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Research Article

Exploring Multivariate Statistics: Unveiling the Power of Eigenvalues in Wishart Distribution Analysis

Randa A. Makled[1,2*], Weihu Cheng[1]

[1]Faculty of mathematics, Statistics and Mechanics, Beijing University of Technology, Beijing 100124, China
[2]Faculty of Commerce, Damietta University, Damietta 34511, Egypt
E-mail: randamakled@du.edu.eg

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Abstract: This research examines how degrees of freedom and covariance matrix configurations affect Wishart distributed matrix eigenvalue distributions. We use the energy distance metric to compare eigenvalue distributions in Identity, Diagonal, and Structured covariance matrices. Through extensive simulations, we show that degrees of freedom and covariance matrix architectures greatly affect eigenvalue dispersion and energy distance distributions. Statistical models are more reliable with smaller, more stable distributions from higher degrees of freedom. We analyze the eigenvalue distributions of NextEra Energy (NEE), Enphase Energy (ENPH), First Solar (FSLR), SunPower (SPWR), and Brookfield Renewable Partners (BEP) stocks using this analytical methodology. Daily returns and covariance matrices were calculated using daily closing prices from January 1, 2018, to January 1, 2023. Our findings support simulation studies indicating larger degrees of freedom lead to more stable energy distance distributions.The findings of this study are useful in finance, genetics, and environmental studies, where stability and variability of the covariance structure are important. The requirement for higher-dimensional settings and real-world datasets to validate the theoretical framework are our research's constraints. This research expands Wishart distribution knowledge and provides a solid analytical foundation for data analysis and model fitting in numerous scientific fields.

Keywords: energy distance, energy statistic, wishart distributions, eigenvalues, high-dimensional data

MSC: 62H10, 62P20, 62H25

Abbreviation

DoF	Degrees of Freedom
ECDF	Empirical Cumulative Distribution Function
ED	Energy Distance
PCA	Principal Component Analysis
PDF	Probability Density Function
WDM	Wishart Distribution Matrices
NEE	NextEra Energ
ENPH	Enphase Energ

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FSLR First Solar
SPWR SunPower
BRP Brookfield Renewable Partners

1. Introduction

The Wishart distribution is a standard tool in multivariate statistics due to its importance as a multivariate probability distribution. John Wishart first presented this distribution in 1928 [1]; it generalizes the matrix-variate gamma distribution straightforwardly, if the data follow a multivariate normal distribution, then the sample covariance matrices will have this distribution. The Wishart distribution has been the subject of thorough investigation within the literature of multivariate analysis concerning its properties and theoretical foundations [2–4] such as spectral decomposition [5] and expected Kullback-Leibler divergence [6] as well as provides practical value such as in [7]. Let $\mathbf{X}$ be a random positive definite matrix with dimensions $p \times p$. If $\mathbf{X}$ satisfies the probability density function (PDF), it is considered to adhere to a Wishart distribution with parameters $\mathbf{V}$ and n. The PDF of $\mathbf{X}$ in the Wishart distribution is given by:

$$f(\mathbf{X};\ \mathbf{V},\ n) = \frac{|X|^{\frac{n-p-1}{2}} \exp\left(-\frac{1}{2}\mathrm{tr}(\mathbf{V}^{-1}\mathbf{X})\right)}{2^{\frac{np}{2}} |\mathbf{V}|^{\frac{n}{2}} \Gamma_p\left(\frac{n}{2}\right)}, \tag{1}$$

where $\mathbf{V}$ is an $n \times n$ positive definite matrix, $v > n - 1$ is the degrees of freedom, $\Gamma_n(\cdot)$ is the multivariate gamma function, and $\mathrm{tr}(\cdot)$ denotes the matrix trace. The probability distribution of the sample covariance matrix follows a Wishart distribution $\mathbf{S} = \mathbf{X}\mathbf{X}'$, where the columns of $\mathbf{X}$ are i.i.d multivariate normal $N_n(\mathbf{0},\ \mathbf{V})$. In this context, $\mathbf{V}$ represents the population covariance matrix.

