\mathscr{H}_t\}$ is a filtration (i.e., an increasing family of sub-σ-algebras of $\mathscr{H}$), and P is a probability measure on $(\Omega,\ \mathscr{H})$. All stochastic processes and random variables considered in this study are adapted to this filtration.

2.1 *Some definitions*

In this section, we review some well-known definitions and properties in the literature of ruined probabilities in insurance.

Definition 1 (Hawkes Process) [6] A Hawkes process $(N(t),\ t>0)$ is a counting process, defined by its conditional intensity $\lambda^*(t)$:

$$\lambda^*(t)=\lambda+\sum_{i=1}^{n}\mu\left(t-t_i\right)$$

where $(t_i)_{1\leq i\leq n}$ is the sequence of dates of occurrence up to t, λ is a real that represents the base intensity and $\mu:\ [0,\ +\infty[\rightarrow [0,\ +\infty[$ is the excitation function .

Definition 2 The time of the risk denoting by τ associated with an initial reserve u, is defined by:

$$\tau=\inf\{t\geq 0;\ R(t)<0\}=\inf\{t\geq 0;\ S(t)>u\} \tag{1}$$

This is the first moment when the reserve process $(R(t))$ becomes negative or, equivalently, the surplus process $(S(t))$ exceeds the u level.

The expected penalty function of Gerber-Shiu, or Gerber-Shiu first appeared in 1998 in the work of [7]. Today, this function is of great use for research. Its analysis remains a central issue in both the field of insurance and finance, a valuable tool not only for studying the probability of ruin, but also for calculating pension and reinsurance premiums, options, etc. It is defined by:

$$\varphi(u) = E\left[e^{-\delta\tau} w\left(R\left(\tau^{-}\right), \, |\mathrm{R}(\tau)|\right) 1_{\tau<\infty} \mid \mathrm{R}(0) = u\right]$$

where:

• τ is the moment of ruin defined by the equation (1);

• τ^{-} is the moment before ruin;

• δ is a force of interest;

• The penalty function $w(x, y)$ is a positive function of the surplus just before the ruin $R\left(\tau^{-}\right)$ and the deficit to ruin $|R(\tau)|$, $\forall x, y \geq 0$.

2.2 *Risk model assumption*

The reserve model is defined by:

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i \tag{2}$$

The surplus process is also defined $S(t)$; $t > 0$ which represents the total loss by:

$$S(t) = u - R(t) = \sum_{i=1}^{N(t)} X_i - ct \tag{3}$$

The ruin assumptions of the new risk model are:

• $u > 0$ is the insurance company's initial reserve. An insurance company has an initial capital, which increases steadily thanks to the contributions of the insured, but decreases from time to time when claims are reimbursed.

• $c > 0$ is the contribution rate or premium. If the premium received by the insurer is linear as a function of time, then ct is the premium received over the time interval $]0;\, t[$.

• The $\{X_i, \, i \in N^*\}$ represent the amount of the company's claims (indemnity) claims or the amount disbursed for the claims and form a series of independent and identically distributed (i.i.d) and processindependent random variables $(N(t)) \cdot F_X$ denotes their common distribution function, f_X their density. $(X_i)_{i\geq 1}$ represents the amount of the i^{eme} loss following an exponential distribution of parameter γ.

• $N(t)$ is a Hawkes process, a continuous variable memory counting process on $[0;\, +\infty[$. In our research $N(t)$ models the number of claims up to time t, density is a function set to R^{+}.

• The $\{T_i, \, i \in N^*\}$ are the arrival times of claims that play an important role in calculating the probability of ruin. It is in this vision that we have dedicated the section 3 especially their probability distribution when the $N(t)$ number of loss follows a Hawkes process.

2.3 *Law of loss arrival time*

The laws of arrival time of a Hawkes process can be established using the intensity function $\lambda(t)$ of the Hawkes process. The intensity function $\lambda(t)$ is a function that describes the infinitesimal probability of an arrival during the time interval $[t, \, t+\Delta t]$, given the history of previous events. The Hawkes process can be interpreted as a non-homogeneous Poisson process. If we note that $w_1, \, w_2, \, \ldots$, the following event dates of the process, we get

$$\lambda^*(t) = \lambda + \sum \mu_{w_{i<t}}(t - w_i)$$

When the event of interest occurs, the intensity of the process is changed by the μ function. In a way, this function can be interpreted as a response to the leap in the process. Its introduction into the expression of intensity makes it possible to extend the possibility of modelling by point processes to a large number of random phenomena. The μ function can be increasing or decreasing. Hawkes considers exponential decay by looking at functions μ form $\forall t > 0$:

$$\mu(t) = \sum_{j=1}^{k} \alpha_j e^{-\beta_j t}$$

With

$$\sum_{j=1}^{k} \alpha_j / \beta_j < 1$$

We consider a Hawkes $N(t)$ counting process, representing the cummul of the number of events up to time t for which:

$$\lambda = \frac{E[dN(t)]}{dt}$$

is a constant. The density function of $N(t)$ is defined in [1] by:

$$f(w) = \frac{1}{2\pi}\left[\lambda + \int_{-\infty}^{+\infty} e^{-itw}\mu(t)dt\right]$$

This simplified expression is given by the equation (4). In this particular case, an analytic expression of the density function $f(w)$ is obtained by [1] and is written:

$$f(w) = \frac{\lambda}{2\pi}\left[1 + \frac{\alpha\lambda(2\beta - \alpha)}{(\beta - \alpha)^2 + w^2}\right] \tag{4}$$

This equation (4) represents the density function of the variable $(w_i)_{i\geq 1}$ of the loss arrival times. In the following, we use it to determine the integro-differential equation associated with the risk model.

