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System of Nonlinear (k, ψ)-Hilfer Fractional Order Hybrid Boundary Value Problems

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Abstract: This paper focuses on the existence and uniqueness of solutions for a new class of coupled fractional differential equations subject to specific boundary conditions. We introduce a novel approach that incorporates the (k, ψ)-Hilfer fractional derivative, a recent advancement in fractional calculus. By leveraging the properties of this operator, we develop a comprehensive framework to analyze the proposed system. Our findings, established through the application of Schauder's fixed point theorem and the Banach contraction principle, contribute to the understanding of complex phenomena modeled by fractional differential equations. This research expands the boundaries of fractional calculus and offers potential applications in various scientific and engineering domains.

***Keywords*:** Cauchy-type problem, ψ-Hilfer fractional derivative, hybrid equations, coupled systems, existence, uniqueness

MSC: 26A33, 34A08, 34B27

1. Introduction

Fractional calculus has become a game-changer in recent years. It's a mathematical framework that extends the familiar concepts of differentiation and integration beyond integer orders to encompass non-integer ones. This seemingly simple extension unlocks a powerful tool for scientists across diverse disciplines. Fractional calculus allows us to formulate fractional order differential equations (FODEs). FODEs offer a significant advantage over classical differential equations. They excel at modeling complex systems that exhibit memory effects or intricate dynamics, where traditional integer-order models struggle to capture the full picture. FODEs have found remarkable success in various fields like physics, engineering, biology, and even finance.

The development of Fractional Ordinary Differential Equations (FODEs) and their theoretical underpinnings has been significantly advanced by numerous researchers. For instance: [1]: This study explores the existence of solutions for fractional differential equations using a fixed point approach, contributing to the theoretical framework of FODEs. [2]: The authors present novel methods for solving boundary value problems involving fractional differential equations,

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extending the applicability of FODEs to more complex systems. [3]: This paper provides new insights into the stability analysis of FODEs, offering techniques to ensure the robustness of solutions under various conditions. [4]: The research delves into numerical methods for solving FODEs, particularly focusing on the efficiency and accuracy of different algorithms. [5]: This work investigates the asymptotic behavior of solutions to FODEs, contributing to the understanding of long-term dynamics in fractional systems. [6]: The authors develop new existence and uniqueness theorems for nonlinear FODEs, providing a solid theoretical basis for further research in the field. [7]: This paper focuses on the application of FODEs in modeling physical phenomena, highlighting the versatility and practical relevance of fractional calculus. [8]: The study introduces innovative fractional operators and applies them to differential equations, expanding the toolbox available for solving FODEs. [9]: The authors examine the qualitative properties of solutions to FODEs, such as oscillation and periodicity, enhancing the understanding of fractional dynamics. [10]: This research addresses the challenges of nonlocal conditions in FODEs, proposing methods to handle such complexities in real-world applications. [11]: The paper presents a comprehensive analysis of fractional derivatives and their impact on the solution behavior of differential equations, contributing to the mathematical foundation of FODEs. [12]: The authors investigate the role of fractional calculus in control theory, showing how FODEs can be utilized in designing control systems with enhanced performance. [13]: This study introduces fractional differential inequalities and their applications, expanding the theoretical landscape of FODEs. [14]: The research explores the interplay between fractional and classical derivatives in differential equations, offering new perspectives on hybrid systems. [15]: The authors develop a general framework for fractional boundary value problems, providing tools for addressing a wide range of applications in science and engineering. [16]: This paper addresses the exact controllability of Sobolev-type Hilfer fractional systems, which informs our study's focus on fractional differential equations involving (k, ψ)-Hilfer derivatives. [17]: Their investigation into the optimal control of Hilfer fractional neutral stochastic systems supports our exploration of boundary value problems in a similar fractional framework. [18]: This study discusses the partial-approximate controllability of Hilfer fractional systems with nonlocal conditions, complementing our results on the controllability of fractional differential systems. [19]: Their analysis of approximate controllability in Hilfer fractional neutral evolution systems via hemivariational inequality is relevant to our discussion on boundary controllability. [20]: The paper on approximate controllability of Hilfer fractional neutral systems in Hilbert spaces provides a theoretical background that supports our findings on existence and uniqueness. [21]: This study introduces a novel method for solving fractional differential equations, which is relevant for the numerical methods that could extend our theoretical results. [22]: The paper presents an efficient computational method for fractional differential equations, offering potential techniques for extending our work into practical applications. [23]: The study on hyperthermia therapy using fractional models highlights real-world applications of fractional calculus, aligning with the broader impact of our research. [24]: This paper explores the controllability of ψ-Caputo fractional differential equations with impulsive effects, providing a basis for understanding controllability in our study's context of (k, ψ)-Hilfer derivatives. [25]: Their work on Hilfer fractional semilinear integro-differential equations informs our exploration of solution existence and uniqueness within the fractional calculus framework.

Each of these contributions has played a crucial role in advancing the theory and application of FODEs, reflecting the growing interest and research activity in this area. Recent research by Diaz [26] introduced new definitions for the k-gamma and k-beta functions. Readers interested in this topic can explore further through references like [27] and [28]. Building upon these advancements, Sousa et al. [29] presented the ψ-Hilfer fractional derivative (HFD) and its properties related to fractional operators. Inspired by these works, we proposed a novel extension of the HFD, the k-generalized ψ-HFD [30]. This extension allows us to generalize Grönwall's lemma and investigate various forms of Ulam stability. Furthermore, by applying this new generalized fractional operator, we conducted comprehensive studies on qualitative and quantitative aspects of different classes of fractional differential problems (e.g., [31–34]). For a deeper dive, refer to the cited references [35–40].

In [41], Wongcharoen et al. studied a coupled system with the Hilfer fractional order boundary value problem (BVP):

$$
\begin{cases}
{}^{H}D^{\alpha,\,\varepsilon}\xi(\tau) = f(\tau,\, \xi(\tau),\, \gamma(\tau)), & \tau \in [a,\, b], \\
{}^{H}D^{\alpha_1,\,\varepsilon_1}\gamma(\tau) = g(\tau,\, \xi(\tau),\, \gamma(\tau)), & \tau \in [a,\, b], \\
\xi(a) = 0,\ \xi(b) = \sum_{i=1}^{m} \varkappa_i I^{\varphi_i}\gamma(\sigma_i), \\
\gamma(a) = 0,\ \gamma(b) = \sum_{\mho=1}^{n} \zeta_j I^{\psi_j}\xi(z_j),
\end{cases}
$$

where ${}^{H}D^{\alpha,\,\varepsilon}$ and ${}^{H}D^{\alpha_1,\,\varepsilon_1}$ are the HFDs of orders $\alpha,\ \varepsilon;\ \alpha_1,\ \varepsilon_1$, in which $1 < \alpha,\ \alpha_1 < 2$, and $0 \leq \varepsilon,\ \varepsilon_1 \leq 1$, and $I^{\varphi_i},\ I^{\psi_j}$ are the Riemann-Liouville (RL) fractional integrals of order $\varphi_i > 0$ and $\psi_j > 0$, respectively, the points $\sigma_i,\ z_j \in [a,\, b]$, $a \geq 0$, $f,\ g \in C([a,\, b] \times \mathbb{R} \times \mathbb{R},\, \mathbb{R})$ and $\varkappa_i,\ \zeta_j \in \mathbb{R}$, $i = 1,\, 2,\, \ldots,\, m$, $\mho = 1,\, 2,\, \ldots,\, n$ are given real constants. The authors support their arguments using the Leray-Schauder and Krasnoselskii's fixed point theorems, as well as Banach's contraction mapping principle.

Abdo et al. [42] utilized fixed-point theorems from Banach and Krasnoselskii to analyze a system incorporating generalized HFDs:

$$
\begin{cases}
D_{\mathfrak{d}^+}^{\varkappa_1,\,\sigma_1;\,\psi}\gamma(\varepsilon) = f_1(\varepsilon,\, \xi(\varepsilon)), & \mathfrak{d} < \varepsilon \leq \mathfrak{T},\ \mathfrak{d} > 0, \\
D_{\mathfrak{d}^+}^{\varkappa_2,\,\sigma_2;\,\psi}\xi(\varepsilon) = f_2(\varepsilon,\, \gamma(\varepsilon)), & \mathfrak{d} < \varepsilon \leq \mathfrak{T},\ \mathfrak{d} > 0, \\
\gamma(\mathfrak{T}) = \omega_1 \in \mathbb{R}, \\
\xi(\mathfrak{T}) = \omega_2 \in \mathbb{R},
\end{cases}
$$

where $0 < \varkappa_i < 1$, $0 \leq \sigma_i \leq 1$, $D_{\mathfrak{d}^+}^{\varkappa_i,\,\sigma_i\psi}\,(i = 1,\, 2)$ is the HFDs of order $\varkappa_i$ and type σ_i with respect to ψ and $f \in C((\mathfrak{d},\, \mathfrak{T}] \times \mathbb{R},\, \mathbb{R})$. Their focus lies on establishing the existence and stability of unique solutions using the Ulam-Hyers criteria.

To investigate the existence and uniqueness of solutions for a switched coupled implicit ψ-Hilfer fractional differential system, Ahmed et al. [43] employed Banach's contraction principle and Schauder's fixed point theorem:

$$
\begin{cases}
{}_{H}D_{a^+}^{p,\,q;\,\psi}u(\delta) = f\left(\delta,\, u(\delta),\, {}_{H}D_{a^+}^{p,\,q;\,\psi}v(\delta)\right), & \delta \in \mathscr{J} = (a,\, b], \\
{}_{H}D_{a^+}^{p,\,q;\,\psi}v(\delta) = g\left(\delta,\, {}_{H}D_{a^+}^{p,\,q;\,\psi}u(\delta),\, v(\delta)\right), & \xi = p + q - pq, \\
I_{a^+}^{1-\xi;\,\psi}u(\delta)\Big|_{\delta=a} = u_a,\ I_{a^+}^{1-\xi;\,\psi}v(\delta)\Big|_{\delta=a} = v_a, & u_a,\ v_a \in \mathbb{R},
\end{cases}
$$

where ${}_{H}D_{a^+}^{p,\,q}$ represent the ψ-HFD of order p and type q with $p \in (0,\, 1)$, $q \in (0,\, 1]$. $I_{a^+}^{1-\xi;\,\psi}$ denote the ψ-Hilfer fractional integral of order $1-\xi$. Moreover $f,\ g \in C(\mathscr{J} \times \mathscr{X} \times \mathscr{X},\, \mathscr{X})$. The function $\psi : \mathscr{J} \to \mathbb{R}$ is linear and $\psi'(\varepsilon) \neq 0,\ \varepsilon \in \mathscr{J}$.

