Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

# Recursive and Explicit Formulas for Expansion and Connection Coefficients in Series of Classical Orthogonal Polynomial Products

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**Received:** 19 May 2024; **Revised:** 10 October 2024; **Accepted:** 22 October 2024

**Abstract:** Consider a function of two variables $f(\xi,\, t)$. In addition, assume the following expansions for it and its derivatives:

$$f(\xi,\, t) = \sum_{i,\, j=0}^{\infty} a_{i,\, j} P_i(\xi) Q_j(t),$$

$$D_{\xi}^{p_1} D_t^{p_2} f(\xi,\, t) = f^{(p_1,\, p_2)}(\xi,\, t) = \sum_{m,\, \nu=0}^{\infty} a_{m,\, \nu}^{(p_1,\, p_2)}\; P_m(\xi) Q_\nu(t), \;\; a_{i,\, j}^{(0,\, 0)} = a_{i,\, j}.$$

Considering the three variables' function $f(\xi,\, t,\, z)$, and assume the following expansions

$$f(\xi,\, t,\, z) = \sum_{i,\, j,\, k=0}^{\infty} a_{i,\, j,\, k}\; P_i(\xi) Q_j(t) R_k(z),$$

$$D_{\xi}^{p_1} D_t^{p_2} D_z^{p_3}\; f(\xi,\, t,\, z) = f^{(p_1,\, p_2,\, p_3)}(\xi,\, t,\, z) = \sum_{m,\, \nu,\, \ell=0}^{\infty} a_{m,\, \nu,\, \ell}^{(p_1,\, p_2,\, p_3)}\; P_m(\xi) Q_\nu(t) R_\ell(z),$$

where $a_{i,\, j,\, k}^{(0,\, 0,\, 0)} = a_{i,\, j,\, k}$ and $P_i(\xi)$, $Q_j(t)$ and $R_k(z)$ are Hermite, Laguerre, Jacobi and Bessel polynomials. We state and prove explicit formulae of $a_{m,\, \nu}^{(p_1,\, p_2)}$ and $a_{m,\, \nu,\, \ell}^{(p_1,\, p_2,\, p_3)}$ as a linear combination of $a_{i,\, j}$ and $a_{i,\, j,\, k}$, $i,\, j,\, k = 0,\, 1,\, 2,\, \ldots$, respectively. Using the moments of orthogonal polynomial,


DOI: https://doi.org/10.37256/cm.5420244990

$$\xi^{m_1} P_i(\xi) = \sum_{n_1=0}^{2m_1} a_{m_1,\, n_1}(i) P_{i+m_1-n_1}(\xi),$$

$$t^{m_2} Q_j(t) = \sum_{n_2=0}^{2m_2} a_{m_2,\, n_2}(j) Q_{j+m_2-n_2}(t),$$

$$z^{m_3} R_k(z) = \sum_{n_3=0}^{2m_3} a_{m_3,\, n_3}(k) R_{k+m_3-n_3}(z),$$

we find the coefficients $b_{i,\, j}^{(p_1,\, p_2,\, m_1,\, m_2)}$ and $b_{i,\, j,\, k}^{(p_1,\, p_2,\, p_3,\, m_1,\, m_2,\, m_3)}$ in the two expansions

$$\xi^{m_1} t^{m_2} D_\xi^{p_1} D_t^{p_2}\; f(\xi,\, t) = \xi^{m_1} t^{m_2} f^{(p_1,\, p_2)}(\xi,\, t) = \sum_{i,\, j=0}^{\infty} b_{i,\, j}^{(p_1,\, p_2,\, m_1,\, m_2)} P_i(\xi) Q_j(t),$$

$$\xi^{m_1} t^{m_2} z^{m_3} D_\xi^{p_1} D_t^{p_2} D_z^{p_3} f(\xi,\, t,\, z) = \xi^{m_1} t^{m_2} z^{m_3} f^{(p_1,\, p_2,\, p_3)}(\xi,\, t,\, z)$$

$$= \sum_{i,\, j=0}^{\infty} b_{i,j,\, k}^{(p_1,\, p_2,\, p_3,\, m_1,\, m_2,\, m_3)} P_i(\xi) Q_j(t) R_k(z).$$

We apply these findings to reduce partial differential equations by converting polynomial coefficients to recursive formulas in the solution's expansion coefficients.

***Keywords*:** orthogonal polynomials, connection and linearization coefficients, moments formulas, generalized hypergeometric functions

**MSC:** 42C10, 33A50, 33C25, 33D45

## 1. Introduction

Several authors have demonstrated a keen interest in researching special functions and orthogonal polynomials (OPs). These mathematical concepts have numerous applications in diverse domains, such as science, mathematics, engineering, and statistics; see, for example [1–3]. In approximation theory and numerical analysis, the role of OPs arises, particularly for treating differential equations (DEs); see, for example [4–11]. In addition, many contributions employ the different special polynomials to treat some models that arise in the applied science; see for example [12–14]. The derivation of several formulas concerned with the different OPs is the backbone of employing these polynomials in different types of DEs; see, for example [15–17].

Classical orthogonal polynomials (Hermite, Laguerre, Jacobi, and Bessel) are utilized widely in various fields of physics and mathematics [18, 19]. For a long time, these polynomials have been utilized in numerical analysis and approximation theory for problems involving two and three dimensions, see [20, 21]. Furthermore, multivariable polynomials are prevalent in various practical contexts. For example, they are used in the calculation of multiple integrals

[22], in addressing connection problems involving multivariable polynomials such as Appell [23], in the expansion of solid harmonics in $\mathbf{R}_\nu$ in the field of statistical mechanics.

Over the past decades, several researchers have focused on extending the concept of classical orthogonal polynomials (OPs) as solutions of a differential equation (DE) from a single variable to multiple variables. Classical bivariate OPs are the polynomial solutions of second-order partial DEs that include polynomial coefficients. Krall and Sheffer [24] were the first to address this issue when they examined second-order partial DEs with OPs as eigenfunctions. Kim et al. [25] discussed DEs that possess solutions as a product of two well-known OPs in a single variable. Suetin [26] provided a straightforward method for deducing a second-order partial DEs that is fulfilled by the product of two classical OPs, each depending on a single variable. However, certain scholars have shown that these polynomials may fulfill higher-order partial DEs. For example, Fernández et al. [27] have discussed this in their work on OPs. Ronveaux and Rebillard [28] presented a methodology for constructing multivariable polynomials by utilizing several series of OPs in a single variable. Dunkl and Xu [29] provide numerous examples of OPs in multiple variables. The majority of these polynomials, although not all, are expressed using the classical OPs of a single variable. Additionally, many of these cases involve expanding OPs from two variables to a higher number of variables.

Several researchers have utilized spectral methods that rely on classical and double classical OPs to solve numerically partial DEs; see, for example [30–36]. This situation motivated the researchers to develop algorithms to solve the partial differential/difference equations with varying coefficients by transforming them into suitable recursive formulas followed as in [37–40] for the continuous case and as in [41–44] for the discrete case.

As far as we are aware, there are no well-known or identifiable formulae in the literature for the expansion coefficients of general-order derivatives of arbitrary functions of two or three variables or the evaluation of the expansion coefficients of the moments of high-order derivatives of these functions in terms of the product of two or three classical OPs (Hermite, Laguerre, Jacobi, and Bessel polynomials). These formulae are similar to those obtained in [45, 46] for double classical OPs of continuous variables for the various classes of Jacobi polynomials. While previous works, such as those referenced in [47–50], explore the expansion and connection coefficients of classical orthogonal polynomials, our paper introduces a unique approach by deriving recursive and explicit formulas that have not been addressed comprehensively in prior studies. This distinction allows for more efficient calculations and broader applications. Additionally, our work extends the existing results by considering a wider class of polynomial products. Specifically, we provide formulas that can be applied to a diverse set of classical orthogonal polynomials, thereby enhancing the applicability of the results across different mathematical contexts. Another reason is that many mathematical and scientific issues can be studied theoretically and numerically by expanding arbitrary polynomials or two- or three-variable functions into a collection of OPs and their derivatives and moments.

This paper follows the following structure: Section 2 gives relevant properties of classical OPs. In Section 3, we give relevant properties of a product two one-variable classical orthogonal polynomials,

$$\{F_{m,\,\nu}(\xi,\,t) : F_{m,\,\nu}(\xi,\,t) = P_m(\xi)Q_\nu(t),\ P_m(\xi),\ Q_\nu(t) \in T\},$$

where $T = \{P_\nu(\xi)$ : Hermite, Laguerre, Jacobi and Bessel polynomials$\}$. In Section 4, we give and prove a theorem that states three expressions for the coefficients of general-order partial derivatives of expansion in $F_{m\nu}(\xi,\,t)$ in terms of the coefficients of the original expansion. How to generate the recursive formulas $a_{i,\,j}$ in the expansion

$$f(\xi,\,t) = \sum_{i,\,j=0}^{\infty} a_{i,\,j} F_{i,\,j}(\xi,\,t),$$

where $f(\xi,\,t)$ is function of two variables $\xi$ and $t$, is the target of Section 5. Sections 6 and 7 present two uses of the presented study, which provides a symbolic algebraic approach (via Mathematica) to construct the recursive formulas for the coefficients that arise, respectively, in the two problems:

$$(\xi+t)^{\nu}=\sum_{i+j\leq \nu} a_{i,\ j}(\nu)F_{i,\ j}(\xi,\ t),\ \ P_{\nu}(a\xi+bt)=\sum_{i+j\leq \nu} a_{i,\ j}(\nu)F_{i,\ j}(\xi,\ t).$$

Extension to expansion in a product three one-variable classical OPs are also given in Section 8. In Section 9, we discuss how to generate the recursive formulas for $a_{i,\ j,\ k}$, in the expansion

$$f(\xi,\ t,\ z)=\sum_{i,\ j,\ k=0}^{\infty} a_{i,\ j,\ k}F_{i,\ j,\ k}(\xi,\ t,\ z),$$

where

$$\{F_{m,\ \nu,\ \ell}(\xi,\ t,\ z): F_{m,\ \nu,\ \ell}(\xi,\ t,\ z)=P_m(\xi)Q_{\nu}(t)R_{\ell}(z),\ P_m(\xi),\ Q_{\nu}(t),\ R_{\ell}(z)\in T\}.$$

We present two uses of the work that employ an algebraic symbolic method, namely Mathematica, in Sections 10 and 11. The goal of these programs is to build the recursive formulas for the coefficients in the two problems:

$$(\xi+t+z)^{\nu}=\sum_{i+j+k\leq \nu} a_{i,\ j,\ k}(\nu)F_{i,\ j,\ k}(\xi,\ t,\ z),$$

$$P_{\nu}(a\xi+bt+cz)=\sum_{i+j+k\leq \nu} a_{i,\ j,\ k}(\nu)F_{i,\ j,\ k}(\xi,\ t,\ z).$$

Section 12 summarizes the main findings and concludes our investigation, highlighting the benefits, limitations, and potential improvements of our algorithm and suggestions for future work.

## 2. Some properties of classical OPs

Let $\{P_{\nu}(\xi)\}$ is classical orthogonal family (the Hermite, Laguerre, Jacobi and Bessel polynomials) each of degree $\nu$ in $\xi\in[a,\ b]$, then meets the DE [47]

$$\sigma(\xi)y''(\xi)+\tau(\xi)y'(\xi)+\lambda_{\nu}\,y(\xi)=0, \tag{1}$$

where $\tau(\xi)$ and $\sigma(\xi)$ are two polynomials with degrees no greater than one and two, respectively, and $\lambda_{\nu}=-\nu\tau'(\xi)-\frac{1}{2}\nu(\nu-1)\sigma''(\xi)$. These polynomials satisfy the following orthogonality property:

$$\int_a^b \rho(\xi)P_m(\xi)P_k(\xi)\,d\xi=\delta_{m,\ k}h_m,\ m,\ k=0,\ 1,\ 2,\ \ldots, \tag{2}$$

where $\rho(\xi)$ satisfies

$$D[\sigma(\xi)\rho(\xi)] = \tau(\xi)\rho(\xi),$$

assuming the specified condition

$$\sigma(\xi)\rho(\xi)\xi^k|_{\xi=a,\ b} = 0, \quad k \geq 0,$$

is satisfied. In addition, the constant $h_\nu$ is given by

$$h_\nu = (-1)^\nu \nu!\, k_\nu B_\nu \int_a^b (\sigma(\xi))^\nu \rho(\xi) d\xi,$$

where the constant $k_\nu$ is the leading coefficient of $P_\nu(\xi)$ and $B_\nu$ is called the normalization constant, that appears in the Rodrigues formula

$$P_\nu(\xi) = \frac{B_\nu}{\rho(\xi)} D^\nu[(\sigma(\xi))^\nu \rho(\xi)].$$

The four referred polynomials: Hermit $H_\nu(\xi)$, Laguerre $L_\nu^{(\theta)}(\xi)$, Jacobi $J_\nu^{(\theta,\ \zeta)}(\xi)$ and Bessel $Y_\nu^{(\theta)}(\xi)$, can be expressed in terms of the hypergeometric functions as [39]:

$$H_\nu(\xi) = (2x)^\nu\, {}_2F_0 \left[ \begin{matrix} -\nu/2,\ \lambda+\nu,\ -(\nu-1)/2 \\ - \end{matrix} ;\ -1/\xi^2 \right],$$

$$L_\nu^{(\theta)}(\xi) = \frac{(\theta+1)_\nu}{\nu!}\, {}_1F_1 \left[ \begin{matrix} -\nu \\ \theta+1 \end{matrix} ;\ \xi \right],\ \theta > -1,$$

$$J_\nu^{(\theta,\ \zeta)}(\xi) = \frac{(\theta+1)_\nu}{\nu!}\, {}_2F_1 \left[ \begin{matrix} -\nu,\ -\nu+\theta+\zeta+1 \\ \theta+1 \end{matrix} ;\ \frac{1-\xi}{2} \right],\ \theta,\ \zeta > -1,$$

$$Y_\nu^{(\theta)}(\xi) = {}_2F_0 \left[ \begin{matrix} -\nu,\ \nu+\theta+1 \\ - \end{matrix} ;\ -\frac{\xi}{2} \right],\ \xi \neq 0,\ \theta \neq -2,\ -3,\ \ldots,$$

and $(z)_\nu$ is the Pochhammer function defined as:

$$(z)_\nu = \frac{\Gamma(z+\nu)}{\Gamma(z)}.$$

According to Koepf and Schmersau [39], the current work relies heavily on the following two recursive formulas:

$$\xi P_\nu(\xi) = \theta_\nu P_{\nu+1}(\xi) + \zeta_\nu P_\nu(\xi) + \gamma_\nu P_{\nu-1}(\xi), \quad P_{-1}(\xi) = 0,\ P_0(\xi) = 1, \quad \nu \geq 1, \tag{3}$$

$$P_\nu(\xi) = \bar{\theta}_\nu DP_{\nu+1}(\xi) + \bar{\zeta}_\nu DP_\nu(\xi) + \bar{\gamma}_\nu DP_{\nu-1}(\xi), \quad \nu \geq 0. \tag{4}$$

**Remark 1** For the expressions of $\sigma(\xi)$, $\tau(\xi)$, $\rho(\xi)$, $\lambda_\nu$, $h_\nu$, $\theta_\nu$, $\zeta_\nu$, $\gamma_\nu$, $\bar{\theta}_\nu$, $\bar{\zeta}_\nu$, $\bar{\gamma}_\nu$, one may consult [39] to the different OPs.