3. Main result

Consider a filtered probability space $(\Omega, H, \{H_t\}, P)$.

This section leverages the groundwork established in [8]. We begin by presenting Theorem 1, which provides the analytical expression for the integro-differential equation that serves as the cornerstone of this investigation. This equation, which we have previously derived, governs the dynamics of the system. Subsequently, we employ the Hawkes process, specifically tailored to model the inter-arrival times of disasters, to determine the probability of ruin over an infinite time horizon.

Theorem 1 The probability of ruin at the infinite horizon $\psi(u)$ is defined by for $u \geq 0$:

$$\psi(u) = \frac{\beta\theta_1 - \theta_2}{\theta_2 + cR_2} e^{-\beta u} + \frac{R_2\theta_1 + \theta_2}{c\left(R_2 + \beta\right)} e^{R_2 u} \tag{5}$$

With

$$\theta_1 = \frac{A(\gamma+\beta) - \gamma\beta K}{\beta\gamma}$$

$$\theta_2 = \frac{(\gamma+\beta)[A(\gamma+\beta) - \gamma\beta K]}{\beta\gamma} + \beta K - A$$

$$R_2 = \frac{-\gamma}{Q+1}$$

And

$$A = \frac{\lambda\gamma}{2\pi}\left(1 + \frac{\alpha\lambda(2\beta - \alpha)}{\beta - \alpha}\right)$$

$$K = \frac{\lambda^2 c\gamma^2 \alpha(2\beta - \alpha)}{2\pi(\beta - \alpha)}$$

$$Q = \frac{\lambda^2 c^2 \gamma\alpha(2\beta - \alpha)}{2\pi(\beta - \alpha)}$$

To prove Theorem 1, we establish the following results.

Lemma 1 For any $u \geq 0$, the integro-differential equation associated with the probability of ruin at the infinite horizon has the following expression:

$$\varphi'(u) = \frac{\delta}{c}\varphi(u) + \frac{\lambda\gamma}{2\pi c}\left[e^{\frac{\delta}{c}u}\int_0^{+\infty} e^{-\frac{\delta}{c}y}\chi(y-u)\sigma(y)dy - \int_0^{u}\chi(y-u)\sigma(y)dy - \left(1 + \frac{\alpha\lambda(2\beta - \alpha)}{(\beta - \alpha)^2}\right)\right] \tag{6}$$

With

$$\chi(y-u) = \frac{-\alpha\lambda(2\beta - \alpha) \times 2 \times (-(1/c))(((y-u)/c))}{[(\beta - \alpha)^2 + (((y-u)/c))^2]^2}$$

and

$$\sigma(y) = \int_0^y \varphi(y-x)e^{-\gamma x}dx + \int_y^{+\infty} w(y,\ x-y)e^{-\gamma x}dx$$

Proof. Using the Gerber-Shiu penalty function, we have (see [7])

$$\varphi(u) = E\left[e^{-\delta\tau} w\left(R\left(\tau^{-}\right),\ |R(\tau)|\right) 1_{\{\tau<\infty\}} \mid R(0) = u\right]$$

And so

$$\varphi(u) = \int_0^{+\infty}\int_0^{u+ct} e^{-\delta t}\varphi(u+ct-x)dF(x,\ t) + \int_0^{+\infty}\int_{u+ct}^{+\infty} e^{-\delta t} w(u+ct,\ x-u-ct)dF(x,\ t)$$

As a result,

$$\varphi(u) = \int_0^{+\infty}\int_0^{u+ct} e^{-\delta t}\varphi(u+ct-x)f_X(x)f_w(t)dxdt +$$

$$\int_0^{+\infty}\int_{u+ct}^{+\infty} e^{-\delta\delta t} w(u+ct,\ x-u-ct)f_X(x)f_w(t)dxdt$$

Let X follows the exponential distribution of parameter γ and $f_w(t)$ is defined by the equation this gives us:

$$\varphi(u) = \int_0^{+\infty}\int_0^{u+ct} e^{-\delta t}\varphi(u+ct-x)\gamma e^{-\gamma x}\frac{\lambda}{2\pi}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right]dxdt +$$

$$\int_0^{+\infty}\int_{u+ct}^{+\infty} e^{-\delta t} w(u+ct,\ x-u-ct)\gamma e^{-\gamma x}\frac{\lambda}{2\pi}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right]dxdt$$

And

$$\varphi(u) = \frac{\lambda\gamma}{2\pi}\int_0^{+\infty} e^{-\delta t}\left(\int_0^{u+ct}\varphi(u+ct-x)e^{-\gamma x}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right]dx\right)dt$$

$$+\frac{\lambda\gamma}{2\pi}\int_0^{+\infty} e^{-\delta t}\left(\int_{u+ct}^{+\infty} w(u+ct,\ x-u-ct)e^{-\gamma x}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right]dx\right)dt$$

