Convergence of the Rosenau-Korteweg-de Vries Equation to the Korteweg-de Vries One

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Abstract: The Rosenau-Korteweg-de Vries equation describes the wave-wave and wave-wall interactions. In this paper, we prove that, as the diffusion parameter is near zero, it coincides with the Korteweg-de Vries equation. The proof relies on deriving suitable a priori estimates together with an application of the Aubin-Lions Lemma.
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1. Introduction

The study of nonlinear wave phenomena is an important area of scientific research. In the last years several mathematical models describing wave behavior have been proposed. Some of them are the Korteweg-de Vries (KdV), the regularized long wave, the Rosenau, Rosenau-Kawahara, the Rosenau-KdV and the Rosenau-KdV-RLW equations (see [1-6]).

The KdV equation [7]

$$\partial_t u + au\partial_x u + \alpha\partial_x^3 u = 0,\ a,\ \alpha \in \mathbb{R}, \tag{1}$$

has a wide range of applications, such as magnetic fluid waves, ion sound waves, and longitudinal astigmatic waves.

From a mathematical point of view, in [8-11], the Cauchy problem for (1) is studied, while in [12], the author reviews the travelling wave solutions for (1). Moreover, in [13-15], the convergence of the solution of (1) to the unique entropy one of the Burgers equation is proven.

The KdV equation cannot describe the wave-wave and wave-wall interactions that appear in compact discrete systems. To overcome the shortcoming of KdV equation, Rosenau proposed the equation (see [3, 16-18])

$$\partial_t u + au\partial_x u + \alpha\partial_x^3 u + \beta^2\partial_t\partial_x^4 u = 0,\ a,\ \alpha,\ \beta \in \mathbb{R}. \tag{2}$$

From a mathematical point of view, the well-posedness of the classical solution of the Cauchy problem of (2) is studied in [19-20]. In particular, in [19], the well-posedness of the solution of the (2) is proven, under the assumption:

$$u_0(x) \in H^2(\mathbb{R}). \tag{3}$$

In [21], Zuo discussed the solitary wave solutions of (2). In [3], a conservative linear finite difference scheme for the numerical solution of an initial-boundary value problem for (2) was considered. Esfahani in [16] and Razborova, Triki, Biswas in [22] studied the solitary solutions for (2) with the solitary ansatz method, and also gave two invariants for (2). In particular, in [22], the two types of soliton solutions were analyzed. One is a solitary wave and the other is a singular soliton. In [23], Zheng and Zhou proposed an average linear finite difference scheme for the numerical approximation of the solutions of the initial-boundary value problem for (2). Finally, in [13, 24], the convergence of the solution of (2) to the unique entropy one of the Burgers equation is proven.

Observe that, if we send $\beta \to 0$ in (2), we obtain (1). Therefore, if β is near 0, the wave-wave and wave-wall interactions are described by (1).

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The aim of this paper is to prove that, when β goes to 0, the solution of (2) converges to the unique one of (1). In other to do this, first, we augment (2) with the initial datum $u_0(x)$, on which assume (3). After, following [24] and fixing two small numbers $0 < \varepsilon, \beta < 1$, we consider the following Cauchy problem:

$$\begin{cases} \partial_t u_{\varepsilon,\beta} + a u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta} + \alpha\partial_x^3 u_{\varepsilon,\beta} + \beta^2\partial_t\partial_x^4 u_{\varepsilon,\beta} = \varepsilon\partial_x^2 u_{\varepsilon,\beta}, & t > 0,\ x \in \mathbb{R}, \\ u_{\varepsilon,\beta}(0, x) = u_{\varepsilon,\beta,0}(x), & x \in \mathbb{R}, \end{cases} \tag{4}$$

where $u_{\varepsilon,\beta,0}$ is a C^∞ is a Capproximation of u_0 such that

$$\begin{aligned} &\|u_{\varepsilon,\beta,0}\|_{H^2(\mathbb{R})} \le \|u_0\|_{H^2(\mathbb{R})},\ \beta\left\|\partial_x^2 u_{\varepsilon,\beta,0}\right\|_{L^2(\mathbb{R})} \le C_0, \\ &\beta\left\|\partial_x^3 u_{\varepsilon,\beta,0}\right\|_{L^2(\mathbb{R})} + \varepsilon\beta\left\|\partial_x u_{\varepsilon,\beta,0}\right\|_{L^2(\mathbb{R})} + \varepsilon\beta\left\|\partial_x^2 u_{\varepsilon,\beta,0}\right\|_{L^2(\mathbb{R})} \le C_0, \\ &\beta\left\|\partial_x^4 u_{\varepsilon,\beta,0}\right\|_{L^2(\mathbb{R})} + \varepsilon\left\|\partial_x u_{\varepsilon,\beta,0}\right\|_{L^2(\mathbb{R})} + \varepsilon\beta^{\frac{1}{2}}\left\|\partial_x^2 u_{\varepsilon,\beta,0}\right\|_{L^2(\mathbb{R})} \le C_0, \end{aligned} \tag{5}$$

where C_0 is a positive constant, independent on ε and β. In what follows we denote with C_0 the constants which depend only on the initial data, and with $C(T)$ the constants which depend also on T.

