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Research Article

A Numerical Scheme of a Fractional Coupled System of Volterra Integro-Differential Equations with the Caputo Fabrizio Fractional Derivative

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Abstract: In this paper, we have developed a new computational scheme for solving coupled systems of fractional order Volterra-type integro-differential equation (FVIDE). We construct new operational matrices, which serve as building blocks for converting the FVIDE into a Sylvester-type algebraic structure that is more easily solvable. The fractional derivatives and their inverses (integrals) are considered in the Caputo-Fabrizio sense. This computational scheme is based on Legendre polynomials. The assessment of the proposed method's convergence involves utilizing appropriate error norms to measure the level of convergence. The algorithm is designed in such a way that it can be simulated using any computational package; we have used Matlab for simulating the proposed scheme. Graphical visualization and tabulation of data are performed to confirm convergence and to analyze errors.

Keywords: orthogonal polynomials, fractional calculus, numerical analysis, approximation

MSC: 34A08, 41A10

1. Introduction

Fractional calculus is the theory of non-integer order ordinary differentiation and integration. This field of mathematics finds its application in a lot of physical and natural processes. This field is observed to describe nonlocal dynamics in computer and electrical engineering [1, 2], economic growth model [3, 4], dynamics systems [5], bioengineering [6], image processing [7] and Chemistry [8]. It is also observed that these fractional order differential equations (FDEs) can be used as an alternative to nonlinear ordinary differential equations [9]. The interested readers may find some recent advances in this field in [10–16].

Recently researchers discovered that these derivatives provide more accurate results as compared to ordinary derivatives in a wide variety of physical problems. The only disadvantage is the high computational complexities in the definitions of fractional derivatives, which makes these tools difficult or sometime impossible to make a suitable and compatible understanding of the fractional order systems of FDE. These sources highlight the ongoing research in numerical methods for fractional differential equations, focusing on various techniques and applications in different fields [17, 18]. in This paper [17] introduces new fourth-degree hat functions (FDHFs) and studies their properties for solving fractional integro-differential equations. The method is tested using numerical examples and shows good results

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In the current field of fractional calculus, different author studied different aspects of coupled system of FDEs. Modeling and describing a given phenomena in terms of FDEs [19–23], study of existence and conformation of uniqueness of solution [24–28] and development of computational strategies to find approximation to a given FDEs [29, 30] are the most devoted aspects of this field.

The objective of this article is to create a computational scheme for solving a system of coupled fractional Volterra integro-differential equations with the given structure.

$$^{CF}D^{\sigma}X(x) = a(x)X(x) + b(x)Y(x) + \int_0^x K_1(x,\ r)X(r)dr + \int_0^x K_2(x,\ r)Y(r)dr + f(x)$$

$$^{CF}D^{\sigma}Y(x) = c(x)X(x) + d(x)Y(x) + \int_0^x K_3(x,\ r)X(r)dr + \int_0^x K_4(x,\ r)Y(r)dr + g(x)$$

The function $K_i \in C([0,\ 1]) \times C([0,\ 1])$ and $f(x)$ and $g(x)$ are predefined source terms while the functions $X(x)$ and $Y(x)$ are the required unknown solution. The coefficients $a,\ b,\ c$ and d are bounded, continuous, and well-defined functions on the domain $[0,\ 1]$. On the left hand side of the equation $^{CF}D^{\sigma}$ represent the famous Caputo Fabrizio fractional derivative.

In the recent years, various efforts have made to construct approximate solution to FVIDE. Among others some of the well known method which are established for solution of FVIDE is Homotopy perturbation is applied in [31], wavelet approaches [32], a semi analytic numeric technique [33], a Chebyshev pseudo-spectral method [34]. The existence and uniquness of these problems are investigated in [35–39].

The main theme objective of the proposed paper is to solve coupled system of FVIDE by converting it to algebraic system of equations. Our approach is based on orthogonal polynomials. We construct some new operational matrices for Caputo Fabrizio fractional derivative and integral. We also construct a new operational matrix which can be used to replace $\int_0^t K(x,\ r)X(r)dr$ to its equivalent matrix representation. Based on these matrices we developed a computational scheme to convert the above system to Sylvester type matrix equation. We use the Lypanovo exponent method for the solution of Sylvester type matrix equations and re-transform the solution to get the equivalent solution to the required coupled system of FVIDE.

The paper is organized as followed: In section 2, we presents some terminology and definitions from fractional calculus and approximation theory. In section 3, We develop some new operational matrices and related theorems. In section 4, the method is designed and discussed. In section 5 presents some test problems are presented and solved with the proposed method. In section 6, we presents results and discussion of the proposed method. The conclusion and future direction are presented in the last section.

2. Preliminaries

In this section, we present basic definitions and important results from fractional calculus, which are briefly used in this paper.

Definition 1 [40] Let $X \in H^1(0,\ b),\ 0 < b,\ 0 < \sigma < 1$, then the fractional Caputo Fabrizio fractional differential operator of order σ is defined as

$$^{\mathrm{CF}}D_x^{\sigma}X(x) = \frac{(2-\sigma)W(\sigma)}{2(1-\sigma)}\int_0^x \exp\left[-\frac{\sigma(x-\tau)}{1-\sigma}\right]X'(\tau)\ d\tau,\ 0 \le x,\ 0 < \sigma < 1.$$

with a normalization function $W(\sigma)$ which depends on σ and $W(0) = W(1) = 1$. As like the usual Caputo derivative, if X is a constant function then this new operator gives ${}^{\mathrm{CF}}D_x^{\sigma}X(x) = 0$.

Definition 2 [41] The Caputo Fabrizio fractional integral operator of order σ is defined as

$$
{}^{\mathrm{CF}}I_x^{\sigma}X(x) = \frac{2(1-\sigma)}{(2-\sigma)W(\sigma)}X(x) + \frac{2\sigma}{(2-\sigma)W(\sigma)}\int_0^x X(\tau)\ d\tau. \tag{1}
$$

The main advantage of the Caputo Fabrizio operator over the traditional operator is the absence of singularity kernel.

2.1 *Shifted legendre polynomials (SLPs)*

The recurrence relation for Legendre polynomials defined on the interval $[-1,\ 1]$, as stated in the reference [42], is:

$$
T_{j+1}(r) = \frac{2j+1}{j+1}rT_j(r) - \frac{j}{j+1}T_{j-1}(r),\ j = 1,\ 2,\ 3,\ \ldots \text{ where } T_0(r) = 0,\ T_1(r) = r.
$$

The transformation $x = \dfrac{(r+1)}{2}$ transforms the interval form $[-1,\ 1]$ to $[0,\ 1]$. The general exprression which defines the shifted Legendre polynomials is given as

$$
\daleth_j(x) = \textstyle\sum_{k=0}^{j}\zeta_{(j,\ k)}x^k \ \text{ where } \zeta_{(j,\ k)} = (-1)^{j+k}\dfrac{(j+k)!}{(j-k)!(k!)^2}. \tag{2}
$$

These polynomials enjoys the precise property of orthogonality. We can express the orthogonality property in the mathematical form as

$$
\int_0^1 \daleth_i(r)\daleth_j(r)ds = \begin{cases} \dfrac{1}{2i+1} & \text{if } i = j, \\ \\ 0 & \text{if } i \neq j. \end{cases}
$$

The orthogonality condition allows us to express any $f(x) \in C[0,\ 1]$ in a linear combination of $\daleth_r(x)$,

$$
f(x) = \textstyle\sum_{r=0}^{\infty}c_r\ \daleth_r(x),\ \text{where } c_r = (2r+1)\displaystyle\int_0^1 f(x)\daleth_r(x)dx. \tag{3}
$$

The truncated series of the function $f(x) \in C[0,\ 1]$ can easily be expressed in vector notation as

$$
f(x)\ \approx\ X_M\Psi_M(x), \tag{4}
$$

where

$$
\Psi_M(x) = [\ \daleth_0(x) \quad \daleth_1(x) \quad \ldots \quad \daleth_m(x)\ ]^T,
$$

and

$$X_M = [\ c_0 \quad c_1 \quad \ldots \quad c_m\ ].$$

In the case of two variable function that is, $k \in C([0,\ 1] \times [0,\ 1])$, we can write as

$$k(x,\ t) = \sum_{i=0}^{m}\sum_{j=0}^{m} c_{(i,\ j)}\ \beth_i(x)\ \beth_j(t),$$

where

$$c_{(i,\ j)} = (2i+1)(2j+1)\int_0^1\int_0^1 k(x,\ t)\beth_i(x)\beth_j(t)dxdt. \tag{5}$$

The orthogonality condition of $\beth_i(x)\beth_j(t)$ is given as

$$\int_0^1\int_0^1 \beth_i(x)\beth_j(t)\beth_a(x)\beth_b(t)dxdt = \begin{cases} \frac{1}{2i+1}\frac{1}{2j+1} & \text{if } a = i,\ b = j, \\ 0 & \text{Otherwise.} \end{cases}$$

In vector notation

$$k(x,\ t) \approx \Psi_M^T(x)C_{M\times M}\Psi_M(t). \tag{6}$$

$C_{(M\times M)}$ is the coefficient matrix whose entries are obtained from (5).