The expected value and variance of $\mathbf{S}$ are given by:

$$\mathbb{E}[\mathbf{S}] = v\mathbf{V} \tag{2}$$

$$\mathrm{Var}(\mathbf{S}) = v(\mathbf{V} \otimes \mathbf{V}) \tag{3}$$

where $\otimes$ denotes the Kronecker product.

Multivariate statistics use the Wishart distribution for covariance matrix estimation, uncertainty analysis, and inference [8–10].The Wishart distributed data is a prior that pairs well with the inverse covariance matrix of a multivariate normal random vector in Bayesian statistics [9]. It is commonly used in random matrix spectral theory [11]. Furthermore, The Wishart distribution helps statisticians and machine learners to identify covariance matrices and uncertainty. Various machine learning methods employ Wishart distributions to represent covariance patterns in multivariate data and evaluate Rayleigh fading MIMO wireless channel performance in wireless communications [12, 13]. In finance, Wishart-based covariance forecasts improve portfolio risk assessment [14]. Wishart-based stochastic volatility models are used to analyze changes in financial time series covariance matrices [15]. Székely developed energy statistics to assess distribution differences using "potential energy" [16]. The energy distance E between Wishart-distributed matrix eigenvalue distributions is:

$$E = \frac{1}{2n}\sum_{i=1}^{n}(F_1(\lambda_i) - F_2(\lambda_i))^2 \tag{4}$$

The empirical cumulative distribution functions (ECDFs) of eigenvalues from two Wishart-distributed matrices are $F_1(\lambda)$ and $F_2(\lambda)$, with λ_i representing the i-th eigenvalue. This energy distance measure quantifies structural differences in eigenvalue distributions, making Wishart-distributed data comparisons robust.

In this sense, the points of data that are statistically closer together have less “potential energy”, while those that are farther away have more [17]. The metric of energy distance is applied to distinguish distributions and underlying statistical analysis and inference [18]. Energy statistics can evaluate differences between samples or proposed distributions based on observation distances [17]. These techniques are used for feature selection, nonparametric distribution equality tests, and independence testing [19].

Energy distance based on Wishart distribution eigenvalue distribution may be beneficial for distribution comparisons. Instead of comparing individual components in the covariance matrices, this method measures the disparity between the underlying structures of the distributions, as expressed by their eigenvalue distributions. High-dimensional Wishart distributions may make conventional methods computationally costly or unintelligible because of to the “curse of dimensionality”. The energy distance technique leverages the unique properties of eigenvalue distributions to provide a precise and useful measure of dissimilarity across Wishart distributions, facilitating for comparisons and analysis in a variety of application fields.

There are some problems with using Székely’s energy statistics directly with pairwise distances for Wishart distributions, even though they provide a strong foundation. First, Euclidean distance is unsuitable because covariance matrices, which express linear interactions between features, fall into non-Euclidean space. Additionally, by ignoring the underlying structure and connections of the covariance matrices and instead computing element-wise differences many valuable information would be lost. These distances also struggle to capture substantial differences across Wishart distributions due to scaling sensitivity and transformation invariance. These difficulties need alternate ways like the suggested energy distance based on eigenvalue distribution, which overcomes these constraints and uses distinctive properties of eigenvalue distributionsfor discrepancy measurement.

The energy distance based on the eigenvalue distribution eliminates pairwise distance drawbacks. To address issues with non-Euclidean space, such as information loss and scale sensitivity, we focus on Wishart distribution eigenvalue distributions. By comparing empirical cumulative distribution functions (ECDFs) of eigenvalues, the energy distance measures Wishart sample structural dissimilarity. Dissimilarities may be difficult to discover with pairwise distance.