Equation (4) is known as the Rosenau-Korteweg-de Vries-Burgers equation (see [25-26]). The well-posedness of (4) can be proven as in [19]. In [25], the initial-boundary value problem for (4) is studied, while, in [26], the well-posedeness of the periodic solutions for (4) is proven.

The main result of this paper is the following theorem.

Theorem 1.1 Fix $T > 0$ and consider (4). Assume (3) and (5). If

$$\beta = \mathcal{O}(\varepsilon^4), \tag{6}$$

then there exist two sequences $\{\varepsilon_n\}_{k\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$, with $\varepsilon_n, \beta_n \to 0$, and a limit function

$$u \in L^\infty(0, T; H^2(\mathbb{R})), \tag{7}$$

which is the unique solution of (1). Moreover, if u_1 and u_2 are two solutions of (1), we have that

$$\|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^2(\mathbb{R})} \le e^{C(T)t}\|u_{1,0} - u_{2,0}\|_{L^2(\mathbb{R})}, \tag{8}$$

for some suitable $C(T) > 0$, and every $0 \le t \le T$.

The assumption (6) consists in

$$\beta \le D^2\varepsilon^4, \tag{9}$$

where D is a positive constant such that

$$D < \min\left\{D_0, \frac{1}{|\alpha|\sqrt{2C_0}}\right\}, \tag{10}$$

where the positive constant D_0 is the unique zero of the function $F(X)$, defined in (33). Note that in (6) we have ε^4 and not ε^2 because we have β^2 and not β in (2).

From a physical point of view, Theorem 1.1, whose proof is based on the Aubin-Lions Lemma (see [27-30]), says that, when β is near 0, Equations (2) or (4) coincides with (1). From a mathematical point of view, compared to [9], Theorem 1.1 gives a new method, to prove well-posedness of the Cauchy problem of (1).

The paper is organized as follows. In Section 2, we prove several a priori estimates on (4). Those play a key role in the proof of our main result, that is given in Section 3.

2. A priori estimates

This section is devoted to some a priori estimates on $u_{\varepsilon,\beta}$. We begin by proving the following result.

Lemma 2.1 For each $t \geq 0$,

$$\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \beta^2 \left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \varepsilon\int_0^t \left\|\partial_x u_{\varepsilon,\beta}(s,\cdot)\right\|_{L^2(\mathbb{R})}^2 ds \leq C_0. \tag{11}$$

In particular, we have

$$\beta^{\frac{1}{2}} \left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})} \leq C_0. \tag{12}$$

Moreover,

$$\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^\infty(\mathbb{R})} \leq C_0\beta^{-\frac{1}{4}} \tag{13}$$

$$\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^\infty(\mathbb{R})} \leq C_0\beta^{-\frac{3}{4}}. \tag{14}$$

Proof. Arguing as in [24, Lemma 2.1] or [13, Lemma 2.2], by (5), we have (11).

We prove (12). Observe that

$$\beta\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 = \beta\int_{\mathbb{R}} \partial_x u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta} dx = -\beta\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta} dx.$$

Therefore, by (11) and the Young inequality,

$$\begin{aligned}\beta\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 &\leq \int_{\mathbb{R}} | u_{\varepsilon,\beta} || \beta\partial_x^2 u_{\varepsilon,\beta} | dx \\ &\leq \frac{1}{2}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{\beta^2}{2}\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 \leq C_0,\end{aligned}$$

which gives (12).

We prove (13). Thanks to the Hölder inequality,

$$\begin{aligned}u_{\varepsilon,\beta}^2(t,x) &= 2\int_{-\infty}^x u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta} dy \leq 2\int_{\mathbb{R}} | u_{\varepsilon,\beta} || \partial_x u_{\varepsilon,\beta} | dx \\ &\leq 2\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}.\end{aligned} \tag{15}$$

Consequently, by (11) and (12), we have that

$$\left\| u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^{\infty}(\mathbb{R})}^{2} \le C_0 \beta^{-\frac{1}{2}},$$

which gives (13).

Finally, we prove (14). Again by the Hölder inequality,

$$(\partial_x u_{\varepsilon,\beta}(t,x))^2 = 2\int_{-\infty}^{x} \partial_x u_{\varepsilon,\beta} \partial_x^2 u_{\varepsilon,\beta} dy \le 2\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\beta}| |\partial_x^2 u_{\varepsilon,\beta}| dx$$

$$\le 2\left\| \partial_x u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})} \left\| \partial_x^2 u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})}. \tag{16}$$

It follows from (11) and (12) that

$$\left\| \partial_x u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^{\infty}(\mathbb{R})}^{2} \le C_0 \beta^{-\frac{3}{2}},$$

which gives (14).