Lemma 1 Definite integral of any three shifted Legendre polynomials on $[0,\ 1]$ is defined by a constant value $£_{(p,\ q,\ r)}^{(a,\ b,\ c)}$ i.e.

$$\int_0^1 \beth_a(x)\beth_b(x)\beth_c(x)dx = £_{(p,\ q,\ r)}^{(a,\ b,\ c)}, \tag{7}$$

where

$$£_{(p,\ q,\ r)}^{(a,\ b,\ c)} = \sum_{p=0}^{a}\sum_{q=0}^{b}\sum_{r=0}^{c}\frac{\zeta_{(a,\ p)}\zeta_{(b,\ q)}\zeta_{(c,\ r)}}{p+q+r+1}$$

The proof of the lemma is given in [43].

3. Development of operational matrices

In this section we will derive important operational matrices. These matrices are building blocks of the proposed computational scheme.

Theorem 1 Consider $X(x)$ and $g(x)$ as arbitrary functions belonging to the space $C[0,\ 1]$. Then

$$g(x)X(x) = X_M G^{g(x)}_{(M\times M)} \Psi_M(x)$$

where X_M is the SLPs coefficients vector of $X(x)$

$$G^{g(x)}_{(M\times M)} = \begin{pmatrix} \sqcup_{(0,\ 0)} & \sqcup_{(0,\ 1)} & \cdots & \sqcup_{(0,\ s)} & \cdots & \sqcup_{(0,\ m)} \\ \sqcup_{(1,\ 0)} & \sqcup_{(1,\ 1)} & \cdots & \sqcup_{(1,\ s)} & \cdots & \sqcup_{(1,\ m)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \sqcup_{(r.0)} & \sqcup_{(r,\ 1)} & \cdots & \sqcup_{(r,\ s)} & \cdots & \sqcup_{(r,\ m)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqcup_{(m,\ 0)} & \sqcup_{(m,\ 1)} & \cdots & \sqcup_{(m,\ s)} & \cdots & \sqcup_{(m,\ m)} \end{pmatrix} \tag{8}$$

and

$$\bigsqcup_{(r,\ s)} = (2s+1)\sum_{i=0}^{m} c_i \pounds^{(i,\ r,\ s)}_{(p,\ q,\ r)}$$

Proof. Let consider the vector form of $Y(x) \simeq X_M \Psi^T_M(x)$, we can express the following:

$$g(x)Y(x) = X_M \overbrace{\Psi^T_M(x)}.$$

where

$$\overbrace{\Psi(x)} = [Z_0(x)Z_1(x) \quad \ldots \quad Z_r(x) \quad \ldots \quad Z_m(x)]^T$$

and

$$Z_r(x) = g(x)\beth_r(x)$$

By M-terms Legendre approximating a function $g(x)$, we get

$$g(x) = \sum_{i=0}^{m} c_i \beth_i(x).$$

By using these results, we can express the following

$$Z_r(x) = \sum_{i=0}^{m} c_i \beth_i(x) \beth_r(x)$$

We express the approximation of the quantity using SLPs (i.e., $Z_r(x)$) as follows:

$$Z_r(x) = \sum_{s=0}^{m} h_{(r,\ s)} \beth_s(x).$$

where

$$h_{(r,\ s)} = (2s+1) \int_0^1 Z_r(x) \beth_s(x) dt.$$

Which gives us

$$h_{(r,\ s)} = (2s+1) \sum_{i=0}^{m} c_i \int_0^1 \beth_i(x) \beth_r(x) \beth_s(x) dx.$$

Using the above results, we can write

$$h_{(r,\ s)} = (2s+1) \sum_{i=0}^{m} c_i £_{(p,\ q,\ r)}^{(i,\ r,\ s)}.$$

Using $h_{(r,\ s)} = \bigsqcup_{r,\ s}$ we get

$$\begin{pmatrix} Z_0(x) \\ Z_1(x) \\ \vdots \\ Z_r(x) \\ \vdots \\ Z_m(x) \end{pmatrix} = \begin{pmatrix} \bigsqcup_{(0,\ 0)} & \bigsqcup_{(0,\ 1)} & \cdots & \bigsqcup_{(0,\ s)} & \cdots & \bigsqcup_{(0,\ m)} \\ \bigsqcup_{(1,\ 0)} & \bigsqcup_{(1,\ 1)} & \cdots & \bigsqcup_{(1,\ s)} & \cdots & \bigsqcup_{(1,\ m)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \bigsqcup_{(r.0)} & \bigsqcup_{(r,\ 1)} & \cdots & \bigsqcup_{(r,\ s)} & \cdots & \bigsqcup_{(r,\ m)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bigsqcup_{(m,\ 0)} & \bigsqcup_{(m,\ 1)} & \cdots & \bigsqcup_{(m,\ s)} & \cdots & \bigsqcup_{(m,\ m)} \end{pmatrix} \begin{pmatrix} \beth_0(x) \\ \beth_1(x) \\ \vdots \\ \beth_r(x) \\ \vdots \\ \beth_m(x) \end{pmatrix} \tag{9}$$

So we can write the equation as

$$\overbrace{\Psi_M(x)} = G_{(M \times M)}^{g(x)} \Psi_M(x).$$

Which completes the proof. □

Theorem 2 Let $k(x,\ t)$ be a function of two variable, and a function $g(t) \in C[0,\ 1]$. Then

$$\int_0^x k(x,\ t)g(t)dt = X_M Q_{(M\times M)}^{g(t)} \Psi_M(x).$$

where X_M is the SLPs coefficients vector of $g(t)$

$$Q_{(M\times M)}^{g(t)} = \begin{pmatrix} \urcorner_{(0,\ 0)} & \urcorner_{(0,\ 1)} & \cdots & \urcorner_{(0,\ r)} & \cdots & \urcorner_{(0,\ m)} \\ \urcorner_{(1,\ 0)} & \urcorner_{(1,\ 1)} & \cdots & \urcorner_{(1,\ r)} & \cdots & \urcorner_{(1,\ m)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \urcorner_{(s.0)} & \urcorner_{(s,\ 1)} & \cdots & \urcorner_{(s,\ r)} & \cdots & \urcorner_{(s,\ m)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \urcorner_{(m,\ 0)} & \urcorner_{(m,\ 1)} & \cdots & \urcorner_{(m,\ r)} & \cdots & \urcorner_{(m,\ m)} \end{pmatrix} \tag{10}$$

and

$$\urcorner_{(s,\ r)} = (2r+1)\sum_{i=0}^m \sum_{j=0}^m \sum_{k=0}^j \sum_{l=0}^s \sum_{h=0}^i \sum_{n=0}^r \frac{c_{(i,\ j)}\zeta_{(j,\ k)}\zeta_{(s,\ l)}\zeta_{(i,\ h)}\zeta_{(r,\ n)}}{(k+l+h+r+2)(k+l+1)}$$