2. Eigenvalue decomposition for wishart-distributed matrices

Understanding eigenvalue decomposition is crucial for understanding linear transformations in Wishart-distributed matrices $\mathbf{W}$ with dimensions $(n \times n)$, where n indicates data dimensionality. The eigenvalues of $\mathbf{W}$, $\lambda_1, \lambda_2, \ldots, \lambda_n$, are linked to their corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in non-increasing order, forming the eigenvector matrix $\mathbf{V}$ with a dimension of $(n \times n)$. The decomposition procedure starts with the determination of the characteristic equation, $|\mathbf{W} - \lambda\mathbf{I}| = 0$, which results in the n eigenvalues, where λ represents an eigenvalue and $\mathbf{I}$ is the identity matrix matching the dimensions of $\mathbf{W}$.

Subsequently, for each derived eigenvalue λ_i, its corresponding eigenvector $\mathbf{v}_i$ is determined by solving $(\mathbf{W} - \lambda_i\mathbf{I})\mathbf{v}_i = \mathbf{0}$, a system ensuring non-trivial solutions due to the determinant condition of $(\mathbf{W} - \lambda_i\mathbf{I})$ being zero. Given the real and symmetric nature of $\mathbf{W}$ in Wishart distributions, all λ_i are real and non-negative, and eigenvectors associated with distinct eigenvalues are orthogonal, i.e., $\mathbf{v}_i^T\mathbf{v}_j = 0$ for $i \neq j$. Ultimately, the eigenvalue decomposition of $\mathbf{W}$ is formulated as $\mathbf{W} = \mathbf{V}\blacksquare\mathbf{V}^T$, where $\blacksquare = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mathbf{V}$ consists of columns of eigenvectors. Decomposing the data represented by $\mathbf{W}$ reveals its structure, including the variance (eigenvalues) and orientation (eigenvectors) of its major components.

2.1 *Eigenvalues and eigenvectors in covariance matrix*

Eigenvalue decomposition is essential for understanding matrix behaviors, especially in covariance matrices, where eigenvalues and eigenvectors reveal data structure and variability. A covariance matrix, Σ, outlines linear relationships between dataset features, forming a $p \times p$ square matrix for p features, where each element Σ_{ij} denotes the covariance between the i-th and j-th features. Eigenvalues (λ_i) of Σ, derived from solving the characteristic equation $|\Sigma - \lambda I| = 0$ where I is the identity matrix of corresponding dimensions, represent the variance amounts captured by linear data transformations, highlighting the spread or dispersion. Each eigenvalue is associated with a non-zero eigenvector ($\mathbf{v}_i$), defining the direction of maximum variance, obtained by solving $(\Sigma - \lambda_i I)\mathbf{v}_i = \mathbf{0}$, indicative of a homogeneous system. Eigenvectors corresponding to distinct eigenvalues are orthogonal, satisfying $\mathbf{v}_i^T \mathbf{v}_j = 0$ for $i \neq j$, thereby ensuring independent variance directions in data space. The eigenvalue decomposition of Σ is $\Sigma = V \Lambda V^T$, where V is the matrix of eigenvectors and Λ is the diagonal matrix containing eigenvalues, which reveals the structure of the covariance matrix. Further, Principal Component Analysis (PCA) uses eigenvalues and eigenvectors to emphasize the most significant variance directions of the data, reducing dimensionality while preserving important information.

2.2 *Energy distance for discrepancy between eigenvalue distributions*

Energy distance, conceptualized by Székely, provides a quantitative measure for assessing the discrepancy between probability distributions, adapted in this context to evaluate the differences between eigenvalue distributions from Wishart-distributed matrices. Denote λ_i^1 and λ_i^2 as the i-th eigenvalues from two distinct Wishart-distributed matrices $\mathbf{W}_1$ and $\mathbf{W}_2$, respectively, ordered non-increasingly. Let $F_1(\lambda)$ and $F_2(\lambda)$ represent the empirical cumulative distribution functions (ECDFs) for the eigenvalues of $\mathbf{W}_1$ and $\mathbf{W}_2$, respectively. The ECDF at a specific value λ signifies the probability of an eigenvalue being less than or equal to λ, with n indicating the dimensionality of the matrices. The energy distance E between the two eigenvalue distributions is mathematically defined as in equation 4 where the discrepancy at each eigenvalue level is accentuated by the differences in ECDFs at each λ_i, with larger discrepancies indicating greater differences. The squared differences emphasize significant gaps, while summing across all n eigenvalues furnishes a comprehensive assessment.