Following [24, Lemma 3.2], we prove the following result.

Lemma 2.2 Fix $T > 0$. There exists a constant $C_0 > 0$, independent on β and ε, such that

$$\left\| u_{\varepsilon,\beta} \right\|_{L^{\infty}((0,T)\times\mathbb{R})} \le C_0. \tag{17}$$

In particular, we have

$$\begin{aligned}
\left\| \partial_x u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})} &\le C_0,\\
\beta \left\| \partial_x^3 u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})} &\le C_0,\\
\varepsilon\beta \left\| \partial_x u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})} &\le C_0,\\
\varepsilon\beta \left\| \partial_x^2 u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})} &\le C_0,\\
\varepsilon \int_0^t \left\| \partial_x^2 u_{\varepsilon,\beta}(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 ds &\le C_0,\\
\varepsilon\beta \int_0^t \left\| \partial_t u_{\varepsilon,\beta}(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 ds &\le C_0,\\
\varepsilon\beta^2 \int_0^t \left\| \partial_t \partial_x u_{\varepsilon,\beta}(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 ds &\le C_0,\\
\varepsilon\beta^3 \int_0^t \left\| \partial_t \partial_x^2 u_{\varepsilon,\beta}(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 ds &\le C_0,\\
\varepsilon\beta^4 \int_0^t \left\| \partial_t \partial_x^3 u_{\varepsilon,\beta}(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 ds &\le C_0,
\end{aligned} \tag{18}$$

for every $0 \le t \le T$. Moreover,

$$\left\| \partial_x^2 u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^{\infty}(\mathbb{R})} \le C_0 \beta^{-1}, \tag{19}$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. Consider three real constants A, B and C, which will be specified.

Multiplying (2) by

$$-2\partial_x^2 u_{\varepsilon,\beta} + Au_{\varepsilon,\beta}^2 - B^2\varepsilon\beta^2\partial_t\partial_x^2 u_{\varepsilon,\beta} + C^2\varepsilon\beta\partial_t u_{\varepsilon,\beta},$$

we have that

$$\begin{aligned}
&(-2\partial_x^2 u_{\varepsilon,\beta} + Au_{\varepsilon,\beta}^2 - B^2\varepsilon\beta^2\partial_t\partial_x^2 u_{\varepsilon,\beta} + C^2\varepsilon\beta\partial_t u_{\varepsilon,\beta})\partial_t u_\varepsilon \\
&+ a(-2\partial_x^2 u_{\varepsilon,\beta} + Au_\varepsilon^2 - B^2\varepsilon\beta^2\partial_t\partial_x^2 u_{\varepsilon,\beta} + C^2\varepsilon\beta\partial_t u_{\varepsilon,\beta})u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta} \\
&+ \alpha(-2\partial_x^2 u_{\varepsilon,\beta} + Au_{\varepsilon,\beta}^2 - B^2\varepsilon\beta^2\partial_t\partial_x^2 u_{\varepsilon,\beta} + C^2\varepsilon\beta\partial_t u_{\varepsilon,\beta})\partial_x^3 u_{\varepsilon,\beta} \\
&+ \beta^2(-2\partial_x^2 u_{\varepsilon,\beta} + Au_{\varepsilon,\beta}^2 - B^2\varepsilon\beta^2\partial_t\partial_x^2 u_{\varepsilon,\beta} + C^2\varepsilon\beta\partial_t u_{\varepsilon,\beta})\partial_t\partial_x^4 u_\varepsilon \\
&= \varepsilon(-2\partial_x^2 u_{\varepsilon,\beta} + Au_{\varepsilon,\beta}^2 - B^2\varepsilon\beta^2\partial_t\partial_x^2 u_{\varepsilon,\beta} + C^2\varepsilon\beta\partial_t u_{\varepsilon,\beta})\partial_x^2 u_\varepsilon.
\end{aligned} \tag{20}$$

Since

$$\begin{aligned}
&\int_{\mathbb{R}}\left(-2\partial_x^2 u_{\varepsilon,\beta} + Au_{\varepsilon,\beta}^2 - B^2\varepsilon\beta^2\partial_t\partial_x^2 u_{\varepsilon,\beta} + C^2\varepsilon\beta\partial_t u_\varepsilon\right)\partial_t u_\varepsilon dx \\
&= \frac{d}{dt}\left(\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{A}{3}\int_{\mathbb{R}} u_{\varepsilon,\beta}^3 dx\right) + B^2\varepsilon\beta^2\left\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + C^2\varepsilon\beta\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2,
\end{aligned}$$

$$\begin{aligned}
&a\int_{\mathbb{R}}\left(-2\partial_x^2 u_{\varepsilon,\beta} + Au_\varepsilon^2 - B^2\varepsilon\beta^2\partial_t\partial_x^2 u_{\varepsilon,\beta} + C^2\varepsilon\beta\partial_t u_{\varepsilon,\beta}\right)u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta} dx \\
&= -2a\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta} dx + B^2 a\varepsilon\beta^2\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2\partial_t\partial_x u_{\varepsilon,\beta} dx \\
&+ B^2 a\varepsilon\beta^2\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}\partial_{tx}^2 u_{\varepsilon,\beta} dx + C^2 a\varepsilon\beta\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_t u_\varepsilon dx,
\end{aligned}$$