Proof. Consider the expansions of k and g as

$$k(x,\ t) = \sum_{i=0}^m \sum_{j=0}^m c_{(i,\ j)}\ \beth_i(x)\ \beth_j(t)$$

$$g(t) = \sum_{s=0}^m g_s\ \beth_s(t)$$

$$\int_0^x k(x,\ t)g(t)dt = \int_0^x \sum_{i=0}^m \sum_{j=0}^m c_{(i,\ j)} \beth_i(x) \beth_j(t)\ \sum_{s=0}^m g_s\ \beth_s(t)dt$$

$$\int_0^x k(x,\ t)g(t)dt = \sum_{s=0}^m g_s \sum_{i=0}^m \sum_{j=0}^m c_{(i,\ j)} \beth_i(x) \int_0^x \beth_j(t)\ \beth_s(t)dt$$

we know from equation (2),

$$\beth_j(t) = \sum_{k=0}^j \zeta_{(j,\ k)} t^k \ \text{ and } \beth_s(t) = \sum_{l=0}^s \zeta_{(s,\ l)} t^l$$

$$\int_0^x \beth_j(t)\beth_s(t)dt = \sum_{k=0}^j \sum_{l=0}^s \zeta_{(j,\ k)}\zeta_{(s,\ l)} \int_0^x t^{k+l}$$

$$\int_0^x \beth_j(t)\beth_s(t)dt = \sum_{k=0}^j \sum_{l=0}^s \zeta_{(j,\ k)}\zeta_{(s,\ l)} \frac{x^{k+l+1}}{k+l+1}$$

$$\int_0^x k(x,\ t)g(t)dt = \sum_{s=0}^{m} g_s \sum_{i=0}^{m} \sum_{j=0}^{m} c_{(i,\ j)} \daleth_i(x) \sum_{k=0}^{j} \sum_{l=0}^{s} \zeta_{(j,\ k)} \zeta_{(s,\ l)} \frac{x^{k+l+1}}{k+l+1},$$

Let assume

$$G_{(i,\ j,\ s)}(x) = \sum_{i=0}^{m} \sum_{k=0}^{j} \sum_{j=0}^{m} \sum_{l=0}^{s} \sum_{h=0}^{i} c_{(i,\ j)} \zeta_{(j,\ k)} \zeta_{(s,\ l)} \zeta_{(i,\ h)} \frac{x^{k+l+h+1}}{k+l+1}$$

The function $G_{(i,\ j,\ s)}(x)$ can be approximated with M terms of SLPs as

$$G_{(i,\ j,\ s)}(x) = \sum_{r=0}^{m} h_r^{(i,\ j,\ s)} \daleth_r(x).$$

where

$$h_r^{(i,\ j,\ s)} = (2r+1) \int_0^1 G_{(i,\ j,\ s)}(x) \daleth_r(x) dx.$$

Which gives us

$$h_r^{(i,\ j,\ s)} = (2r+1) \sum_{i=0}^{m} \sum_{j=0}^{m} \sum_{k=0}^{j} \sum_{l=0}^{s} \sum_{h=0}^{i} c_{(i,\ j)} \zeta_{(j,\ k)} \zeta_{(s,\ l)} \zeta_{(i,\ h)} \int_0^1 \frac{x^{k+l+h+1}}{k+l+1} \daleth_r(x) dx.$$

Using $h_{(r)}^{(i,\ j,\ s)} = \urcorner_{(s,\ r)}$

$$\urcorner_{(s,\ r)} = (2r+1) \sum_{i=0}^{m} \sum_{j=0}^{m} \sum_{k=0}^{j} \sum_{l=0}^{s} \sum_{h=0}^{i} \sum_{n=0}^{r} \frac{c_{(i,\ j)} \zeta_{(j,\ k)} \zeta_{(s,\ l)} \zeta_{(i,\ h)} \zeta_{(r,\ n)}}{(k+l+h+r+2)(k+l+1)}.$$

simulating the above results for all value of r and s. □

Theorem 3 Let $\Psi_M(x)$ be a vector function then the generalization of Caputo Fabrizio fractional integration of order α is defined as,

$$^{CF}I^{\alpha}(\Psi_M(x)) \simeq \mathbb{P}^{\alpha}(\Psi_M(x)) \tag{11}$$

The operational matrix of integration of order α, is denoted by $\mathbb{P}^{\alpha}$, and its order is $(M \times M)$ which is defined as follows:

$$\mathbb{P}^{\alpha} = \begin{pmatrix} \triangle_{(0,\ 0)} & \triangle_{(0,\ 1)} & \cdots & \triangle_{(0,\ j)} & \cdots & \triangle_{(0,\ m)} \\ \triangle_{(1,\ 0)} & \triangle_{(1,\ 1)} & \cdots & \triangle_{(1,\ j)} & \cdots & \triangle_{(1,\ m)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \triangle_{(i,\ 0)} & \triangle_{(i,\ 1)} & \cdots & \triangle_{(i,\ j)} & \cdots & \triangle_{(i,\ m)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \triangle_{(m,\ 0)} & \triangle_{(m,\ 1)} & \cdots & \triangle_{(m,\ j)} & \cdots & \triangle_{(m,\ m)} \end{pmatrix} \tag{12}$$

Proof. By applying the Caputo Fabrizio fractional integration on Legendre Polynomials, we get

$$^{CF}I^{\alpha}\daleth_i(x) = a_0\,\daleth_i(x) + b_0\int_0^x \daleth_i(t)dt$$

where

$$a_0 = \frac{1-\alpha}{B(\alpha)} \quad \text{and} \quad b_0 = \frac{\alpha}{B(\alpha)}$$

$$^{CF}I^{\alpha}\daleth_i(x) = a_0\sum_{k=0}^{i}\frac{(i+k)!(-1)^{i+k}}{(k!)^2(i-k)!}x^k + b_0\sum_{k=0}^{i}\frac{(i+k)!(-1)^{i+k}}{(k!)^2(i-k)!}\frac{x^{k+1}}{k+1}$$

$$^{CF}I^{\alpha}\daleth_i(x) = \sum_{k=0}^{i}\frac{(i+k)!(-1)^{i+k}}{(k!)^2(i-k)!}[a_0x^k + b_0\frac{x^{k+1}}{k+1}]$$

Now the function $^{CF}I^{\alpha}\daleth_i(x)$ we can be approximated in SLPs as,

$$^{CF}I^{\alpha}\daleth_i(x) = \sum_{j=0}^{m}\triangle_{(i,\ j)}\daleth_j(x)$$

we can find the values of $\triangle_{(i,\ j)}$ from orthogonality condition as,

$$\triangle_{(i,\ j)} = (2j+1)\int_0^1 {}^{CF}I^{\alpha}\daleth_i(x)\daleth_j(x)dx$$

$$\triangle_{(i,\ j)} = (2j+1)\int_0^1 \sum_{k=0}^{i}\frac{(i+k)!(-1)^{i+k}}{(k!)^2(i-k)!}[a_0x^k + b_0\frac{x^{k+1}}{k+1}]\sum_{l=0}^{j}\frac{(j+l)!(-1)^{j+l}}{(l!)^2(j-l)!}t^l dx$$

further simplification

$$\triangle_{(i,\ j)} = (2j+1)\sum_{k=0}^{i}\sum_{l=0}^{j}(-1)^{i+k+j+l}\frac{(i+k)!(j+l)!}{(i-k)!(k!)^2(j-l)!(l!)^2}[a_0\frac{1}{k+l+1} + b_0\frac{1}{k+l+2}]$$

which implies that,

$$^{CF}I^{\alpha}\daleth_i(x) = \triangle_{(i,\ 0)}\daleth_0(x) + \triangle_{(i,\ 1)}\daleth_1(x) + \triangle_{(i,\ 2)}\daleth_2(x) + \triangle_{(i,\ 3)}\daleth_3(x) + \ldots + \triangle_{(i,\ m)}\daleth_m(x)$$

Combining all the equation for $i = 0 : m$ completes the proof. □

4. Method of solution

The operational matrices play important role in the design of the computational schemes. We will now begin by considering the following system.

$$\begin{aligned} ^{CF}D^{\sigma}X(x) &= a(x)X(x) + b(x)Y(x) + \int_0^x \kappa_1(x,\ r)X(r)dr + \int_0^x \kappa_2(x,\ r)Y(r)dr + f(x), \\ ^{CF}D^{\sigma}Y(x) &= c(x)X(x) + d(x)Y(x) + \int_0^x \kappa_3(x,\ r)X(r)dr + \int_0^x \kappa_4(x,\ r)Y(r)dr + g(x). \end{aligned} \tag{13}$$