This paper delves into the existence and uniqueness of solutions for a novel class of coupled, nonlinear fractional hybrid differential equations with boundary conditions. Our work is inspired by above mentioned recent advancements in fractional differential equations, particularly those involving coupled systems and $(k,\ \psi)$-Hilfer derivatives.

$$\begin{cases} \left({}_{k}^{H}\mathscr{D}_{\varkappa_1+}^{\vartheta_1,\ r_1;\ \psi}\Theta\xi\right)(\delta) = \varsigma_1(\delta,\ \xi(\delta),\ \gamma(\delta)),\ \ \delta \in (\varkappa_1,\ \varkappa_2], \\ \left({}_{k}^{H}\mathscr{D}_{\varkappa_1+}^{\vartheta_2,\ r_2;\ \psi}\widehat{\Theta}\gamma\right)(\delta) = \varsigma_2(\delta,\ \xi(\delta),\ \gamma(\delta)),\ \ \delta \in (\varkappa_1,\ \varkappa_2], \end{cases} \tag{1}$$

$$\begin{cases} \alpha_1\left(\mathscr{J}_{\varkappa_1+}^{k(1-\sigma_1),\ k;\ \psi}\Theta\xi\right)(\varkappa_1{}^{+}) + \alpha_2\left(\mathscr{J}_{\varkappa_1+}^{k(1-\sigma_1),\ k;\ \psi}\Theta\xi\right)(\varkappa_2) = \alpha_3, \\ \beta_1\left(\mathscr{J}_{\varkappa_1+}^{k(1-\sigma_2),\ k;\ \psi}\widehat{\Theta}\gamma\right)(\varkappa_1{}^{+}) + \beta_2\left(\mathscr{J}_{\varkappa_1+}^{k(1-\sigma_2),\ k;\ \psi}\widehat{\Theta}\gamma\right)(\varkappa_2) = \beta_3, \end{cases} \tag{2}$$

where for $i = 1,\ 2$, ${}_{k}^{H}\mathscr{D}_{\varkappa_1+}^{\vartheta_i,\ r_i;\ \psi}$ and $\mathscr{J}_{\varkappa_1+}^{k(1-\sigma_i),\ k;\ \psi}$ are, respectively, the $(k,\ \psi)$-HFD of order $\vartheta_i \in (0,\ k)$ and type $r_i \in [0,\ 1]$, and k-generalized ψ-fractional integral of order $k(1-\sigma_i)$, where $\sigma_i = \frac{1}{k}(r_i(k-\vartheta_i)+\vartheta_i)$, $k > 0$, $\varsigma_i : [\varkappa_1,\ \varkappa_2] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are given functions, $\Theta,\ \widehat{\Theta} \in C([\varkappa_1,\ \varkappa_2],\ \mathbb{R}\backslash\{0\})$ and $\alpha_1,\ \alpha_2,\ \alpha_3,\ \beta_1,\ \beta_2,\ \beta_3 \in \mathbb{R}$ such that $\alpha_1 + \alpha_2 \neq 0$ and $\beta_1 + \beta_2 \neq 0$.

2. Motivation of the paper

2.1 *Advancement of fractional calculus*

The development of fractional calculus has opened new avenues for modeling complex systems in various scientific fields. However, there is still a need to explore different types of fractional derivatives to understand their potential applications better [44].

2.2 *Introduction of (k, ψ)-Hilfer fractional derivatives*

The $(k,\ \psi)$-Hilfer fractional derivative is a relatively new concept in fractional calculus. Investigating its properties and applications can lead to a deeper understanding of fractional differential equations, particularly those involving coupled systems [45].

2.3 *Need for new solution techniques*

Existing methods for solving fractional differential equations often face limitations, especially when dealing with nonlinear and coupled systems. There is a need to develop and apply new techniques that can provide existence and uniqueness results for such equations [46].

2.4 *Relevance to boundary value problems*

Boundary value problems are crucial in many practical applications, such as physics, engineering, and finance. Extending fractional calculus to address boundary conditions in coupled nonlinear systems can enhance the modeling accuracy of these applications [47].

2.5 *Application of fixed point theorems*

The use of fixed point theorems, such as Schauder's and Banach's, in fractional differential equations is a powerful tool. Demonstrating their applicability to (k, ψ)-Hilfer fractional derivatives can provide a robust framework for future research in this area [48].

2.6 *Contribution to the field*

This study aims to contribute to the growing body of research in fractional calculus by establishing new results for coupled nonlinear fractional differential equations with boundary conditions. The findings can potentially inspire further investigations into more complex systems and different types of fractional operators.

The structure of the paper is as follows. A few notations, basic information, and auxiliary results are introduced in Section 2. Our primary findings for problems (1)-(2) are presented in Section 3 and are based on Schauder's fixed point theorem and the Banach contraction principle. Finally, we provide an example demonstrating the practicality of our findings.

3. Preliminaries

Initially, we introduce the weighted spaces, notations, definitions, and preliminary concepts pertinent to this study. Let $0 < \varkappa_1 < \varkappa_2 < \infty$, $\mho = [\varkappa_1, \varkappa_2]$, and consider $\lambda \in (0, k)$, $\rho \in [0, 1]$, $k > 0$, and $\sigma = \frac{1}{k}(\rho(k-\lambda)+\lambda)$.

The space $C(\mho, \mathbb{R})$ denotes the Banach space comprising all continuous functions mapping from J to $\mathbb{R}$ with the norm defined as

$$\|\xi\|_\infty = \sup\{|\xi(\delta)| : \delta \in \mho\}.$$

Let $AC^n(\mho, \mathbb{R})$ and $C^n(\mho, \mathbb{R})$ be the spaces of n-times absolutely continuous and n-times continuously differentiable functions on J, respectively.

Consider the weighted Banach space

$$C_{\sigma;\,\psi}(\mho) = \left\{\xi : (\varkappa_1, \varkappa_2] \to \mathbb{R} : \delta \to \Phi_\sigma^\psi(\delta, \varkappa_1)\xi(\delta) \in C(\mho, \mathbb{R})\right\},$$

where $\Phi_\sigma^\psi(\delta, \varkappa_1) = (\psi(\delta) - \psi(\varkappa_1))^{1-\sigma}$, with the norm

$$\|\xi\|_{C_{\sigma;\,\psi}} = \sup_{\delta\in\mho}\left|\Phi_\sigma^\psi(\delta, \varkappa_1)\xi(\delta)\right|,$$

and

$$C^n_{\sigma;\,\psi}(\mho) = \left\{\xi \in C^{n-1}(\mho) : \xi^{(n)} \in C_{\sigma;\,\psi}(\mho)\right\}, \quad n \in \mathbb{N},$$

$$C^0_{\sigma;\,\psi}(\mho) = C_{\sigma;\,\psi}(\mho),$$

with the norm

$$\|\xi\|_{C^n_{\sigma;\,\psi}} = \sum_{i=0}^{n-1} \|\xi^{(i)}\|_\infty + \|\xi^{(n)}\|_{C_{\sigma;\,\psi}}.$$

Consider the Banach space

$$\mathscr{F}_{\sigma_1,\,\sigma_2} := C_{\sigma_1;\,\psi}(\mho) \times C_{\sigma_2;\,\psi}(\mho),$$

with the norm

$$\|(\xi,\,\gamma)\|_{\mathscr{F}_{\sigma_1,\,\sigma_2}} = \max\left\{\|\xi\|_{C_{\sigma_1;\,\psi}},\ \|\gamma\|_{C_{\sigma_2;\,\psi}}\right\},$$

where $0 < \sigma_1,\ \sigma_2 \leq 1$.

Consider the space $X^p_\psi(\varkappa_1,\ \varkappa_2)$ of those real-valued Lebesgue measurable functions μ on $[\varkappa_1,\ \varkappa_2]$ with $\|\mu\|_{X^p_\psi} < \infty$, and the norm

$$\|\mu\|_{X^p_\psi} = \left(\int_{\varkappa_1}^{\varkappa_2} \psi'(\delta)|\mu(\delta)|^p d\delta\right)^{\frac{1}{p}},$$

where ψ is an increasing and positive function on $[\varkappa_1,\ \varkappa_2]$ where ψ' is continuous on $[\varkappa_1,\ \varkappa_2]$ and $1 \leq p \leq \infty$.

Definition 1 ([26]) The k-gamma function is given by

$$\Gamma_k(\varkappa) = \int_0^\infty s^{\varkappa-1} e^{-\frac{s^k}{k}} ds, \quad \varkappa > 0.$$

When $k \to 1$ then $\Gamma(\varkappa) = \Gamma_k(\varkappa)$, and some other useful relations are $\Gamma_k(\varkappa) = k^{\frac{\varkappa}{k}-1}\Gamma\left(\frac{\varkappa}{k}\right)$, $\Gamma_k(\varkappa+k) = \varkappa\Gamma_k(\varkappa)$ and $\Gamma_k(k) = 1$. Moreover, the k-beta function is given as

$$B_k(\varkappa,\ \xi) = \frac{1}{k}\int_0^1 s^{\frac{\varkappa}{k}-1}(1-s)^{\frac{\xi}{k}-1} ds,$$

so that $B_k(\varkappa,\ \xi) = \frac{1}{k}B\left(\frac{\varkappa}{k},\ \frac{\xi}{k}\right)$ and $B_k(\varkappa,\ \xi) = \frac{\Gamma_k(\varkappa)\Gamma_k(\xi)}{\Gamma_k(\varkappa+\xi)}$.

Definition 2 (k-generalized ψ-fractional integral [49]) Consider $\mu \in X^p_\psi(\varkappa_1,\ \varkappa_2)$, where $\psi(\delta) > 0$ is an increasing function on $(\varkappa_1,\ \varkappa_2]$ and $\psi'(\delta) > 0$ is continuous on $(\varkappa_1,\ \varkappa_2)$. We define the generalized k-fractional integral operators of a function μ of order $\lambda > 0$ as follows:

$$\mathscr{I}^{\lambda,\,k;\,\psi}_{\varkappa_1+}\mu(\delta) = \int_{\varkappa_1}^\delta \bar{\Phi}^{k;\,\psi}_\lambda(\delta,\ s)\psi'(s)\mu(s)ds,$$

with $k>0$ and $\bar{\Phi}_{\lambda}^{k,\,\psi}(\delta,\, s)=\dfrac{(\psi(\delta)-\psi(s))^{\frac{\lambda}{k}-1}}{k\Gamma_k(\lambda)}$.

Theorem 3 ([31, 32]) Let $\mu\in X_{\psi}^{p}(\varkappa_1,\,\varkappa_2)$, $\lambda>0$ and $k>0$. Then $\mathscr{I}_{\varkappa_1+}^{\lambda,\,k;\,\psi}\mu\in C([\varkappa_1,\,\varkappa_2],\,\mathbb{R})$.

Lemma 4 ([31, 32]) Let $\lambda>0$, $\rho>0$ and $k>0$. Then, the semigroup properties that follow are met:

$$\mathscr{I}_{\varkappa_1+}^{\lambda,\,k;\,\psi}\mathscr{I}_{\varkappa_1+}^{\rho,\,k;\,\psi}\mu(\delta)=\mathscr{I}_{\varkappa_1+}^{\lambda+\rho,\,k;\,\psi}\mu(\delta)=\mathscr{I}_{\varkappa_1+}^{\rho,\,k;\,\psi}\mathscr{I}_{\varkappa_1+}^{\lambda,\,k;\,\psi}\mu(\delta).$$

Lemma 5 ([31, 32]) Let $\lambda,\,\rho>0$ and $k>0$. Then, we get

$$\mathscr{I}_{\varkappa_1+}^{\lambda,\,k;\,\psi}\bar{\Phi}_{\rho}^{k,\,\psi}(\delta,\,\varkappa_1)=\bar{\Phi}_{\lambda+\rho}^{k,\,\psi}(\delta,\,\varkappa_1).$$

Theorem 6 ([31, 32]) Let $0<\varkappa_1<\varkappa_2<\infty$, $\lambda,\,\rho>0$, $0\leq\sigma=\dfrac{1}{k}(\rho(k-\lambda)+\lambda)<1$, $k>0$ and $\xi\in C_{\sigma;\,\psi}(\mho)$. If $\dfrac{\lambda}{k}>1-\sigma$, then

$$\left(\mathscr{I}_{\varkappa_1+}^{\lambda,\,k;\,\psi}\xi\right)(\varkappa_1)=\lim_{\delta\to\varkappa_1^+}\left(\mathscr{I}_{\varkappa_1+}^{\lambda,\,k;\,\psi}\xi\right)(\delta)=0.$$

Definition 7 ($(k,\,\psi)$-Hilfer derivative [31, 32]) Let $m-1<\dfrac{\lambda}{k}\leq m$ with $m\in\mathbb{N}$, $-\infty\leq\varkappa_1<\varkappa_2\leq\infty$ and $\mu,\,\psi\in C^m([\varkappa_1,\,\varkappa_2],\,\mathbb{R})$ where ψ is increasing and $\psi'(\delta)\neq0$, for all $\delta\in[\varkappa_1,\,\varkappa_2]$. The $(k,\,\psi)$-HFDs ${}_{k}^{H}\mathscr{D}_{\varkappa_1+}^{\lambda,\,\rho;\,\psi}(\cdot)$ of a function μ of order λ and type $0\leq\rho\leq1$, with $k>0$ is given by:

$$\begin{aligned}{}_{k}^{H}\mathscr{D}_{\varkappa_1+}^{\lambda,\,\rho;\,\psi}\mu(\delta)&=\left(\mathscr{I}_{\varkappa_1+}^{\rho(km-\lambda),\,k;\,\psi}\left(\frac{1}{\psi'(\delta)}\frac{d}{d\delta}\right)^m\left(k^m\mathscr{I}_{\varkappa_1+}^{(1-\rho)(km-\lambda),\,k;\,\psi}\mu\right)\right)(\delta)\\&=\left(\mathscr{I}_{\varkappa_1+}^{\rho(km-\lambda),\,k;\,\psi}\delta_{\psi}^{m}\left(k^m\mathscr{I}_{\varkappa_1+}^{(1-\rho)(km-\lambda),\,k;\,\psi}\mu\right)\right)(\delta),\end{aligned}$$

where $\delta_{\psi}^{m}=\left(\dfrac{1}{\psi'(\delta)}\dfrac{d}{d\delta}\right)^m$.