We have

$$\varphi(u)=\frac{\lambda\gamma}{2\pi}\int_0^{+\infty}e^{-\delta t}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right]$$

$$\left(\int_0^{u+ct}\varphi(u+ct-x)e^{-\gamma x}dx+\int_{u+ct}^{+\infty}w(u+ct,\ x-u-ct)e^{-\gamma x}dx\right)dt$$

$$\varphi(u)=\frac{\lambda\gamma}{2\pi}\int_0^{+\infty}e^{-\delta t}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right]\sigma(u+ct)dt$$

With

$$\sigma(u+ct)=\int_0^{u+ct}\varphi(u+ct-x)e^{-\gamma x}dx+\int_{u+ct}^{+\infty}w(u+ct,\ x-u-ct)e^{-\gamma x}dx$$

Ask $y=u+ct$, then $t=((y-u)/c)$. And this implies that $dt=(1/c)dy$. if $t=0$, then $y=u$ and if $t=+\infty$, then $y=+\infty$, So we have:

$$\varphi(u)=\frac{\lambda\gamma}{2\pi c}\int_u^{+\infty}e^{-\delta\left(\frac{y-u}{c}\right)}\left[1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+\left(\frac{y-u}{c}\right)^2}\right]\sigma(y)dy$$

The Gerber-Shiu penalty function $\varphi(u)$ is continuous on R^+ and admits as a derivative $\varphi'(u)$ and using Leibniz's formula we get:

$$\varphi'(u)=\frac{\lambda\gamma}{2\pi c}\left[\int_u^{+\infty}\frac{d}{du}\left(e^{-\delta\left(\frac{y-u}{c}\right)}\left(1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+\left(\frac{y-u}{c}\right)^2}\right)\sigma(y)\right)dy-e^{-\delta(0)}\left(1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+0}\right)\right]$$

$$=\frac{\lambda\gamma}{2\pi c}\int_u^{+\infty}\frac{\delta}{c}e^{-\delta\left(\frac{y-u}{c}\right)}\left(1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+\left(\frac{y-u}{c}\right)^2}\right)\sigma(y)dy$$

$$+\frac{\lambda\gamma}{2\pi c}\int_u^{+\infty}e^{-\delta\left(\frac{y-u}{c}\right)}\left(\frac{-\alpha\lambda(2\beta-\alpha)\times 2\times\left(\frac{-1}{c}\right)\left(\frac{y-u}{c}\right)}{\left[(\beta-\alpha)^2+\left(\frac{y-u}{c}\right)^2\right]^2}\right)\sigma(y)dy-\frac{\lambda\gamma}{2\pi c}\left(1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2}\right)$$

$$=\frac{\delta}{c}\varphi(u)+\frac{\lambda\gamma}{2\pi c}\int_{u}^{+\infty}e^{-\delta\left(\frac{y-u}{c}\right)}\left(\frac{-\alpha\lambda(2\beta-\alpha)\times 2\times\left(\frac{-1}{c}\right)\left(\frac{y-u}{c}\right)}{\left[(\beta-\alpha)^{2}+\left(\frac{y-u}{c}\right)^{2}\right]^{2}}\right)\sigma(y)dy-\frac{\lambda\gamma}{2\pi c}\left(1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^{2}}\right)$$

$$=\frac{\delta}{c}\varphi(u)+\frac{\lambda\gamma}{2\pi c}\int_{0}^{+\infty}e^{-\delta\left(\frac{y-u}{c}\right)}\left(\frac{-\alpha\lambda(2\beta-\alpha)\times 2\times\left(\frac{-1}{c}\right)\left(\frac{y-u}{c}\right)}{\left[(\beta-\alpha)^{2}+\left(\frac{y-u}{c}\right)^{2}\right]^{2}}\right)\sigma(y)dy$$

$$-\frac{\lambda\gamma}{2\pi c}\int_{0}^{u}\left(\frac{-\alpha\lambda(2\beta-\alpha)\times 2\times\left(\frac{-1}{c}\right)\left(\frac{y-u}{c}\right)}{\left[(\beta-\alpha)^{2}+\left(\frac{y-u}{c}\right)^{2}\right]^{2}}\right)\sigma(y)dy-\frac{\lambda\gamma}{2\pi c}\left(1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^{2}}\right)$$