Writing the system in matrix form we get,

$$\begin{bmatrix} ^{CF}D^{\sigma}X(x) \\ ^{CF}D^{\sigma}Y(x) \end{bmatrix} = \begin{bmatrix} a(x)X(x) \\ d(x)Y(x) \end{bmatrix} + \begin{bmatrix} b(x)Y(x) \\ c(x)X(x) \end{bmatrix} + \begin{bmatrix} \int_0^t \kappa_1(t,\ s)X(s)ds \\ \int_0^t \kappa_4(t,\ s)Y(s)ds \end{bmatrix} + \begin{bmatrix} \int_0^t \kappa_2(t,\ s)Y(s)ds \\ \int_0^t \kappa_3(t,\ s)X(s)ds \end{bmatrix} + \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}.$$

Let us assume that

$$^{CF}D^{\sigma}X(x) = X_M\Psi_M(x), \quad ^{CF}D^{\sigma}Y(x) = Y_M\Psi_M(x).$$

On integration of order σ and using the operational matrices we get

$$X(x) - U_0 = X_M\,\mathbb{P}^{\alpha}\Psi_M(x), \qquad Y(x) - V_0 = Y_M\mathbb{P}^{\alpha}\Psi_M(x).$$

Let

$$U_0 = F_1\Psi_M(x), \qquad V_0 = F_2\Psi_M(x).$$

$$f(x) = F_3\Psi_M(x), \qquad g(x) = F_4\Psi_M(x).$$

we get

$$X(x) = X_M \mathbb{P}^{\alpha} \Psi_M(x) + F_1 \Psi_M(x), \qquad Y(x) = Y_M \mathbb{P}^{\alpha} \Psi_M(x) + F_2 \Psi_M(x).$$

$$X(x) = Ł_1 \Psi_M(x), \qquad Y(x) = Ł_2 \Psi_M(x).$$

and

$$a(x)X(x) = Ł_1\, G_{M\times M}^{a(x)}\, \Psi_M(x), \qquad b(x)Y(x) = Ł_2\, G_{M\times M}^{b(x)}\, \Psi_M(x).$$

$$c(x)X(x) = Ł_1\, G_{M\times M}^{c(x)}\, \Psi_M(x), \qquad d(x)Y(x) = Ł_2\, G_{M\times M}^{d(x)}\, \Psi_M(x).$$

by using the using the above results

$$\int_0^t \kappa_1(t,\ s)X(s)ds = Ł_1 Q_{M\times M}^{\kappa_1,\ X(s)} \Psi_M(x), \quad \int_0^t \kappa_2(t,\ s)Y(s)ds = Ł_2 Q_{M\times M}^{\kappa_2,\ Y(s)} \Psi_M(x).$$

$$\int_0^t \kappa_3(t,\ s)X(s)ds = Ł_1 Q_{M\times M}^{\kappa_3,\ X(s)} \Psi_M(x), \quad \int_0^t \kappa_4(t,\ s)Y(s)ds = Ł_2 Q_{M\times M}^{\kappa_4,\ Y(s)} \Psi_M(x).$$

$$\begin{bmatrix} X_M\, \Psi_M(x) \\ Y_M\, \Psi_M(x) \end{bmatrix} = \begin{bmatrix} Ł_1\, G_{M\times M}^{a(x)}\, \Psi_M(x) \\ Ł_2\, G_{M\times M}^{d(x)}\, \Psi_M(x) \end{bmatrix} + \begin{bmatrix} Ł_2\, G_{M\times M}^{b(x)}\, \Psi_M(x) \\ Ł_1\, G_{M\times M}^{c(x)}\, \Psi_M(x) \end{bmatrix} +$$

$$\begin{bmatrix} Ł_1 Q_{M\times M}^{\kappa_1,\ X(s)} \Psi_M(x) \\ Ł_2 Q_{M\times M}^{\kappa_4,\ G(s)} \Psi_M(x) \end{bmatrix} + \begin{bmatrix} Ł_2 Q_{M\times M}^{\kappa_3,\ X(s)} \Psi_M(x) \\ Ł_1 Q_{M\times M}^{\kappa_2,\ Y(s)} \Psi_M(x) \end{bmatrix} + \begin{bmatrix} F_3 \Psi_M(x) \\ F_4 \Psi_M(x) \end{bmatrix}.$$

by taking $\Psi_M(x)$ as common from both sides and putting the values of $Ł_1$ and $Ł_2$, we get

$$\begin{bmatrix} X_M \\ Y_M \end{bmatrix} = \begin{bmatrix} (X_M\, \mathbb{P}^{\alpha} + F_1)\, G_{M\times M}^{a(x)} \\ (Y_M\, \mathbb{P}^{\alpha} + F_2)\, G_{M\times M}^{d(x)} \end{bmatrix} + \begin{bmatrix} (Y_M\, \mathbb{P}^{\alpha} + F_2)\, G_{M\times M}^{b(x)} \\ (X_M\, \mathbb{P}_{(M\times M)} + F_1)\, G_{M\times M}^{c(x)} \end{bmatrix} +$$

$$\begin{bmatrix} (X_M\, \mathbb{P}^{\alpha} + F_1) Q_{M\times M}^{\kappa_1,\ X(s)} \\ (Y_M\, \mathbb{P}^{\alpha} + F_2) Q_{M\times M}^{\kappa_4,\ Y(s)} \end{bmatrix} + \begin{bmatrix} (Y_M\, \mathbb{P}^{\alpha} + F_2) Q_{M\times M}^{\kappa_3,\ Y(s)} \\ (X_M\, \mathbb{P}^{\alpha} + F_1) Q_{M\times M}^{\kappa_2,\ X(s)} \end{bmatrix} + \begin{bmatrix} F_3 \\ F_4 \end{bmatrix}.$$

$$\begin{bmatrix} X_M \\ Y_M \end{bmatrix} = \begin{bmatrix} X_M\, \mathbb{P}^{\alpha}\, G_{M\times M}^{a(x)} + F_1 G_{M\times M}^{a(x)} \\ Y_M\, \mathbb{P}^{\alpha}\, G_{M\times M}^{d(x)} + F_2 G_{M\times M}^{d(x)} \end{bmatrix} + \begin{bmatrix} Y_M\, \mathbb{P}^{\alpha}\, G_{M\times M}^{b(x)} + F_2 G_{M\times M}^{b(x)} \\ X_M\, \mathbb{P}^{\alpha}\, G_{M\times M}^{c(x)} + F_1 G_{M\times M}^{c(x)} \end{bmatrix} +$$

$$\begin{bmatrix} X_M\, \mathbb{P}^{\alpha}\, Q_{M\times M}^{\kappa_1,\ X(s)} + F_1 Q_{M\times M}^{\kappa_1,\ X(s)} \\ Y_M\, \mathbb{P}^{\alpha}\, Q_{M\times M}^{\kappa_4,\ Y(s)} + F_2 Q_{M\times M}^{\kappa_4,\ Y(s)} \end{bmatrix} + \begin{bmatrix} Y_M\, \mathbb{P}^{\alpha}\, Q_{M\times M}^{\kappa_2,\ Y(s)} + F_2 Q_{M\times M}^{\kappa_2,\ Y(s)} \\ X_M\, \mathbb{P}^{\alpha}\, Q_{M\times M}^{\kappa_3,\ X(s)} + F_1 Q_{M\times M}^{\kappa_3,\ X(s)} \end{bmatrix} + \begin{bmatrix} F_3 \\ F_4 \end{bmatrix}.$$

now by taking transpose and arranging the terms we get,

$$\begin{bmatrix} X_M & Y_M \end{bmatrix} = \begin{bmatrix} X_M & Y_M \end{bmatrix} \begin{bmatrix} \mathbb{P}^\alpha G_{M\times M}^{a(x)} & O_{M\times M} \\ O_{M\times M} & \mathbb{P}^\alpha G_{M\times M}^{d(x)} \end{bmatrix} + \begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} G_{M\times M}^{a(x)} & O_{M\times M} \\ O_{M\times M} & G_{M\times M}^{d(x)} \end{bmatrix}$$