Lemma 8 ([31, 32]) Let $\delta>\varkappa_1$, $0<\dfrac{\lambda}{k}<1$, $0\leq\rho\leq1$, $k>0$. Then for $0<\sigma<1$; $\sigma=\dfrac{1}{k}(\rho(k-\lambda)+\lambda)$, we have

$$\left[{}_{k}^{H}\mathscr{D}_{\varkappa_1+}^{\lambda,\,\rho;\,\psi}\left(\Phi_{\sigma}^{\psi}(s,\,\varkappa_1)\right)^{-1}\right](\delta)=0.$$

Theorem 9 ([31, 32]) If $\mu\in C_{\sigma;\,\psi}^{m}[\varkappa_1,\,\varkappa_2]$, $m-1<\dfrac{\lambda}{k}<m$, $0\leq\rho\leq1$, where $m\in\mathbb{N}$ and $k>0$, then

$$\left(\mathscr{I}_{\varkappa_1+}^{\lambda,\,k;\,\psi}\,{}_{k}^{H}\mathscr{D}_{\varkappa_1+}^{\lambda,\,\rho;\,\psi}\mu\right)(\delta)=\mu(\delta)-\sum_{i=1}^{m}\frac{(\psi(\delta)-\psi(\varkappa_1))^{\sigma-i}}{k^{i-m}\Gamma_k(k(\sigma-i+1))}\left\{\delta_{\psi}^{m-i}\left(\mathscr{I}_{\varkappa_1+}^{k(m-\sigma),\,k;\,\psi}\mu(\varkappa_1)\right)\right\},$$

where

$$\sigma = \frac{1}{k}\left(\rho(km-\lambda)+\lambda\right).$$

If $m = 1$, we have

$$\left(\mathscr{I}_{\varkappa_1+}^{\lambda,\ k;\ \psi}\ {}_{k}^{H}\mathscr{D}_{\varkappa_1+}^{\lambda,\ \rho;\ \psi}\mu\right)(\delta) = \mu(\delta) - \frac{(\psi(\delta)-\psi(\varkappa_1))^{\sigma-1}}{\Gamma_k(\rho(k-\lambda)+\lambda)}\mathscr{I}_{\varkappa_1+}^{(1-\rho)(k-\lambda),\ k;\ \psi}\mu(\varkappa_1).$$

Lemma 10 ([31, 32]) Let $\lambda > 0,\ 0 \le \rho \le 1$, and $\xi \in C^1_{\sigma;\ \psi}(\mho)$, where $k > 0$. Then for $\delta \in (\varkappa_1,\ \varkappa_2]$, we have

$$\left({}_{k}^{H}\mathscr{D}_{\varkappa_1+}^{\lambda,\ \rho;\ \psi}\ \mathscr{I}_{\varkappa_1+}^{\lambda,\ k;\ \psi}\xi\right)(\delta) = \xi(\delta).$$

4. Existence of solutions

To facilitate the analysis of system (1)-(2), we first establish a key theorem that allows us to convert it into a coupled system of fractional integral equations.

Theorem 11 Let $\sigma = \dfrac{\rho(k-\lambda)+\lambda}{k}$, where $k > 0$, $0 < \lambda < k$, $0 \le \rho \le 1$, and let $\varphi(\cdot) \in C(\mho,\ \mathbb{R})$, $\Theta(\cdot) \in C(\mho,\ \mathbb{R}\backslash\{0\})$. The function ξ satisfies the boundary value problem for $(k,\ \psi)$-Hilfer fractional differential equations:

$$\left({}_{k}^{H}\mathscr{D}_{\varkappa_1+}^{\lambda,\ \rho;\ \psi}\Theta\xi\right)(\delta) = \varphi(\delta),\ \ \delta \in (\varkappa_1,\ \varkappa_2], \tag{3}$$

$$\alpha_1\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\ k;\ \psi}\Theta\xi\right)({\varkappa_1}^{+}) + \alpha_2\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\ k;\ \psi}\Theta\xi\right)(\varkappa_2) = \alpha_3, \tag{4}$$

if and only if it verifies the following integral equation:

$$\xi(\delta) = \frac{1}{\Theta(\delta)}\left[\frac{\alpha_3 - \alpha_2\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma)+\lambda,\ k;\ \psi}\varphi\right)(\varkappa_2)}{(\alpha_1+\alpha_2)\Gamma_k(k\sigma)\Phi_{\sigma}^{\psi}(\delta,\ \varkappa_1)} + \left(\mathscr{I}_{\varkappa_1+}^{\lambda,\ k;\ \psi}\varphi\right)(\delta)\right],\ \ \delta \in (\varkappa_1,\ \varkappa_2], \tag{5}$$

where $\alpha_1,\ \alpha_2,\ \alpha_3 \in \mathbb{R}$ such that $\alpha_1 + \alpha_2 \ne 0$.

Proof. Assume that ξ satisfies the equations (3)-(4). We apply $\mathscr{I}_{\varkappa_1+}^{\lambda,\ k;\ \psi}(\cdot)$ on both sides of equation (3) to obtain

$$\left(\mathscr{I}_{\varkappa_1+}^{\lambda,\ k;\ \psi}\ {}_{k}^{H}\mathscr{D}_{\varkappa_1+}^{\lambda,\ \rho;\ \psi}\Theta\xi\right)(\delta) = \left(\mathscr{I}_{\varkappa_1+}^{\lambda,\ k;\ \psi}\varphi\right)(\delta),$$

and using Theorem 9, we get

$$\xi(\delta) = \frac{1}{\Theta(\delta)}\left[\frac{\mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\ k;\ \psi}\xi(\varkappa_1)}{\Phi_\sigma^\psi(\delta,\ \varkappa_1)\Gamma_k(k\sigma)} + \left(\mathscr{I}_{\varkappa_1+}^{\lambda,\ k;\ \psi}\varphi\right)(\delta)\right]. \tag{6}$$

Applying $\mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\ k;\ \psi}(\cdot)$ on both sides of (6), using Lemma 4, Lemma 5 and taking $\delta = \varkappa_2$, we have

$$\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\ k;\ \psi}\Theta\xi\right)(\varkappa_2) = \left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\ k;\ \psi}\Theta\xi\right)(\varkappa_1) + \left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma)+\lambda,\ k;\ \psi}\varphi\right)(\varkappa_2). \tag{7}$$

Multiplying both sides of (7) by α_2, we get

$$\alpha_2\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\ k;\ \psi}\Theta\xi\right)(\varkappa_2) = \alpha_2\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\ k;\ \psi}\Theta\xi\right)(\varkappa_1) + \alpha_2\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma)+\lambda,\ k;\ \psi}\varphi\right)(\varkappa_2).$$

Using condition (4), we obtain

$$\alpha_2\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\ k;\ \psi}\Theta\xi\right)(\varkappa_2) = \alpha_3 - \alpha_1\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\ k;\ \psi}\Theta\xi\right)(\varkappa_1{}^+).$$

Thus

$$\alpha_3 - \alpha_1\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\ k;\ \psi}\Theta\xi\right)(\varkappa_1{}^+) = \alpha_2\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\ k;\ \psi}\Theta\xi\right)(\varkappa_1) + \alpha_2\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma)+\lambda,\ k;\ \psi}\varphi\right)(\varkappa_2).$$

Then

$$\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\ k;\ \psi}\Theta\xi\right)(\varkappa_1{}^+) = \frac{\alpha_3}{\alpha_1+\alpha_2} - \frac{\alpha_2}{\alpha_1+\alpha_2}\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma)+\lambda,\ k;\ \psi}\varphi\right)(\varkappa_2). \tag{8}$$

Substituting (8) into (6), we obtain (5).

For the converse, let us now prove that if ξ satisfies equation (5), then it satisfies (3)-(4). We apply the operator ${}_k^H\mathscr{D}_{\varkappa_1+}^{\lambda,\ \rho;\ \psi}(\cdot)$ on equation (5) to get

$$\left({}_k^H\mathscr{D}_{\varkappa_1+}^{\lambda,\ \rho;\ \psi}\Theta\xi\right)(\delta) = {}_k^H\mathscr{D}_{\varkappa_1+}^{\lambda,\ \rho;\ \psi}\left(\frac{\alpha_3 - \alpha_2\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma)+\lambda,\ k;\ \psi}\varphi\right)(\varkappa_2)}{(\alpha_1+\alpha_2)\Gamma_k(k\sigma)\Phi_\sigma^\psi(\delta,\ \varkappa_1)}\right) + \left({}_k^H\mathscr{D}_{\varkappa_1+}^{\lambda,\ \rho;\ \psi}\ \mathscr{I}_{\varkappa_1+}^{\lambda,\ k;\ \psi}\varphi\right)(\delta).$$

Using Lemma 8 and Lemma 10, we get (3). Now we apply the operator $\mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\ k;\ \psi}(\cdot)$ to equation (5), to obtain

$$\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\, k;\, \psi}\Theta\xi\right)(\delta) = \mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\, k;\, \psi}\left(\frac{\alpha_3 - \alpha_2\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma)+\lambda,\, k;\, \psi}\varphi\right)(\varkappa_2)}{(\alpha_1+\alpha_2)\Gamma_k(k\sigma)\Phi_\sigma^\psi(\delta,\, \varkappa_1)}\right)$$

$$+\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\, k;\, \psi}\mathscr{I}_{\varkappa_1+}^{\lambda,\, k;\, \psi}\varphi\right)(\delta).$$

Now, using Lemma 4 and 5, we get

$$\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\, k;\, \psi}\Theta\xi\right)(\delta) = \frac{\alpha_3}{\alpha_1+\alpha_2} - \frac{\alpha_2}{\alpha_1+\alpha_2}\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma)+\lambda,\, k;\, \psi}\varphi\right)(\varkappa_2) + \left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma)+\lambda,\, k;\, \psi}\varphi\right)(\delta). \tag{9}$$

Using Theorem 6 with $\delta \to \varkappa_1$, we obtain

$$\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\, k;\, \psi}\Theta\xi\right)(\varkappa_1{}^+) = \frac{\alpha_3}{\alpha_1+\alpha_2} - \frac{\alpha_2}{\alpha_1+\alpha_2}\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma)+\lambda,\, k;\, \psi}\varphi\right)(\varkappa_2). \tag{10}$$