We deduce that

$$\varphi'(u)=\frac{\delta}{c}\varphi(u)+\frac{\lambda\gamma}{2\pi c}e^{\delta\left(\frac{u}{c}\right)}\int_{0}^{+\infty}e^{-\delta\left(\frac{y}{c}\right)}\left(\frac{-\alpha\lambda(2\beta-\alpha)\times 2\times\left(\frac{-1}{c}\right)\left(\frac{y-u}{c}\right)}{\left[(\beta-\alpha)^{2}+\left(\frac{y-u}{c}\right)^{2}\right]^{2}}\right)\sigma(y)dy$$

$$-\frac{\lambda\gamma}{2\pi c}\int_{0}^{u}\left(\frac{-\alpha\lambda(2\beta-\alpha)\times 2\times\left(\frac{-1}{c}\right)\left(\frac{y-u}{c}\right)}{\left[(\beta-\alpha)^{2}+\left(\frac{y-u}{c}\right)^{2}\right]^{2}}\right)\sigma(y)dy-\frac{\lambda\gamma}{2\pi c}\left(1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^{2}}\right)$$

$$=\frac{\delta}{c}\varphi(u)+\frac{\lambda\gamma}{2\pi c}e^{\delta\left(\frac{u}{c}\right)}\int_{0}^{+\infty}e^{-\delta\left(\frac{y}{c}\right)}\chi(y-u)\sigma(y)dy-\frac{\lambda\gamma}{2\pi c}\int_{0}^{u}\chi(y-u)\sigma(y)dy-\frac{\lambda\gamma}{2\pi c}\left(1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^{2}}\right)$$

And thus

$$\varphi'(u)=\frac{\delta}{c}\varphi(u)+\frac{\lambda\gamma}{2\pi c}\left[e^{\delta\left(\frac{u}{c}\right)}\int_{0}^{+\infty}e^{-\delta\left(\frac{y}{c}\right)}\chi(y-u)\sigma(y)dy-\int_{0}^{u}\chi(y-u)\sigma(y)dy-\left(1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^{2}}\right)\right]$$

With

$$\chi(y-u)=\frac{-\alpha\lambda(2\beta-\alpha)\times 2\times\left(\frac{-1}{c}\right)\left(\frac{y-u}{c}\right)}{\left[(\beta-\alpha)^2+\left(\frac{y-u}{c}\right)^2\right]^2}$$

The Laplace Transform of the Integro-Differential Equation is given by the following lemma.
Lemma 2 The Laplace transformation of the Gerber Shiu function is defined by:

$$\mathrm{L}_{\varphi}(s)=\frac{\mathrm{c}\varphi(0)+\frac{\lambda\gamma\mathrm{c}}{2\pi}\times\frac{\alpha\lambda(2\beta-\alpha)(\delta-\mathrm{cs})\sin[(\beta-\alpha)(\delta-\mathrm{cs})]\left(\frac{\gamma\mathrm{c}}{\gamma\mathrm{c}+\delta}\mathrm{L}_{\varphi}(\mathrm{z})+\mathrm{L}_{\varphi\left(\frac{\delta}{\mathrm{c}}\right)}\right)}{\beta-\alpha}}{(\mathrm{cs}-\delta)\left(1+\frac{\lambda^2\gamma^2\alpha(2\beta-\alpha)\,\mathrm{css}\sin[(\beta-\alpha)(\delta-\mathrm{cs})]}{2\pi(\mathrm{s}+\gamma)(\beta-\alpha)}\right)}$$
$$+\frac{\frac{-\alpha\lambda^2\gamma(2\beta-\alpha)\,\mathrm{cssin}[(\beta-\alpha)\mathrm{cs}]\mathrm{L}_{\mathrm{w}}(\mathrm{s})}{\beta-\alpha}-\frac{\lambda\gamma}{2\pi\mathrm{s}}\left(1+\frac{\alpha\lambda(2\beta-\alpha)}{\beta-\alpha}\right)}{(\mathrm{cs}-\delta)\left(1+\frac{\lambda^2\gamma^2\alpha(2\beta-\alpha)\,\mathrm{css}\sin[(\beta-\alpha)(\delta-\mathrm{cs})]}{2\pi(\mathrm{s}+\gamma)(\beta-\alpha)}\right)} \tag{7}$$

Proof. Using the Integro-Differential Equation and the formulas of the Laplace transform, we get

$$\varphi'(\mathrm{u})=\frac{\delta}{c}\varphi(u)+\frac{\lambda\gamma}{2\pi c}\left[e^{\delta\left(\frac{u}{c}\right)}\int_0^{+\infty}e^{-\delta\left(\frac{y}{c}\right)}\chi(y-u)\sigma(y)dy-\int_0^u\chi(y-u)\sigma(y)dy-\left(1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2}\right)\right]$$

$$=\frac{\delta}{c}\varphi(u)+\frac{\lambda y}{2\pi c}\left[e^{\delta\left(\frac{u}{c}\right)}L_{\Lambda}\left(\frac{\delta}{c}\right)-\int_0^u\chi(y-u)\sigma(y)dy-\left(1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2}\right)\right]$$

With

$$\Lambda(\mathrm{y})=\chi(\mathrm{y}-\mathrm{u})\sigma(\mathrm{y})$$

And

$$\chi(y-u)=\frac{-\alpha\lambda(2\beta-\alpha)\times 2\times\left(\frac{-1}{c}\right)\left(\frac{y-u}{c}\right)}{\left[(\beta-\alpha)^2+\left(\frac{y-u}{c}\right)^2\right]^2}$$

$$\sigma(y)=\int_0^y\varphi(y-x)e^{-\gamma x}dx+\int_y^{+\infty}w(y,\,x-y)e^{-\gamma x}dx$$