The normalization factor $\frac{1}{2}$ ensures that the energy distance E remains non-negative and proportional to the count of eigenvalues. A higher value of E suggests significant differences between the eigenvalue distributions of the two matrices, indicative of a substantial discrepancy in their Wishart distributions. Conversely, a lower E reflects closer similarity between distributions. This metric, distinct from the Frobenius norm, specifically addresses structures inherent to eigenvalues, circumventing limitations associated with non-Euclidean space or loss of information, and thus provides a robust measure for matrix dissimilarity by leveraging the unique properties of eigenvalue distributions.

This study is driven by the need for strong statistical techniques that appropriately represent covariance structure variability and stability. This work uses the energy distance metric to thoroughly analyze how degrees of freedom and covariance matrix structure impact eigenvalue distributions. This study helps construct robust statistical models for finance, genetics, and environmental research by revealing how degrees of freedom and covariance structures impact eigenvalue dispersion.

The discipline benefits conceptually and practically from this study. By explaining how degrees of freedom and covariance matrix configurations affect eigenvalue dispersion, it improves Wishart distribution knowledge. It provides useful insights for risk management and portfolio optimization in finance, genetics, and environmental research, where stable covariance structures are crucial.

The energy distance metric is used to evaluate differences in Wishart-distributed matrix eigenvalue distributions in this research. Significant simulations and real-world data analysis from the renewable energy industry corroborate theoretical conclusions, linking theory and practice.

3. Simulation study

This simulation study examines Wishart distribution eigenvalue distribution energy distance under many circumstances. Different degrees of freedom (ν) will be studied to understand how Wishart distribution and eigenvalue dispersion affect energy distance. We used the `dcor` Python package developed by [20] for precise and efficient energy distance calculations. An analysis will be conducted to assess the influence of various configurations of the theoretical covariance matrix

($\mathbf{W}_0$), including the identity matrix ($\mathbf{W}_0 = \mathbf{I}$) for uncorrelated features with identical eigenvalues, a diagonal matrix ($\mathbf{W}_0 = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ for distinct variances per feature, and structured matrices with specific correlation patterns. We will create $n = 300$ Wishart distributed matrices for each configuration and combination of parameters for statistical simulation.

3.1 *Exploration of scenarios impacting energy distance*

Here, we focus on the impact of degrees of freedom (v) and theoretical covariance matrix structures W_0 on energy distance.

3.1.1 *Degrees of freedom (ν)*

We investigate how seven degrees of freedom impact Wishart distribution and eigenvalue dispersion as follow:

$\nu = 3$: Less degree of freedom leads to a more skewed distribution with a higher eigenvalue dispersion, which means a less stable eigenvalue spectrum.

$\nu = 10$: At this moderate degree of freedom, a predicted equilibrium between the eigenvalue spread and distribution stability balances distribution features.

$\nu = 30$: Higher degree of freedom denotes a distribution that approximates normality with a smaller eigenvalue variance, indicating a more stable and less variable distribution.

3.1.2 *Theoretical covariance matrix (Σ_0)*

We investigate three covariance matrix configurations to determine their effect on energy distance in different data interrelationships:

Identity Matrix ($\Sigma_0 = I$): Symbolizes uncorrelated features with equal variance (diagonal elements equal to 1, and off-diagonal elements being 0), leading to identical eigenvalues due to absent feature correlations.

Diagonal Matrix with Varied Elements ($\Sigma_0 = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p)$): This configuration, varying diagonal elements (σ_i), introduces distinct variances per feature, from $\sigma_1 = 2$ down to $\sigma_p = 0.1$, aiming to scrutinize the effect of a gradient in variance on the eigenvalue distribution dynamics.