Next, taking $\delta = \varkappa_2$ in (9), we have

$$\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma),\, k;\, \psi}\Theta\xi\right)(\varkappa_2) = \frac{\alpha_3}{\alpha_1+\alpha_2} - \frac{\alpha_2}{\alpha_1+\alpha_2}\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma)+\lambda,\, k;\, \psi}\varphi\right)(\varkappa_2) + \left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma)+\lambda,\, k;\, \psi}\varphi\right)(\varkappa_2). \tag{11}$$

Based on the results from equations (10) and (11), we can arrive at equation (4). This concludes the proof of the theorem. □

Leveraging Theorem 11, we can establish the following lemma:

Lemma 12 Let $i = 1,\ 2,\ \sigma_i = \dfrac{r_i(k-\vartheta_i)+\vartheta_i}{k}$ where $0 < \vartheta_i < k$ and $0 \le r_i \le 1$, and let $\varsigma_i : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous functions. Then $(\xi,\ \gamma) \in \mathscr{F}_{\sigma_1,\ \sigma_2}$ satisfies the coupled system (1)-(2) if and only if $(\xi,\ \gamma)$ is the fixed point of the operator $\mathscr{H} : \mathscr{F}_{\sigma_1,\ \sigma_2} \to \mathscr{F}_{\sigma_1,\ \sigma_2}$ defined by:

$$\mathscr{H}(\xi,\ \gamma)(\delta) = (\mathscr{H}_1(\xi,\ \gamma)(\delta),\ \mathscr{H}_2(\xi,\ \gamma)(\delta)),\ \ \delta \in (\varkappa_1,\ \varkappa_2], \tag{12}$$

the operators $\mathscr{H}_1$ and $\mathscr{H}_2$ defined for $\delta \in (\varkappa_1,\ \varkappa_2]$ as:

$$\begin{aligned}\mathscr{H}_1(\xi,\ \gamma)(\delta) = \frac{1}{\Theta(\delta)}\Bigg[&\frac{\alpha_3 - \alpha_2\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma_1)+\vartheta_1,\, k;\, \psi}\varsigma_1(s,\ \xi(s),\ \gamma(s))\right)(\varkappa_2)}{(\alpha_1+\alpha_2)\Gamma_k(k\sigma_1)\Phi_{\sigma_1}^\psi(\delta,\ \varkappa_1)} \\ &+\left(\mathscr{I}_{\varkappa_1+}^{\vartheta_1,\, k;\, \psi}\varsigma_1(s,\ \xi(s),\ \gamma(s))\right)(\delta)\Bigg],\end{aligned} \tag{13}$$

and

$$\mathscr{H}_2(\xi,\ \gamma)(\delta) = \frac{1}{\widehat{\Theta}(\delta)}\left[\frac{\beta_3 - \beta_2\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma_2)+\vartheta_2,\ k;\ \psi}\varsigma_2\left(s,\ \xi(s),\ \gamma(s)\right)\right)(\varkappa_2)}{(\beta_2+\beta_2)\Gamma_k(k\sigma_2)\Phi_{\sigma_2}^{\psi}(\delta,\ \varkappa_1)}\right. \tag{14}$$

$$\left. + \left(\mathscr{I}_{\varkappa_1+}^{\vartheta_2,\ k;\ \psi}\varsigma_2\left(s,\ \xi(s),\ \gamma(s)\right)\right)(\delta)\right].$$

Building upon Theorem 3, we can readily show that for any $(\xi,\ \gamma) \in \mathscr{F}_{\sigma_1,\ \sigma_2}$, the operator $\mathscr{H}$ defined in equation (12) maps this element to a member within the set $\mathscr{F}_{\sigma_1,\ \sigma_2}$.

The hypotheses

(Cd_1) The functions $\varsigma_i : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R};\ i = 1,\ 2$, are continuous.

(Cd_2) There exist constants $\zeta_i,\ \varpi_i > 0$ such that

$$|\varsigma_1(\delta,\ \xi_1,\ \gamma_1) - \varsigma_1(\delta,\ \xi_2,\ \gamma_2)| \leq \zeta_1|\xi_1 - \xi_2| + \varpi_1\Phi_{\sigma_2}^{\psi}(\delta,\ \varkappa_1)|\gamma_1 - \gamma_2|$$

and

$$|\varsigma_2(\delta,\ \xi_1,\ \gamma_1) - \varsigma_2(\delta,\ \xi_2,\ \gamma_2)| \leq \zeta_2\Phi_{\sigma_1}^{\psi}(\delta,\ \varkappa_1)|\xi_1 - \xi_2| + \varpi_2|\gamma_1 - \gamma_2|$$

for any $\xi_i,\ \gamma_i \in \mathbb{R}$ and $\delta \in (\varkappa_1,\ \varkappa_2]$, where $i = 1,\ 2$.

(Cd_3) There exist functions $q_i,\ p_i \in C(\mho,\ \mathbb{R}_+)$ with

$$q_i^* = \sup_{\delta\in\mho} q_i(\delta);\ \ i = 1,\ 2,$$

and

$$p_i^* = \sup_{\delta\in\mho} p_i(\delta);\ \ i = 1,\ 2,$$

such that

$$(1 + |\xi| + |\gamma|)|\varsigma_1(\delta,\ \xi,\ \gamma)| \leq q_1(\delta)|\xi| + q_2(\delta)|\gamma|$$

and

$$(1 + |\xi| + |\gamma|)|\varsigma_2(\delta,\ \xi,\ \gamma)| \leq p_1(\delta)|\xi| + p_2(\delta)|\gamma|$$

for any $\xi,\ \gamma \in \mathbb{R}$, $\delta \in (\varkappa_1,\ \varkappa_2]$ and $i = 1,\ 2$.

(Cd_4) The functions $\Theta,\ \widehat{\Theta}$ are continuous on J and there exist $\mathrm{Q},\ \widehat{\mathrm{Q}} > 0$ where

$$|\Theta(\delta)| \geq \mathrm{Q} \ \text{ and } \ |\widehat{\Theta}(\delta)| \geq \widehat{\mathrm{Q}},$$

for all $\delta \in \mho$.

We are now prepared to establish our first existence result for problem (1)-(2), leveraging the power of Banach's fixed point theorem [50].

Theorem 13 Suppose that (Cd_1), (Cd_2) and (Cd_4) hold. If

$$\Upsilon = \max\{\mathscr{A}_1,\ \mathscr{B}_1\} + \max\{\mathscr{A}_2,\ \mathscr{B}_2\} < 1, \tag{15}$$

where

$$\mathscr{A}_1 := \frac{\zeta_1(\psi(\varkappa_2)-\psi(\varkappa_1))^{\frac{\vartheta_1}{k}}}{\mathrm{Q}}\left[\frac{|\alpha_2|}{|\alpha_1+\alpha_2|\Gamma_k(k+\vartheta_1)} + \frac{\Gamma_k(k\sigma_1)}{\Gamma_k(\vartheta_1+k\sigma_1)}\right],$$

$$\mathscr{A}_2 := \frac{\varpi_1(\psi(\varkappa_2)-\psi(\varkappa_1))^{1-\sigma_1+\frac{\vartheta_1}{k}}}{\mathrm{Q}}\left[\frac{|\alpha_2|}{|\alpha_1+\alpha_2|\Gamma_k(k\sigma_1)\Gamma_k(2k-k\sigma_1+\vartheta_1)} + \frac{1}{\Gamma_k(\vartheta_1+k)}\right],$$

and

$$\mathscr{B}_1 := \frac{\zeta_2(\psi(\varkappa_2)-\psi(\varkappa_1))^{1-\sigma_2+\frac{\vartheta_2}{k}}}{\widehat{\mathrm{Q}}}\left[\frac{|\beta_2|}{|\beta_1+\beta_2|\Gamma_k(k\sigma_2)\Gamma_k(2k-k\sigma_2+\vartheta_2)} + \frac{1}{\Gamma_k(\vartheta_2+k)}\right],$$

$$\mathscr{B}_2 := \frac{\varpi_2(\psi(\varkappa_2)-\psi(\varkappa_1))^{\frac{\vartheta_2}{k}}}{\widehat{\mathrm{Q}}}\left[\frac{|\beta_2|}{|\beta_1+\beta_2|\Gamma_k(k+\vartheta_2)} + \frac{\Gamma_k(k\sigma_2)}{\Gamma_k(\vartheta_2+k\sigma_2)}\right],$$

then the problem (1)-(2) has a unique solution in $\mathscr{F}_{\sigma_1,\ \sigma_2}$.

Proof. We demonstrate that $\mathscr{H}$ defined in (12) has a unique fixed point in $\mathscr{F}_{\sigma_1,\ \sigma_2}$.

Let $(\xi,\ \gamma),\ (\bar{\xi},\ \bar{\gamma}) \in \mathscr{F}_{\sigma_1,\ \sigma_2}$. Then for any $\delta \in (\varkappa_1,\ \varkappa_2]$ we have

$$\begin{aligned}
&\left|\mathscr{H}_1(\xi,\ \gamma)(\delta) - \mathscr{H}_1(\bar{\xi},\ \bar{\gamma})(\delta)\right| \\
&\leq \frac{1}{|\Theta(\delta)|}\left[\frac{|\alpha_2|\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma_1)+\vartheta_1,\ k;\ \psi}\left|\varsigma_1(s,\ \xi(s),\ \gamma(s)) - \varsigma_1\left(s,\ \bar{\xi}(s),\ \bar{\gamma}(s)\right)\right|\right)(\varkappa_2)}{|\alpha_1+\alpha_2|\Gamma_k(k\sigma_1)\Phi_{\sigma_1}^{\psi}(\delta,\ \varkappa_1)}\right. \\
&\left. + \left(\mathscr{I}_{\varkappa_1+}^{\vartheta_1,\ k;\ \psi}\left|\varsigma_1(s,\ \xi(s),\ \gamma(s)) - \varsigma_1\left(s,\ \bar{\xi}(s),\ \bar{\gamma}(s)\right)\right|\right)(\delta)\right].
\end{aligned}$$

By the hypothesis (Cd_2), we have for each $\delta \in (\varkappa_1,\ \varkappa_2]$

$$\left|\mathscr{H}_1(\xi,\ \gamma)(\delta) - \mathscr{H}_1(\bar{\xi},\ \bar{\gamma})(\delta)\right|$$

$$\leq \frac{1}{|\Theta(\delta)|}\left[\frac{|\alpha_2|\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma_1)+\vartheta_1,\ k;\ \psi}\zeta_1|\xi(s)-\bar{\xi}(s)|+\varpi_1\Phi_{\sigma_2}^{\psi}(s,\ \varkappa_1)|\gamma(s)-\bar{\gamma}(s)|\right)(\varkappa_2)}{|\alpha_1+\alpha_2|\Gamma_k(k\sigma_1)\Phi_{\sigma_1}^{\psi}(\delta,\ \varkappa_1)}\right.$$

$$\left.+\left(\mathscr{I}_{\varkappa_1+}^{\vartheta_1,\ k;\ \psi}\zeta_1|\xi(s)-\bar{\xi}(s)|+\varpi_1\Phi_{\sigma_2}^{\psi}(s,\ \varkappa_1)|\gamma(s)-\bar{\gamma}(s)|\right)(\delta)\right].$$