$$\begin{aligned}
&\alpha\int_{\mathbb{R}}\left(-2\partial_x^2 u_{\varepsilon,\beta} + Au_{\varepsilon,\beta}^2 - B^2\varepsilon\beta^2\partial_t\partial_x^2 u_{\varepsilon,\beta} + C^2\varepsilon\beta\partial_t u_{\varepsilon,\beta}\right)\partial_x^3 u_{\varepsilon,\beta} dx \\
&= -2A\alpha\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta} dx + B^2 a\varepsilon\beta^2 dx + B^2\alpha\varepsilon\beta^2\int_{\mathbb{R}}\partial_x^2 u_{\varepsilon,\beta}\partial_t\partial_x^3 u_{\varepsilon,\beta} dx - C^2\alpha\varepsilon\beta\int_{\mathbb{R}}\partial_x^2 u_{\varepsilon,\beta}\partial_t\partial_x u_{\varepsilon,\beta} dx,
\end{aligned}$$

$$\begin{aligned}
&\beta^2\int_{\mathbb{R}}\left(-2\partial_x^2 u_{\varepsilon,\beta} + Au_{\varepsilon,\beta}^2 - B^2\varepsilon\beta^2\partial_t\partial_x^2 u_{\varepsilon,\beta} + C^2\varepsilon\beta\partial_t u_\varepsilon\right)\partial_t\partial_x^4 u_{\varepsilon,\beta} dx \\
&= \beta^2\frac{d}{dt}\left\|\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 - 2A\beta^2\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_t\partial_x^3 u_{\varepsilon,\beta} dx + B^2\beta^4\varepsilon\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 \\
&+ C^2\varepsilon\beta^3\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 \\
&= \beta^2\frac{d}{dt}\left\|\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 - 4A\beta^2\int_{\mathbb{R}}\partial_x u_{\varepsilon,\beta}\partial_x^2 u_\varepsilon\partial_t\partial_x u_\varepsilon dx
\end{aligned}$$

$$+2A\beta^2\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_{\varepsilon,\beta}\partial_t\partial_x^2 u_{\varepsilon,\beta}dx + B^2\beta^4\varepsilon\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C^2\varepsilon\beta^3\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$\varepsilon\int_{\mathbb{R}}\left(-2\partial_x^2 u_{\varepsilon,\beta} + Au^2_{\varepsilon,\beta} - B^2\varepsilon\beta^2\partial_t\partial_x^2 u_{\varepsilon,\beta} + C^2\varepsilon\beta\partial_t u_{\varepsilon,\beta}\right)\partial_x^2 u_{\varepsilon,\beta}$$

$$= -2\varepsilon\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2A\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^2 dx - \frac{B^2\varepsilon^2\beta^2}{2}\frac{d}{dt}\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\varepsilon\int_{\mathbb{R}}\left(-2\partial_x^2 u_{\varepsilon,\beta} + Au^2_{\varepsilon,\beta} - B^2\varepsilon\beta^2\partial_t\partial_x^2 u_{\varepsilon,\beta} + C^2\varepsilon\beta\partial_t u_{\varepsilon,\beta}\right)\partial_x^2 u_{\varepsilon,\beta}$$

$$= -2\varepsilon\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2A\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^2 dx - \frac{B^2\varepsilon^2\beta^2}{2}\frac{d}{dt}\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$-\frac{C^2\varepsilon^2\beta}{2}\frac{d}{dt}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

an integration on $\mathbb{R}$ of (20) gives

$$\frac{d}{dt}\left(\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{A}{3}\int_{\mathbb{R}} u^3_{\varepsilon,\beta}dx + \beta^2\left\|\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)$$

$$+\frac{d}{dt}\left(\frac{C^2\varepsilon^2\beta}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{B^2\varepsilon^2\beta^2}{2}\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)$$

$$+2\varepsilon\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C^2\varepsilon\beta\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+B^2\varepsilon\beta^2\left\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C^2\varepsilon\beta^3\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + B^2\varepsilon\beta^4\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$= 2\left(a + A\alpha\right)\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}dx - B^2 a\varepsilon\beta^2\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2\partial_t\partial_x u_{\varepsilon,\beta}dx$$

$$-B^2 a\varepsilon\beta^2\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}\partial_{tx}^2 u_{\varepsilon,\beta}dx - C^2 a\varepsilon\beta\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_t u_\varepsilon dx$$

$$-B^2 a\varepsilon\beta^2\int_{\mathbb{R}}\partial_x^2 u_{\varepsilon,\beta}\partial_t\partial_x^3 u_{\varepsilon,\beta}dx - C^2 a\varepsilon\beta\int_{\mathbb{R}}\partial_x^3 u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta}dx$$

$$+4A\beta^2\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_t\partial_x u_{\varepsilon,\beta}dx - 2A\beta^2\int_{\mathbb{R}} u_\varepsilon\partial_x^2 u_{\varepsilon,\beta}\partial_t\partial_x^2 u_{\varepsilon,\beta}dx - 2A\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^2 dx.$$