$$+ \begin{bmatrix} X_M & Y_M \end{bmatrix} \begin{bmatrix} O_{M\times M} & \mathbb{P}^\alpha G_{M\times M}^{c_1(x)} \\ \mathbb{P}^\alpha G_{M\times M}^{b_1(x)} & O_{M\times M} \end{bmatrix} + \begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} O_{M\times M} & G_{M\times M}^{c(x)} \\ G_{M\times M}^{b_1(x)} & O_{M\times M} \end{bmatrix}$$

$$+ \begin{bmatrix} X_M & Y_M \end{bmatrix} \begin{bmatrix} \mathbb{P}^\alpha Q_{M\times M}^{\kappa_1,\, X(s)} & O_{M\times M} \\ O_{M\times M} & \mathbb{P}^\alpha Q_{M\times M}^{\kappa_4,\, Y(s)} \end{bmatrix} + \begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} Q_{M\times M}^{\kappa_1,\, X(s)} & O_{M\times M} \\ O_{M\times M} & Q_{M\times M}^{\kappa_4,\, Y(s)} \end{bmatrix}$$

$$+ \begin{bmatrix} X_M & Y_M \end{bmatrix} \begin{bmatrix} O_{M\times M} & \mathbb{P}^\alpha Q_{M\times M}^{\kappa_3,\, X(s)} \\ \mathbb{P}^\alpha Q_{M\times M}^{\kappa_2,\, Y(s)} & O_{M\times M} \end{bmatrix} + \begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} O_{M\times M} & Q_{M\times M}^{3,\, X(s)} \\ Q_{M\times M}^{2,\, Y(s)} & O_{M\times M} \end{bmatrix}$$

$$+ \begin{bmatrix} F_3 & F_4 \end{bmatrix}.$$

Which can be further simplifies as,

$$\begin{bmatrix} X_M & Y_M \end{bmatrix} = \begin{bmatrix} X_M & Y_M \end{bmatrix} \begin{bmatrix} \mathbb{P}^\alpha G_{M\times M}^{a(x)} + \mathbb{P}^\alpha_{(M\times M)} Q_{M\times M}^{\kappa_1,\, X(s)} & \mathbb{P}^\alpha G_{M\times M}^{c(x)} + \mathbb{P}^\alpha Q_{M\times M}^{\kappa_3,\, X(s)} \\ \mathbb{P}^\alpha_{(M\times M)} G_{M\times M}^{b(x)} + \mathbb{P}^\alpha Q_{M\times M}^{\kappa_2,\, Y(s)} & \mathbb{P}^\alpha G_{M\times M}^{d(x)} + \mathbb{P}^\alpha Q_{M\times M}^{\kappa_4,\, Y(s} \end{bmatrix} +$$

$$\begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} G_{M\times M}^{a(x)} + Q_{M\times M}^{\kappa_1,\, X(s)} & G_{M\times M}^{c(x)} + Q_{M\times M}^{\kappa_3,\, X(s)} \\ G_{M\times M}^{b(x)} + Q_{M\times M}^{\kappa_2,\, Y(s)} & G_{M\times M}^{d(x)} + Q_{M\times M}^{\kappa_4,\, Y(s)} \end{bmatrix} + \begin{bmatrix} F_3 & F_4 \end{bmatrix}. \tag{14}$$

5. Test problems

To assess the applicability and convergence of the proposed scheme, we utilize the following test problems.
Test Problem 1 Considered the Coupled system of FVIDE

$$^{CF}D^\sigma U(x) = \int_0^x (t-x)U(t)dt + \int_0^x (2t-x)V(t)dt + x^2U(x) + x^3V(x) + f(x),$$

$$^{CF}D^\sigma U(x) = \int_0^x txU(t)dt + \int_0^x (2t-3x)V(t)dt + (x+2)U(x) + x^8V(x) + g(x).$$

subject to

$$U(0) = -\frac{21}{20} \quad \text{and} \quad V(0) = -\frac{8}{5}.$$

Where the source term is selected, and

$$U(x) = 10e^x(x - \frac{5}{10})(x - \frac{7}{10})(x - \frac{3}{10}),$$

$$V(x) = 10e^x(x - \frac{4}{10})(x - \frac{8}{10})(x - \frac{5}{10}).$$

Test Problem 2 Considered the Coupled system of FVIDE

$$^{CF}D^{\sigma}U(x) = \int_0^x (\sin(t-x)x^3)\,U(t)dt + \int_0^1 (t^2 - \cos(x))V(t)dt$$

$$+ (\cos(x) + \sin(x))U(x) + (\sin(x) - x^3)V(x) + f(x),$$

$$^{CF}D^{\sigma}V(x) = \int_0^x (\cos(t+x) + \sin(t-x))\,U(t)dt + \int_0^x (\cos(x)\sin(t+x)V(t)dt$$

$$+ (\cos(x)\sin(x))U(x) + (\sin(x)(x+2)^4)V(x) + g(x).$$

With the initial condition

$$U(0) = 6 \text{ and } V(0) = 5.$$

Where the source term is selected and

$$U(x) = 70x^5 - 140x^4 + 90x^3 - 20x^2 + x + 6,$$

$$V(x) = 252x^6 - 630x^5 + 560x^4 - 210x^3 + 30x^2 - x + 5.$$

Test Problem 3 Considered the Coupled system of FVIDE

$$^{CF}D^{\sigma}U(x) = \int_0^x (x+t)\sin(t-x)U(t)dt + \int_0^x (e^{(x+t)} + (x-t))V(t)dt$$

$$+ (e^x - x^4)U(x) + \cos(x)V(x) + f(x),$$

$$^{CF}D^{\sigma}V(x) = \int_0^x (e^{(t-x)} + \sin(x))U(t)dt + \int_0^x (e^{(x+t)} + \cos(x))V(t)dt$$

$$+ x^2\sin(x)U(x) + (\sin(x) + (x+1)^2)V(x) + g(x).$$

Note that

$$U(0) = 4 \text{ and } V(0) = \frac{27}{200}.$$

Where the source term is selected, and

$$U(x) = (x - \frac{8}{10})e^x sin(x)(x - \frac{5}{10}) + 4,$$

$$V(x) = e^x(10x - 3)(x - 0.5)(x - 0.9)(x - 0.1).$$

6. Results and discussion

We employ the proposed method to solve the test examples. We approximate the solution for different scale level, ranging from 3 to 6. As the scale level increases, we observe that the approximate solution becomes increasingly accurate. Figure 1 illustrates the comparison between the exact solutions $U(x)$ and $V(x)$ and the approximate solution obtained at different scales. The red dots represent the exact solution of the problem. In this illustration, we fix the derivative order at 0.5. It is evident that the approximate solution closely aligns with the exact solution, Similarly, we fix the derivative order at 0.8 and compute the approximation error at $M = 8,\ 10,\ 12,\ 14$ using the following relationship.

$$u_m^e = |U(x) - U_M(x)|. \tag{15}$$

The error of approximation for Test problem 1 is presented in Figure 2. We can easily see that at $M = 14$ the error of approximation is less than 10^-14. In order to conform the convergence of solution for all value of σ, we use the following error norm

$$||U_e||_1 = \int_0^1 |U(x) - U_M(x)|dt. \tag{16}$$

We computed the $||U_e||_1$ and $||V_e||_1$ norms for a range of values of σ (0.1 to 0.9) and M (2 to 8). The resulting errors are presented in Figure 3. We observed that the error converges to zero for almost all values of σ. The figure uses a reverse logarithmic scale to enhance visibility. To assess the accuracy of our method, we conducted a numerical comparison between the approximate and exact solutions for Test Problem 1. The results are displayed in Table 1 and Table 2, demonstrating that the approximate solutions agree with the exact solutions up to 15 decimal places.For Test Problem 2, the comparison between the exact and approximate solutions is shown in Figure 4, while the error at different scaling levels is illustrated in Figure 7. Similarly, Figure 5 and Figure 8 display the error of approximation for Test Problem 2 and Test Problem 3, respectively. We performed a parametric analysis of the $||U_e||_1$ and $||V_e||_1$ norms for Test Problem 2 and Test Problem 3, presented in Figure 6 and Figure 10, respectively. Our analysis indicates that the method exhibits convergence for nearly all choices of the parameter σ.Furthermore, we compared the exact and approximate solutions for Test Problem 2 in Table 3 and Table 4, while the corresponding comparisons for Test Problem 3 are shown in Table 5 and Table 6.