If we consider a function $f(\xi)$ that may be expressed as an infinite series of classical OPs $P_\nu(\xi)$ as

$$f(\xi) = \sum_{\nu=0}^{\infty} a_\nu P_\nu(\xi), \tag{5}$$

then we can express $D^p f(\xi) = \dfrac{d^p f(\xi)}{dx^p}$ as

$$f^{(p)}(\xi) = D^p f(\xi) = \sum_{\nu=0}^{\infty} a_\nu^{(p)} P_\nu(\xi),\ \ a_\nu^{(0)} = a_\nu, \tag{6}$$

subsequently, a recursive formula incorporating the expansion coefficients of successive derivatives of $f(\xi)$ can be derived. Now, we have

$$D\left[\sum_{\nu=0}^{\infty} a_\nu^{(p-1)} P_\nu(\xi)\right] = \sum_{\nu=0}^{\infty} a_\nu^{(p)} P_\nu(\xi). \tag{7}$$

Based on (7), we get the following recursive formula

$$\overline{\theta}_{\nu-1} a_{\nu-1}^{(p+1)} + \overline{\zeta}_\nu\, a_\nu^{(p+1)} + \overline{\gamma}_{\nu+1} a_{\nu+1}^{(p+1)} = a_\nu^{(p)},\ \ p \geq 0,\ \nu \geq 1. \tag{8}$$

**Lemma 1** For $\nu,\ p \in \mathbb{Z}$ with $\nu \geq p$, we have

$$D^p P_\nu(\xi) = \sum_{k=0}^{\nu-p} C_{k,\ p}(\nu) P_k(\xi), \tag{9}$$

which is equivalent to

$$a_\nu^{(p)} = \sum_{k=0}^{\infty} C_{p,\ \nu}(\nu + k + p)\, a_{\nu+k+p}, \tag{10}$$

and $C_{k,\ p}(\nu)$ are known coefficients.

**Proof.** Starting with (9), and $D^p$ to (5), we get

$$D^p f(\xi) = \sum_{\nu=p}^{\infty} a_\nu D^p P_\nu(\xi). \tag{11}$$

Making use of (9) and performing some calculations lead to

$$D^p f(\xi) = \sum_{\nu=0}^{\infty} \left[ \sum_{k=0}^{\infty} C_{p,\,\nu}(\nu+k+p)\, a_{\nu+k+p} \right] P_\nu(\xi). \tag{12}$$

The two formulas (6) and (12) imply (10).

Now, inserting (10) into (6) gives (12). Performing some computations on (12) and identifying the result with (11), formula (9) can be obtained. This completes the proof. □

The expressions of $C_{k,\,p}(\nu)$ for Laguerre $L_\nu^{(\theta)}(\xi)$, Jacobi $J_\nu^{(\theta,\,\zeta)}(\xi)$, Hermit $H_\nu(\xi)$ and Bessel $Y_\nu^{(\theta)}(\xi)$, are given in Doha [48, 49, 51] and Doha and Ahmed [52], respectively. In addition, explicit formulas of $a_{m,\,\nu}(j)$ in the expansion

$$\xi^m P_j(\xi) = \sum_{\nu=0}^{2m} a_{m,\,\nu}(j) P_{j+m-\nu}(\xi), \quad j \geq 0,\ m \geq 0, \tag{13}$$

are given there. Moreover, it is proved in these papers that $a_{m,\,\nu}(j)$ meet the recursive formula

$$a_{m,\,\nu}(j) = \theta_{j+m-\nu-1} a_{m-1,\,\nu}(j) + \zeta_{j+m-\nu} a_{m-1,\,\nu-1}(j) + \gamma_{j+m-\nu+1} a_{m-1,\,\nu-2}(j), \quad \nu = 0,\ 1,\ \ldots,\ 2m, \tag{14}$$

with $a_{0,\,0}(j) = 1,\ a_{m-1,\,-\ell}(j) = 0,\ \forall\, \ell > 0,\ a_{m-1,\,r}(j) = 0,\ r = 2m-1,\ 2m.$

Consider a classical OP $P_j(\xi)$, the expansion coefficients of the moments of a general order derivative of any function, expressed in terms of its original expansion, are shown by the following theorem:

**Theorem 1** [48, 49, 51, 52] Let $f(\xi)$, $f^{(p)}(\xi)$ and $\xi^m P_j(\xi)$ have the expansions (5), (6) and (13) respectively, and consider the formula

$$\xi^m \left( \sum_{i=0}^{\infty} a_i^{(p)} P_i(\xi) \right) = \sum_{i=0}^{\infty} b_i^{(p,\,m)} P_i(\xi), \tag{15}$$

then $b_i^{(p,\,m)}$ are expressed as

$$
b_i^{(p,\, m)} = \begin{cases} \displaystyle\sum_{k=0}^{m-1} a_{m,\, k+m-i}(k) a_k^{(p)} + \sum_{k=0}^{i} a_{m,\, k+2m-i}(k+m) a_{k+m}^{(p)}, & 0 \le i \le m, \\ \displaystyle\sum_{k=i-m}^{m-1} a_{m,\, k+m-i}(k) a_k^{(p)} + \sum_{k=0}^{i} a_{m,\, k+2m-i}(k+m) a_{k+m}^{(p)}, & m+1 \le i \le 2m-1, \\ \displaystyle\sum_{k=i-2m}^{i} a_{m,\, k+2m-i}(k+m) a_{k+m}^{(p)}, & i \ge 2m. \end{cases} \tag{16}
$$

**Corollary 1** It is easy to derive the formula

$$
b_i^{(p,\, m)} = \sum_{r=0}^{2m} a_{m,\, r}(r+i-m)\, a_{r+i-m}^{(p)}, \quad i \ge 0. \tag{17}
$$

## 3. Some properties of a product two one-variable classical OPs

Let $\{P_m(\xi)\}$ and $\{Q_\nu(t)\}$ are two classical orthogonal families of degree $m$ and $\nu$ in the variables $\xi \in I$ and $t \in J$ respectively. A product two one-variable classical OPs is defined as follows

$$
F_{m,\, \nu}(\xi,\, t) = P_m(\xi) Q_\nu(t). \tag{18}
$$

These polynomials are satisfying the orthogonality relation [26, p.37-41]

$$
\int \int_G F_{m,\, \nu}(\xi,\, t) F_{k,\, s}(\xi,\, t) \Omega(\xi,\, t) d\xi\, dy = \delta_{m,\, k} \delta_{\nu,\, s} \overset{\mathrm{P}}{h}_m \overset{\mathrm{Q}}{h}_\nu, \quad m,\, \nu,\, k,\, s = 0,\, 1,\, \ldots, \tag{19}
$$

where $G = \{(\xi,\, t) : \xi \in I \text{ and } t \in J\}$ and, $\Omega(\xi,\, t) = \overset{\mathrm{P}}{\rho}(\xi) \overset{\mathrm{Q}}{\rho}(t),\ (\xi,\, t) \in G.$ Suetin [26, p.38-41] gave a simple way to conclude a second order PDE satisfied by $F_{m,\, \nu}(\xi,\, t)$. In view of this conclusion, we can put this PDE-depending on the polynomials coefficients $\overset{\mathrm{P}}{\sigma}(\xi),\ \overset{\mathrm{Q}}{\sigma}(t),\ \overset{\mathrm{P}}{\tau}(\xi)$ and $\overset{\mathrm{Q}}{\tau}(t)$, and the constants $\overset{\mathrm{P}}{\lambda}_m,\ \overset{\mathrm{Q}}{\lambda}_\nu$-in the form [26, p.39-41]

$$
\left[ \overset{\mathrm{P}}{\sigma}(\xi) \frac{\partial_2}{\partial \xi^2} + \overset{\mathrm{Q}}{\sigma}(t) \frac{\partial_2}{\partial t^2} + \overset{\mathrm{P}}{\tau}(\xi) \frac{\partial}{\partial x} + \overset{\mathrm{Q}}{\tau}(t) \frac{\partial}{\partial y} + (\overset{\mathrm{P}}{\lambda}_m + \overset{\mathrm{Q}}{\lambda}_\nu) \right] F_{m,\, \nu}(\xi,\, t) = 0. \tag{20}
$$

Also, he gave a convenient name for these polynomials, which consists of two sections related to the names of $P_m(\xi)$ and $Q_\nu(t)$, i.e. Name of $P_\nu(\xi)$-Name of $Q_\nu(t)$.

Let $f(\xi,\, t)$ be a continuous function defined on the domain $G$, and let it have continuous and bounded partial derivatives of any order concerning its variables $\xi$ and $t$. We can write

$$f(\xi, t)=\sum_{m, \nu=0}^{\infty} a_{m, \nu} P_m(\xi) Q_\nu(t), \tag{21}$$

$$f^{(p, q)}(\xi, t)=D_\xi^p D_t^q f(\xi, t)=\sum_{m, \nu=0}^{\infty} a_{m, \nu}^{(p, q)} P_m(\xi) Q_\nu(t), \quad a_{m, \nu}^{(0, 0)}=a_{m, \nu}. \tag{22}$$

In view of relation (4), with assumptions that

$$D_\xi \sum_{m, \nu=0}^{\infty} a_{m, \nu}^{(p-1, q)} P_m(\xi) Q_\nu(t)=\sum_{m, \nu=0}^{\infty} a_{m, \nu}^{(p, q)} P_m(\xi) Q_\nu(t), \tag{23}$$

and

$$D_t \sum_{m, \nu=0}^{\infty} a_{m, \nu}^{(p, q-1)} P_m(\xi) Q_\nu(t)=\sum_{m, \nu=0}^{\infty} a_{m, \nu}^{(p, q)} P_m(\xi) Q_\nu(t), \tag{24}$$

we derive the recurrences

$$\overset{\mathrm{P}}{\bar{\theta}}_{m-1}\, a_{m-1, \nu}^{(p, q)}+\overset{\mathrm{P}}{\bar{\zeta}}_m\, a_{m, \nu}^{(p, q)}+\overset{\mathrm{P}}{\bar{\gamma}}_{m+1}\, a_{m+1, \nu}^{(p, q)}=a_{m, \nu}^{(p-1, q)}, \quad m, p \geq 1, \nu, q \geq 0, \tag{25}$$

and

$$\overset{\mathrm{Q}}{\bar{\theta}}_{\nu-1}\, a_{m, \nu-1}^{(p, q)}+\overset{\mathrm{Q}}{\bar{\zeta}}_\nu\, a_{m, \nu}^{(p, q)}+\overset{\mathrm{Q}}{\bar{\gamma}}_{\nu+1}\, a_{m, \nu+1}^{(p, q)}=a_{m, \nu}^{(p, q-1)}, \quad \nu, q \geq 1, m, p \geq 0. \tag{26}$$

**Remark 2** The symbols $\overset{\mathrm{P}}{\bar{\theta}}_m, \overset{\mathrm{P}}{\bar{\zeta}}_m, \overset{\mathrm{P}}{\bar{\gamma}}_m, \overset{\mathrm{P}}{\lambda}_m, \overset{\mathrm{P}}{\sigma}(\xi), \overset{\mathrm{P}}{\tau}(\xi)$ and $\overset{\mathrm{P}}{\rho}(\xi)$ are the corresponding data regarding to the polynomials $P_m(\xi)$.

## 4. Relation between the coefficients $a_{m, \nu}^{(p, q)}$ and $a_{m, \nu}$ and explicit formula for the expansion coefficients of $\xi^{m_1} t^{m_2} f^{(p, q)}(\xi, t)$

The principal aim of this section is to prove two results. The first gives the expression of $a_{m, \nu}^{(p, q)}$ in terms of $a_{m, \nu}$. The second expression represents the expansion coefficients of the moments of general order derivatives of any function of two variables in terms of its original expansion, $P_m(\xi) Q_\nu(t)$.

**Theorem 2** The coefficients $a_{m, \nu}^{(p, q)}$ are related to the coefficients $a_{m, \nu}^{(0, q)}$, $a_{m, \nu}^{(p, 0)}$ and the original coefficients $a_{m, \nu}$ by

$$a_{m,\ \nu}^{(p,\ q)} = \sum_{i=0}^{\infty} \overset{\mathrm{P}}{C}_{pm}\ (p+m+i)a_{p+m+i,\ \nu}^{(0,\ q)},\ p \geq 1, \tag{27}$$

$$a_{m,\ \nu}^{(p,\ q)} = \sum_{j=0}^{\infty} \overset{\mathrm{Q}}{C}_{q\nu}\ (q+\nu+j)a_{m,\ q+\nu+j}^{(p,\ 0)},\ q \geq 1, \tag{28}$$

$$a_{m,\ \nu}^{(p,\ q)} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \overset{\mathrm{P}}{C}_{pm}\ (p+m+i)\ \overset{\mathrm{Q}}{C}_{q\nu}\ (q+\nu+j)a_{p+m+i,\ q+\nu+j},\ \ p,\ q \geq 1,\ m,\ \nu \geq 0, \tag{29}$$

where the formulae of $\overset{\mathrm{P}}{C}_{pm}\ (p+m+i)$ and $\overset{\mathrm{Q}}{C}_{q\nu}\ (q+\nu+j)$ can be defined according to the expressions of $C_{k,\ p}(\nu)$ regarding to Laguerre $L_\nu^{(\theta)}(\xi)$, Jacobi $J_\nu^{(\theta,\ \zeta)}(\xi)$, Hermit $H_\nu(\xi)$ and Bessel $Y_\nu^{(\theta)}(\xi)$ polynomials which are given in Doha [48, 49, 51] and Doha and Ahmed [52], respectively.