In addition, using the Laplace transform of the convolution product, we have

$$\varphi'(\mathrm{u}) = \frac{\delta}{\mathrm{c}}\varphi(\mathrm{u}) + \frac{\lambda\gamma}{2\pi\mathrm{c}}\left[\mathrm{e}^{\delta\left(\frac{\mathrm{u}}{\mathrm{c}}\right)}\mathrm{L}_{\Lambda}\left(\frac{\delta}{\mathrm{c}}\right) - (\sigma * \chi)(\mathrm{u}) - \left(1 + \frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2}\right)\right]$$

$$\mathrm{sL}_{\varphi}(\mathrm{s}) - \varphi(0) = \frac{\delta}{\mathrm{c}}\mathrm{L}_{\varphi}(\mathrm{s}) + \frac{\lambda\gamma}{2\pi\mathrm{c}}\left[\mathrm{L}_{\mathrm{e}^{\delta\left(\frac{\mathrm{u}}{\mathrm{c}}\right)}}(s)\mathrm{L}_{\Lambda}\left(\frac{\delta}{\mathrm{c}}\right) - \mathrm{L}_{\sigma(\mathrm{s})}\mathrm{L}_{\chi(\mathrm{s})} - \left(1 + \frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2}\right)\right]$$

$$\mathrm{L}_{e^{\delta\left(\frac{u}{c}\right)}}(s) = \int_0^{+\infty} e^{\left(\frac{\delta}{c}-s\right)u}du = \frac{1}{s-\frac{\delta}{c}} = \frac{c}{cs-\delta}$$

Therefore

$$\left(\mathrm{s} - \frac{\delta}{\mathrm{c}}\right)\mathrm{L}_{\varphi}(\mathrm{s}) = \varphi(0) + \frac{\lambda\gamma}{2\pi\mathrm{c}}\left[\frac{\mathrm{c}}{\mathrm{cs}-\delta}\mathrm{L}_{\Lambda}\left(\frac{\delta}{\mathrm{c}}\right) - \mathrm{L}_{\sigma(\mathrm{s})}\mathrm{L}_{\chi}(\mathrm{s}) - \left(1 + \frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2}\right)\right]$$

$$\mathrm{L}_{\varphi}(\mathrm{s}) = \frac{\mathrm{c}}{\mathrm{cs}-\delta}\varphi(0) + \frac{\lambda\gamma}{2\pi\mathrm{c}} \times \frac{\mathrm{c}}{\mathrm{cs}-\delta}\left[\frac{\mathrm{c}}{\mathrm{cs}-\delta}\mathrm{L}_{\Lambda}\left(\frac{\delta}{\mathrm{c}}\right) - \mathrm{L}_{\sigma}(\mathrm{s})\mathrm{L}_{\chi}(\mathrm{s}) - \left(1 + \frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2}\right)\right]$$

we have

$$\begin{aligned}\mathrm{L}_{\varphi}(\mathrm{s}) &= \frac{\mathrm{c}}{\mathrm{cs}-\delta}\varphi(0) + \frac{\lambda\gamma}{2\pi\mathrm{c}} \times \frac{\mathrm{c}}{\mathrm{cs}-\delta}\left[\frac{\mathrm{c}}{\mathrm{cs}-\delta}\mathrm{L}_{\Lambda}\left(\frac{\delta}{\mathrm{c}}\right) - \mathrm{L}_{\sigma}(\mathrm{s})\mathrm{L}_{x}(\mathrm{s}) - \frac{1}{\mathrm{s}}\left(1 + \frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2}\right)\right] \\ &= \frac{\mathrm{c}\varphi(0) + \frac{\lambda\gamma}{2\pi} \times \frac{\mathrm{c}}{\mathrm{cs}-\delta}\mathrm{L}_{\Lambda}\left(\frac{\delta}{\mathrm{c}}\right) - \frac{\lambda\gamma}{2\pi}\mathrm{L}_{\sigma}(\mathrm{s})\mathrm{L}_{\mathrm{x}}(\mathrm{s}) - \frac{\lambda\gamma}{2\pi} \times \frac{1}{\mathrm{s}}\left(1 + \frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2)}\right)}{\mathrm{cs}-\delta}\end{aligned} \tag{8}$$

Using Laplace transforms, we get:

$$\sigma(\mathrm{u}) = \int_0^u \varphi(\mathrm{u}-\mathrm{x})\mathrm{e}^{-\gamma\mathrm{x}}\mathrm{dx} + \int_{\mathrm{u}}^{+\infty} \mathrm{w}(\mathrm{u},\ \mathrm{x}-\mathrm{u})\mathrm{e}^{-\gamma\mathrm{r}x}dx$$

$$\sigma(\mathrm{u}) = (\mathrm{f}_{\mathrm{x}} * \varphi)(\mathrm{u}) + \mathrm{w}(\mathrm{u})$$

This gives

$$\mathrm{L}_{\sigma}(s) = \frac{\gamma}{\gamma+\mathrm{s}}\mathrm{L}_{\varphi}(\mathrm{s}) + \mathrm{L}_{\mathrm{w}}(\mathrm{s}) \tag{9}$$