Taking $A = -\frac{a}{\alpha}$,we have that

$$\frac{d}{dt}\left(\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \frac{a}{3\alpha}\int_{\mathbb{R}} u^3_{\varepsilon,\beta}dx + \beta^2\left\|\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)$$

$$+\frac{d}{dt}\left(\frac{C^2\varepsilon^2\beta}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{B^2\varepsilon^2\beta^2}{2}\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)$$

$$+2\varepsilon\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C^2\varepsilon\beta\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+B^2\varepsilon\beta^2\left\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C^2\varepsilon\beta^3\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+B^2\varepsilon\beta^4\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$=-B^2a\varepsilon\beta^2\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2\partial_t\partial_x u_{\varepsilon,\beta}dx-B^2a\varepsilon\beta^2\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}\partial_{tx}^2 u_{\varepsilon,\beta}dx$$

$$-C^2a\varepsilon\beta\int_{\mathbb{R}}u_\varepsilon\partial_x u_\varepsilon\partial_t u_\varepsilon dx-B^2\alpha\varepsilon\beta^2\int_{\mathbb{R}}\partial_x^2 u_{\varepsilon,\beta}\partial_t\partial_x^3 u_{\varepsilon,\beta}dx$$

$$+C^2\alpha\varepsilon\beta\int_{\mathbb{R}}\partial_x^2 u_{\varepsilon,\beta}\partial_t\partial_x u_{\varepsilon,\beta}dx-\frac{4a\beta^2}{\alpha}\int_{\mathbb{R}}\partial_x u_{\varepsilon,\beta}\partial_x^2 u_\varepsilon\partial_t\partial_x u_{\varepsilon,\beta}dx$$

$$+\frac{2a\beta^2}{\alpha}\int_{\mathbb{R}}u_\varepsilon\partial_x^2 u_{\varepsilon,\beta}\partial_t\partial_x^2 u_{\varepsilon,\beta}dx+\frac{2a\varepsilon}{\alpha}\int_{\mathbb{R}}u_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^2dx. \tag{21}$$

Since $0<\varepsilon,\beta<1$, due to (10), (11), (13), (14) and the Young inequality,

$$B^2|a|\varepsilon\beta^2\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2|\partial_t\partial_x u_{\varepsilon,\beta}|dx$$

$$\le\frac{B^2a^2\varepsilon\beta^2}{2}\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4dx+\frac{B^2\varepsilon\beta^2}{2}\left\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\le\frac{B^2\varepsilon\beta^2}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{B^2\varepsilon\beta^2}{2}\left\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\le\frac{B^2\varepsilon\beta^{\frac{1}{2}}}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{B^2\varepsilon\beta^2}{2}\left\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\le C_0B^2D\varepsilon^3\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{B^2\varepsilon\beta^2}{2}\left\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\le C_0B^2D\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{B^2\varepsilon\beta^2}{2}\left\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$B^2|a|\varepsilon\beta^2\int_{\mathbb{R}}|u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}||\partial_{tx}^2 u_{\varepsilon,\beta}|dx$$

$$\le\frac{3B^2a^2\varepsilon\beta^2}{4}\int_{\mathbb{R}}u_{\varepsilon,\beta}^2(\partial_x^2 u_{\varepsilon,\beta}|)^2dx+\frac{B^2\varepsilon\beta^2}{3}\left\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\le\frac{3B^2a^2\varepsilon\beta^2}{4}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{B^2\varepsilon\beta^2}{3}\left\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq C_0 B^2 \varepsilon \beta^{\frac{3}{2}} \left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{B^2 \varepsilon \beta^2}{3} \left\|\partial_t \partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2$$

$$\leq C_0 B^2 D^3 \varepsilon^3 \left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{B^2 \varepsilon \beta^2}{3} \left\|\partial_t \partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2$$

$$\leq C_0 B^2 D^3 \varepsilon \left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{B^2 \varepsilon \beta^2}{3} \left\|\partial_t \partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2,$$

$$C^2 |a| \varepsilon \beta \int_{\mathbb{R}} |u_\varepsilon \partial_x u_\varepsilon| |\partial_t u_\varepsilon| dx$$

$$\leq \frac{C^2 a^2 \varepsilon \beta}{2} \int_{\mathbb{R}} u_\varepsilon^2 (\partial_x u_\varepsilon)^2 dx + \frac{C^2 \varepsilon \beta}{2} \left\|\partial_t u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}^2$$