**Proof.** Eq. (22) can be written as

$$f^{(p,\ q)}(\xi,\ t) = \sum_{m=0}^{\infty} b_m^{(p,\ q)}(t)P_m(\xi), \tag{30}$$

where

$$b_m^{(p,\ q)}(t) = \sum_{\nu=0}^{\infty} a_{m,\ \nu}^{(p,\ q)} Q_\nu(t), \tag{31}$$

while holding $y$ and $q$ constant. Lemma 1 allows us to conclude that

$$b_m^{(p,\ q)}(t) = \sum_{i=0}^{\infty} \overset{\mathrm{P}}{C}_{pm}\ (p+m+i)b_{p+m+i}^{(0,\ q)}(t). \tag{32}$$

In virtue of (31) and (32), the following formula can be obtained

$$\sum_{\nu=0}^{\infty} a_{m,\ \nu}^{(p,\ q)} Q_\nu(t) = \sum_{\nu=0}^{\infty}\left[\sum_{i=0}^{\infty} \overset{\mathrm{P}}{C}_{pm}\ (p+m+i)a_{p+m+i,\ \nu}^{(0,\ q)}\right]Q_\nu(t), \tag{33}$$

which implies that

$$a_{m,\ \nu}^{(p,\ q)} = \sum_{i=0}^{\infty} \overset{\mathrm{P}}{C}_{pm}\ (p+m+i)a_{p+m+i,\ \nu}^{(0,\ q)},\ p \geq 1.$$

Formula (27) is now proved.

By maintaining $\xi$ and $p$ constant and repeating the same steps with Eq. (22), it can be demonstrated that formula (28) is also true. Formula (29) can be obtained by just plugging (4.1) into (28). With this, we have proved Theorem 2. □

**Remark 3** The corresponding theorems of Doha [45, 50, 53] and Doha et al. [46] in the cases of double Chebyshev, Legender, ultraspherical and Jacobi polynomials, respectively, can be obtained from our Theorem 2 by taking the suitable polynomials $P_m(\xi)$ and $Q_\nu(t)$.

**Corollary 2** Let $f(\xi, t)$ and $f^{(p, q)}(\xi, t)$ be expanded as in (21) and (22), respectively, and also assume that

$$\xi^{m_1} P_i(\xi) = \sum_{\ell_1=0}^{2m_1} \overset{\mathrm{P}}{a}_{m_1, \ell_1}(i) P_{i+m_1-\ell_1}(\xi), \tag{34}$$

$$t^{m_2} Q_j(t) = \sum_{\ell_2=0}^{2m_2} \overset{\mathrm{Q}}{a}_{m_2, \ell_2}(j) Q_{j+m_2-\ell_2}(t), \tag{35}$$

and

$$\xi^{m_1} t^{m_2} \left( \sum_{i, j=0}^{\infty} a_{i, j}^{(p, q)} P_i(\xi) Q_j(t) \right) = \sum_{i, j=0}^{\infty} b_{i, j}^{(p, q, m_1, m_2)} P_i(\xi) Q_j(t), \tag{36}$$

then $b_{i, j}^{(p, q, m_1, m_2)}$ are expressed as

$$b_{i, j}^{(p, q, m_1, m_2)} = \sum_{\ell_1=0}^{2m_1} \sum_{\ell_2=0}^{2m_2} \overset{\mathrm{P}}{a}_{m_1, \ell_1}(\ell_1+i-m_1) \overset{\mathrm{Q}}{a}_{m_2, \ell_2}(\ell_2+j-m_2) a_{\ell_1+i-m_1, \ell_2+j-m_2}^{(p, q)}, \quad i, j \geq 0, \tag{37}$$

where the formulae of $\overset{\mathrm{P}}{a}_{m_1, \ell_1}(\ell_1+i-m_1)$ and $\overset{\mathrm{Q}}{a}_{m_2, \ell_2}(\ell_2+j-m_2)$ can be defined according to the expressions of $a_{m, \nu}(j)$ regarding to Laguerre $L_\nu^{(\theta)}(\xi)$, Jacobi $J_\nu^{(\theta, \zeta)}(\xi)$, Hermit $H_\nu(\xi)$ and Bessel $Y_\nu^{(\theta)}(\xi)$ polynomials which are given in Doha [48, 49, 51] and Doha and Ahmed [52], respectively.

**Proof.** Corollary 1, together with formula (34), yields

$$I^{(p, q, m_1, m_2)} = t^{m_2} \sum_{j=0}^{\infty} \left( \sum_{i=0}^{\infty} a_{i, j}^{(p, q)} \xi^{m_1} P_i(\xi) \right) P_j(t) = t^{m_2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{i, j}^{(p, q, m_1)} P_i(\xi) P_j(t), \tag{38}$$

where

$$b_{i, j}^{(p, q, m_1)} = \sum_{\ell_1=0}^{2m_1} \overset{\mathrm{P}}{a}_{m_1, \ell_1}(\ell_1+i-m_1) a_{\ell_1+i-m_1, j}^{(p, q)}. \tag{39}$$

Using Corollary 1 and formula (35) enables one to write (38) as

$$I^{(p,\,q,\,m_1,\,m_2)} = \sum_{i=0}^{\infty} P_i(\xi)\left(\sum_{j=0}^{\infty} b_{i,\,j}^{(p,\,q,\,m_1)} t^{m_2} P_j(t)\right) = \sum_{i,\,j=0}^{\infty} b_{i,\,j}^{(p,\,q,\,m_1,\,m_2)} P_i(\xi) P_j(t), \tag{40}$$

where

$$b_{i,\,j}^{(p,\,q,\,m_1,\,m_2)} = \sum_{\ell_2=0}^{2m_2} \overset{Q}{a}_{m_2,\,\ell_2}\,(\ell_2+j-m_2)\, b_{i,\,\ell_2+j-m_2}^{(p,\,q,\,m_1)}. \tag{41}$$

By substituting (39) into (41), we obtain (37) and complete the proof of corollary. □

## 5. Establishing the recursive formulas for the expansion coefficients in series of a product two of classical OPs

Assume that $f(\xi,\,t)$ can be expanded as in (21), and suppose it meets the linear non-homogeneous partial DE

$$\sum_{i=0}^{m}\sum_{j=0}^{\nu} p_{i,\,j}(\xi,\,t) f^{(i,\,j)}(\xi,\,t) = g(\xi,\,t), \tag{42}$$

where $\{p_{i,\,j}(\xi,\,t)\}_{0\le i,\,j\le \nu}$ are polynomials in $\xi$ and $t$ with $p_{m,\,0},\; p_{0,\,\nu} \neq 0$, and

$$g(\xi,\,t) = \sum_{i,\,j=0}^{\infty} g_{i,\,j} P_i(\xi) Q_j(t), \tag{43}$$

with known $g_{i,\,j}$, then applying Theorem 2 and Corollary 2 or repeated use of relations (25) and (26) and using (42), we obtain the following recursive formula of order $(d_1,\,d_2)$,

$$\sum_{i=0}^{d_1}\sum_{j=0}^{d_2} \theta_{i,\,j}(r,\,k) a_{r+i,\,j+k} = \zeta(r,\,k), \quad r,\,k \ge 0, \tag{44}$$

where $\theta_{i,\,j}(r,\,k)$, $i = 0,\,1,\,\ldots,\,d_1,\; j = 0,\,1,\,\ldots,\,d_2$, are polynomials in $r$ and $k$ with $\theta_{d_1,\,0}(r,\,k),\,\theta_{0,\,d_2}(r,\,k) \neq 0$.

An example dealing with a non-homogeneous partial DE is considered to clarify the application of the results obtained.

**Example 1** Consider the non-homogeneous partial DE

$$\xi u_\xi - t u_t + (\xi - t)u = \xi^2 - t^2 + \xi - t, \quad u(0,\,t) = y,\; u(\xi,\,0) = \xi. \tag{45}$$

If $\xi^2 - t^2 + \xi - t$ and $u^{(p,\,q)}(\xi,\,t)$ are expanded as follows

$$\xi^2 - t^2 + \xi - t = \sum_{i+j\leq 2} d_{i,\, j} P_i(\xi) Q_j(t), \tag{46}$$

$$u^{(p,\, q)}(\xi,\, t) = \sum_{i,\, j=0}^{\infty} a_{i,\, j}^{(p,\, q)} P_i(\xi) Q_j(t), \quad p,\, q = 0,\, 1, \tag{47}$$

then our ability to apply partial DE (45) on (47) and using Corollary 2 lead to the following equation

$$\begin{aligned} &\overset{\mathrm{P}}{\theta}_{i-1}\, a_{i-1,\, j}^{(1,\, 0)} + \overset{\mathrm{P}}{\zeta}_i\, a_{i,\, j}^{(1,\, 0)} + \overset{\mathrm{P}}{\gamma}_{i+1}\, a_{i+1,\, j}^{(1,\, 0)} - \overset{\mathrm{Q}}{\theta}_{j-1}\, a_{i,\, j-1}^{(0,\, 1)} - \overset{\mathrm{Q}}{\zeta}_j\, a_{i,\, j}^{(0,\, 1)} - \overset{\mathrm{Q}}{\gamma}_{j+1}\, a_{i,\, j+1}^{(0,\, 1)} \\ &+ \overset{\mathrm{P}}{\theta}_{i-1}\, a_{i-1,\, j} + \overset{\mathrm{P}}{\gamma}_{i+1}\, a_{i+1,\, j} + (\overset{\mathrm{P}}{\zeta}_i - \overset{\mathrm{Q}}{\zeta}_j)\, a_{i,\, j} - \overset{\mathrm{Q}}{\theta}_{j-1}\, a_{i,\, j-1} - \overset{\mathrm{Q}}{\gamma}_{j+1}\, a_{i,\, j+1} = d_{ij}. \end{aligned} \tag{48}$$

In the next, we find the recursive formula satisfied by the expansion coefficients $a_{i,\, j}$ in two different cases for $P_i(\xi)$ and $Q_j(t)$:

**Case 1** The expansion of $u(\xi,\, t)$ in Bessel-Bessel polynomials, $Y_i^{(\theta)}(\xi).Y_j^{(\zeta)}(t)$ In this problem, Eq. (48) takes the form

$$\begin{aligned} &\frac{2(i+\theta)}{(2i+\theta-1)_2}(a_{i-1,\, j}^{(1,\, 0)} + a_{i-1,\, j}) - \frac{2\theta}{(2i+\theta-2)(2i+\theta)} a_{i,\, j}^{(1,\, 0)} \\ &- \frac{2i}{(2i+\theta+2)_2}(a_{i+1,\, j}^{(1,\, 0)} + a_{i+1,\, j}) - \frac{2(j+\zeta)}{(2j+\zeta-1)_2}(a_{i,\, j-1}^{(0,\, 1)} + a_{i,\, j-1}) \\ &+ \frac{2\zeta}{(2j+\zeta-2)(2j+\zeta)} a_{i,\, j}^{(0,\, 1)} + \frac{2j}{(2j+\zeta+2)_2}(a_{i,\, j+1}^{(0,\, 1)} + a_{i,\, j+1}) \\ &- \left(\frac{2\theta}{(2i+\theta-2)(2i+\theta)} - \frac{2\zeta}{(2j+\zeta-2)(2j+\zeta)}\right) a_{i,\, j} = d_{ij}, \end{aligned} \tag{49}$$

where

$$
d_{i,\ j}=\begin{cases}\dfrac{2(\zeta+1)}{(\zeta+2)_2}-\dfrac{2(\theta+1)}{(\theta+2)_2}, & i=0,\ j=0,\\[2ex] \dfrac{-2\zeta}{(\zeta+2)(\zeta+4)}, & i=0,\ j=1,\\[2ex] \dfrac{-4}{(\zeta+3)_2}, & i=0,\ j=2,\\[2ex] \dfrac{2\theta}{(\theta+2)(\theta+4)}, & i=1,\ j=0,\\[2ex] \dfrac{4}{(\theta+3)_2}, & i=2,\ j=0,\\[2ex] 0, & \text{otherwise.}\end{cases}
$$

Based on (27) and (28) with (49) and after doing some calculations, the following recursive formula can be obtained:

$$
\begin{aligned}
&(\eta_{0i}-\gamma_{0j})a_{i,\ j}+\eta_{1i}a_{i+1,\ j}+\eta_{2i}a_{i+2,\ j}+\eta_{3i}a_{i+3,\ j}+\eta_{4i}a_{i+4,\ j}\\
&-\gamma_{1j}a_{i,\ j+1}-\gamma_{2j}a_{i,\ j+2}-\gamma_{3j}a_{i,\ j+3}-\gamma_{4j}a_{i,\ j+4}=c_{i,\ j},
\end{aligned}
\tag{50}
$$

where

$$
\begin{aligned}
\eta_{0i}=&2[(2i+\theta+1)_5]^{-1}(i+\theta+1)_3,\\
\eta_{1i}=&[(2i+\theta+2)_5]^{-1}(i+\theta+2)_2[4i^3+(20+4\theta)i^2+(32+12\theta+\theta^2)i+(16+6\theta+\theta^2)],\\
\eta_{2i}=&-[(2i+\theta+3)_5]^{-1}(i+\theta+3)(i+2)\\
&\times\left[(8+4\theta)i^2+(40+28\theta+4\theta^2)i+(48+47\theta+12\theta^2+\theta^3)\right],\\
\eta_{3i}=&-[(2i+\theta+4)_5]^{-1}(i+2)_2\\
&\times\left[4i^3+(40+8\theta)i^2+(144+52\theta+5\theta^2)i+(144+86\theta+16\theta^2+\theta^3)\right],\\
\eta_{4i}=&-2[(2i+\theta+5)_5]^{-1}(i+2)_3,
\end{aligned}
$$

$\gamma_{0j}$, $\gamma_{1j}$, $\gamma_{2j}$, $\gamma_{3j}$ and $\gamma_{4j}$ are obtained from $\eta_{0i}$, $\eta_{1i}$, $\eta_{2i}$, $\eta_{3i}$ and $\eta_{4i}$, respectively, by replacing each of $i$ and $\theta$ with $j$ and $\zeta$ respectively, and the coefficients $c_{i,\ j}$ are given by

$$c_{i,\ j} = \begin{cases} 2\dfrac{(\zeta+3)}{(\zeta+4)_2} - 2\dfrac{(\theta+3)}{(\theta+4)_2}, & i=0,\ j=0, \\[2ex] \dfrac{-2(\zeta+2)}{(\zeta+4)(\zeta+6)}, & i=0,\ j=1, \\[2ex] \dfrac{-4}{(\zeta+5)_2}, & i=0,\ j=2, \\[2ex] \dfrac{2(\theta+2)}{(\theta+4)(\theta+6)}, & i=1,\ j=0, \\[2ex] \dfrac{4}{(\theta+5)_2}, & i=2,\ j=0, \\[2ex] 0, & \text{otherwise.} \end{cases}$$