Let us now determine the transformation of the Laplace of $\chi(u)$ defined by:

$$\chi(u)=\frac{\alpha\lambda(2\beta-\alpha)\times 2\times\left(\frac{1}{c}\right)\left(\frac{u}{c}\right)}{\left[(\beta-\alpha)^2+\left(\frac{u}{c}\right)^2\right]^2}$$

Let's apply the following rule for the inverse Laplace transformation.
If $\mathrm{L}^{-1}\{\mathrm{F(s)}\}=\mathrm{f(t)}$ then $\mathrm{L}^{-1}\{\mathrm{sF(s)}\}=(\mathrm{d}/(\mathrm{dt}))\mathrm{f(t)}+\mathrm{f(0)}$.
We have the following result:

$$\mathrm{L}_x(s)=\frac{\alpha\lambda(2\beta-\alpha)\,\mathrm{cssin}[(\beta-\alpha)cs]}{(\beta-\alpha)} \tag{10}$$

The relation (10) allows us to deduce the transformation of the Laplace of $\chi\left(\frac{\delta}{c}-u\right)$

$$L_x\left(\frac{\delta}{c}-s\right)=\frac{\alpha\lambda(2\beta-\alpha)(\delta-cs)\sin[(\beta-\alpha)(\delta-cs)]}{(\beta-\alpha)} \tag{11}$$

and

$$\begin{aligned}\mathrm{L}_\Lambda\left(\frac{\delta}{\mathrm{c}}\right)&=\mathrm{L}_\sigma\left(\frac{\delta}{\mathrm{c}}\right)\mathrm{L}_\chi\left(\frac{\delta}{\mathrm{c}}-\mathrm{u}\right)\\ \mathrm{L}_\Lambda\left(\frac{\delta}{\mathrm{c}}\right)&=\frac{\alpha\lambda(2\beta-\alpha)(\delta-\mathrm{cs})\sin[(\beta-\alpha)(\delta-\mathrm{cs})]}{(\beta-\alpha)}\left(\frac{\gamma}{\gamma+\mathrm{s}}\mathrm{L}_\varphi\left(\frac{\delta}{\mathrm{c}}\right)+\mathrm{L_w}\left(\frac{\delta}{\mathrm{c}}\right)\right)\end{aligned} \tag{12}$$

Using (8), (9), (10), (11) and (12) we get

$$\mathrm{L}_\varphi(\mathrm{s})=\frac{\mathrm{c}\varphi(0)+\dfrac{\lambda\gamma\mathrm{c}}{2\pi}\times\dfrac{\alpha\lambda(2\beta-\alpha)(\delta-\mathrm{cs})\sin[(\beta-\alpha)(\delta-\mathrm{cs})]\left(\dfrac{\gamma\mathrm{c}}{\gamma\mathrm{c}+\delta}\mathrm{L}_\varphi(\mathrm{z})+\mathrm{L}_\varphi\left(\dfrac{\delta}{\mathrm{c}}\right)\right)}{\beta-\alpha}}{(\mathrm{cs}-\delta)\left(1+\dfrac{\lambda^2\gamma^2\alpha(2\beta-\alpha)\,\mathrm{cssin}[(\beta-\alpha)(\delta-\mathrm{cs})]}{2\pi(\mathrm{s}+\gamma)(\beta-\alpha)}\right)}$$

$$+\frac{\dfrac{-\alpha\lambda^2\gamma(2\beta-\alpha)\,\mathrm{cssin}[(\beta-\alpha)\mathrm{cs}]\mathrm{L}_{\mathrm{w(s)}}}{\beta-\alpha}-\dfrac{\lambda\gamma}{2\pi\mathrm{s}}\left(1+\dfrac{\alpha\lambda(2\beta-\alpha)}{\beta-\alpha}\right)}{(\mathrm{cs}-\delta)\left(1+\dfrac{\lambda^2\gamma^2\alpha(2\beta-\alpha)\,\mathrm{cssin}[(\beta-\alpha)(\delta-\mathrm{cs})]}{2\pi(\mathrm{s}+\gamma)(\beta-\alpha)}\right)}$$

Lemma 3 The Laplace transformation of the probability of ruin is defined by:

$$L_{\psi}(s)=\frac{\left[\left(c\psi(0)+H\psi(z)+\frac{H}{\beta}\right)\left(s^{2}+\beta s\right)-As-A\beta-Ks^{2}\right][\gamma+s]}{cs^{2}(s+\beta)[(Q+1)s+\gamma]} \tag{13}$$

With

$$\mathrm{A}=\frac{\lambda\gamma}{2\pi}\left(1+\frac{\alpha\lambda(2\beta-\alpha)}{\beta-\alpha}\right);$$

$$\mathrm{H}=\frac{\lambda^{2}\mathrm{c}^{2}\gamma\alpha(2\beta-\alpha)}{2\pi};$$

$$\mathrm{K}=\frac{\lambda^{2}\mathrm{c}\gamma^{2}\alpha(2\beta-\alpha)}{2\pi(\beta-\alpha)};$$

$$\mathrm{Q}=\frac{\lambda^{2}\mathrm{c}^{2}\gamma\alpha(2\beta-\alpha)}{2\pi(\beta-\alpha)}.$$