$$\leq \frac{C^2 a^2 \varepsilon \beta}{2} \left\|u_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{R})}^2 \left\|\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{C^2 \varepsilon \beta}{2} \left\|\partial_t u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}^2$$

$$\leq C_0 C \varepsilon \beta^{\frac{1}{2}} \left\|\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{C^2 \varepsilon \beta}{2} \left\|\partial_t u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}^2$$

$$\leq C_0 C^2 D \varepsilon^2 \left\|\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{C^2 \varepsilon \beta}{2} \left\|\partial_t u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}^2$$

$$\leq C_0 C^2 D \varepsilon \left\|\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{C^2 \varepsilon \beta}{2} \left\|\partial_t u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}^2,$$

$$B^2 |\alpha| \varepsilon \beta^2 \int_{\mathbb{R}} |\partial_x^2 u_{\varepsilon,\beta}| |\partial_t \partial_x^3 u_{\varepsilon,\beta}| dx \leq \frac{B^2 \alpha^2 \varepsilon}{2} \left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{B^2 \varepsilon \beta^4}{2} \left\|\partial_t \partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2,$$

$$C^2 |\alpha| \varepsilon \beta \int_{\mathbb{R}} |\partial_x^2 u_{\varepsilon,\beta}| |\partial_t \partial_x u_{\varepsilon,\beta}| dx \leq \frac{7C^4 \alpha^2 \varepsilon}{4B^2} \left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{B^2 \varepsilon \beta^2}{7} \left\|\partial_t \partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2,$$

$$4\left|\frac{a\beta^2}{\alpha}\right| \int_{\mathbb{R}} |\partial_x u_{\varepsilon,\beta} \partial_x^2 u_\varepsilon| |\partial_t \partial_x u_{\varepsilon,\beta}| dx \leq \frac{43 a^2 \beta^2}{B^2 \alpha^2 \varepsilon} \int_{\mathbb{R}} (\partial_x u_{\varepsilon,\beta})^2 (\partial_x^2 u_{\varepsilon,\beta})^2 dx + \frac{B^2 \varepsilon \beta^2}{43} \left\|\partial_t \partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2$$

$$\leq \frac{43 a^2 \beta^2}{B^2 \alpha^2 \varepsilon} \left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^\infty(\mathbb{R})}^2 \left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{B^2 \varepsilon \beta^2}{43} \left\|\partial_t \partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2$$

$$\leq \frac{C_0 \beta^{\frac{1}{2}}}{B^2 \varepsilon} \left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{B^2 \varepsilon \beta^2}{43} \left\|\partial_t \partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2$$

$$\leq \frac{C_0 D \varepsilon}{B^2} \left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{B^2 \varepsilon \beta^2}{43} \left\|\partial_t \partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2,$$

$$2\left|\frac{a\beta^2}{\alpha}\right| \int_{\mathbb{R}} |u_{\varepsilon,\beta} \partial_x^2 u_{\varepsilon,\beta}| \partial_t \partial_x^2 u_{\varepsilon,\beta}| dx \leq \frac{2a^2 \beta}{\alpha^2 C^2 \varepsilon} \int_{\mathbb{R}} u_{\varepsilon,\beta}^2 (\partial_x^2 u_{\varepsilon,\beta})^2 dx + \frac{C^2 \varepsilon \beta^3}{2} \left\|\partial_t \partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2$$

$$\leq \frac{2a^2\beta}{\alpha^2 C^2\varepsilon}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{C^2\varepsilon\beta^3}{2}\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq \frac{2a^2\beta^{\frac{1}{2}}}{\alpha^2 C^2\varepsilon}\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{C^2\varepsilon\beta^3}{2}\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq \frac{2a^2 D\varepsilon}{\alpha^2 C^2}\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{C^2\varepsilon\beta^3}{2}\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Therefore, by (21),

$$\frac{d}{dt}\left(\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-\frac{a}{3\alpha}\int_{\mathbb{R}}u^3_{\varepsilon,\beta}dx+\beta^2\left\|\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)$$

$$+\frac{d}{dt}\left(\frac{C^2\varepsilon^2\beta}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{B^2\varepsilon^2\beta^2}{2}\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)$$

$$+\left(2-C_0B^2D^3-\frac{B^2\alpha^2}{2}-\frac{7C^4\alpha^2}{4B^2}-\frac{C_0D}{B^2}-\frac{2a^2D}{\alpha^2C^2}\right)\varepsilon\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+\frac{C^2\varepsilon\beta}{2}\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{B^2\varepsilon\beta^2}{1806}\left\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+\frac{C^2\varepsilon\beta^3}{2}\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{B^2\beta^4\varepsilon}{2}\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq C_0D\left(B^2+C^2\right)\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon\left|\frac{a}{\alpha}\right|\int_{\mathbb{R}}|u_{\varepsilon,\beta}|(\partial_x u_{\varepsilon,\beta})^2dx$$

$$\leq C_0\left(D\left(B^2+C^2\right)+\left\|u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}\right)\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \tag{22}$$