To reach the solution of the example, we solve the recursive formula of (50) which is given explicitly by

$$a_{i,\ j} = \begin{cases} A_0 B_0 - \dfrac{2(\theta+\zeta+4)}{(\theta+2)(\zeta+2)}, & i=0,\ j=0, \\[2ex] A_1 B_0 + \dfrac{2}{(\theta+2)}, & i=1,\ j=0, \\[2ex] A_0 B_1 + \dfrac{2}{(\zeta+2)}, & i=0,\ j=1, \\[2ex] A_i B_j, & \text{otherwise,} \end{cases} \tag{51}$$

where $A_i = I_i(\theta),\ B_i = I_i(\zeta),\ i \geq 0$,

$$I_i(\theta) = \begin{cases} 2(\theta+2)^{-1}\,{}_0F_1\left[-;\ \theta+3;\ 2\right], & i=0, \\[2ex] \dfrac{(-1)^{i+1}2^i}{(i-1)!(i+\theta+1)_i}\,{}_1F_2\left[i+1;\ i,\ 2i+\theta+2;\ 2\right], & i \geq 1. \end{cases}$$

**Case 2** The expansion of $u(\xi,\ t)$ in Laguerre-Laguerre polynomials, $L_i^{(\theta)}(\xi).L_j^{(\zeta)}(t)$

In this case, Eq. (48) takes the form

$$-(i+\theta+1)(a_{i+1,\ j}^{(1,\ 0)}+a_{i+1,\ j})+(2i+\theta+1)a_{i,\ j}^{(1,\ 0)}-i(a_{i-1,\ j}^{(1,\ 0)}+a_{i-1,\ j})$$

$$+(j+\zeta+1)(a_{i,\ j+1}^{(0,\ 1)}+a_{i,\ j+1})-(2j+\zeta+1)a_{i,\ j}^{(0,\ 1)}+j(a_{i,\ j-1}^{(0,\ 1)}+a_{i,\ j-1}) \tag{52}$$

$$+(2i-2j+\theta-\zeta)a_{i,\ j}=d_{i,\ j},$$

where

$$d_{i,\ j}=\begin{cases}(\theta+1)(\theta+3)-(\zeta+1)(\zeta+3), & i=0,\ j=0,\\ 2\zeta+5, & i=0,\ j=1,\\ -2, & i=0,\ j=2,\\ 2\theta+5, & i=1,\ j=0,\\ 2, & i=2,\ j=0,\\ 0 & \text{otherwise.}\end{cases}$$

Eqs. (25) and (26) become

$$a_{i,\ j}^{(p,\ q)}-a_{i-1,\ j}^{(p,\ q)}=a_{i,\ j}^{(p-1,\ q)},\ p\geq 1,\ q\geq 0, \tag{53}$$

$$a_{i,\ j}^{(p,\ q)}-a_{i,\ j-1}^{(p,\ q)}=a_{i,\ j}^{(p,\ q-1)},\ \ p\geq 0,\ q\geq 1. \tag{54}$$

Now, repeated use of relations (53) and (54) to eliminate the coefficients $a_{i-1,\ j}^{(1,\ 0)}$, $a_{i,\ j}^{(1,\ 0)}$, $a_{i+1,\ j}^{(1,\ 0)}$, $a_{i,\ j-1}^{(0,\ 1)}$, $a_{i,\ j}^{(0,\ 1)}$ and $a_{i,\ j+1}^{(0,\ 1)}$ yields:

$$(3i-3j+\theta-\zeta)a_{i,\ j}-2(i+\theta+1)a_{i+1,\ j}-ia_{i-1,\ j}+2(j+\zeta+1)a_{i,\ j+1}+ja_{i,\ j-1}=f_{ij},\ \ i,\ j\geq 0, \tag{55}$$

where

$$f_{ij}=d_{i,\ j}-d_{i-1,\ j}-d_{i,\ j-1}+d_{i-1,\ j-1},\ \ i,\ j\geq 0.$$

The solution of (55) is

$$a_{i,\ j} = \begin{cases} 2^{-(\theta+\zeta+4)}(\theta+1)(\zeta+1)+(\theta+\zeta+2), & i=0,\ j=0, \\ 2^{-(\theta+\zeta+5)}\theta(\zeta+1)-1, & i=1,\ j=0, \\ 2^{-(\theta+\zeta+5)}\zeta(\theta+1)-1, & i=0,\ j=1, \\ 2^{-(\theta+\zeta+i+j+4)}(\theta-i+1)(\zeta-j+1), & \text{otherwise}. \end{cases} \tag{56}$$

**Lemma 4** Hermite and Jacobi polynomials can also solve the case above; however, the details are not provided here.

## 6. The expansion of $(\xi + t)^{\nu}$ as a multiple series in a product two of classical OPs

In this problem

$$(\xi+t)^{\nu} = \sum_{i+j\leq\nu} a_{i,\ j}(\nu)P_i(\xi)Q_j(t), \tag{57}$$

we have $u(\xi,\ t) = (\xi+t)^{\nu}$ satisfies the homogeneous partial differential equation

$$\xi u_{\xi} + t\, u_t - \nu\, u = 0, \tag{58}$$

then our ability to apply partial DE (58) on (57) and apply Corollary 2 leads to the following equation:

$$\overset{\text{P}}{\theta}_{i-1}\, a^{(1,\ 0)}_{i-1,\ j} + \overset{\text{P}}{\zeta}_i\, a^{(1,\ 0)}_{i,\ j} + \overset{\text{P}}{\gamma}_{i+1}\, a^{(1,\ 0)}_{i+1,\ j} + \overset{\text{Q}}{\theta}_{j-1}\, a^{(0,\ 1)}_{i,\ j-1} + \overset{\text{Q}}{\zeta}_j\, a^{(0,\ 1)}_{i,\ j} + \overset{\text{Q}}{\gamma}_{j+1}\, a^{(0,\ 1)}_{i,\ j+1} - n a_{i,\ j} = 0. \tag{59}$$

### 6.1 *The connection between $(\xi + t)^{\nu}$ and Double Hermite polynomials*

Consider the connection formula

$$(\xi+t)^{\nu} = \sum_{i+j\leq\nu} a_{i,\ j}(\nu)H_i(\xi)H_j(t), \tag{60}$$

Eq. (59) becomes

$$\frac{1}{2}a^{(1,\ 0)}_{i-1,\ j}(\nu) + (i+1)a^{(1,\ 0)}_{i+1,\ j} + \frac{1}{2}a^{(0,\ 1)}_{i,\ j-1}(\nu) + (j+1)a^{(0,\ 1)}_{i,\ j+1} - \nu\, a_{i,\ j}(\nu) = 0. \tag{61}$$

In the case of Hermite-Hermite polynomials, formula (29) becomes

$$a^{(p,\ q)}_{m,\ \nu} = 2^{p+q}\ p!q!\binom{p+m}{p}\binom{q+\nu}{q}a_{m+p,\ \nu+q}, \quad p,\ q \geq 0. \tag{62}$$

Application of formula (62) with (61), gives

$$(\nu-i-j)a_{i,\ j}(\nu)-2(i+1)_2a_{i+2,\ j}(\nu)-2(j+1)_2\,a_{i,\ j+2}(\nu)=0,\ \ i,\ j=\nu-1,\ \nu-2,\ \dots,\ 0, \tag{63}$$

with $a_{i,\ j}(\nu)=0$, $i+j>\nu$, $a_{-1,\ j}(\nu)=a_{i,\ -1}(\nu)=0$, $a_{\nu,\ 0}(\nu)=2^{-\nu}$ and $a_{0,\ \nu}(\nu)=2^{-\nu}$. Eq. (63) can be solved to give

$$a_{i,\ j}(\nu)=\begin{cases}\dfrac{\nu!2^{-\nu}2^{(\nu-i-j)/2}}{i!\,j!\left(\dfrac{\nu-i-j}{2}\right)!}, & (\nu-i-j)\text{ even},\\[2ex] 0, & (\nu-i-j)\,\text{odd}.\end{cases} \tag{64}$$

Specifically, Eq. (60) becomes for the case $y=0$.

$$\xi^{\nu}=\sum_{\substack{i=0\\(\nu-i)\ even}}^{\nu}\frac{\nu!}{2^{\nu}i!\left(\dfrac{\nu-i}{2}\right)!}H_i(\xi),$$

which coincides with the result obtained by Rainville [54, p.194] and Sánchez-Ruiz and Dehesa [55, p.159].

## 6.2 *The link between $(\xi+t)^{\nu}$ and Double Laguerre polynomials*

In this problem

$$(\xi+t)^{\nu}=\sum_{i+j\leq\nu}a_{i,\ j}(\nu)L_i^{(\theta)}(\xi)L_j^{(\zeta)}(t), \tag{65}$$

Eq. (59) turns into

$$\begin{aligned}&(i+\theta+1)a_{i+1,\ j}^{(1,\ 0)}(\nu)-(2i+\theta+1)a_{i,\ j}^{(1,\ 0)}(\nu)+i\,a_{i-1,\ j}^{(1,\ 0)}(\nu)+(j+\zeta+1)a_{i,\ j+1}^{(0,\ 1)}(\nu)\\ &-(2j+\zeta+1)a_{i,\ j}^{(0,\ 1)}(\nu)+j\,a_{i,\ j-1}^{(0,\ 1)}(\nu)+\nu\,a_{i,\ j}^{(0,\ 0)}(\nu)=0.\end{aligned} \tag{66}$$

Now, repeated use of relations (53) and (54) to eliminate $a_{i,\ j-1}^{(0,\ 1)}(\nu)$, $a_{i,\ j}^{(0,\ 1)}$, $a_{i,\ j+1}^{(0,\ 1)}(\nu)$, $a_{i,\ j}^{(1,\ 0)}$, $a_{i-1,\ j}^{(1,\ 0)}(\nu)$ and $a_{i+1,\ j}^{(1,\ 0)}(\nu)$ yields

$$(\nu-i-j)a_{i,\ j}(\nu)+(i+\theta+1)a_{i+1,\ j}(\nu)+(j+\zeta+1)a_{i,\ j+1}(\nu)=0,\ \ i,\ j=\nu-1,\ \nu-2,\ \dots,\ 0, \tag{67}$$

with $a_{i,\ j}(\nu)=0$, $i+j>\nu$, $a_{-1,\ j}(\nu)=a_{i,\ -1}(\nu)=0$ and $a_{\nu,\ 0}(\nu)=a_{0,\ \nu}(\nu)=\dfrac{(-1)^{\nu}}{\nu!}$. Eq. (67) can be solved to give

$$a_{i,\ j}(\nu)=\begin{cases}\dfrac{(-\nu)_{i+j}(\theta+\zeta+2)_{\nu}}{(\theta+\zeta+2)_{i+j}}, & i+j\leq\nu,\\ \\ 0, & \text{otherwise.}\end{cases} \tag{68}$$

For the special case $y=0$, Eq. (65), after some calculations lead to into

$$\xi^{\nu}=\sum_{i=0}^{\nu}\frac{(-1)^{i}\,\nu!\,\Gamma(\nu+\theta+1)}{(\nu-i)!\,\Gamma(i+\theta+1)}L_{i}^{(\theta)}(\xi),$$

which coincides with the result in Rainville [54, p.207] and Sánchez-Ruiz and Dehesa [55, p.159].

In view of the relation [47, p.51]

$$H_{2\nu}(\xi)=(-1)^{\nu}2^{2\nu}\nu!L_{\nu}^{(-1/2)}(\xi^{2}), \tag{69}$$

we can obtain the following corollary.

**Corollary 3** In the problem

$$(\xi^{2}+t^{2})^{\nu}=\sum_{i+j\leq\nu}a_{i,\ j}(\nu)H_{2i}(\xi)H_{2j}(t), \tag{70}$$

$a_{i,\ j}(\nu)$ can be expressed as

$$a_{i,\ j}(\nu)=\begin{cases}\dfrac{(\nu!)^{2}4^{-i-j}}{(\nu-i-j)!(i+j)!\,i!\,j!}, & i+j\leq\nu,\\ \\ 0, & \text{otherwise.}\end{cases}$$

## 7. Connection problem in the sense of two variables

We consider the following connection problem

$$F_{\nu}(a\xi+bt)=\sum_{i+j\leq\nu}a_{i,\ j}(\nu)P_{i}(\xi)Q_{j}(t), \tag{71}$$

where $F_{\nu}$, $P_i$ and $Q_j$ are classical OPs. The work developed in Section 4 permits us to obtain a recursive formula satisfied by the coefficients $a_{i,\ j}(\nu)$, if we know a partial differential operator that cancels the left-hand side of (71).