Then, for each $\delta \in (\varkappa_1,\ \varkappa_2]$, we obtain

$$\left|\mathscr{H}_1(\xi,\ \gamma)(\delta) - \mathscr{H}_1(\bar{\xi},\ \bar{\gamma})(\delta)\right|$$

$$\leq \frac{1}{|\Theta(\delta)|}\left[\frac{|\alpha_2|\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma_1)+\vartheta_1,\ k;\ \psi}\zeta_1(\psi(s)-\psi(\varkappa_1))^{\sigma_1-1}\Phi_{\sigma_1}^{\psi}(s,\ \varkappa_1)|\xi(s)-\bar{\xi}(s)|\right)(\varkappa_2)}{|\alpha_1+\alpha_2|\Gamma_k(k\sigma_1)\Phi_{\sigma_1}^{\psi}(\delta,\ \varkappa_1)}\right.$$

$$+\frac{|\alpha_2|\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma_1)+\vartheta_1,\ k;\ \psi}\varpi_1\Phi_{\sigma_2}^{\psi}(s,\ \varkappa_1)|\gamma(s)-\bar{\gamma}(s)|\right)(\varkappa_2)}{|\alpha_1+\alpha_2|\Gamma_k(k\sigma_1)\Phi_{\sigma_1}^{\psi}(\delta,\ \varkappa_1)}$$

$$+\left(\mathscr{I}_{\varkappa_1+}^{\vartheta_1,\ k;\ \psi}\zeta_1(\psi(s)-\psi(\varkappa_1))^{\sigma_1-1}\Phi_{\sigma_1}^{\psi}(s,\ \varkappa_1)|\xi(s)-\bar{\xi}(s)|\right)(\delta)$$

$$\left.+\left(\mathscr{I}_{\varkappa_1+}^{\vartheta_1,\ k;\ \psi}\varpi_1\Phi_{\sigma_2}^{\psi}(s,\ \varkappa_1)|\gamma(s)-\bar{\gamma}(s)|\right)(\delta)\right].$$

Thus,

$$\left|\mathscr{H}_1(\xi,\ \gamma)(\delta)-\mathscr{H}_1(\bar{\xi},\ \bar{\gamma})(\delta)\right| \leq \frac{1}{|\Theta(\delta)|}\left[\frac{|\alpha_2|\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma_1)+\vartheta_1,\ k;\ \psi}\zeta_1(\psi(s)-\psi(\varkappa_1))^{\sigma_1-1}\right)(\varkappa_2)}{|\alpha_1+\alpha_2|\Gamma_k(k\sigma_1)\Phi_{\sigma_1}^{\psi}(\delta,\ \varkappa_1)}\|\xi-\bar{\xi}\|_{C_{\sigma_1;\ \psi}}\right.$$

$$+\frac{|\alpha_2|\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma_1)+\vartheta_1,\ k;\ \psi}\varpi_1\right)(\varkappa_2)}{|\alpha_1+\alpha_2|\Gamma_k(k\sigma_1)\Phi_{\sigma_1}^{\psi}(\delta,\ \varkappa_1)}\|\gamma-\bar{\gamma}\|_{C_{\sigma_2;\ \psi}}$$

$$+\|\xi-\bar{\xi}\|_{C_{\sigma_1;\ \psi}}\left(\mathscr{I}_{\varkappa_1+}^{\vartheta_1,\ k;\ \psi}\zeta_1(\psi(s)-\psi(\varkappa_1))^{\sigma_1-1}\right)(\delta)$$

$$\left.+\|\gamma-\bar{\gamma}\|_{C_{\sigma_2;\ \psi}}\left(\mathscr{I}_{\varkappa_1+}^{\vartheta_1,\ k;\ \psi}\varpi_1\right)(\delta)\right].$$

Lemma 5 implies

$$\left|\mathscr{H}_1(\xi,\ \gamma)(\delta)-\mathscr{H}_1(\bar{\xi},\ \bar{\gamma})(\delta)\right| \leq \frac{1}{|\Theta(\delta)|}\left[\frac{\zeta_1|\alpha_2|(\psi(\varkappa_2)-\psi(\varkappa_1))^{\frac{\vartheta_1}{k}}}{|\alpha_1+\alpha_2|\Gamma_k(k+\vartheta_1)\Phi_{\sigma_1}^{\psi}(\delta,\ \varkappa_1)}\|\xi-\bar{\xi}\|_{C_{\sigma_1;\ \psi}}\right.$$

$$+\frac{\varpi_1|\alpha_2|(\psi(\varkappa_2)-\psi(\varkappa_1))^{1-\sigma_1+\frac{\vartheta_1}{k}}}{|\alpha_1+\alpha_2|\Gamma_k(k\sigma_1)\Gamma_k(2k-k\sigma_1+\vartheta_1)\Phi_{\sigma_1}^{\psi}(\delta,\ \varkappa_1)}\|\gamma-\bar{\gamma}\|_{C_{\sigma_2;\ \psi}}$$

$$+\frac{\zeta_1\Gamma_k(k\sigma_1)(\psi(\delta)-\psi(\varkappa_1))^{\frac{\vartheta_1}{k}+\sigma_1-1}}{\Gamma_k(\vartheta_1+k\sigma_1)}\|\xi-\bar{\xi}\|_{C_{\sigma_1;\ \psi}}$$

$$\left.+\frac{\varpi_1(\psi(\delta)-\psi(\varkappa_1))^{\frac{\vartheta_1}{k}}}{\Gamma_k(\vartheta_1+k)}\|\gamma-\bar{\gamma}\|_{C_{\sigma_2;\ \psi}}\right].$$

Hence,

$$\left|\Phi^{\psi}_{\sigma_1}(\delta,\ \varkappa_1)\mathscr{H}_1(\xi,\ \gamma)(\delta)-\mathscr{H}_1(\bar{\xi},\ \bar{\gamma})(\delta)\right| \leq \|\xi-\bar{\xi}\|_{C_{\sigma_1;\ \psi}}\left[\frac{\zeta_1|\alpha_2|(\psi(\varkappa_2)-\psi(\varkappa_1))^{\frac{\vartheta_1}{k}}}{Q|\alpha_1+\alpha_2|\Gamma_k(k+\vartheta_1)}\right.$$

$$\left.+\frac{\zeta_1\Gamma_k(k\sigma_1)(\psi(\delta)-\psi(\varkappa_1))^{\frac{\vartheta_1}{k}}}{Q\Gamma_k(\vartheta_1+k\sigma_1)}\right]$$

$$+\|\gamma-\bar{\gamma}\|_{C_{\sigma_2;\ \psi}}\left[\frac{\varpi_1|\alpha_2|(\psi(\varkappa_2)-\psi(\varkappa_1))^{1-\sigma_1+\frac{\vartheta_1}{k}}}{Q|\alpha_1+\alpha_2|\Gamma_k(k\sigma_1)\Gamma_k(2k-k\sigma_1+\vartheta_1)}\right.$$

$$\left.+\frac{\varpi_1(\psi(\delta)-\psi(\varkappa_1))^{1-\sigma_1+\frac{\vartheta_1}{k}}}{Q\Gamma_k(\vartheta_1+k)}\right],$$

which implies that

$$\|\mathscr{H}_1(\xi,\ \gamma)-\mathscr{H}_1(\bar{\xi},\ \bar{\gamma})\|_{C_{\sigma_1;\ \psi}} \leq \mathscr{A}_1\|\xi-\bar{\xi}\|_{C_{\sigma_1;\ \psi}}+\mathscr{A}_2\|\gamma-\bar{\gamma}\|_{C_{\sigma_2;\ \psi}}. \tag{16}$$

By following the same approach for obtaining (16), and for $(\xi,\ \gamma),\ (\bar{\xi},\ \bar{\gamma})\in\mathscr{F}_{\sigma_1,\ \sigma_2}$ and for any $\delta\in(\varkappa_1,\ \varkappa_2]$, we can obtain

$$\left|\Phi^{\psi}_{\sigma_2}(\delta,\ \varkappa_1)\mathscr{H}_2(\xi,\ \gamma)(\delta)-\mathscr{H}_2(\bar{\xi},\ \bar{\gamma})(\delta)\right| \leq \|\xi-\bar{\xi}\|_{C_{\sigma_1;\ \psi}}\left[\frac{\zeta_2|\beta_2|(\psi(\varkappa_2)-\psi(\varkappa_1))^{1-\sigma_2+\frac{\vartheta_2}{k}}}{\widehat{Q}|\beta_1+\beta_2|\Gamma_k(k\sigma_2)\Gamma_k(2k-k\sigma_2+\vartheta_2)}\right.$$

$$\left.+\frac{\zeta_2(\psi(\delta)-\psi(\varkappa_1))^{1-\sigma_2+\frac{\vartheta_2}{k}}}{\widehat{Q}\Gamma_k(\vartheta_2+k)}\right]$$

$$+\|\gamma-\bar{\gamma}\|_{C_{\sigma_2;\ \psi}}\left[\frac{\varpi_2|\beta_2|(\psi(\varkappa_2)-\psi(\varkappa_1))^{\frac{\vartheta_2}{k}}}{\widehat{Q}|\beta_1+\beta_2|\Gamma_k(k+\vartheta_2)}\right.$$

$$\left.+\frac{\varpi_2\Gamma_k(k\sigma_2)(\psi(\delta)-\psi(\varkappa_1))^{\frac{\vartheta_2}{k}}}{\widehat{Q}\Gamma_k(\vartheta_2+k\sigma_2)}\right],$$

which implies that

$$\|\mathscr{H}_2(\xi,\ \gamma)-\mathscr{H}_2(\bar{\xi},\ \bar{\gamma})\|_{C_{\sigma_2;\ \psi}} \leq \mathscr{B}_1\|\xi-\bar{\xi}\|_{C_{\sigma_1;\ \psi}}+\mathscr{B}_2\|\gamma-\bar{\gamma}\|_{C_{\sigma_2;\ \psi}}. \tag{17}$$

We can deduce now that by (16) and (17), we have

$$
\begin{aligned}
&\|\mathscr{H}(\xi,\ \gamma)-\mathscr{H}(\bar{\bar{\xi}},\ \bar{\gamma})\|_{\mathscr{F}_{\sigma_1,\ \sigma_2}}\\
&=\max\left\{\|\mathscr{H}_1(\xi,\ \gamma)-\mathscr{H}_1(\bar{\bar{\xi}},\ \bar{\gamma})\|_{C_{\sigma_1;\ \psi}},\ \|\mathscr{H}_2(\xi,\ \gamma)-\mathscr{H}_2(\bar{\bar{\xi}},\ \bar{\gamma})\|_{C_{\sigma_2;\ \psi}}\right\}\\
&\leq(\max\{\mathscr{A}_1,\ \mathscr{B}_1\}+\max\{\mathscr{A}_2,\ \mathscr{B}_2\})\,\|(\xi,\ \gamma)-(\bar{\bar{\xi}},\ \bar{\gamma})\|_{\mathscr{F}_{\sigma_1,\ \sigma_2}}\\
&\leq\Upsilon\|(\xi,\ \gamma)-(\bar{\bar{\xi}},\ \bar{\gamma})\|_{\mathscr{F}_{\sigma_1,\ \sigma_2}}.
\end{aligned}
$$

By (15), the operator $\mathscr{H}$ is a contraction on $\mathscr{F}_{\sigma_1,\ \sigma_2}$. Thus, by Banach's contraction principle, $\mathscr{H}$ has a unique fixed point $(\xi,\ \gamma)\in\mathscr{F}_{\sigma_1,\ \sigma_2}$, which is a solution to our problem (1)-(2). □

Our next existence result for the problem (1)-(2) is based on based on Schauder's fixed point theorem [50].

Theorem 14 Assume (Cd_1), (Cd_3) and (Cd_4) hold. Then problem (1)-(2) has at least one solution in $\mathscr{F}_{\sigma_1,\ \sigma_2}$.

Proof. Consider the operator $\mathscr{H}$ defined in (12).

Claim 1 The operator $\mathscr{H}$ is continuous.