We search B, C, D such that

$$2-C_0B^2D^3-\frac{B^2\alpha^2}{2}-\frac{7C^4\alpha^2}{4B^2}-\frac{C_0D}{B^2}-\frac{2a^2D}{\alpha^2C^2}>0, \tag{23}$$

that is

$$C_0B^2D^3+\frac{\alpha^2C_0C^2+2a^2B^2}{\alpha^2B^2C^2}D+\frac{B^2\alpha^2}{2}+\frac{7C^4\alpha^2}{4B^2}-2<0. \tag{24}$$

We search B, C such that

$$\frac{B^2\alpha^2}{2}+\frac{7C^4\alpha^2}{4B^2}=1, \tag{25}$$

that is

$$\frac{2\alpha^2B^4-4B^2+7\alpha^2C^4}{4B^2}=0. \tag{26}$$

(26) is verified when

$$2\alpha^2B^4-4B^2+7\alpha^2C^4=0. \tag{27}$$

B does exist if and only if

$$2-7\alpha^2C^4>0. \tag{28}$$

Choosing

$$C^2=\frac{1}{\sqrt{7\alpha^2}}, \tag{29}$$

(28) is verified. Moreover, thanks to (28) and (29), by (27), we have that

$$B^2=\frac{2-\sqrt{2}}{2\alpha^2} \text{ or } B^2=\frac{2+\sqrt{2}}{2\alpha^2}. \tag{30}$$

Thanks to (29) and (30), we can define the following positive constant:

$$k_1^2:=C_0B^2,\ k_2^2:=\frac{\alpha^2C_0C^2+2a^2B^2}{\alpha^2B^2C^2}. \tag{31}$$

It follows from (24), (25) and (31) that

$$k_1^2D^3+k_2^2D-1<0. \tag{32}$$

Let consider the following function:

$$F(\mathrm{X}):=k_1^2X^3+k_2^2X-1,\ X\geq 0. \tag{33}$$

Observe that

$$F(0)=-1,\ \lim_{X\to\infty}F(X)=\infty. \tag{34}$$

Moreover,

$$F'(X)=3k_1^2X^3+k_2^2>0. \tag{35}$$

Then, it follows from (34) and (35) that the function F has an only zero $D_0>0$. Therefore, the following inequality

$$k_1^2X^3+k_2^2X-1<0,$$

is verified when

$$0<X<D_0.$$

Taking $X = D$, we have (10).

Consequently, by (10), (22), (23), (29) and (30),

$$\frac{d}{dt}\left(\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \frac{a}{3\alpha}u^3_{\varepsilon,\beta}dx + \beta^2\left\|\partial^3_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)$$

$$+\frac{d}{dt}\left(\frac{C^2\varepsilon^2\beta}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{B^2\varepsilon^2\beta^2}{2}\left\|\partial^2_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)$$

$$+K_1^2\varepsilon\left\|\partial^2_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{C^2\varepsilon\beta}{2}\left\|\partial_t\partial^2_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+\frac{B^2\varepsilon\beta^2}{1806}\left\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{C^2\varepsilon\beta^3}{2}\left\|\partial_t\partial^2_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{B^2\varepsilon\beta^4}{2}\left\|\partial_t\partial^3_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq C_0\left(K_2^2 + \left\|u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}\right)\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

for some constants K_1^2, K_2^2.

(5), (11) and an integration on $(0, t)$ give

$$\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \frac{a}{3\alpha}\int_{\mathbb{R}} u^3_{\varepsilon,\beta}dx + \beta^2\left\|\partial^3_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+\frac{C^2\varepsilon^2\beta}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{B^2\varepsilon^2\beta^2}{2}\left\|\partial^2_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+K_1^2\varepsilon\int_0^t\left\|\partial^2_x u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds + \frac{C^2\varepsilon\beta}{2}\int_0^t\left\|\partial_t u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds$$

$$+\frac{B^2\varepsilon\beta^2}{1806}\int_0^t\left\|\partial_t\partial_x u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds + \frac{C^2\varepsilon\beta^3}{2}\int_0^t\left\|\partial_t\partial^2_x u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds$$

$$+\frac{B^2\varepsilon\beta^4}{2}\int_0^t\left\|\partial_t\partial^3_x u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds$$

$$\leq C_0\left(K_2^2 + \left\|u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}\right)\varepsilon\int_0\left\|\partial_x u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds$$

$$\leq C_0\left(1 + \left\|u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}\right).$$