### 7.1 *The Hermite-Double Hermite connection problem*

In this problem

$$H_\nu\left(\frac{\xi+t}{\sqrt{2}}\right)=\sum_{i+j\leq\nu}a_{i,\ j}(\nu)H_i(\xi)H_j(t), \tag{72}$$

where $H_\nu((\xi+t)/\sqrt{2})$ satisfy the DE

$$[D^2-4(\xi+t)D+4\nu]H_\nu((\xi+t)/\sqrt{2})=0, \tag{73}$$

the coefficients $a_{i,\ j}(\nu)$ meet the recursive formula

$$\begin{aligned}&(\nu-i)a_{i,\ j}(\nu)-(i+1)a_{i+1,\ j-1}(\nu)-2(i+1)(j+1)a_{i+1,\ j+1}(\nu)-(i+1)_2a_{i+2,\ j}(\nu)=0,\\ &i,\ j=\nu-1,\ \nu-2,\ \ldots,\ 0,\end{aligned} \tag{74}$$

with $a_{i,\ j}(\nu)=0$, $i+j>\nu$, $a_{-1,\ j}(\nu)=a_{i,\ -1}(\nu)=0$ and $a_{\nu,\ 0}(\nu)=a_{0,\ \nu}(\nu)=2^{-\nu/2}$. Eq. (74) can be solved to give

$$a_{i,\ j}(\nu)=\begin{cases}2^{-\nu/2}\dfrac{\nu!}{i!j!}, & i+j=\nu,\\ \\ 0, & i+j\neq\nu,\end{cases} \tag{75}$$

which is coherent with the result found in Abramowitz and Stegun [56, formula (22.12.8)] and Hansen [57, formula (49.7.1)].

$$H_\nu\left(\frac{\xi+t}{\sqrt{2}}\right)=2^{-\nu/2}\sum_{i=0}^{\nu}\binom{\nu}{i}H_{\nu-i}(\xi)H_i(t). \tag{76}$$

## 7.2 *The Laguerre-Double Laguerre connection problem*

In this problem

$$L_\nu^{(\theta+\zeta+1)}(\xi+t)=\sum_{i+j\leq\nu}a_{i,\ j}(\nu)L_i^{(\theta)}(\xi)L_j^{(\zeta)}(t), \tag{77}$$

where $L_\nu^{(\theta+\zeta+1)}(\xi+t)$ satisfy the differential equation

$$[(\xi+t)D_\xi^2+(2+\theta+\zeta-\xi-y)D_\xi+\nu]L_\nu^{(\theta+\zeta+1)}(\xi+t)=0, \tag{78}$$

the coefficients $a_{i,\ j}(\nu)$ satisfy the recursive formula

$$
\begin{aligned}
&2(\nu-i-j)a_{i,\,j}(\nu)+(j+\zeta+1)a_{i,\,j+1}(\nu)+ja_{i,\,j-1}(\nu)-(\nu-i+1)a_{i-1,\,j}(\nu)\\
&-(\nu+\zeta-i)a_{i+1,\,j}(\nu)=0, \quad i,\ j=\nu-1,\ \nu-2,\ \ldots,\ 0,
\end{aligned}
\tag{79}
$$

with $a_{i,\,j}(\nu)=0$, $i+j>\nu$, $a_{-1,\,j}(\nu)=a_{i,\,-1}(\nu)=0$ and $a_{\nu,\,0}(\nu)=a_{0,\,\nu}(\nu)=1$. Eq. (79) can be solved to give

$$
a_{i,\,j}(\nu)=\begin{cases}1, & i+j=\nu,\\ 0, & i+j\neq\nu,\end{cases}
\tag{80}
$$

which is coherent with the result found in Abromowitz and Stegun [56, formula (22.12.6)], Rounveaux and Rebillard [28, formula (41)] and Hansen [57, formula (48.24.1)].

$$
L_{\nu}^{(\theta+\zeta+1)}(\xi+t)=\sum_{i=0}^{\nu}L_i^{(\theta)}(\xi)L_{\nu-i}^{(\zeta)}(t).
\tag{81}
$$

**Remark 5** Multiple integrals using OPs in distinct variables, such as the integral in hydrogen, can be generated via quantum mechanics [58].

$$
J(p,\ m,\ a,\ b)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\xi+\mathbf{i}y)^m L_{p+m}^{(m)}(\xi^2+t^2)H_a(\xi)H_b(t)e^{-(\xi^2+t^2)}d\xi\,dy,
\tag{82}
$$

related to the representation of the hydrogen atom wave functions into the Hermite oscillator wave function is where $a,\ b,\ p$, and $m$ are the positive integers.

The explicit form of $J(p,\ m.a,\ b)$ is unknown. But firstly, the function $(\xi+\mathbf{i}y)^m$, $\mathbf{i}=\sqrt{-1}$, can be immediately written in Hermite polynomials [28, p.411],

$$
(\xi+\mathbf{i}y)^m=2^{-m}\sum_{\nu=0}^{m}\binom{m}{\nu}\mathbf{i}^{\nu}H_{m-\nu}(\xi)H_{\nu}(t).
$$

Secondly, $L_{p+m}^{(m)}(\xi^2+t^2)$ can be expanded into a product of Laguerre polynomials, $L_i^{(-1/2)}(\xi^2)$, $L_j^{(-1/2)}(t^2)$, using (69), it can be expressed in terms of $H_{\ell}(\xi)H_k(t)$. This manipulation implies that the integral (81) can be written in terms of the integrals $\int_{-\infty}^{\infty}\prod_{i=1}^{4}H_{n_i}(s)e^{-s^2}ds$, which can be computed as in Azor et al. [59, formula (58)].

## 8. Extension to a product three one-variable classical orthogonal polynomials

The product of three classical OPs is defined as follows

$$
F_{m,\ \nu,\ r}(\xi,\ t,\ z)=P_m(\xi)Q_{\nu}(t)R_r(z),
\tag{83}
$$

where $P_m(\xi)$, $Q_\nu(t)$ and $R_r(z)$ are three classical OPs of degrees $m$, $\nu$ and $r$ in the variables $\xi \in I$, $y \in J$ and $z \in K$ respectively. These polynomials satisfy the orthogonality relation

$$\int\int\int_G F_{m,\,\nu,\,r}(\xi,\,t,\,z)F_{i,\,j,\,k}(\xi,\,t,\,z)\Omega(\xi,\,t,\,z)d\xi\,dy\,dz = \delta_{mi}\delta_{\nu j}\delta_{rk}\,\overset{\mathrm{P}}{h}_m\overset{\mathrm{Q}}{h}_\nu\overset{\mathrm{R}}{h}_r,\ k,\ \nu,\ r = 0,\ 1,\ 2,\ \ldots, \tag{84}$$

where

$$G = \{(\xi,\,t,\,z) : \xi \in I,\ t \in J \text{ and } z \in K\}, \tag{85}$$

$$\Omega(\xi,\,t,\,z) = \overset{\mathrm{P}}{\rho}(\xi)\,\overset{\mathrm{Q}}{\rho}(t)\,\overset{\mathrm{R}}{\rho}(z),\ (\xi,\,t,\,z) \in G. \tag{86}$$

A partial DE satisfied by $F_{m,\,\nu,\,r}(\xi,\,t,\,z)$ has the form

$$\begin{aligned}&\Bigg[\overset{\mathrm{P}}{\sigma}(\xi)\frac{\partial_2}{\partial\xi^2}+\overset{\mathrm{Q}}{\sigma}(t)\frac{\partial_2}{\partial t^2}+\overset{\mathrm{R}}{\sigma}(z)\frac{\partial_2}{\partial z^2}+\overset{\mathrm{P}}{\tau}(\xi)\frac{\partial}{\partial x}+\overset{\mathrm{Q}}{\tau}(t)\frac{\partial}{\partial y}+\overset{\mathrm{R}}{\tau}(z)\frac{\partial}{\partial z}\\ &+(\overset{\mathrm{P}}{\lambda}_m+\overset{\mathrm{Q}}{\lambda}_\nu+\overset{\mathrm{R}}{\lambda}_r)\Bigg]F_{m,\,\nu,\,r}(\xi,\,t,\,z) = 0.\end{aligned} \tag{87}$$

A convenient name for the polynomials $F_{m,\,\nu,\,r}(\xi,\,t,\,z)$ may be consist of three sections related to the names of $P_m(\xi)$, $Q_\nu(t)$ and $R_\nu(z)$, i.e., Name of $P_m(\xi)$-Name of $Q_\nu(t)$-Name of $R_r(z)$.

Let $f(\xi,\,t,\,z)$ be a continuous function defined on the domain $G$, and have continuous and bounded partial derivatives of any order concerning its variables $\xi$, $t$ and $z$, then we can write

$$f(\xi,\,t,\,z) = \sum_{m,\,\nu,\,\ell=0}^{\infty} a_{m,\,\nu,\,\ell}P_m(\xi)Q_\nu(t)R_\ell(z), \tag{88}$$

$$f^{(p,\,q,\,r)}(\xi,\,t,\,z) = D_\xi^p D_t^q D_z^r f(\xi,\,t,\,z) = \sum_{m,\,\nu,\,\ell=0}^{\infty} a_{m,\,\nu,\,\ell}^{(p,\,q,\,r)}P_m(\xi)Q_\nu(t)R_\ell(z),\ a_{m,\,\nu,\,\ell}^{(0,\,0,\,0)} = a_{m,\,\nu,\,\ell}, \tag{89}$$

where $a_{m,\,\nu,\,\ell}^{(p,\,q,\,r)}$ denote the expansion coefficients of $f^{(p,\,q,\,r)}(\xi,\,t,\,z)$ in terms of the product $P_m(\xi)Q_\nu(t)R_\ell(z)$. In view of relation (4), with assumptions that

$$D_\xi \sum_{m,\,\nu,\,\ell=0}^{\infty} a_{m,\,\nu,\,\ell}^{(p-1,\,q,\,r)}P_m(\xi)Q_\nu(t)R_\ell(z) = \sum_{m,\,\nu,\,\ell=0}^{\infty} a_{m,\,\nu,\,\ell}^{(p,\,q,\,r)}P_m(\xi)Q_\nu(t)R_\ell(z), \tag{90}$$

$$D_t \sum_{m,\,\nu,\,\ell=0}^{\infty} a_{m,\,\nu,\,\ell}^{(p,\,q-1,\,r)}P_m(\xi)Q_\nu(t)R_\ell(z) = \sum_{m,\,\nu,\,\ell=0}^{\infty} a_{m,\,\nu,\,\ell}^{(p,\,q,\,r)}P_m(\xi)Q_\nu(t)R_\ell(z), \tag{91}$$

and

$$D_z \sum_{m,\ \nu,\ \ell=0}^{\infty} a_{m,\ \nu,\ \ell}^{(p,\ q,\ r-1)} P_m(\xi) Q_\nu(t) R_\ell(z) = \sum_{m,\ \nu,\ \ell=0}^{\infty} a_{m,\ \nu,\ \ell}^{(p,\ q,\ r)} P_m(\xi) Q_\nu(t) R_\ell(z), \tag{92}$$

the following recursive formulas can be obtained:

$$\overset{\mathrm{P}}{\bar{\theta}}_{m-1}\, a_{m-1,\ \nu,\ \ell}^{(p,\ q,\ r)} + \overset{\mathrm{P}}{\bar{\zeta}}_{m}\, a_{m,\ \nu,\ \ell}^{(p,\ q,\ r)} + \overset{\mathrm{P}}{\bar{\gamma}}_{m+1}\, a_{m+1,\ \nu,\ \ell}^{(p,\ q,\ r)} = a_{m,\ \nu,\ \ell}^{(p-1,\ q,\ r)},\ m,\ p \geq 1,\ \nu,\ \ell,\ q,\ r \geq 0, \tag{93}$$

$$\overset{\mathrm{Q}}{\bar{\theta}}_{\nu-1}\, a_{m,\ \nu-1,\ \ell}^{(p,\ q,\ r)} + \overset{\mathrm{Q}}{\bar{\zeta}}_{\nu}\, a_{m,\ \nu,\ \ell}^{(p,\ q,\ r)} + \overset{\mathrm{Q}}{\bar{\gamma}}_{\nu+1}\, a_{m,\ \nu+1,\ \ell}^{(p,\ q,\ r)} = a_{m,\ \nu,\ \ell}^{(p,\ q-1,\ r)},\ \nu,\ q \geq 1,\ m,\ \ell,\ p,\ r \geq 0, \tag{94}$$

$$\overset{\mathrm{R}}{\bar{\theta}}_{\ell-1}\, a_{m,\ \nu,\ \ell-1}^{(p,\ q,\ r)} + \overset{\mathrm{R}}{\bar{\zeta}}_{\ell}\, a_{m,\ \nu,\ \ell}^{(p,\ q,\ r)} + \overset{\mathrm{R}}{\bar{\gamma}}_{\ell+1}\, a_{m,\ \nu,\ \ell+1}^{(p,\ q,\ r)} = a_{m,\ \nu,\ \ell}^{(p,\ q,\ r-1)},\ \ell,\ r \geq 1,\ m, \nu,\ p,\ q \geq 0. \tag{95}$$

The following theorem extends Theorem 2.

**Theorem 3** The coefficients $a_{m,\ \nu,\ \ell}^{(p,\ q,\ r)}$ are related to the coefficients with the different superscripts and the original ones $a_{m,\ \nu,\ \ell}$ as

$$a_{m,\ \nu,\ \ell}^{(p,\ q,\ r)} = \sum_{i=0}^{\infty} \overset{\mathrm{P}}{C}_{pm}\,(p+m+i) a_{p+m+i,\ \nu,\ \ell}^{(0,\ q,\ r)}, \tag{96}$$

$$a_{m,\ \nu,\ \ell}^{(p,\ q,\ r)} = \sum_{j=0}^{\infty} \overset{\mathrm{Q}}{C}_{q\nu}\ (q+\nu+j)\ a_{m,\ q+\nu+j,\ \ell}^{(p,\ 0,\ r)}, \quad q \geq 1, \tag{97}$$

$$a_{m,\ \nu,\ \ell}^{(p,\ q,\ r)} = \sum_{k=0}^{\infty} \overset{\mathrm{R}}{C}_{r\ell}\ (r+\ell+k)\, a_{m,\ \nu,\ r+\ell+k}^{(p,\ q,\ 0)}, \quad r \geq 1, \tag{98}$$

$$a_{m,\ \nu,\ \ell}^{(p,\ q,\ r)} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \overset{\mathrm{P}}{C}_{pm}\,(p+m+i)\, \overset{\mathrm{Q}}{C}_{q\nu}\ (q+\nu+j)\, a_{p+m+i,\ q+\nu+j,\ \ell}^{(0,\ 0,\ r)}, \quad p,\ q \geq 1, \tag{99}$$

$$a_{m,\ \nu,\ \ell}^{(p,\ q,\ r)} = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \overset{\mathrm{P}}{C}_{pm}\,(p+m+i)\, \overset{\mathrm{R}}{C}_{r\ell}\ (r+\ell+k)\, a_{p+m+i,\ \nu,\ r+\ell+k}^{(0,\ q,\ 0)}, \quad p,\ r \geq 1, \tag{100}$$

$$a_{m,\ \nu,\ \ell}^{(p,\ q,\ r)} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \overset{\mathrm{Q}}{C}_{q\nu}\ (q+\nu+j)\, \overset{\mathrm{R}}{C}_{r\ell}\ (r+\ell+k)\, a_{m,\ q+\nu+j,\ r+\ell+k}^{(p,\ 0,\ 0)}, \quad r \geq 1, \tag{101}$$

$$a_{m,\ \nu,\ \ell}^{(p,\ q,\ r)} = \sum_{i,\ j,\ k=0}^{\infty} \overset{\mathrm{P}}{C}_{pm}\,(p+m+i)\, \overset{\mathrm{Q}}{C}_{q\nu}\ (q+\nu+j)\, \overset{\mathrm{R}}{C}_{r\ell}\ (r+\ell+k)\, a_{p+m+i,\ q+\nu+j,\ r+\ell+k}, \quad p,\ q,\ r \geq 1. \tag{102}$$

**Proof.** Eq. (88) can be written as

$$f^{(p,\,q,\,r)}(\xi,\,t,\,z)=\sum_{m=0}^{\infty}b_m^{(p,\,q,\,r)}(y,\,z)P_m(\xi), \tag{103}$$

where

$$b_m^{(p,\,q,\,r)}(y,\,z)=\sum_{\nu,\,\ell=0}^{\infty}a_{m,\,\nu,\,\ell}^{(p,\,q,\,r)}Q_\nu(t)R_\ell(z), \tag{104}$$

while holding $y,\ z,\ q$, and $r$ constant. Based on Lemma 1, we can infer that

$$b_m^{(p,\,q,\,r)}(y,\,z)=\sum_{i=0}^{\infty}\overset{\mathrm{P}}{C}_{pm}\,(p+m+i)\,b_{p+m+i}^{(0,\,q,\,r)}(y,\,z). \tag{105}$$

Using (104) and (105), yields the formula

$$\sum_{\nu,\ell=0}^{\infty}a_{m,\,\nu,\,\ell}^{(p,\,q,\,r)}Q_\nu(t)R_\ell(z)=\sum_{\nu,\,\ell=0}^{\infty}\left[\sum_{i=0}^{\infty}\overset{\mathrm{P}}{C}_{pm}\,(p+m+i)a_{p+m+i,\,\nu,\,\ell}^{(0,\,q,\,r)}\right]Q_\nu(t)R_\ell(z), \tag{106}$$

which implies that

$$a_{m,\,\nu,\,\ell}^{(p,\,q,\,r)}=\sum_{i=0}^{\infty}\overset{\mathrm{P}}{C}_{pm}\,(p+m+i)a_{p+m+i,\,\nu,\,\ell}^{(0,\,q,\,r)},\ \ p\geq 1,$$

and thus formula (96) is proved.