3.1.3 *Structured matrix with correlation patterns (W_0)*

Instills specific feature dependencies via a predefined structured covariance matrix, exemplified by:

$$\mathbf{W}_0 = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix},$$

introducing discernible positive correlations between select features, thereby influencing the resultant eigenvalue distributions compared to the initial scenarios.

Through these scenarios, we aim to unravel the nuanced responses of energy distance to alterations in the Wishart distributions' structure and parameterization.

Figure 1 shows the integrated visual analysis of the energy distance distributions of the eigenvalues in the Identity, Diagonal and Structured scenarios and the selected degrees of freedom (DoF: 5, 10, 20). While Figure 2 provides a detailed visualization of energy distance between eign values distributions for a broader range of degrees of freedom. The degree of freedom (DoF) has a substantial impact on the dispersion and central tendency of distribution since bigger DoFs lead to more reduced distributions. The structures of covariance matrices significantly impact the diversity of energy distance, as the Diagonal and Structured scenarios reveal larger distributions at lower DoFs. Visualizing Wishart-distributed matrices helps statistical modeling and analysis of high-dimensional data.

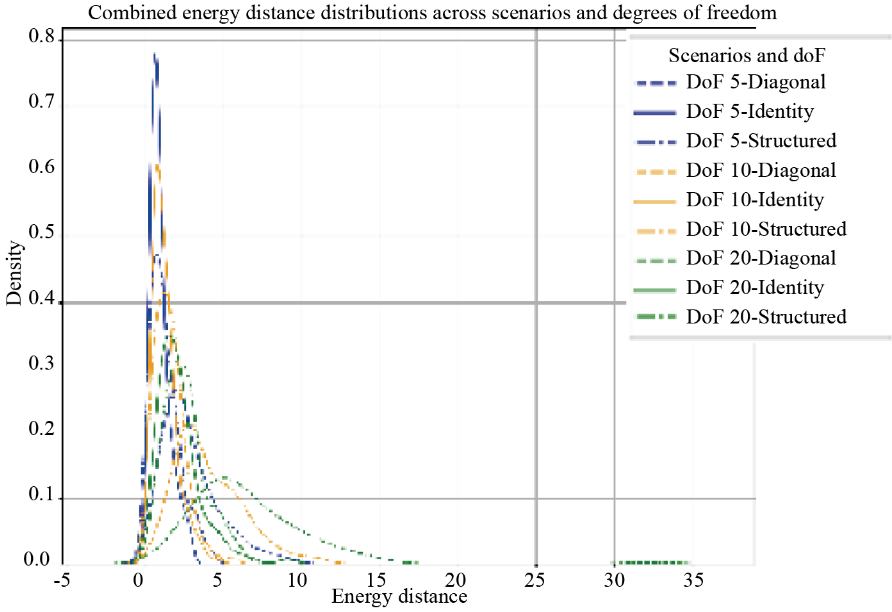

Figure 1. Energy distance distributions across scenarios and selected degrees of freedom