Proof. We find the Laplace transform of the probability of ruin for the penalty function $w(x;\ y)=\frac{1}{s+\beta}$ and $\delta=0$. This simplifies equation (7) and gives:

$$\mathrm{L}_{\varphi}(\mathrm{s})=\frac{\mathrm{c}\varphi(0)+\frac{\lambda\gamma\mathrm{c}}{2\pi}\times\frac{\alpha\lambda(2\beta-\alpha)(\mathrm{cs})\sin[(\beta-\alpha)(\mathrm{cs})]\left(\mathrm{L}_{\varphi}(z)+\frac{1}{\mathrm{s}\beta}\right)}{\beta-\alpha}}{(\mathrm{cs})\left(1+\frac{\lambda^{2}\gamma^{2}\alpha(2\beta-\alpha)\,\mathrm{css}\sin[(\beta-\alpha)(\mathrm{cs})]}{2\pi(\mathrm{s}+\gamma)(\beta-\alpha)}\right)}$$

$$+\frac{\frac{-\alpha\lambda^{2}\gamma(2\beta-\alpha)\,\mathrm{cssin}[(\beta-\alpha)\mathrm{cs}]}{\beta-\alpha}\times\frac{1}{\mathrm{s}+\beta}-\frac{\lambda\gamma}{2\pi\mathrm{s}}\left(1+\frac{\alpha\lambda(2\beta-\alpha)}{\beta-\alpha}\right)}{(\mathrm{cs})\left(1+\frac{\lambda^{2}\gamma^{2}\alpha(2\beta-\alpha)\,\mathrm{css}\sin[(\beta-\alpha)(\mathrm{cs})]}{2\pi(s+\gamma)(\beta-\alpha)}\right)}$$

To simplify the cacules, we ask:

$$A = \frac{\lambda\gamma}{2\pi}\left(1+\frac{\alpha\lambda(2\beta-\alpha)}{\beta-\alpha}\right);$$

$$H = \frac{\lambda^2 c^2 \gamma\alpha(2\beta-\alpha)}{2\pi};$$

$$K = \frac{\lambda^2 c\gamma^2 \alpha(2\beta-\alpha)}{2\pi(\beta-\alpha)};$$

$$Q = \frac{\lambda^2 c^2 \gamma\alpha(2\beta-\alpha)}{2\pi(\beta-\alpha)}.$$

which are all constants.

This gives:

$$L_\psi(s) = \frac{c\Psi(0)+H\left(\frac{\psi(z)}{s}+\frac{1}{s\beta}\right)s-K\left(\frac{s}{s+\beta}\right)-A\left(\frac{1}{s}\right)}{cs\left[1+Q\left(\frac{s}{s+\gamma}\right)\right]}$$

$$= \frac{\left[\left(c\Psi(0)+H\Psi(z)+\frac{H}{\beta}\right)(s^2+\beta s)-As-A\beta-Ks^2\right][\gamma+s]}{cs^2(s+\beta)[(Q+1)s+\gamma]}$$

From the above, we can deduce the proof of the theorem.

Proof. We now turn our attention to equation 13. Notably, the numerator is a third-degree polynomial, while the denominator is a fourth-degree polynomial in the variable sss. However, the denominator also includes the original denominator itself, effectively introducing additional complexity. This composite structure suggests the presence of poles within the solution domain.

$$R_0 = 0;\ R_1 = -\beta;\ R_2 = \frac{-\gamma}{Q+1} \tag{14}$$

Decomposing $L_\psi(s)$ into simple elements gives:

$$L_\psi(s) = \frac{a}{s}+\frac{b}{s^2}+\frac{d}{s+\beta}+\frac{h}{s-R_2} \tag{15}$$

By returning formula 15 to the same denominator, we obtain:

$$L_\psi(s) = \frac{c(a+d+h)s^3+c(a\beta+b-aR_2-dR_2+h\beta)s^2+c(b\beta-a\beta R_2+bR_2)}{cs^2(s+\beta)(s-R_2)} \tag{16}$$

By identifying equations (13) and (16) we have:

$$c(a+d+h) = c\psi(0) + H\psi(z) - K + \frac{H}{\beta} = \theta_1 \tag{17}$$

$$c\left(a\beta + b - aR^2 - dR^2 + h\beta\right) = c\gamma\psi(0) + H\gamma\psi(z) - K\gamma + \frac{\gamma H}{\beta} + \beta c\psi(0) + \beta H\psi(z) + H\psi(z) - A = \theta_2 \tag{18}$$

$$c\left(b\beta - a\beta R_2 + bR_2\right) = \beta c\gamma\psi(0) + \beta H\gamma\psi(z) + H\gamma\psi(z) - \gamma A - \beta A = 0 \tag{19}$$

Using the inverse of the Laplace transform of equation (15) we get:

$$\Psi(u) = a + bu + de^{-\beta u} + he^{R_2 u} \tag{20}$$

Like

$$\lim_{u \to +\infty} \Psi(u) = 0$$

and $R_2 < 0$, We have:

$$a = 0 \tag{21}$$

$$b = 0 \tag{22}$$

From equations (17), (18), (19), (21) and (22) we get

$$\begin{cases} d + h = \dfrac{\theta_1}{c} \\ \\ -R_2 d + h\beta = \dfrac{\theta_2}{c} \end{cases}$$