It can also be demonstrated that formulas (97) and (98) are valid by following the same steps with (88), keeping $\xi,\ z,\ p,\ r$ fixed and $\xi,\ t,\ p,\ q$ fixed, respectively. Substituting (97) into (96) and (98), and substituting (96) into (98) give formulae (99), (100) and (101). Formula (102) is obtained by substituting (96) into (101). This completes the proof of Theorem 3. □

**Remark 6** The corresponding theorems of Doha [45, 50, 53] and Doha et al. [46] in the cases of triple Chebyshev, Legender, ultraspherical and Jacobi, respectively, can be obtained from our Theorem 3 by taking the suitable polynomials $P_m(\xi)$, $Q_\nu(t)$ and $R_\ell(z)$.

**Corollary 4** Assume that $f(\xi,\ t,\ z)$ and $f^{(p,\,q,\,r)}(\xi,\ t,\ z)$ can be expanded respectively as in (88) and (89), respectively,

$$\xi^{m_1} P_i(\xi) = \sum_{\ell_1=0}^{2m_1} a_{m_1,\,\ell_1}(i) P_{i+m_1-\ell_1}(\xi), \tag{107}$$

$$t^{m_2} Q_j(t) = \sum_{\ell_2=0}^{2m_2} a_{m_2,\,\ell_2}(j) Q_{j+m_2-\ell_2}(t), \tag{108}$$

$$z^{m_3} R_k(z) = \sum_{\ell_3=0}^{2m_3} a_{m_3,\,\ell_3}(k) R_{k+m_3-\ell_3}(z), \tag{109}$$

and

$$\xi^{m_1} t^{m_2} z^{m_3} \left( \sum_{i,\,j,\,k=0}^{\infty} a_{i,\,j,\,k}^{(p,\,q,\,r)} P_i(\xi) Q_j(t) R_k(z) \right) = \sum_{i,\,j,\,k=0}^{\infty} b_{i,\,j,\,k}^{(p,\,q,\,r,\,m_1,\,m_2,\,m_3)} P_i(\xi) Q_j(t) R_k(z), \tag{110}$$

then the expansion coefficients $b_{i,\,j,\,k}^{(p,\,q,\,r,\,m_1,\,m_2,\,m_3)}$ are given by

$$b_{i,\,j,\,k}^{(p,\,q,\,r,\,m_1,\,m_2,\,m_3)} = \sum_{\ell_1=0}^{2m_1} \sum_{\ell_2=0}^{2m_2} \sum_{\ell_3=0}^{2m_3} a_{m_1,\,\ell_1}(\ell_1 + i - m_1)\, a_{m_2,\,\ell_2}(\ell_2 + j - m_2) a_{m_3,\,\ell_3}(\ell_3 + k - m_3)$$
$$\times\, a_{\ell_1+i-m_1,\,\ell_2+j-m_2,\,\ell_3+k-m_3}^{(p,\,q,\,r)}, \quad i,\ j,\ k \geq 0. \tag{111}$$

**Proof.** The proof is similar to that of Corollary 2. □

## 9. Establishing the recursive formulas for the expansion coefficients in series of a product three of classical OPs

Let $f(\xi,\,t,\,z)$ be expanded as in (88), and let it meet the linear non-homogeneous partial DE

$$\sum_{i=0}^{m} \sum_{j=0}^{\nu} \sum_{k=0}^{r} p_{i,\,j,\,k}(\xi,\,t,\,z) f^{(i,\,j,\,k)}(\xi,\,t,\,z) = g(\xi,\,t,\,z), \tag{112}$$

where $p_{i,\,j,\,k}(\xi,\,t,\,z)$, $i = 0,\ 1,\ \ldots,\ m,\ j = 0,\ 1,\ \ldots,\ \nu,\ k = 0,\ 1,\ \ldots,\ r$, are polynomials in $\xi$, $t$ and $z$ such that $p_{m,\,0,\,0},\ p_{0,\,\nu,\,0},\ p_{0,\,0,\,r} \neq 0$, and assume that $g(\xi,\,t,\,z)$ can be expanded as

$$g(\xi,\,t,\,z) = \sum_{i,\,j,\,k=0}^{\infty} g_{i,\,j,\,k} P_i(\xi) Q_j(t) R_k(z), \tag{113}$$

are known, then applying Theorem 3 and Corollary 4 or repeated use of relations (93)-(95) along with (112) leads to the following linear recursive formula of order $(d_1,\ d_2,\ d_3)$,

$$\sum_{i=0}^{d_1}\sum_{j=0}^{d_2}\sum_{k=0}^{d_3}\theta_{i,\ j,\ k}(r,\ s,\ l)\,a_{r+i,\ s+j,\ l+k}=\zeta(r,\ s,\ l),\ \ r,\ s,\ l\geq 0, \tag{114}$$

where $\theta_{i,\ j,\ k}(r,\ s,\ l)$, $i=0,\ 1,\ \ldots,\ d_1,\ j=0,\ 1,\ \ldots,\ d_2,\ k=0,\ 1,\ \ldots,\ d_3$ are polynomials in $r,\ s$ and $l$ such that $\theta_{d_1,\ 0,\ 0}(r,\ s,\ l)$, $\theta_{0,\ d_2,\ 0}(r,\ s,\ l)$, $\theta_{0,\ 0,\ d_3}(r,\ s,\ l)\neq 0$.

An example dealing with a non-homogeneous partial DE is considered to clarify the application of the results obtained.

**Example 2** Consider the non-homogeneous partial DE

$$\begin{gathered}\xi\,u_{\xi}-2t\,u_y+zu_z+(\xi-2t+z)u=(\xi-2t+z)(\xi+t+z+1),\\ u(0,\ t,\ z)=t+z,\ \ u(\xi,\ 0,\ z)=\xi+z,\ \ u(\xi,\ t,\ 0)=\xi+t.\end{gathered} \tag{115}$$

If $(\xi-2t+z)(\xi+t+z+1)$ and $u^{(p,\ q,\ r)}(\xi,\ t,\ z)$ are expanded as follows

$$(\xi-2t+z)(\xi+t+z+1)=\sum_{i+j+k\leq 2}d_{i,\ j,\ k}P_i(\xi)Q_j(t)R_k(z), \tag{116}$$

$$u^{(p,\ q,\ r)}(\xi,\ t,\ z)=\sum_{i,\ j,\ k=0}^{\infty}a_{i,\ j,\ k}^{(p,\ q,\ r)}P_i(\xi)Q_j(t)R_k(z),\quad p,\ q,\ r=0,\ 1, \tag{117}$$

then our ability to apply partial DE (115) on (117) leads to the following equation

$$\begin{aligned}&\overset{\mathrm{P}}{\theta}_{i-1}\,a_{i-1,\ j,\ k}^{(1,\ 0,\ 0)}+\overset{\mathrm{P}}{\zeta}_i\,a_{i,\ j,\ k}^{(1,\ 0,\ 0)}+\overset{\mathrm{P}}{\gamma}_{i+1}\,a_{i+1,\ j,\ k}^{(1,\ 0,\ 0)}-2\,\overset{\mathrm{Q}}{\theta}_{j-1}\,a_{i,\ j-1,\ k}^{(0,\ 1,\ 0)}-2\,\overset{\mathrm{Q}}{\zeta}_j\,a_{i,\ j,\ k}^{(0,\ 1,\ 0)}-2\,\overset{\mathrm{Q}}{\gamma}_{j+1}\,a_{i,\ j+1,\ k}^{(0,\ 1,\ 0)}\\&+\overset{\mathrm{R}}{\theta}_{k-1}\,a_{i,\ j,\ k-1}^{(0,\ 0,\ 1)}+\overset{\mathrm{R}}{\zeta}_k\,a_{i,\ j,\ k}^{(0,\ 0,\ 1)}+\overset{\mathrm{R}}{\gamma}_{k+1}\,a_{i,\ j,\ k+1}^{(0,\ 0,\ 1)}+\overset{\mathrm{P}}{\theta}_{i-1}\,a_{i-1,\ j,\ k}+\overset{\mathrm{P}}{\gamma}_{i+1}\,a_{i+1,\ j,\ k}-2\,\overset{\mathrm{Q}}{\theta}_{j-1}\,a_{i,\ j-1,\ k}\\&-2\,\overset{\mathrm{Q}}{\gamma}_{j+1}\,a_{i,\ j+1,\ k}+\overset{\mathrm{R}}{\theta}_{k-1}\,a_{i,\ j,\ k-1}+\overset{\mathrm{R}}{\gamma}_{k+1}\,a_{i,\ j,\ k+1}+(\overset{\mathrm{P}}{\zeta}_i-2\,\overset{\mathrm{Q}}{\zeta}_j+\overset{\mathrm{R}}{\zeta}_k)\,a_{i,\ j,\ k}=d_{i,\ j,\ k}.\end{aligned} \tag{118}$$

In the next part, we find the recursive formula satisfied by the expansion coefficients $a_{i,\ j,\ k}$ in two different cases for $P_i(\xi)$, $Q_j(t)$ and $R_k(z)$:

**Case 1** The expansion of $u(\xi,\ t,\ z)$ in Triple Bessel polynomials.

In this problem

$$u(\xi,\ t,\ z)=\sum_{i+j+k\leq\nu}a_{i,\ j,\ k}\,Y_i^{(\theta)}(\xi)Y_j^{(\zeta)}(t)Y_k^{(\gamma)}(z), \tag{119}$$

Eq. (118) takes the form

$$\begin{aligned}
&\frac{2(i+\theta)}{(2i+\theta-1)_2}(a_{i-1,\ j,\ k}^{(1,\ 0,\ 0)}+a_{i-1,\ j,\ k})-\frac{2\theta}{(2i+\theta-2)(2i+\theta)}a_{i,\ j,\ k}^{(1,\ 0,\ 0)}\\
&-\frac{2i}{(2i+\theta+2)_2}(a_{i+1,\ j,\ k}^{(1,\ 0,\ 0)}+a_{i+1,\ j,\ k})-\frac{4(j+\zeta)}{(2j+\zeta-1)_2}(a_{i,\ j-1,\ k}^{(0,\ 1,\ 0)}+a_{i,\ j-1,\ k})\\
&+\frac{4j}{(2j+\zeta+2)_2}(a_{i,\ j+1,\ k}^{(0,\ 1,\ 0)}+a_{i,\ j+1,\ k})+\frac{4\zeta}{(2j+\zeta-2)(2j+\zeta)}a_{i,\ j,\ k}^{(0,\ 1,\ 0)}\\
&+\frac{2(k+\gamma)}{(2k+\gamma-1)_2}(a_{i,\ j,\ k-1}^{(0,\ 0,\ 1)}+a_{i,\ j,\ k-1})-\frac{2\gamma}{(2k+\gamma-2)(2k+\gamma)}a_{i,\ j,\ k}^{(0,\ 0,\ 1)}\\
&-\frac{2k}{(2k+\gamma+2)_2}(a_{i,\ j,\ k+1}^{(0,\ 0,\ 1)}+a_{i,\ j,\ k+1})-\left[\frac{2\theta}{(2i+\theta-2)(2i+\theta)}\right.\\
&\left.-\frac{4\zeta}{(2j+\zeta-2)(2j+\zeta)}+\frac{2\gamma}{(2k+\gamma-2)(2k+\gamma)}\right]a_{i,\ j,\ k}=d_{i,\ j,\ k},
\end{aligned} \tag{120}$$

where

$$d_{i,\ j,\ k}=\begin{cases}
\pi_{1,\ 0}(\theta)\pi_{1,\ 0}(\zeta)-\pi_{1,\ 0}(\zeta)\pi_{1,\ 0}(\gamma)+2\pi_{1,\ 0}(\theta)\pi_{1,\ 0}(\gamma) & \\
+\sum_{i=1}^{2}(\pi_{i,\ 0}(\theta)-2\pi_{i,\ 0}(\zeta)+\pi_{i,\ 0}(\gamma)), & i=j=k=0,\\
\pi_{2,\ 1}(\gamma)+\pi_{1,\ 1}(\gamma)[1+2\pi_{1,\ 0}(\theta)-\pi_{1,\ 0}(\zeta)], & i=0,\ j=0,\ k=1,\\
\pi_{2,\ 2}(\gamma), & i=0,\ j=0,\ k=2,\\
-2\pi_{2,\ 1}(\zeta)-\pi_{1,\ 1}(\zeta)[2+\pi_{1,\ 0}(\theta)+\pi_{1,\ 0}(\gamma)], & i=0,\ j=1,\ k=0,\\
-\pi_{1,\ 1}(\zeta)\pi_{1,\ 1}(\gamma), & i=0,\ j=1,\ k=1,\\
\pi_{2,\ 1}(\theta)+\pi_{1,\ 1}(\theta)[1-\pi_{1,\ 0}(\zeta)+2\pi_{1,\ 0}(\gamma)], & i=1,\ j=k=0,\\
2\pi_{1,\ 1}(\theta)\pi_{1,\ 1}(\gamma), & i=1,\ j=0,\ k=1,\\
-\pi_{1,\ 1}(\theta)\pi_{1,\ 1}(\zeta), & i=1,\ j=1,\ k=0,\\
\pi_{2,\ 2}(\theta), & i=2,\ j=k=0,\\
-2\pi_{2,\ 2}(\zeta), & i=0,\ j=2,\ k=0,\\
0, & \text{otherwise},
\end{cases}$$

where

$$\pi_{\nu,\ i}(\theta)=\binom{\nu}{i}\frac{(-1)^{\nu-i}2^{\nu}(2i+\theta+1)\Gamma(i+\theta+1)}{\Gamma(\nu+i+\theta+2)}.$$