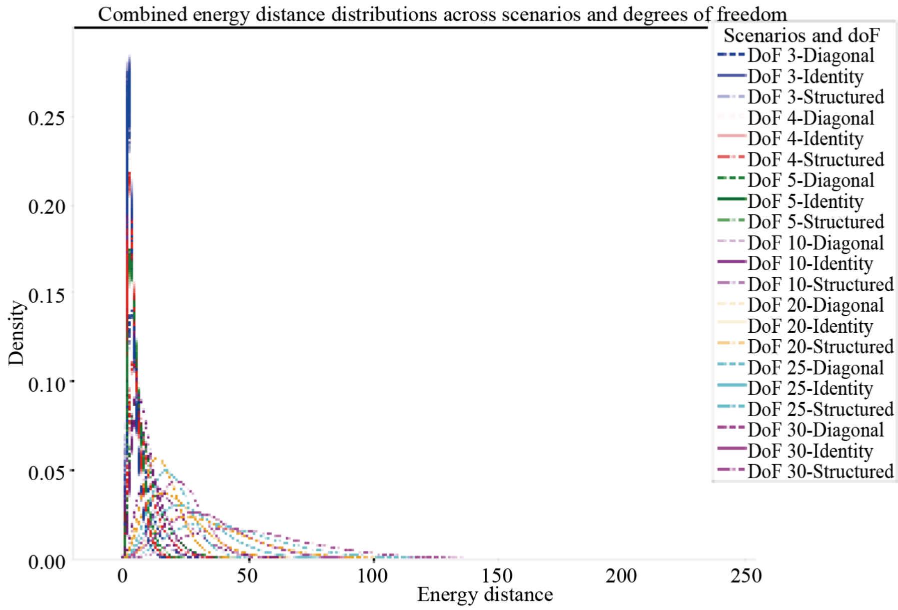

Figure 2. Energy distance distributions across scenarios and seven degrees of freedom

The covariance matrix structure and degrees of freedom in dealing with energy distances in Wishart-distributed matrices are shown to be strongly related. Specifically, matrices with fewer degrees of freedom have more energy distance variability, which predicts more eigenvalue distribution deviations. Energy distance distributions decrease with increasing degrees of freedom, thereby maintaining eigenvalue dispersion.In degrees of freedom, identity matrices vary little, whereas diagonal and structured matrices are more sensitive. This work shows a detailed relationship between matrix structure and statistical properties, shedding light on Wishart distributions in multivariate statistical research.

3.2 *Simulation methodology*

We build Wishart distributed matrices with 300 samples for each scenario and degree of freedom for (n = 100). These matrices' eigenvalues are calculated, and Székely's concept is used to calculate the energy distance between configurations. This adequate technique offers a realistic statistical simulation that captures diversity in eigenvalue distributions across settings as seen in Figure 3.

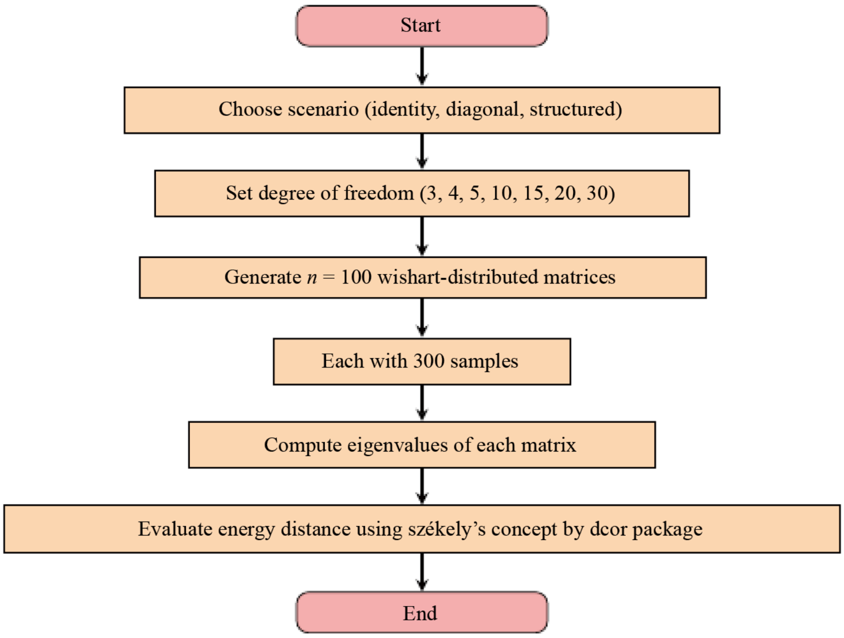

Figure 3. Flowchart of the simulation process

4. Real data analysis

4.1 *Data collection and preparation*

We examined the energy distance of renewable energy industry stock eigenvalue distributions in this research. The companies analyzed were NextEra Energy (NEE), Enphase Energy (ENPH), First Solar (FSLR), SunPower (SPWR), and Brookfield Renewable Partners (BRP). Yahoo Finance provided daily closing prices from January 1, 2018, until January 1, 2023. The percentage change in adjusted closing prices was used to generate daily returns for covariance matrix calculations.