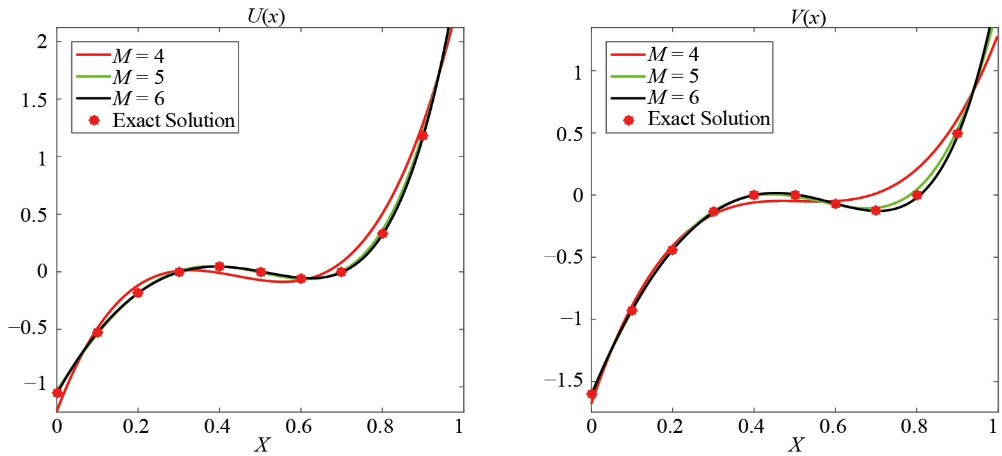

Figure 1. Exact and approximate solutions comparison for Test Problem 1

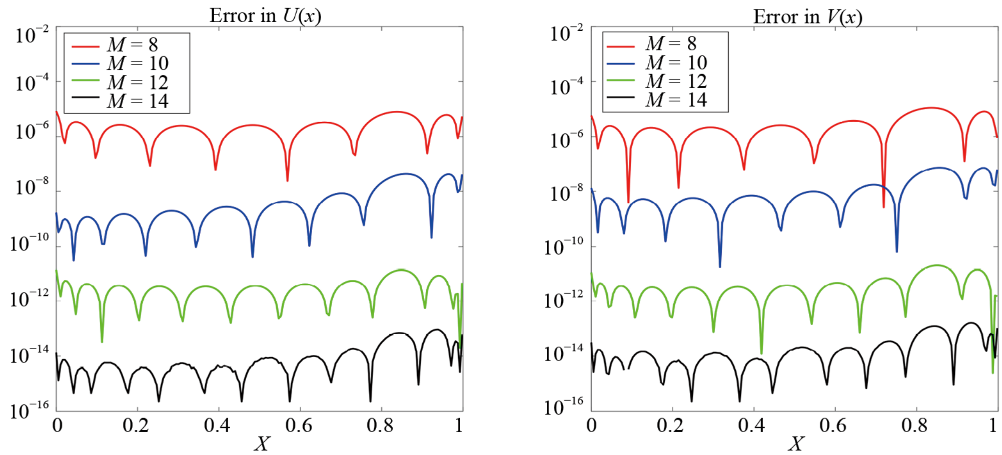

Figure 2. Absolute error at different scale level (M = 8: 1: 11) for Test Problem 1

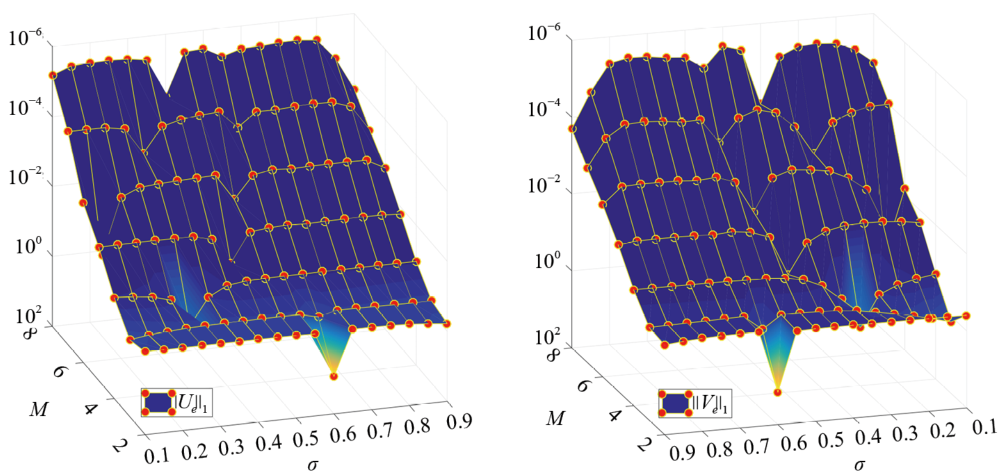

Figure 3. Norm of error at different Values of σ and M for test problem 2

Table 1. Exact and approximate solution comparison for Test Problem 1

$t\backslash M$	Exact Solution $X(x)$	$M = 7$	$M = 9$	$M = 11$
0.0	-1.050000000000000e + 00	-1.049817176485376e + 00	-1.049999703053302e + 00	-1.049999999360436e + 00
0.1	-5.304820406763109e-01	-5.305246639817243e-01	-5.304819723185904e-01	-5.304820404890429e-01
0.2	-1.832104137240255e-01	-1.831485667428360e-01	-1.832104490133078e-01	-1.832104138812085e-01
0.3	0	-8.578017710317281e-06	-3.481857210865717e-08	1.713653827379562e-10
0.4	4.475474092923811e-02	4.469452744556440e-02	4.475482520485269e-02	4.475474079315606e-02
0.5	0	1.112066286295606e-05	-4.855022581315649e-08	1.517925070148650e-11
0.6	-5.466356401171527e-02	-5.458987746413100e-02	-5.466361141640007e-02	-5.466356386451286e-02
0.7	0	-1.841516728523174e-05	1.268422496540141e-07	-2.878492064265952e-10
0.8	3.338311392738702e-01	3.336744480274148e-01	3.338309228132566e-01	3.338311397647572e-01
0.9	1.180609493355336e + 00	1.180569475998093e + 00	1.180609281255596e + 00	1.180609493608147e + 00
1.0	2.854195919881998e + 00	2.854187367584006e + 00	2.854195384035017e + 00	2.854195920192352e + 00

Table 2. Exact and approximate solution comparison for Test Problem 1

$t\backslash M$	Exact Solution $Y(x)$	$M = 7$	$M = 9$	$M = 11$
0.0	-1.600000000000000e + 00	-1.599836638960835e + 00	-1.599999865915273e + 00	-1.599999999313162e + 00
0.1	-9.283435711835441e-01	-9.283780374502194e-01	-9.283435161793993e-01	-9.283435709792716e-01
0.2	-4.397049929376612e-01	-4.396455168396983e-01	-4.397050470895551e-01	-4.397049931128872e-01
0.3	-1.349858807576003e-01	-1.350022768667661e-01	-1.349858636196690e-01	-1.349858805573919e-01
0.4	0	-6.107839157226036e-05	5.989360771858777e-08	-1.675338491998126e-10
0.5	0	2.147347679021570e-05	-9.506702737131825e-08	2.253956025569065e-11
0.6	-7.288475201562035e-02	-7.280060786157006e-02	-7.288474183192982e-02	-7.288475182442865e-02
0.7	-1.208251624482286e-01	-1.208594818547644e-01	-1.208250036882733e-01	-1.208251628473461e-01
0.8	0	-2.215262721118198e-04	-4.038117291591408e-07	7.473764054985464e-10
0.9	4.919206222313900e-01	4.918133632073060e-01	4.919203381577246e-01	4.919206227980424e-01
1.0	1.630969097075428e + 00	1.631161219290524e + 00	1.630968549459984e + 00	1.630969098173236e + 00