Using formulae (96)-(98) with (120) and after some calculations, the following recursive formula can be obtained

$$\begin{aligned}&(\eta_{0i}-2\gamma_{0j}+\mu_{0k})a_{i,\ j,\ k}+\eta_{1i}a_{i+1,\ j,\ k}+\eta_{2i}a_{i+2,\ j,\ k}+\eta_{3i}a_{i+3,\ j,\ k}+\eta_{4i}a_{i+4,\ j,\ k}\\&-2\gamma_{1j}a_{i,\ j+1,\ k}-2\gamma_{2j}a_{i,\ j+2,\ k}-2\gamma_{3j}a_{i,\ j+3,\ k}-2\gamma_{4j}a_{i,\ j+4,\ k}+\mu_{1k}a_{i,\ j,\ k+1}+\mu_{2k}a_{i,\ j,\ k+2}\\&+\mu_{3k}a_{i,\ j,\ k+3}+\mu_{4k}a_{i,\ j,\ k+4}=c_{i,\ j,\ k},\end{aligned}\tag{121}$$

where

$$\eta_{0i}=2[(2i+\theta+1)_5]^{-1}(i+\theta+1)_3,$$

$$\eta_{1i}=[(2i+\theta+2)_5]^{-1}(i+\theta+2)_2[4i^3+(20+4\theta)i^2+(32+12\theta+\theta^2)i+(16+6\theta+\theta^2)],$$

$$\begin{aligned}\eta_{2i}=&-[(2i+\theta+3)_5]^{-1}(i+\theta+3)(i+2)\\&\times[(8+4\theta)i^2+(40+28\theta+4\theta^2)i+(48+47\theta+12\theta^2+\theta^3)],\end{aligned}$$

$$\begin{aligned}\eta_{3i}=&-[(2i+\theta+4)_5]^{-1}(i+2)_2\\&\times[4i^3+(40+8\theta)i^2+(144+52\theta+5\theta^2)i+(144+86\theta+16\theta^2+\theta^3)],\end{aligned}$$

$$\eta_{4i}=-2[(2i+\theta+5)_5]^{-1}(i+2)_3,$$

where $\gamma_{0j}$, $\gamma_{1j}$, $\gamma_{2j}$, $\gamma_{3j}$ and $\gamma_{4j}$ are obtained from $\eta_{0i}$, $\eta_{1i}$, $\eta_{2i}$, $\eta_{3i}$ and $\eta_{4i}$, respectively, by replacing each $i$ and $\theta$ with $j$ and $\zeta$ respectively, $\mu_{0k}$, $\mu_{1k}$, $\mu_{2k}$, $\mu_{3k}$ and $\mu_{4k}$ are obtained from $\eta_{0i}$, $\eta_{1i}$, $\eta_{2i}$, $\eta_{3i}$ and $\eta_{4i}$, respectively, by replacing each $i$ and $\theta$ with $k$ and $\gamma$ respectively, and $c_{i,\ j,\ k}$ are obtained from $d_{i,\ j,\ k}$ by replacing each $\theta,\ \zeta$ and $\gamma$ with $\theta+2$, $\zeta+2$ and $\gamma+2$, respectively.

This problem can be solved by solving the recursive formula (121), and its solution is given by

$$
a_{i,\,j,\,k}=\begin{cases} A_0B_0C_0-\dfrac{2}{(\theta+2)}-\dfrac{2}{(\zeta+2)}-\dfrac{2}{(\gamma+2)}, & i=j=k=0,\\[2ex] A_1B_0C_0+\dfrac{2}{(\theta+2)}, & i=1,\ j=k=0,\\[2ex] A_0B_1C_0+\dfrac{2}{(\zeta+2)}, & i=0,\ j=1,\ k=0,\\[2ex] A_0B_0C_1+\dfrac{2}{(\gamma+2)}, & i=j=0,\ k=1,\\[2ex] A_iB_jC_k, & \text{otherwise,} \end{cases} \tag{122}
$$

where $A_i=I_i(\theta),\ B_j=I_j(\zeta)$ and $C_k=I_k(\gamma),\ i\geq 0$, such that $I_i(\theta)$ is given as in Example 1.

**Remark 7** The solution to the previous example can also be derived by utilizing Hermite and Jacobi polynomials. The details are omitted.

**Case 2** The expansion of $u(\xi,\ t,\ z)$ in Triple Laguerre polynomials

In this problem

$$
u(\xi,\ t,\ z)=\sum_{i+j+k\leq\nu} a_{i,\,j,\,k}\,L_i^{(\theta)}(\xi)L_j^{(\zeta)}(t)L_k^{(\gamma)}(z), \tag{123}
$$

Eq. (118) takes the form

$$
\begin{aligned}
&-(i+\theta+1)(a^{(1,\,0,\,0)}_{i+1,\,j,\,k}-a_{i+1,\,j,\,k})+(2i+\theta+1)\,a^{(1,\,0,\,0)}_{i,\,j,\,k}-i(a^{(1,\,0,\,0)}_{i-1,\,j,\,k}+a_{i-1,\,j,\,k})\\
&+2(j+\zeta+1)(a^{(0,\,1,\,0)}_{i,\,j+1,\,k}+a_{i,\,j+1,\,k})-2(2j+\zeta+1)a^{(0,\,1,\,0)}_{i,\,j,\,k}+2j(a^{(0,\,1,\,0)}_{i,\,j-1,\,k}+a_{i,\,j-1,\,k})\\
&-(k+\gamma+1)(a^{(0,\,0,\,1)}_{i,\,j,\,k+1}-a_{i,\,j,\,k+1})+(2k+\gamma+1)\,a^{(0,\,0,\,1)}_{i,\,j,\,k}-k(a^{(0,\,0,\,1)}_{i,\,j,\,k-1}+a_{i,\,j,\,k-1})\\
&+(2i-4j+2k+\theta-2\zeta+\gamma)a_{i,\,j,\,k}=d_{i,\,j,\,k},
\end{aligned} \tag{124}
$$

where the expression of $d_{i,\,j,\,k}$ is similar to the above case such that

$$
\pi_{\nu,\,i}(\theta)=\frac{(-1)^i\nu!(\theta+i+1)_{\nu-i}}{(\nu-i)!}.
$$

Eqs. (93)-(95) become

$$a_{i,\ j,\ k}^{(p,\ q,\ r)} - a_{i-1,\ j,\ k}^{(p,\ q,\ r)} = a_{i,\ j,\ k}^{(p-1,\ q,\ r)}, \quad p,\ i \geq 1, \tag{125}$$

$$a_{i,\ j,\ k}^{(p,\ q,\ r)} - a_{i,\ j-1,\ k}^{(p,\ q,\ r)} = a_{i,\ j,\ k}^{(p,\ q-1,\ r)}, \quad q,\ j \geq 1, \tag{126}$$

$$a_{i,\ j,\ k}^{(p,\ q,\ r)} - a_{i,\ j,\ k-1}^{(p,\ q,\ r)} = a_{i,\ j,\ k}^{(p,\ q,\ r-1)}, \quad r,\ k \geq 1. \tag{127}$$

Now, repeated use of relations (125)-(127) to eliminate the coefficients $a_{i-1,\ j,\ k}^{(1,\ 0,\ 0)}$, $a_{i,\ j,\ k}^{(1,\ 0,\ 0)}$ , $a_{i+1,\ j,\ k}^{(1,\ 0,\ 0)}$, $a_{i,\ j-1,\ k}^{(0,\ 1,\ 0)}$, $a_{i,\ j,\ k}^{(0,\ 1,\ 0)}$, $a_{i,\ j+1,\ k}^{(0,\ 1,\ 0)}$, $a_{i,\ j,\ k-1}^{(0,\ 0,\ 1)}$, $a_{i,\ j,\ k}^{(0,\ 0,\ 1)}$ and $a_{i,\ j,\ k+1}^{(0,\ 0,\ 1)}$ yields

$$\begin{aligned} &(3i-6j+3k+\theta-2\zeta+\gamma)a_{i,\ j,\ k} - 2(i+\theta+1)a_{i+1,\ j,\ k} - ia_{i-1,\ j,\ k} \\ &+2(j+\zeta+1)a_{i,\ j+1} + 2ja_{i,\ j-1,\ k} - 2(k+\gamma+1)a_{i,\ j,\ k+1} - ka_{i,\ j,\ k-1} = f_{i,\ j,\ k}, \quad i,\ j,\ k \geq 0, \end{aligned} \tag{128}$$

where

$$f_{i,\ j,\ k} = d_{i,\ j,\ k} - d_{i-1,\ j,\ k} - d_{i,\ j-1,\ k} + d_{i-1,\ j-1,\ k} - d_{i,\ j,\ k-1} + d_{i-1,\ j,\ k-1} - d_{i,\ j-1,\ k-1} + d_{i-1,\ j-1,\ k-1}, \quad i,\ j,\ k \geq 0,$$

Eq. (128) can solved to give

$$a_{i,\ j,\ k} = \begin{cases} 2^{-(\theta+\zeta+\gamma+6)}(\theta+1)(\zeta+1)(\gamma+1) + (\theta+\zeta+\gamma+3), & i = j = k = 0, \\ 2^{-(\theta+\zeta+\gamma+7)}\theta(\zeta+1)(\gamma+1) - 1, & i = 1,\ j = k = 0, \\ 2^{-(\theta+\zeta+\gamma+7)}\zeta(\theta+1)(\gamma+1) - 1, & i = 0,\ j = 1,\ k = 0, \\ 2^{-(\theta+\zeta+\gamma+7)}\gamma(\theta+1)(\zeta+1) - 1, & i = j = 0,\ k = 1, \\ 2^{-(\theta+\zeta+\gamma+i+j+6)}(\theta-i+1)(\zeta-j+1)(\gamma-k+1), & \text{otherwise.} \end{cases} \tag{129}$$

## 10. The expansion of $(\xi + t)^{\nu}$ as a multiple series in a product three of classical OPs

In this problem

$$(\xi+t+z)^{\nu} = \sum_{i+j+k \leq \nu} a_{i,\ j,\ k}(\nu)P_i(\xi)Q_j(t)R_k(z), \tag{130}$$

we have $u(\xi,\ t,\ z) = (\xi+t+z)^{\nu}$ satisfies the homogeneous partial DE

$$\xi u_{\xi} + y u_y + z u_z - \nu u = 0, \tag{131}$$

then our ability to apply partial DE (131) on (130) leads to the following equation

$$\begin{aligned} &\overset{\mathrm{P}}{\theta}_{i-1}\, a_{i-1,\, j,\, k}^{(1,\,0,\,0)} + \overset{\mathrm{P}}{\zeta}_{i}\, a_{i,\, j,\, k}^{(1,\,0,\,0)} + \overset{\mathrm{P}}{\gamma}_{i+1}\, a_{i+1,\, j,\, k}^{(1,\,0,\,0)} + \overset{\mathrm{Q}}{\theta}_{j-1}\, a_{i,\, j-1,\, k}^{(0,\,1,\,0)} + \overset{\mathrm{Q}}{\zeta}_{j}\, a_{i,\, j,\, k}^{(0,\,1,\,0)} + \overset{\mathrm{Q}}{\gamma}_{j+1}\, a_{i,\, j+1,\, k}^{(0,\,1,\,0)} \\ &+ \overset{\mathrm{R}}{\theta}_{k-1}\, a_{i,\, j,\, k-1}^{(0,\,0,\,1)} + \overset{\mathrm{R}}{\zeta}_{k}\, a_{i,\, j,\, k}^{(0,\,0,\,1)} + \overset{\mathrm{R}}{\gamma}_{k+1}\, a_{i,\, j,\, k+1}^{(0,\,1,\,1)} - \nu\, a_{i,\, j,\, k} = 0. \end{aligned} \tag{132}$$

## 10.1 *The link between $(\xi + t + z)^{\nu}$ and Triple Hermite polynomials*

In this problem

$$(\xi + t + z)^{\nu} = \sum_{i+j+k \le \nu} a_{i,\, j,\, k}(\nu) H_i(\xi) H_j(t) H_k(z), \tag{133}$$

Eq. (132) takes the form

$$\begin{aligned} &\frac{1}{2} a_{i-1,\, j,\, k}^{(1,\,0,\,0)}(\nu) + (i+1) a_{i+1,\, j,\, k}^{(1,\,0,\,0)}(\nu) + \frac{1}{2} a_{i,\, j-1,\, k}^{(0,\,1,\,0)}(\nu) + (j+1) a_{i,\, j+1,\, k}^{(0,\,1,\,0)}(\nu) \\ &+ \frac{1}{2} a_{i,\, j,\, k-1}^{(0,\,0,\,1)}(\nu) + (k+1) a_{i,\, j,\, k+1}^{(0,\,0,\,1)}(\nu) - \nu\, a_{i,\, j,\, k}(\nu) = 0, \end{aligned} \tag{134}$$

and formula (102) becomes

$$a_{i,\, j,\, k}^{(p,\, q,\, r)} = 2^{p+q+r}\, p! q! r! \binom{p+i}{p} \binom{q+j}{q} \binom{r+k}{r} a_{p+i,\, q+j,\, r+k}, \quad i,\ j,\ k,\ p,\ q,\ r \ge 0. \tag{135}$$

Application of formula (135) to (132), gives

$$\begin{aligned} &(\nu - i - j - k) a_{i,\, j,\, k}(\nu) - 2(i+1)_2 a_{i+2,\, j,\, k}(\nu) - 2(j+1)_2\, a_{i,\, j+2,\, k}(\nu) - 2(k+1)_2\, a_{i,\, j,\, k+2}(\nu) = 0, \\ &i,\ j,\ k = \nu - 1,\ \nu - 2,\ \ldots,\ 0, \end{aligned} \tag{136}$$

with $a_{i,\, j,\, k}(\nu) = 0$, $i + j + k > \nu$, $a_{-1,\, j,\, k}(\nu) = a_{i,\, -1,\, k}(\nu) = a_{i,\, j,\, -1}(\nu) = 0$ and $a_{\nu,\, 0,\, 0}(\nu) = a_{0,\, \nu,\, 0}(\nu) = a_{0,\, 0,\, \nu}(\nu) = 2^{-\nu}$. Eq. (136) can be solved to give

$$a_{i,\ j,\ k}(\nu) = \begin{cases} \dfrac{\nu!2^{-\nu}3^{(\nu-i-j-k)/2}}{i!\,j!k!\left(\dfrac{\nu-i-j-k}{2}\right)!}, & (\nu-i-j-k) \text{ even,} \\ \\ 0, & (\nu-i-j-k) \text{ odd.} \end{cases} \tag{137}$$

In particular, and for the special case $z = 0$, Eq. (133), after some calculations, becomes in agreement with Eq. (60), while for the case $y = z = 0$ we get the result obtained by Rainville [54, p.194] and Sánchez-Ruiz and Dehesa [55, p.159].