Let $\{(\xi_n,\ \gamma_n)\}$ be a sequence where $(\xi_n,\ \gamma_n)\longrightarrow(\xi,\ \gamma)$ in $\mathscr{F}_{\sigma_1,\ \sigma_2}$. For each $\delta\in(\varkappa_1,\ \varkappa_2]$, we have

$$
\begin{aligned}
&|\mathscr{H}_1(\xi_n,\ \gamma_n)(\delta)-\mathscr{H}_1(\xi,\ \gamma)(\delta)|\\
&\leq\frac{1}{|\Theta(\delta)|}\left[\frac{|\alpha_2|\left(\mathscr{J}_{\varkappa_1+}^{k(1-\sigma_1)+\vartheta_1,\ k;\ \psi}|\varsigma_1(s,\ \xi_n(s),\ \gamma_n(s))-\varsigma_1(s,\ \xi(s),\ \gamma(s))|\right)(\varkappa_2)}{|\alpha_1+\alpha_2|\Gamma_k(k\sigma_1)\Phi_{\sigma_1}^{\psi}(\delta,\ \varkappa_1)}\right.\\
&\left.+\left(\mathscr{J}_{\varkappa_1+}^{\vartheta_1,\ k;\ \psi}|\varsigma_1(s,\ \xi_n(s),\ \gamma_n(s))-\varsigma_1(s,\ \xi(s),\ \gamma(s))|\right)(\delta)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&|\mathscr{H}_2(\xi_n,\ \gamma_n)(\delta)-\mathscr{H}_2(\xi,\ \gamma)(\delta)|\\
&\leq\frac{1}{|\widehat{\Theta}(\delta)|}\left[\frac{|\beta_2|\left(\mathscr{J}_{\varkappa_1+}^{k(1-\sigma_2)+\vartheta_2,\ k;\ \psi}|\varsigma_2(s,\ \xi_n(s),\ \gamma_n(s))-\varsigma_2(s,\ \xi(s),\ \gamma(s))|\right)(\varkappa_2)}{|\beta_1+\beta_2|\Gamma_k(k\sigma_2)\Phi_{\sigma_2}^{\psi}(\delta,\ \varkappa_1)}\right.\\
&\left.+\left(\mathscr{J}_{\varkappa_1+}^{\vartheta_2,\ k;\ \psi}|\varsigma_2(s,\ \xi_n(s),\ \gamma_n(s))-\varsigma_2(s,\ \xi(s),\ \gamma(s))|\right)(\delta)\right].
\end{aligned}
$$

Since $(\xi_n,\ \gamma_n)\to(\xi,\ \gamma)$, then we get

$$\varsigma_1\left(s,\ \xi_n(s),\ \gamma_n(s)\right) \to \varsigma_1\left(s,\ \xi(s),\ \gamma(s)\right)$$

and

$$\varsigma_2\left(s,\ \xi_n(s),\ \gamma_n(s)\right) \to \varsigma_2\left(s,\ \xi(s),\ \gamma(s)\right),$$

as $n \to \infty$ for each $\delta \in (\varkappa_1,\ \varkappa_2]$, and since ς_1 and ς_2 are continuous, then we have

$$\|\mathscr{H}_1(\xi_n,\ \gamma_n) - \mathscr{H}_1(\xi,\ \gamma)\|_{C_{\sigma_1;\ \psi}} \to 0 \ \text{ and } \ \|\mathscr{H}_2(\xi_n,\ \gamma_n) - \mathscr{H}_2(\xi,\ \gamma)\|_{C_{\sigma_2;\ \psi}} \to 0 \ \text{ as } \ n \to \infty.$$

Thus, for each $\delta \in (\varkappa_1,\ \varkappa_2]$, we get

$$\|\mathscr{H}(\xi_n,\ \gamma_n) - \mathscr{H}(\xi,\ \gamma)\|_{\mathscr{F}_{\sigma_1,\ \sigma_2}} \to 0 \ \text{ as } \ n \to \infty.$$

Consequently, $\mathscr{H}$ is continuous.

Claim 2 $\mathscr{H}(B_\Omega) \subset B_\Omega$.

Let $\Omega \geq \max\{\Omega_1,\ \Omega_2\}$, where

$$\Omega_1 = \frac{|\alpha_3|}{\mathrm{Q}|\alpha_1+\alpha_2|\Gamma_k(k\sigma_1)} + (q_1^*+q_2^*)\left(\psi(\varkappa_2)-\psi(\varkappa_1)\right)^{1-\sigma_1+\frac{\vartheta_1}{k}}$$

$$\times\left[\frac{1}{\mathrm{Q}\Gamma_k(\vartheta_1+k)} + \frac{|\alpha_2|}{\mathrm{Q}|\alpha_1+\alpha_2|\Gamma_k(k\sigma_1)\Gamma_k(2k-k\sigma_1+\vartheta_1)}\right],$$

and

$$\Omega_2 = \frac{|\beta_3|}{\widehat{\mathrm{Q}}|\beta_1+\beta_2|\Gamma_k(k\sigma_2)} + (p_1^*+p_2^*)\left(\psi(\varkappa_2)-\psi(\varkappa_1)\right)^{1-\sigma_2+\frac{\vartheta_2}{k}}$$

$$\times\left[\frac{1}{\widehat{\mathrm{Q}}\Gamma_k(\vartheta_2+k)} + \frac{|\beta_2|}{\widehat{\mathrm{Q}}|\beta_1+\beta_2|\Gamma_k(k\sigma_2)\Gamma_k(2k-k\sigma_2+\vartheta_2)}\right].$$

We define the following bounded closed set

$$B_\Omega = \left\{(\xi,\ \gamma) \in \mathscr{F}_{\sigma_1,\ \sigma_2} : \|(\xi,\ \gamma)\|_{\mathscr{F}_{\sigma_1,\ \sigma_2}} \leq \Omega\right\}.$$

For each $\delta \in (\varkappa_1,\ \varkappa_2]$, (13) and (14) imply that we have

$$|\mathscr{H}_1(\xi,\ \gamma)(\delta)| \leq \frac{|\alpha_3| + |\alpha_2| \left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma_1)+\vartheta_1,\ k;\ \psi} |\varsigma_1(s,\ \xi(s),\ \gamma(s))| \right)(\varkappa_2)}{\mathrm{Q}|\alpha_1 + \alpha_2|\Gamma_k(k\sigma_1)\Phi_{\sigma_1}^{\psi}(\delta,\ \varkappa_1)} + \frac{1}{\mathrm{Q}} \left(\mathscr{I}_{\varkappa_1+}^{\vartheta_1,\ k;\ \psi} |\varsigma_1(s,\ \xi(s),\ \gamma(s))| \right)(\delta), \tag{18}$$

and

$$|\mathscr{H}_2(\xi,\ \gamma)(\delta)| \leq \frac{|\beta_3| + |\beta_2| \left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma_2)+\vartheta_2,\ k;\ \psi} |\varsigma_2(s,\ \xi(s),\ \gamma(s))| \right)(\varkappa_2)}{\widehat{\mathrm{Q}}|\beta_2 + \beta_2|\Gamma_k(k\sigma_2)\Phi_{\sigma_2}^{\psi}(\delta,\ \varkappa_1)} + \frac{1}{\widehat{\mathrm{Q}}} \left(\mathscr{I}_{\varkappa_1+}^{\vartheta_2,\ k;\ \psi} |\varsigma_2(s,\ \xi(s),\ \gamma(s))| \right)(\delta). \tag{19}$$

By the hypothesis (Cd_3), for $\delta \in (\varkappa_1,\ \varkappa_2]$, we have

$$|\Phi_{\sigma_1}^{\psi}(\delta,\ \varkappa_1)\mathscr{H}_1(\xi,\ \gamma)(\delta)| \leq \frac{|\alpha_3|}{\mathrm{Q}|\alpha_1 + \alpha_2|\Gamma_k(k\sigma_1)} + \frac{|\alpha_2|(q_1(\delta) + q_2(\delta))}{\mathrm{Q}|\alpha_1 + \alpha_2|\Gamma_k(k\sigma_1)} \left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma_1)+\vartheta_1,\ k;\ \psi}(1) \right)(\varkappa_2) + \frac{(q_1(\delta) + q_2(\delta))}{\mathrm{Q}} \Phi_{\sigma_1}^{\psi}(\delta,\ \varkappa_1) \left(\mathscr{I}_{\varkappa_1+}^{\vartheta_1,\ k;\ \psi}(1) \right)(\delta),$$

and

$$|\Phi_{\sigma_2}^{\psi}(\delta,\ \varkappa_1)\mathscr{H}_2(\xi,\ \gamma)(\delta)| \leq \frac{|\beta_3|}{\widehat{\mathrm{Q}}|\beta_1 + \beta_2|\Gamma_k(k\sigma_2)} + \frac{|\beta_2|(p_1(\delta) + p_2(\delta))}{\widehat{\mathrm{Q}}|\beta_1 + \beta_2|\Gamma_k(k\sigma_2)} \left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma_2)+\vartheta_2,\ k;\ \psi}(1) \right)(\varkappa_2) + \frac{(p_1(\delta) + p_2(\delta))}{\widehat{\mathrm{Q}}} \Phi_{\sigma_2}^{\psi}(\delta,\ \varkappa_1) \left(\mathscr{I}_{\varkappa_1+}^{\vartheta_2,\ k;\ \psi}(1) \right)(\delta).$$

Lemma 5 implies

$$|\Phi_{\sigma_1}^{\psi}(\delta,\ \varkappa_1)\mathscr{H}_1(\xi,\ \gamma)(\delta)| \leq \frac{|\alpha_3|}{\mathrm{Q}|\alpha_1 + \alpha_2|\Gamma_k(k\sigma_1)} + (q_1^* + q_2^*) \left[\frac{|\alpha_2| (\psi(\varkappa_2) - \psi(\varkappa_1))^{1-\sigma_1+\frac{\vartheta_1}{k}}}{\mathrm{Q}|\alpha_1 + \alpha_2|\Gamma_k(k\sigma_1)\Gamma_k(2k - k\sigma_1 + \vartheta_1)} + \frac{(\psi(\delta) - \psi(\varkappa_1))^{1-\sigma_1+\frac{\vartheta_1}{k}}}{\mathrm{Q}\Gamma_k(\vartheta_1 + k)} \right],$$

and

$$|\Phi^{\psi}_{\sigma_2}(\delta,\ \varkappa_1)\mathscr{H}_2(\xi,\ \gamma)(\delta)| \leq \frac{|\beta_3|}{\widehat{\mathrm{Q}}|\beta_1+\beta_2|\Gamma_k(k\sigma_2)} + (p_1^* + p_2^*)$$

$$\left[\frac{|\beta_2|\left(\psi(\varkappa_2)-\psi(\varkappa_1)\right)^{1-\sigma_2+\frac{\vartheta_2}{k}}}{\widehat{\mathrm{Q}}|\beta_1+\beta_2|\Gamma_k(k\sigma_2)\Gamma_k(2k-k\sigma_2+\vartheta_2)} + \frac{\left(\psi(\delta)-\psi(\varkappa_1)\right)^{1-\sigma_2+\frac{\vartheta_2}{k}}}{\widehat{\mathrm{Q}}\Gamma_k(\vartheta_2+k)}\right].$$