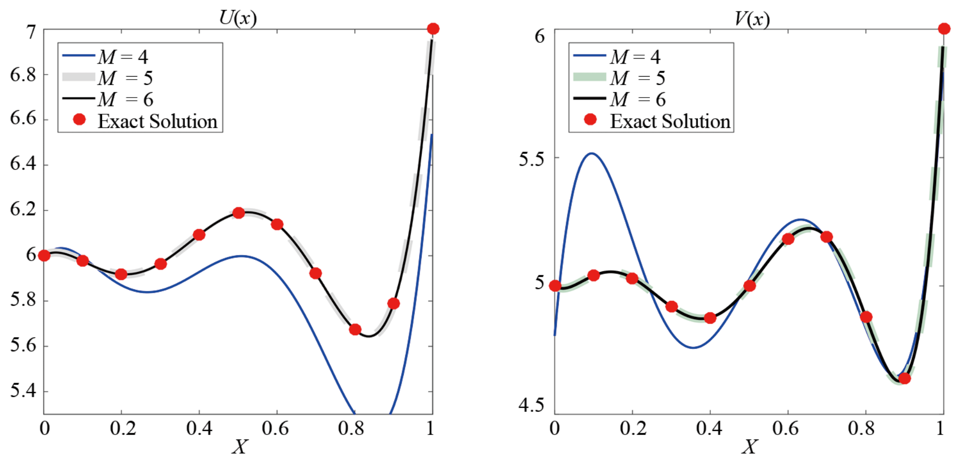

Figure 4. Exact and approximate solution comparison for Test Problem 2

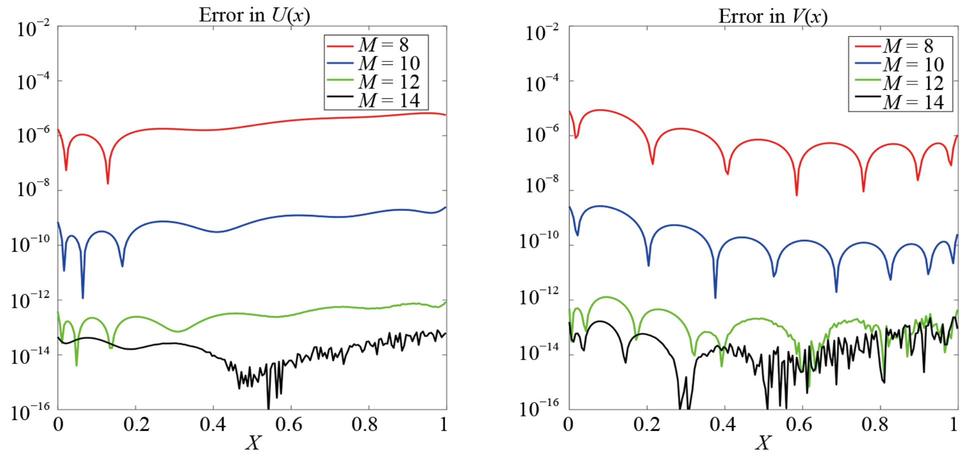

Figure 5. Absolute error at different scale level (M = 8: 2: 14) for Test Problem 2

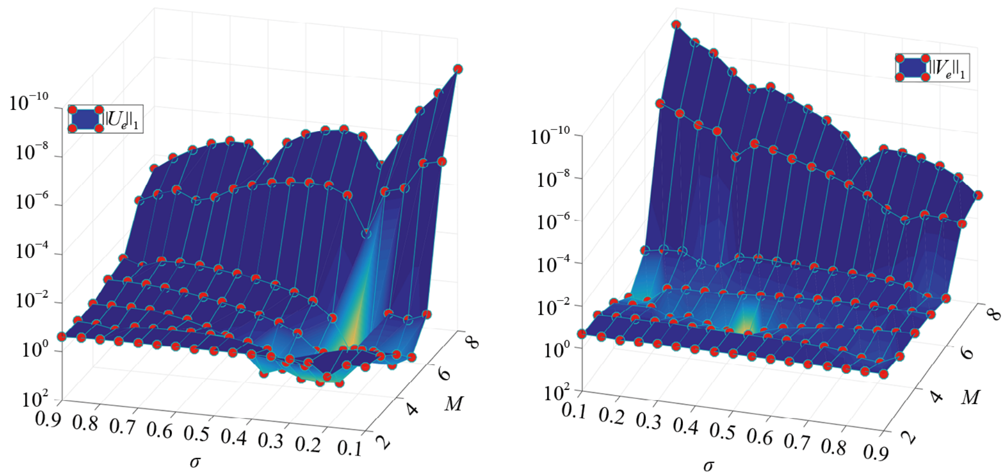

Figure 6. Norm of error at different values of σ and M for test problem 2

Table 3. Exact and approximate solution comparison for Test Problem 2

$t\backslash M$	Exact Solution $X(x)$	$M=7$	$M=9$	$M=11$
0.0	6.000000000000000e + 00	6.000048422872937e + 00	6.000000011416646e + 00	6.000000000017529e + 00
0.1	5.976700000000000e + 00	5.976677777198168e + 00	5.976700015799415e + 00	5.976700000008484e + 00
0.2	5.918400000000000e + 00	5.918421627737461e + 00	5.918399961007246e + 00	5.918399999989387e + 00
0.3	5.966100000000000e + 00	5.966143246953616e + 00	5.966099911651690e + 00	5.966099999993973e + 00
0.4	6.092800000000000e + 00	6.092834472753316e + 00	6.092799912707160e + 00	6.092799999994899e + 00
0.5	6.187500000000000e + 00	6.187529056678925e + 00	6.187499904336642e + 00	6.187499999983416e + 00
0.6	6.139200000000000e + 00	6.139248858192607e + 00	6.139199850805736e + 00	6.139199999986335e + 00
0.7	5.920900000000000e + 00	5.920982885052759e + 00	5.920899796730442e + 00	5.920899999983369e + 00
0.8	5.673600000000000e + 00	5.673699179782752e + 00	5.673599778353224e + 00	5.673599999971427e + 00
0.9	5.790300000000000e + 00	5.790389552232000e + 00	5.790299731991063e + 00	5.790299999977563e + 00
1.0	7.000000000000000e + 00	7.000147158229317e + 00	6.999999756248317e + 00	6.999999999986124e + 00

Table 4. Exact and approximate solution comparison for Test Problem 2

$t\backslash M$	Exact Solution $Y(x)$	$M = 7$	$M = 9$	$M = 11$
0.0	5.000000000000000e + 00	5.000048858279064e + 00	4.999999527527705e + 00	4.999999999984985e + 00
0.1	5.039952000000000e + 00	5.039836919409483e + 00	5.039952372884792e + 00	5.039952000045952e + 00
0.2	5.030528000000000e + 00	5.030492567913836e + 00	5.030528005500454e + 00	5.030527999996310e + 00
0.3	4.918808000000000e + 00	4.918832937890545e + 00	4.918807941821071e + 00	4.918807999992990e + 00
0.4	4.876992000000000e + 00	4.877012659053330e + 00	4.876992006471041e + 00	4.876992000005513e + 00
0.5	5.000000000000000e + 00	4.999993432514009e + 00	5.000000010054277e + 00	4.999999999997150e + 00
0.6	5.184512000000000e + 00	5.184498342797133e + 00	5.184511989669357e + 00	5.184511999998727e + 00
0.7	5.189448000000000e + 00	5.189450906086453e + 00	5.189447999857047e + 00	5.189448000003148e + 00
0.8	4.877888000000000e + 00	4.877898854703253e + 00	4.877888005935147e + 00	4.877887999996943e + 00
0.9	4.640432000000000e + 00	4.640422657816498e + 00	4.640431995911891e + 00	4.640432000003269e + 00
1.0	6.000000000000000e + 00	6.000028778384841e + 00	5.999999991405440e + 00	6.000000000009438e + 00