This implies that:

$$d = \frac{\beta\theta_1 - \theta_2}{c\left(R_2 + \beta\right)}; \quad h = \frac{R_2\theta_1 + \theta_2}{c\left(R_2 + \beta\right)} \tag{23}$$

According to (17), (20), (21) and (22) we have:

$$\Psi(0) = d + h = \Psi(0) + \frac{H\Psi(z)}{c} - \frac{K}{c} + \frac{H}{c\beta} = \Psi(0)$$

This, gives us:

$$\psi(z) = \frac{\beta \mathrm{K} - \mathrm{H}}{\beta \mathrm{H}} \tag{24}$$

Using formulas (19), (21), (22) and (24) we have:

$$\beta\gamma c\psi(0) + (\beta\gamma \mathrm{H} + \gamma \mathrm{H})\psi(\mathrm{z}) = \gamma \mathrm{A} + \beta \mathrm{A}$$

This, in turn, makes it possible to obtain:

$$\psi(0) = \frac{A(\gamma+\beta) - \gamma\beta K}{\beta\gamma c} \tag{25}$$

Using equations (17), (24) and (26) we get:

$$\theta_1 = \frac{A(\gamma+\beta) - \gamma\beta K}{\beta\gamma} \tag{26}$$

In the same way with equations (20), (24) and (25), we have:

$$\theta_2 = \frac{(\gamma+\beta)[A(\gamma+\beta) - \gamma\beta K]}{\beta\gamma} + \beta K - A \tag{27}$$

So the formulas (20), (21) and (23) give us:

$$\psi(u) = \frac{\beta\theta_1 - \theta_2}{c\,(R_2+\beta)} e^{-\beta u} + \frac{R_2\theta_1 + \theta_2}{c\,(R_2+\beta)} e^{R_2 u}$$

Example 1 We have to run a simulation against different values of the reserve u (in millions of euros as units) of this new probability of ruin obtained 5. The choice is made essentially this parameter u because the insurance company sets the value per and that reality is the probability of ruin. We run simulations to $\beta = 0.7$, $\alpha = 0.5$, $\lambda = 0.2$, $\gamma = 0.3$, $\pi = 3.14$ and $c = 10$. This simution was performed from the commands of the *R* software. We vary the value of u increasing to observe the behavior of the probability of ruin on the infinite horizon. The results of this simulation can be found in the table.

Table 1. Table of result of the simulation of ruin probability for different values of the reserve

u	0	2	3	4	5	10
$\Psi(u)$	0.098976	0.069943	0.053249	0.049261	0.027069	0.009897

The figure below shows a simulation of the probabilities of ruin as a function of the initial reserve u. The contribution rate c is constant throughout the simulation. Thus we obtain a value for the probability of ruin that would be desirable for the insurance company. It is clearly visible from the figure that an increase in leads to a decrease in the probability of ruin. When the initial reserve varies and tends towards plus infinity $(+\ \infty)$, then the probability of ruin $\Psi(u)$ tends to 0, which is quite normal for a risk model.

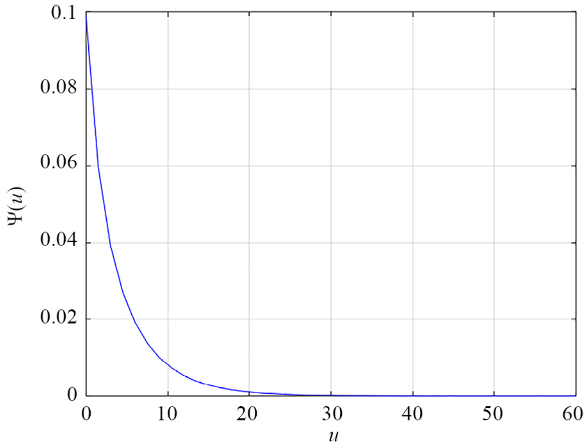

Figure 1. Figure of the simulation of ruin probability

4. Conclusion

In this paper, we have developed a novel reserve model based on a variable-memory counting process, specifically a Hawkes process, as detailed in section (3). Our study effectively achieves its primary objective of exploring the impact of self-excitation on the probability of ruin for insurance companies. We have demonstrated that Hawkes processes, with their self-excitation and memory properties, provide a more realistic representation of claim arrivals, which is crucial for effective risk management in the insurance sector.

We derived the integro-differential equation governing the probability of ruin over an infinite time horizon and subsequently obtained its Laplace transform. Furthermore, we established a simplified expression for the probability of ruin at the infinite horizon when the initial capital u approaches infinity, which allowed us to perform numerical simulations.

The novelty of our study lies in the application of Hawkes processes to model ruin probabilities, offering an innovative approach compared to classical models that do not account for temporal dependence between claims. This contribution enhances our understanding of ruin mechanisms and opens new avenues for risk management strategies in insurance.

Future work will focus on calculating the probability of ruin over a finite horizon using the results obtained in this study, thereby providing further insights into the practical implications for insurance companies.

Conflict of interest

The authors declare there is no conflict of interest at any point with reference to research findings.

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