Thus,

$$\|\mathscr{H}_1(\xi,\ \gamma)(\delta)\|_{C_{\sigma_1;\ \psi}} \leq \frac{|\alpha_3|}{\mathrm{Q}|\alpha_1+\alpha_2|\Gamma_k(k\sigma_1)} + (q_1^*+q_2^*)\left(\psi(\varkappa_2)-\psi(\varkappa_1)\right)^{1-\sigma_1+\frac{\vartheta_1}{k}}\left[\frac{1}{\mathrm{Q}\Gamma_k(\vartheta_1+k)}\right.$$

$$\left. + \frac{|\alpha_2|}{\mathrm{Q}|\alpha_1+\alpha_2|\Gamma_k(k\sigma_1)\Gamma_k(2k-k\sigma_1+\vartheta_1)}\right] := \Omega_1,$$

and

$$\|\mathscr{H}_2(\xi,\ \gamma)(\delta)\|_{C_{\sigma_2;\ \psi}} \leq \frac{|\beta_3|}{\widehat{\mathrm{Q}}|\beta_1+\beta_2|\Gamma_k(k\sigma_2)} + (p_1^*+p_2^*)\left(\psi(\varkappa_2)-\psi(\varkappa_1)\right)^{1-\sigma_2+\frac{\vartheta_2}{k}}\left[\frac{1}{\widehat{\mathrm{Q}}\Gamma_k(\vartheta_2+k)}\right.$$

$$\left. + \frac{|\beta_2|}{\widehat{\mathrm{Q}}|\beta_1+\beta_2|\Gamma_k(k\sigma_2)\Gamma_k(2k-k\sigma_2+\vartheta_2)}\right] := \Omega_2.$$

Thus, for each $\delta \in (\varkappa_1,\ \varkappa_2]$ we get

$$\|\mathscr{H}(\xi,\ \gamma)\|_{\mathscr{F}_{\sigma_1,\ \sigma_2}} \leq \max\{\Omega_1,\ \Omega_2\} \leq \Omega.$$

Claim 3 $\mathscr{H}(B_\Omega)$ is relatively compact.
Let $\tau_1,\ \tau_2 \in (\varkappa_1,\ \varkappa_2]$, $\tau_1 < \tau_2$ and let $(\xi,\ \gamma) \in B_\Omega$. Then

$$\left|\Phi^{\psi}_{\sigma_1}(\tau_1,\ \varkappa_1)\mathscr{H}_1(\xi,\ \gamma)(\tau_1) - \Phi^{\psi}_{\sigma_1}(\tau_2,\ \varkappa_1)\mathscr{H}_1(\xi,\ \gamma)(\tau_2)\right|$$

$$\leq \left|\frac{1}{\Theta(\tau_1)} - \frac{1}{\Theta(\tau_2)}\right| \times \left[\frac{|\alpha_3| + |\alpha_2|\left(\mathscr{I}^{k(1-\sigma_1)+\vartheta_1,\ k;\ \psi}_{\varkappa_1+}|\varsigma_1(s,\ \xi(s),\ \gamma(s))|\right)(\varkappa_2)}{|\alpha_1+\alpha_2|\Gamma_k(k\sigma_1)}\right]$$

$$+ \left|\frac{\Phi^{\psi}_{\sigma_1}(\tau_1,\ \varkappa_1)}{\Theta(\tau_1)}\left(\mathscr{I}^{\vartheta_1,\ k;\ \psi}_{\varkappa_1+}|\varsigma_1(s,\ \xi(s),\ \gamma(s))|\right)(\tau_1)\right.$$

$$
-\frac{\Phi_{\sigma_1}^{\psi}(\tau_2,\ \varkappa_1)}{\Theta(\tau_2)}\left(\mathscr{I}_{\varkappa_1^+}^{\vartheta_1,\ k;\ \psi}|\varsigma_1(s,\ \xi(s),\ \gamma(s))|\right)(\tau_2)\Bigg|
$$

$$
\leq\left|\frac{1}{\Theta(\tau_1)}-\frac{1}{\Theta(\tau_2)}\right|\times\left[\frac{|\alpha_3|+|\alpha_2|\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma_1)+\vartheta_1,\ k;\ \psi}|\varsigma_1(s,\ \xi(s),\ \gamma(s))|\right)(\varkappa_2)}{|\alpha_1+\alpha_2|\Gamma_k(k\sigma_1)}\right]
$$

$$
+\int_{\varkappa_1}^{\tau_1}\left|\frac{\Phi_{\sigma_1}^{\psi}(\tau_1,\ \varkappa_1)}{\Theta(\tau_1)}\bar{\Phi}_{\vartheta_1}^{k,\ \psi}(\tau_1,\ s)-\frac{\Phi_{\sigma_1}^{\psi}(\tau_2,\ \varkappa_1)}{\Theta(\tau_2)}\bar{\Phi}_{\vartheta_1}^{k,\ \psi}(\tau_2,\ s)\right||\psi'(s)\varsigma_1(s,\ \xi(s),\ \gamma(s))|ds
$$

$$
+\left|\frac{\Phi_{\sigma_1}^{\psi}(\tau_2,\ \varkappa_1)}{\Theta(\tau_2)}\left(\mathscr{I}_{\tau_1^+}^{\vartheta_1,\ k;\ \psi}|\varsigma_1(s,\ \xi(s),\ \gamma(s))|\right)(\tau_2)\right|,
$$

and

$$
\left|\Phi_{\sigma_2}^{\psi}(\tau_1,\ \varkappa_1)\mathscr{H}_2(\xi,\ \gamma)(\tau_1)-\Phi_{\sigma_2}^{\psi}(\tau_2,\ \varkappa_1)\mathscr{H}_2(\xi,\ \gamma)(\tau_2)\right|
$$

$$
\leq\left|\frac{1}{\widehat{\Theta}(\tau_1)}-\frac{1}{\widehat{\Theta}(\tau_2)}\right|\times\left[\frac{|\beta_3|+|\beta_2|\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma_2)+\vartheta_2,\ k;\ \psi}|\varsigma_2(s,\ \xi(s),\ \gamma(s))|\right)(\varkappa_2)}{|\beta_1+\beta_2|\Gamma_k(k\sigma_2)}\right]
$$

$$
+\Bigg|\frac{\Phi_{\sigma_2}^{\psi}(\tau_1,\ \varkappa_1)}{\widehat{\Theta}(\tau_1)}\left(\mathscr{I}_{\varkappa_1^+}^{\vartheta_2,\ k;\ \psi}|\varsigma_2(s,\ \xi(s),\ \gamma(s))|\right)(\tau_1)
$$

$$
-\frac{\Phi_{\sigma_2}^{\psi}(\tau_2,\ \varkappa_1)}{\widehat{\Theta}(\tau_2)}\left(\mathscr{I}_{\varkappa_1^+}^{\vartheta_2,\ k;\ \psi}|\varsigma_2(s,\ \xi(s),\ \gamma(s))|\right)(\tau_2)\Bigg|
$$

$$
\leq\left|\frac{1}{\widehat{\Theta}(\tau_1)}-\frac{1}{\widehat{\Theta}(\tau_2)}\right|\times\left[\frac{|\beta_3|+|\beta_2|\left(\mathscr{I}_{\varkappa_1+}^{k(1-\sigma_2)+\vartheta_2,\ k;\ \psi}|\varsigma_2(s,\ \xi(s),\ \gamma(s))|\right)(\varkappa_2)}{|\beta_1+\beta_2|\Gamma_k(k\sigma_2)}\right]
$$

$$
+\int_{\varkappa_1}^{\tau_1}\left|\frac{\Phi_{\sigma_2}^{\psi}(\tau_1,\ \varkappa_1)}{\widehat{\Theta}(\tau_1)}\bar{\Phi}_{\vartheta_2}^{k,\ \psi}(\tau_1,\ s)-\frac{\Phi_{\sigma_2}^{\psi}(\tau_2,\ \varkappa_1)}{\widehat{\Theta}(\tau_2)}\bar{\Phi}_{\vartheta_2}^{k,\ \psi}(\tau_2,\ s)\right||\psi'(s)\varsigma_2(s,\ \xi(s),\ \gamma(s))|ds
$$

$$
+\left|\frac{\Phi_{\sigma_2}^{\psi}(\tau_2,\ \varkappa_1)}{\widehat{\Theta}(\tau_2)}\left(\mathscr{I}_{\tau_1^+}^{\vartheta_2,\ k;\ \psi}|\varsigma_2(s,\ \xi(s),\ \gamma(s))|\right)(\tau_2)\right|,
$$

By Lemma 5, we get

$$\left|\Phi^{\psi}_{\sigma_1}(\tau_1,\, \varkappa_1)\mathscr{H}_1(\xi,\, \gamma)(\tau_1) - \Phi^{\psi}_{\sigma_1}(\tau_2,\, \varkappa_1)\mathscr{H}_1(\xi,\, \gamma)(\tau_2)\right|$$

$$\leq \left|\frac{1}{\Theta(\tau_1)} - \frac{1}{\Theta(\tau_2)}\right| \times \left[\frac{|\alpha_3| + |\alpha_2|\left(\mathscr{I}^{k(1-\sigma_1)+\vartheta_1,\, k;\, \psi}_{\varkappa_1+}|\varsigma_1(s,\, \xi(s),\, \gamma(s))|\right)(\varkappa_2)}{|\alpha_1 + \alpha_2|\Gamma_k(k\sigma_1)}\right]$$

$$+ (q_1^* + q_2^*)\int_{\varkappa_1}^{\tau_1}\left|\frac{\Phi^{\psi}_{\sigma_1}(\tau_1,\, \varkappa_1)}{\Theta(\tau_1)}\bar{\Phi}^{k,\, \psi}_{\vartheta_1}(\tau_1,\, s) - \frac{\Phi^{\psi}_{\sigma_1}(\tau_2,\, \varkappa_1)}{\Theta(\tau_2)}\bar{\Phi}^{k,\, \psi}_{\vartheta_1}(\tau_2,\, s)\right||\psi'(s)|ds$$

$$+ \frac{(q_1^* + q_2^*)\Phi^{\psi}_{\sigma_1}(\tau_2,\, \varkappa_1)(\psi(\tau_2) - \psi(\tau_1))^{\frac{\vartheta_1}{k}}}{|\Theta(\tau_2)|\Gamma_k(\vartheta_1 + k)},$$

and

$$\left|\Phi^{\psi}_{\sigma_2}(\tau_1,\, \varkappa_1)\mathscr{H}_2(\xi,\, \gamma)(\tau_1) - \Phi^{\psi}_{\sigma_2}(\tau_2,\, \varkappa_1)\mathscr{H}_2(\xi,\, \gamma)(\tau_2)\right|$$

$$\leq \left|\frac{1}{\widehat{\Theta}(\tau_1)} - \frac{1}{\widehat{\Theta}(\tau_2)}\right| \times \left[\frac{|\beta_3| + |\beta_2|\left(\mathscr{I}^{k(1-\sigma_2)+\vartheta_2,\, k;\, \psi}_{\varkappa_1+}|\varsigma_2(s,\, \xi(s),\, \gamma(s))|\right)(\varkappa_2)}{|\beta_1 + \beta_2|\Gamma_k(k\sigma_2)}\right]$$

$$+ (p_1^* + p_2^*)\int_{\varkappa_1}^{\tau_1}\left|\frac{\Phi^{\psi}_{\sigma_2}(\tau_1,\, \varkappa_1)}{\widehat{\Theta}(\tau_1)}\bar{\Phi}^{k,\, \psi}_{\vartheta_2}(\tau_1,\, s) - \frac{\Phi^{\psi}_{\sigma_2}(\tau_2,\, \varkappa_1)}{\widehat{\Theta}(\tau_2)}\bar{\Phi}^{k,\, \psi}_{\vartheta_2}(\tau_2,\, s)\right||\psi'(s)|ds$$

$$+ \frac{(p_1^* + p_2^*)\Phi^{\psi}_{\sigma_2}(\tau_2,\, \varkappa_1)(\psi(\tau_2) - \psi(\tau_1))^{\frac{\vartheta_2}{k}}}{|\widehat{\Theta}(\tau_2)|\Gamma_k(\vartheta_2 + k)}.$$

Letting $\tau_1 \to \tau_2$, the right-hand side of the inequalities converges to zero. This demonstrates the equicontinuity of the operator $\mathscr{H}$ on the set B_Ω.