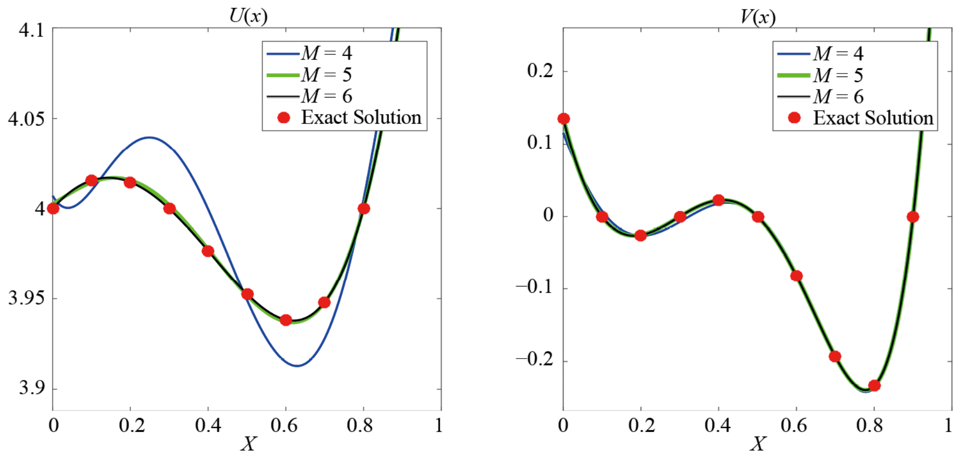

Figure 7. Exact and approximate solution comparison for Test Problem 3

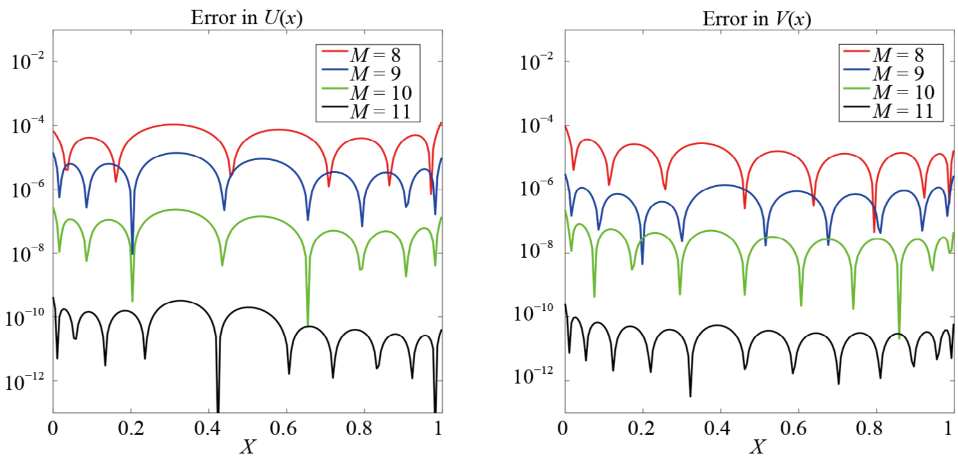

Figure 8. Absolute error at different scale level (M = 8: 1: 11) for Test Problem 3

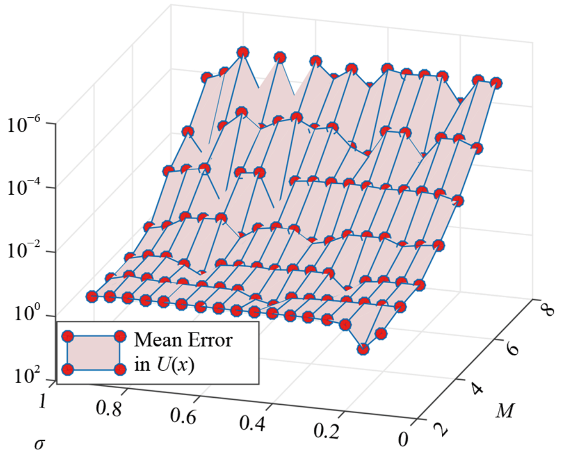

Figure 9. Norm of error at different values of σ and M in $U(x)$ for Test problem 3

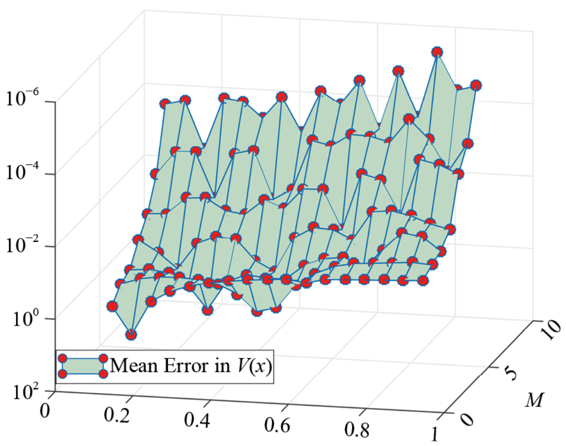

Figure 10. Norm of error at different values of σ and M in $V(x)$ for Test problem 3

Table 5. Comparison of exact and approximate solution of Test Problem 3

$t\backslash M$	Exact Solution $X(x)$	$M = 6$	$M = 8$	$M = 10$
0.0	4.000000000000000e + 00	4.002584470250579e + 00	4.000014234954282e + 00	3.999999879007794e + 00
0.1	4.015446618422229e + 00	4.014369779146812e + 00	4.015448986285839e + 00	4.015446568899046e + 00
0.2	4.014559316115696e + 00	4.015638470663570e + 00	4.014559962110798e + 00	4.014559383363525e + 00
0.3	4.000000000000000e + 00	4.001932304178486e + 00	3.999986527994391e + 00	3.999999810829810e + 00
0.4	3.976762243969177e + 00	3.977540546027928e + 00	3.976756030938303e + 00	3.976762134351098e + 00
0.5	3.952573655007183e + 00	3.951887143133661e + 00	3.952581184477324e + 00	3.952573807316868e + 00
0.6	3.938269260023675e + 00	3.937261855957261e + 00	3.938274850724708e + 00	3.938269294828847e + 00
0.7	3.948108195524989e + 00	3.947895350782296e + 00	3.948105147072294e + 00	3.948108143713309e + 00
0.8	4.000000000000000e + 00	4.000378251324270e + 00	4.000000549138948e + 00	4.000000060888149e + 00
0.9	4.115600398238363e + 00	4.115424149668324e + 00	4.115601405448341e + 00	4.115600333565572e + 00
1.0	4.320229740205038e + 00	4.320976576534706e + 00	4.320241023204595e + 00	4.320229501032460e + 00

Table 6. Exact and approximate solution comparison for Test Problem 3

$t \backslash M$	Exact Solution $Y(x)$	$M=6$	$M=8$	$M=10$
0.0	1.350000000000000e-01	1.360943642699249e-01	1.350030453405216e-01	1.350000349064031e-01
0.1	0	-3.000852167462655e-04	3.286941486757342e-07	1.030263249606851e-08
0.2	-2.564945792136357e-02	-2.539833986610897e-02	-2.564948392494333e-02	-2.564946736253324e-02
0.3	0	-1.610275383883328e-05	1.708354617740650e-08	1.071685961978429e-08
0.4	2.237737046461906e-02	2.201922356959399e-02	2.237873314120855e-02	2.237738447662979e-02
0.5	0	-1.505318100213836e-04	3.037936175431935e-07	-4.350436664818799e-09
0.6	-8.199534601757290e-02	-8.177497409727534e-02	-8.199621144237373e-02	-8.199534566575804e-02
0.7	-1.933202599171658e-01	-1.931974057514690e-01	-1.933199076261127e-01	-1.933202589497289e-01
0.8	-2.336817974917091e-01	-2.338986864886379e-01	-2.336816457910698e-01	-2.336818020791653e-01
0.9	0	1.173096787315217e-04	-5.701651543799729e-07	4.565344016431000e-09
1.0	8.562587759645993e-01	8.555966403642934e-01	8.562555112983435e-01	8.562587997193517e-01

7. Conclusion and future work

The method developed in this paper provides an accurate approximation to the solution of Caputo Fabrizio coupled system of FVIDE. The scheme is based on orthogonal polynomials. The simulation experiments suggest that the method is highly convergent. The error norm used to evaluate the approximation conforms that the method is convergent for all choices of σ. The error of approximation converges to 0, with the increase of the scale level. The theoretical conformation of the convergence of this scheme is however not known to the authors. The analytical convergence of the proposed method will be discussed in our future work. The use of other orthogonal polynomials are also expected to provide more accurate solution. The construction of the operational matrices for other definitions of fractional derivatives is also a part of our future plan. The current scheme have the ability to solve FVIDE with variable coefficients, however this scheme can not handle the nonlinear version of these equations in the current state. The extension of this method for the solution of nonlinear problems is still an open problem.

Conflict of interest

The authors declare there is no conflict of interest at any point with reference to research findings.

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