4.2 *Methodology*

We used the energy distance metric to compare distributions. It compares eigenvalue distributions of covariance matrices formed from stock returns under different circumstances. Our analysis focused on DoF of 5, 10 and 20 and their effects on energy distance distributions.Considered scenarios:

Identity Matrix Shows uncorrelated characteristics with identical eigenvalues.

Diagonal Matrix shows features' unique variances.

Structured Matrix shows correlation patterns with non-diagonal members set to 0.5.

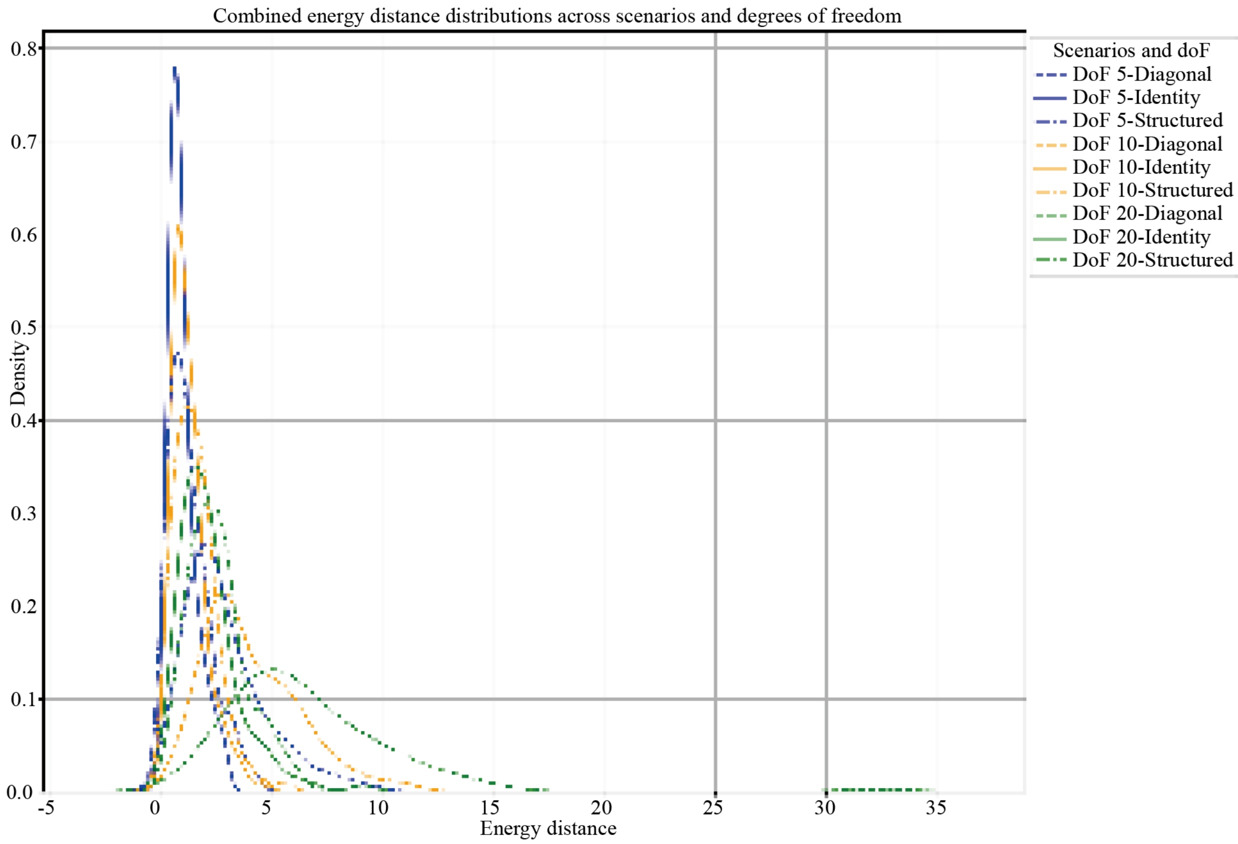

Figure 4. Kernel density calculates energy distance distributions for eigenvalue distributions in Identity, Diagonal, and Structured situations with 5, 10 and 20 degrees of freedom. The graph shows how degrees of freedom and covariance structures affect energy distance measurements stability and variability

In Figure 4, energy distance distributions are shown across several situations and degrees of freedom. Narrower distributions indicate more stable eigenvalue distributions with more degrees of freedom. This figure confirms the simulation findings regarding the influence of degrees of freedom on energy distance distributions. Higher degrees of freedom narrow and stabilize energy distance distributions.