Combining Claims 1-3 with the Arzelà-Ascoli Theorem, we can establish that $\mathscr{H} : \mathscr{F}_{\sigma_1,\, \sigma_2} \to \mathscr{F}_{\sigma_1,\, \sigma_2}$ is a continuous and compact map. As a consequence, a fixed point of $\mathscr{H}$ exists, corresponding to a solution of problem (1)-(2). □

5. Illustrative example

To illustrate the application of our theoretical results, we consider a specific example that can be viewed as a particular case of problem (1)-(2). Throughout this section, we assume the following parameter settings: $\mu = [1,\, e]$, $\sigma_i = \frac{1}{k}(r_i(k - \theta_i) + \theta_i)$ for $i = 1,\, 2$, and the functions

$$\varsigma_1(\delta,\ \xi,\ \gamma)=\frac{\ln(\delta)\sqrt{\delta-1}}{(123+131e^{\delta+1})(1+|\xi|+|\gamma|)},$$

$$\varsigma_2(\delta,\ \xi,\ \gamma)=\frac{\arctan(\delta)}{(56+56e^{\delta+3})(1+|\xi|+|\gamma|)},$$

$$\Theta(\delta)=\frac{2}{13e^{-2}}(2\delta+\cos(\delta)+2),$$

and

$$\widehat{\Theta}(\delta)=\frac{1}{11e^{-1}}(\delta+\sin(\delta)+2),$$

where $\delta\in\mho,\ \xi,\ \gamma\in\mathbb{R}$.

Example 15 Substituting $r_1\to\frac{1}{2}$, $r_2\to 0$, $\vartheta_1=\vartheta_2=\frac{1}{2}$, $k=1$, $\psi(\delta)=\delta$, $\alpha_1=1$, $\alpha_2=0$, $\alpha_3=0$, $\beta_1=1$, $\beta_2=1$, $\sigma_1=\frac{3}{4}$, and $\sigma_2=\frac{1}{2}$ into problem (1)-(2), we obtain a specific instance. This instance comprises a coupled system of two problems: the first one is an initial value problem with a HFD, while the second is a boundary value problem with Riemann-Liouville fractional derivative. This coupled system is expressed as:

$$\begin{cases}\left({}^{H}_{1}\mathscr{D}^{\frac{1}{2},\ \frac{1}{2};\ \psi}_{1+}\xi\right)(\delta)=\left({}^{H}\mathbb{D}^{\frac{1}{2},\ \frac{1}{2}}_{1+}\xi\right)(\delta)=\varsigma_1(\delta,\ \xi(\delta),\ \gamma(\delta)),\ \ \delta\in(1,\ e],\\[2ex] \left({}^{H}_{1}\mathscr{D}^{\frac{1}{2},\ 0;\ \psi}_{1+}\gamma\right)(\delta)=\left({}^{RL}\mathbb{D}^{\frac{1}{2}}_{1+}\xi\right)(\delta)=\varsigma_2(\delta,\ \xi(\delta),\ \gamma(\delta)),\ \ \delta\in(1,\ e],\end{cases}\tag{20}$$

$$\begin{cases}\left(\mathscr{J}^{\frac{1}{4},\ 1;\ \psi}_{1+}\xi\right)(1)=0,\\[2ex] \left(\mathscr{J}^{\frac{1}{2},\ 1;\ \psi}_{1+}\gamma\right)(1)+\left(\mathscr{J}^{\frac{1}{2},\ 1;\ \psi}_{1+}\gamma\right)(e)=1.\end{cases}\tag{21}$$

We have

$$C_{\sigma_1,\ k;\ \psi}(\mho)=C_{\frac{3}{4},\ 1;\ \psi}(\mho)=\left\{\xi:(1,\ e]\to\mathbb{R}:(\delta-1)^{\frac{1}{4}}\xi\in C(\mho,\ \mathbb{R})\right\},$$

and

$$C_{\sigma_2,\ k;\ \psi}(\mho)=C_{\frac{1}{2},\ 1;\ \psi}(\mho)=\left\{\xi:(1,\ e]\to\mathbb{R}:\xi\sqrt{\delta-1}\in C(\mho,\ \mathbb{R})\right\}.$$

Since it is clear to see that the functions ς_1 and ς_2 is continuous, then the condition (Cd_1) is verified. Further, for each $\xi_1,\ \gamma_1,\ \xi_2,\ \gamma_2 \in \mathbb{R}$ and $\delta \in \mho$, we have

$$|\varsigma_1(\delta,\ \xi_1,\ \gamma_1) - \varsigma_1(\delta,\ \xi_2,\ \gamma_2)| \leq \frac{\sqrt{\delta - 1}}{123 + 131e^{\delta+1}} \left(|\xi_1 - \xi_2| + |\gamma_1 - \gamma_2|\right),$$

and

$$|\varsigma_2(\delta,\ \xi_1,\ \gamma_1) - \varsigma_2(\delta,\ \xi_2,\ \gamma_2)| \leq \frac{\pi(\delta - 1)^{\frac{1}{4}}}{56 + 56e^{\delta+3}} \left(|\xi_1 - \xi_2| + |\gamma_1 - \gamma_2|\right).$$

Thus, the condition (Cd_2) is satisfied with

$$\zeta_1 = \frac{\sqrt{e-1}}{123 + 131e^2},\ \varpi_1 = \frac{1}{123 + 131e^2},\ \zeta_2 = \frac{1}{56 + 56e^4},\ \varpi_2 = \frac{(e-1)^{\frac{1}{4}}}{56 + 56e^4}.$$

The functions Θ and $\widehat{\Theta}$ are continuous on J and thus, the condition (Cd_4) is verified with $\mathrm{Q} = \dfrac{8}{13e^{-2}}$ and $\widehat{\mathrm{Q}} = \dfrac{3}{11e^{-1}}$. As illustrated in Figures 1 and 2, these conditions are shown to hold. Further,

$$\Upsilon = \max\{\mathscr{A}_1,\ \mathscr{B}_1\} + \max\{\mathscr{A}_2,\ \mathscr{B}_2\} \approx 0.002569290983 < 1,$$

where

$$\mathscr{A}_1 := \frac{13e^{-2}(e-1)}{123 + 131e^2} \left[\frac{\Gamma\left(\frac{3}{4}\right)}{8\Gamma\left(\frac{5}{4}\right)}\right] \approx 0.000468284898,$$

$$\mathscr{A}_2 := \frac{13e^{-2}(e-1)^{\frac{3}{4}}}{123 + 131e^2} \left[\frac{1}{4\sqrt{\pi}}\right] \approx 0.00034137279,$$

and

$$\mathscr{B}_1 := \frac{11e^{-1}(e-1)}{168 + 168^4} \left[\frac{1}{2\sqrt{\pi}} + \frac{2}{\sqrt{\pi}}\right] \approx 0.00104999698,$$

$$\mathscr{B}_2 := \frac{11e^{-1}(e-1)^{\frac{3}{4}}}{168 + 168e^4} \left[\frac{1}{\sqrt{\pi}} + \sqrt{\pi}\right] \approx 0.001519294003.$$

As all the conditions of Theorem 13 are met, then the problem (20)-(21) has a unique solution in $C_{\frac{3}{4};\,\psi}([1,\ e]) \times C_{\frac{1}{2};\,\psi}([1,\ e])$.

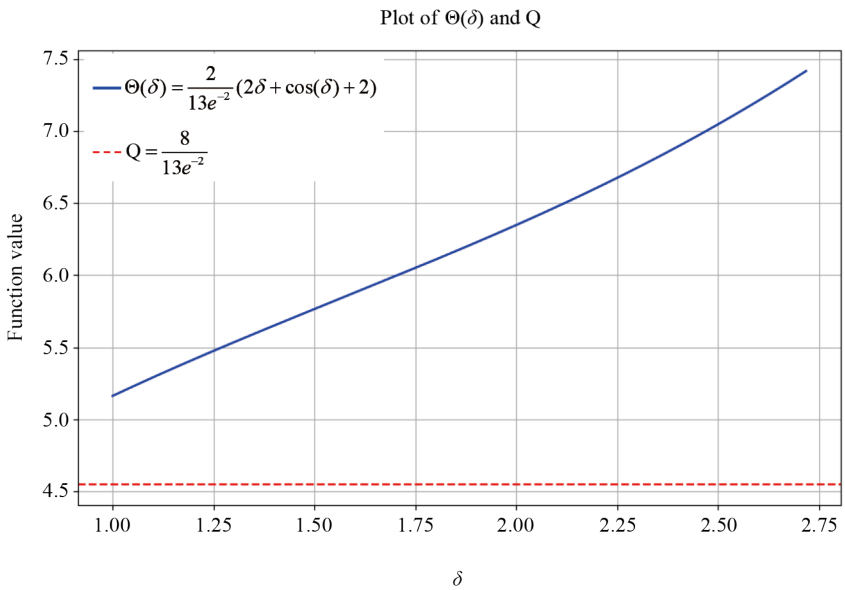

Figure 1. Plot of $\Theta(\delta)$ and Q

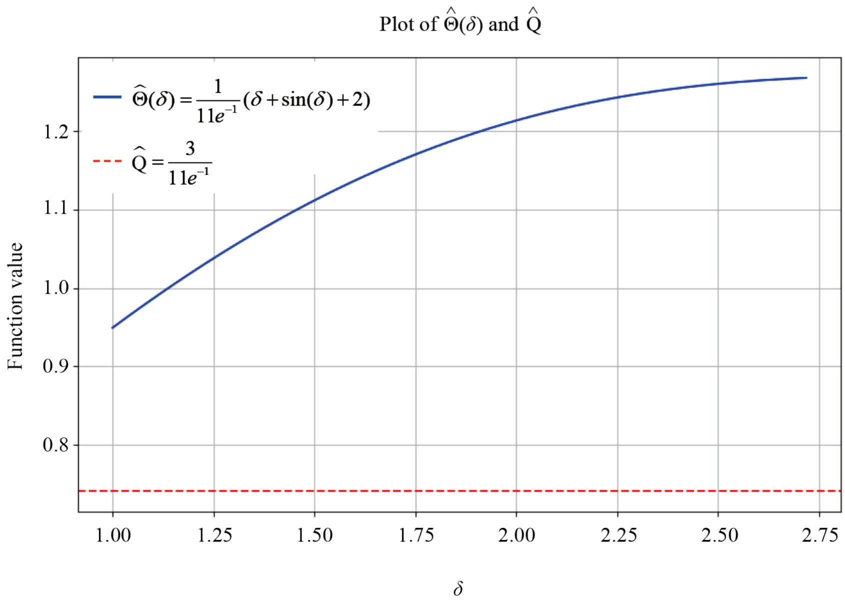

Figure 2. Plot of $\widehat{\Theta}(\delta)$ and $\widehat{\mathrm{Q}}$

6. Conclusion and future scope

This study established the existence and uniqueness of solutions for a novel class of coupled nonlinear fractional differential equations employing the (k, ψ)-Hilfer derivative. Leveraging Schauder's fixed point theorem, we provided a theoretical framework for these solutions under specific boundary conditions.

6.1 *Future scope*

1. Generalization: Extending the proposed methods to other fractional derivatives, such as the Caputo-Hadamard type, can broaden the applicability of the results.

2. Numerical analysis: Developing numerical schemes to approximate solutions will provide practical tools for real-world applications and validate theoretical findings.

3. Interdisciplinary applications: Applying the established framework to model complex systems in physics, biology, or finance can showcase the potential of fractional calculus.

4. Dynamical systems analysis: Investigating the stability and control properties of systems described by these equations will deepen our understanding of their behavior.

5. Complex systems: Exploring multidimensional or more strongly nonlinear systems can further advance the field of fractional differential equations.

This research lays the groundwork for future investigations into the rich dynamics of fractional systems.

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Author's contributions

All authors have equally and significantly contributed to the contents of this manuscript.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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