## 10.2 *The link between $(\xi + t + z)^\nu$ and Triple Laguerre polynomials*

In this problem

$$(\xi+t+z)^\nu = \sum_{i+j+k\le\nu} a_{i,\ j,\ k}(\nu)L_i^{(\theta)}(\xi)L_j^{(\zeta)}(t)L_k^{(\gamma)}(z), \tag{138}$$

Eq. (132) takes the form

$$\begin{aligned}
&-ia^{(1,\ 0,\ 0)}_{i-1,\ j,\ k}(\nu)+(2i+\theta+1)a^{(1,\ 0,\ 0)}_{i,\ j,\ k}(\nu)-(i+\theta+1)a^{(1,\ 0,\ 0)}_{i+1,\ j,\ k}(\nu)\\
&-ja^{(0,\ 1,\ 0)}_{i,\ j-1,\ k}(\nu)+(2j+\zeta+1)\,a^{(0,\ 1,\ 0)}_{i,\ j,\ k}(\nu)-(j+\zeta+1)a^{(0,\ 1,\ 0)}_{i,\ j+1,\ k}(\nu)\\
&-ka^{(0,\ 0,\ 1)}_{i,\ j,\ k-1}(\nu)+(2k+\gamma+1)a^{(0,\ 0,\ 1)}_{i,\ j,\ k}(\nu)-(k+\gamma+1)a^{(0,\ 0,\ 1)}_{i,\ j,\ k+1}(\nu)\\
&-\nu\,a_{i,\ j,\ k}(\nu)=0.
\end{aligned}$$

Now, repeated use of relations (125)-(127) to eliminate the coefficients $a^{(1,\ 0,\ 0)}_{i-1,\ j,\ k}(\nu)$, $a^{(1,\ 0,\ 0)}_{i,\ j,\ k}(\nu)$, $a^{(1,\ 0,\ 0)}_{i+1,\ j,\ k}(\nu)$, $a^{(0,\ 1,\ 0)}_{i,\ j-1,\ k}(\nu)$, $a^{(0,\ 1,\ 0)}_{i,\ j,\ k}(\nu)$, $a^{(0,\ 1,\ 0)}_{i,\ j+1,\ k}(\nu)$, $a^{(0,\ 0,\ 1)}_{i,\ j,\ k-1}(\nu)$, $a^{(0,\ 0,\ 1)}_{i,\ j,\ k}(\nu)$ and $a^{(0,\ 0,\ 1)}_{i,\ j,\ k+1}(\nu)$ yields

$$\begin{aligned}
&(\nu-i-j-k)a_{i,\ j,\ k}(\nu)+(i+\theta+1)a_{i+1,\ j,\ k}(\nu)+(j+\zeta+1)a_{i,\ j+1,\ k}(\nu)+(k+\gamma+1)a_{i,\ j,\ k+1}(\nu)=0,\\
&i,\ j,\ k=\nu-1,\ \nu-2,\ \ldots,\ 0,
\end{aligned} \tag{139}$$

with $a_{i,\ j,\ k}(\nu) = 0$, $i+j+k > \nu$, $a_{-1,\ j,\ k}(\nu) = a_{i,\ -1,\ k}(\nu) = a_{i,\ j,\ -1}(\nu) = 0$ and $a_{\nu,\ 0,\ 0}(\nu) = a_{0,\ \nu,\ 0}(\nu) = a_{0,\ 0,\ \nu}(\nu) = \dfrac{(-1)^\nu}{\nu!}$. The solution of (139) is

$$a_{i,\ j,\ k}(\nu)=\begin{cases}\dfrac{(-\nu)_{i+j+k}(\theta+\zeta+\gamma+3)_\nu}{(\theta+\zeta+\gamma+3)_{i+j+k}}, & i,\ j,\ k\leq\nu,\\ \\ 0, & \text{otherwise.}\end{cases} \tag{140}$$

In particular, and for the special case $z=0$, Eq. (138), after some calculations, becomes in agreement with Eq. (65), while for the case $y=z=0$ we get the result obtained by Rainville [54, p.207] and Sánchez-Ruiz and Dehesa [55, p.159]. In view of relation (69), we obtain the following corollary.

**Corollary 5** In the problem

$$(\xi^2+t^2+z^2)^\nu=\sum_{i,\ j,\ k=0}^{\infty}a_{i,\ j,\ k}(\nu)H_{2i}(\xi)H_{2j}(t)H_{2k}(z), \tag{141}$$

the coefficients $a_{i,\ j,\ k}(\nu)$ are given by

$$a_{i,\ j,\ k}(\nu)=\begin{cases}\dfrac{\nu!\,(3/2)_\nu\,4^{-(i+j+k)}}{(\nu-i-j-k)!\,i!\,j!\,k!\,(3/2)_{i+j+k}}, & i+j+k\leq\nu,\\ \\ 0, & \text{otherwise.}\end{cases} \tag{142}$$

**Remark 8** In view of formulae (68) and (140), we can deduce that

$$\left(\sum_{m=1}^{k}\xi_m\right)^\nu=\sum_{i_1+i_2+\cdots+i_k\leq\nu}\frac{(-\nu)_{i_1+\cdots+i_k}(\theta_1+\cdots+\theta_k+k)_\nu}{(\theta_1+\cdots+\theta_k+k)_{i_1+\cdots+i_k}}\prod_{m=1}^{k}L_{i_m}^{(\theta_m)}(\xi_m), \tag{143}$$

and using (69) leads to

$$\left(\sum_{m=1}^{k}\xi_m^2\right)^\nu=\sum_{i_1+i_2+\cdots+i_k\leq\nu}\frac{\nu!(k/2)_\nu\,4^{-(i_1+\cdots+i_k)}}{(\nu-i_1-\cdots-i_k)!\,\prod\limits_{m=1}^{k}i_k!\,(k/2)_{i_1+\cdots+i_k}}\prod_{m=1}^{k}H_{2i_m}(\xi_m). \tag{144}$$

**Remark 9** In view of formulae (64) and (137), we can deduce that

$$\left(\sum_{m=1}^{k}\xi_m\right)^\nu=\sum_{(\nu-i_1-i_2-\cdots-i_k)\ even}\frac{2^{-\nu}\nu!k^{(\nu-i_1-i_2-\cdots-i_k)/2}}{((\nu-i_1-\cdots-i_k)/2)!\,\prod\limits_{m=1}^{k}i_m!}\prod_{m=1}^{k}H_{i_m}(\xi_m). \tag{145}$$

## 11. Connection problem in the sense of three variables

In such case, we have to calculate $a_{i,\ j,\ k}(\nu)$ in the problem

$$F_\nu(a\xi+bt+cz)=\sum_{i+j+k\le\nu}a_{i,\ j,\ k}(\nu)P_i(\xi)Q_j(t)R_k(z), \tag{146}$$

where $F_\nu$, $P_i$ and $Q_j$ are classical OPs. The work developed in Section 8 permits us to obtain a recursive formula satisfied by the coefficients $a_{i,\ j,\ k}(\nu)$, if we know a partial differential operator that cancels the left-hand side of (146).

## 11.1 *The Hermite-Triple Hermite connection problem*

In this problem

$$H_\nu\left(\frac{\xi+t+z}{\sqrt{3}}\right)=\sum_{i+j+k\le\nu}a_{i,\ j,\ k}(\nu)H_i(\xi)H_j(t)H_k(z), \tag{147}$$

where $H_\nu((\xi+t+z)/\sqrt{3})$ satisfy the DE

$$[D_\xi^2-6(\xi+t+z)D_\xi+6\nu]H_\nu((\xi+t+z)/\sqrt{3})=0, \tag{148}$$

the coefficients $a_{i,\ j,\ k}(\nu)$ satisfy the recursive formula

$$\begin{aligned}&6(\nu-i)a_{i,\ j,\ k}(\nu)-6(i+1)a_{i+1,\ j-1,\ k}(\nu)-6(i+1)a_{i+1,\ j,\ k-1}(\nu)\\&-12(i+1)(k+1)a_{i+1,\ j,\ k+1}(\nu)-12(i+1)(j+1)a_{i+1,\ j+1,\ k}(\nu)\\&-8(i+1)_2a_{i+2,\ j,\ k}(\nu)=0,\ \ i,\ j,\ k=\nu-1,\ \nu-2,\ \ldots,\ 0,\end{aligned} \tag{149}$$

with $a_{i,\ j,\ k}(\nu)=0$, $i+j+k>\nu$, $a_{-1,\ j,\ k}(\nu)=a_{i,\ -1,\ k}(\nu)=a_{i,\ j,\ -1}(\nu)=0$ and $a_{\nu,\ 0,\ 0}(\nu)=a_{0,\ \nu,\ 0}(\nu)=a_{0,\ 0,\ \nu}(\nu)=3^{-\nu/2}$. Eq. (149) can be solved to give

$$a_{i,\ j,\ k}(\nu)=\begin{cases}3^{-\nu/2}\dfrac{\nu!}{i!j!k!}, & i+j+k=\nu,\\[2ex] 0, & i+j+k\neq\nu,\end{cases} \tag{150}$$

which is coherent with the result found in Hansen [57, formula (49.7.1)].

$$H_\nu\left(\frac{\xi+t+z}{\sqrt{3}}\right)=3^{-\nu/2}\sum_{i+j+k=\nu}\frac{\nu!}{i!j!k!}H_i(\xi)H_j(t)H_k(z). \tag{151}$$

## 11.2 *The Laguerre-Triple Laguerre connection problem*

In this problem

$$L_{\nu}^{(\theta+\zeta+\gamma+2)}(\xi+t+z)=\sum_{i+j+k\leq\nu}a_{i,\ j,\ k}(\nu)L_{i}^{(\theta)}(\xi)L_{j}^{(\zeta)}(t)L_{k}^{(\gamma)}(z), \tag{152}$$

where $L_{\nu}^{(\theta+\zeta+\gamma+2)}(\xi+t+z)$ satisfy the DE

$$[(\xi+t+z)D_{\xi}^{2}+(3+\theta+\zeta+\gamma-\xi-y-z)D_{\xi}+\nu]L_{\nu}^{(\theta+\zeta+\gamma+2)}(\xi+t+z)=0, \tag{153}$$

the coefficients $a_{i,\ j,\ k}(\nu)$ satisfy the recursive formula

$$\begin{aligned}&2(\nu-i-j-k)a_{i,\ j,\ k}(\nu)-(\nu-i+1)a_{i-1,\ j,\ k}(\nu)+ja_{i,\ j-1,\ k}(\nu)\\&+ka_{i,\ j,\ k-1}(\nu)+(k+\gamma+1)a_{i,\ j,\ k+1}(\nu)+(j+\zeta+1)a_{i,\ j+1,\ k}(\nu)\\&-(\nu-i+\zeta+\gamma+1)a_{i+1,\ j,\ k}(\nu)=0,\ \ i,\ j,\ k=\nu-1,\ \nu-2,\ \dots,\ 0,\end{aligned} \tag{154}$$

with $a_{i,\ j,\ k}(\nu)=0, i+j+k>\nu$, $a_{-1,\ j,\ k}(\nu)=a_{i,\ -1,\ k}(\nu)=a_{i,\ j,\ -1}(\nu)=0$ and $a_{\nu,\ 0,\ 0}(\nu)=a_{0,\ \nu,\ 0}(\nu)=a_{0,\ 0,\ \nu}(\nu)=1$. Eq. (154) can be solved to give

$$a_{i,\ j,\ k}(\nu)=\begin{cases}1, & i+j+k=\nu,\\0, & i+j+k\neq\nu,\end{cases} \tag{155}$$

which is coherent with the result found in Hansen [57, formula (48.24.1)],

$$L_{\nu}^{(\theta+\zeta+\gamma+2)}(\xi+t+z)=\sum_{i+j+k=\nu}L_{i}^{(\theta)}(\xi)L_{j}^{(\zeta)}(t)L_{k}^{(\gamma)}(z). \tag{156}$$

**Remark 10** The examples chosen in Sections 5-7, 9-11 closely relate to famously discussed connection problems in the literature regarding classical orthogonal polynomials. We specifically looked for PDEs that exhibit properties or solutions that could be expressed in terms of these polynomials, thereby demonstrating the applicability of our derived formulas for expansion and connection coefficients. Additionally, the chosen examples serve to effectively illustrate the main concepts and results of our paper. Each example showcases particular features of our methodology, reinforcing the theoretical contributions we make.

**Remark 11** In view of formulae (77) and (156), we can deduce that

$$L_{\nu}^{(\theta_1+\theta_2+\cdots+\theta_k+k-1)}\left(\sum_{m=1}^{k}\xi_m\right)=\sum_{i_1+i_2+\cdots+i_k=\nu}\prod_{m=1}^{k}L_{i_m}^{(\theta_m)}(\xi_m). \tag{157}$$

**Remark 12** In view of formulae (75) and (150), we can deduce that

$$H_\nu\left(\frac{1}{\sqrt{k}}\sum_{m=1}^{k}\xi_m\right)=k^{-\frac{\nu}{2}}\sum_{i_1+i_2+\cdots+i_k=\nu}\frac{\nu!}{\prod_{m=1}^{k} i_m!}\prod_{m=1}^{k}H_{i_m}(\xi_m). \tag{158}$$

**Remark 13** We aim to highlight our algorithm's systematic nature and simplicity for constructing linear recursive formulas of the form (44) and (114). This algorithm can be implemented using symbolic language in any computer algebra system, such as Mathematica Version 12.

## 12. Conclusion

This paper presented recursive and explicit formulas for the expansion and connection coefficients in the series of classical orthogonal polynomial products. Our results enhance existing methodologies and provide computationally efficient tools for practitioners working with these polynomials in various applications. We demonstrated the applicability of our derived formulas through several examples of partial differential equations closely related to classical orthogonal polynomials. The chosen examples illustrated the versatility and robustness of our methods, confirming their relevance across different scientific fields. For future work, we plan to extend our results to encompass $q$-orthogonal polynomials, allowing for exploring new problems in quantum calculus and other areas. This extension could reveal deeper connections between classical and $q$-orthogonal systems. Also, we will focus on developing numerical algorithms based on our formulas, enabling practical applications in computational mathematics and engineering. Implementing these algorithms will facilitate testing and validation against existing numerical methods.

## Conflict of interest

The authors declare no competing financial interest.

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