5. Discussion

Energy distance evolution in Wishart-distributed matrices with variable degrees of freedom and covariance structures is naturally conveyed through simulations. Parametric enhancements transform energy distances, conveniently appreciated through spatial renditions. Energy distance distributions are larger for reduced degrees of freedom matrices, signifying corresponding larger eigenvalue dispersions. As degrees of freedom increase, this behavior becomes progressively ameliorated, leveling off for higher-degree matrices. The Structure of the matrix identity diagonal is disclosed as mediating these distributions. Higher energy distances for both diagonal- and structured matrices identity matrices implies data correlations and variances affect eigenvalue distributions. These phenomena deliver a comprehensive understanding of differential eigenvalue distribution spread and similarity given influential factors, germane to Wishart distributions and data covariance estimates. Cognizant patterns and narratives conform to theoretical presuppositions and portend distinct multivariate statistical methodologies and data models.

6. Conclusion

Using extensive simulations, we examined how degrees of freedom and covariance matrix architectures affect energy distances in Wishart-distributed matrices. Degrees of freedom and covariance matrix topologies greatly affect eigenvalue dispersion and energy distance distribution, according to our results. In particular, energy distance distributions decrease as degrees of freedom rise, demonstrating eigenvalue spectra stability. Higher parameter values seem to stabilize and consistent distributions, improving statistical model dependability.

In real-world data analysis, we examined the energy distance of the stock eigenvalue distributions of the renewable energy sector. NextEra Energy, Enphase Energy, First Solar, SunPower, and Brookfield Renewable Partners were examined. Yahoo Finance provided daily closing prices from January 1, 2018, until January 1, 2023. Daily covariance matrix returns were calculated using adjusted closing price percentage changes.

The energy distance distributions in Figure 4 show that when there are more degrees of freedom, the eigenvalue distributions get smaller and more stable. This supports the simulation results, showing that degrees of freedom affect energy distance distributions.

These findings affect several disciplines. Financial risk management and portfolio optimization need to understand covariance structure stability and variability. The findings may potentially improve statistical models in genetics, environmental science, and other fields where data covariance is important. This work expands knowledge of Wishart distributions and their applications by giving actual data to support theoretical expectations.

Despite its importance, this study has some drawbacks. Our simulations were limited to specified degrees of freedom and covariance matrix topologies. This research should be extended to higher dimensions and more distributional assumptions. Applying these insights to real-world datasets will also verify and improve this theoretical approach. Wishart distributions were our entire focus to analyze their features and applications. Future research could benefit from performing a comparison study between Wishart distributions with other distributions to further enhance the robustness of the findings.

To confirm the practical application of these results, future research should expand this study to higher-dimensional settings, investigate distributional assumptions, and apply these insights to real-world datasets.Future studies should further examine the complex relationship between the energy discrepancy and eigenvalue Mutual Information (MI) by [20] to comprehend these statistical indicators better that may improve the Wishart-distributed matrix statistical methods by studying these complicated interactions. Expanding the scope of this investigation could broaden multivariate statistical analysis, especially Wishart distribution knowledge. To conclude, this work improves Wishart distribution theory and strengthens simulation studies and current statistical research. We present a rigorous analytical approach to enhance data analysis and model fitting in numerous scientific domains by studying the complex dynamics of eigenvalue distributions under diverse circumstances.

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School of Mathematics, Statistics, and Mechanics, Beijing University of Technology, Beijing 100124, China.

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Confilict of interest

The authors declare no competing financial interest.

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