Therefore, by (11),

$$\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2\left\|\partial^3_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{C^2\varepsilon^2\beta}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{B^2\varepsilon^2\beta^2}{2}\left\|\partial^2_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+K_1^2\varepsilon\int_0^t\left\|\partial^2_x u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds + \frac{C^2\varepsilon\beta}{2}\int_0^t\left\|\partial_t u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds + \frac{B^2\varepsilon\beta^2}{1806}\int_0^t\left\|\partial_t\partial_x u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds$$

$$+\frac{C^2\varepsilon\beta^3}{2}\int_0^t\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds+\frac{B^2\varepsilon\beta^4}{2}\int_0^t\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds$$

$$\leq C_0\left(1+\left\|u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}\right)+\frac{a}{3\alpha}\int_{\mathbb{R}}u^3_{\varepsilon,\beta}dx$$

$$\leq C_0\left(1+\left\|u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}\right)+\left|\frac{a}{3\alpha}\right|\int_{\mathbb{R}}|u_{\varepsilon,\beta}|^3dx$$

$$\leq C_0\left(1+\left\|u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}\right)+\left|\frac{a}{3\alpha}\right|\left\|u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq C_0\left(1+\left\|u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}\right). \tag{36}$$

We prove (17). Thanks to (11), (15) and (36).

$$u^2_{\varepsilon,\beta}(t,x)\leq C_0\sqrt{\left(1+\left\|u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}\right)}.$$

Therefore,

$$\left\|u_{\varepsilon,\beta}\right\|^4_{L^\infty((0,T)\times\mathbb{R})}-C_0\left\|u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}-C_0\leq 0.$$

Arguing as in [31, Lemma 2.4], we have (17).
(18) follows from (17) and (36).
Finally, we prove (19). Due to (11), (18) and the Hölder inequality,

$$(\partial_x^2 u_{\varepsilon,\beta}(t,x))^2=2\int_{-\infty}^x\partial_x^2 u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}dy\leq 2\int_{\mathbb{R}}|\partial_x^2 u_{\varepsilon,\beta}||\partial_x^3 u_{\varepsilon,\beta}|dx$$

$$\leq 2\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}\leq C_0\beta^{-2}.$$

Hence,

$$\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\leq C_0\beta^{-2},$$

which gives (19)
Lemma 2.3 Fix $T > 0$. There exists a constant $C(T) > 0$, independent on β and ε, such that

$$\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}\leq C(T). \tag{37}$$

In particular, we have that

$$\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}\leq C(T),$$

$$\beta\left\|\partial_x^4 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})} \leq C(T),$$

$$\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})} \leq C(T),$$

$$\varepsilon\beta^{\frac{1}{2}}\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})} \leq C(T),$$

$$\varepsilon\int_0^t\left\|\partial_t u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C(T),$$

$$\varepsilon\beta\int_0^t\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C(T),$$

$$\varepsilon\beta^2\int_0^t\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(s,\cdot)\right\|(s,\cdot)^2_{L^2(\mathbb{R})} ds \leq C(T),$$

$$\varepsilon\beta^3\int_0^t\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C(T),$$

$$\varepsilon\int_0^t\left\|\partial_x^3 u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C(T), \tag{38}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Consider four real constants E, F, G and H, which will be specified later. Multiplying (2) by

$$2\partial_x^4 u_{\varepsilon,\beta} + E(\partial_x u_{\varepsilon,\beta})^2 + Fu_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta} + \varepsilon G^2\partial_t u_{\varepsilon,\beta} - \varepsilon\beta H^2\partial_t\partial_x^2 u_{\varepsilon,\beta},$$

we have

$$(2\partial_x^4 u_{\varepsilon,\beta} + E(\partial_x u_{\varepsilon,\beta})^2 + Fu_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta})\partial_t u_{\varepsilon,\beta}$$

$$+ (\varepsilon G^2\partial_t u_{\varepsilon,\beta} - \varepsilon\beta H^2\partial_t\partial_x^2 u_{\varepsilon,\beta})\partial_t u_{\varepsilon,\beta}$$

$$+ a(2\partial_x^4 u_{\varepsilon,\beta} + E(\partial_x u_{\varepsilon,\beta})^2 + Fu_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta})u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}$$

$$+ a(\varepsilon G^2\partial_t u_{\varepsilon,\beta} - \varepsilon\beta H^2\partial_t\partial_x^2 u_{\varepsilon,\beta})u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}$$

$$+ \alpha(2\partial_x^4 u_{\varepsilon,\beta} + E(\partial_x u_{\varepsilon,\beta})^2 + Fu_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta})\partial_x^3 u_{\varepsilon,\beta}$$

$$+ \alpha(\varepsilon G^2\partial_t u_{\varepsilon,\beta} - \varepsilon\beta H^2\partial_t\partial_x^2 u_{\varepsilon,\beta})\partial_x^3 u_{\varepsilon,\beta}$$

$$+ \beta^2(2\partial_x^4 u_{\varepsilon,\beta} + E(\partial_x u_{\varepsilon,\beta})^2 + Fu_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta})\partial_t\partial_x^4 u_{\varepsilon,\beta}$$

$$+ \beta^2(\varepsilon G^2\partial_t u_{\varepsilon,\beta} - \varepsilon\beta H^2\partial_t\partial_x^2 u_{\varepsilon,\beta})\partial_t\partial_x^4 u_{\varepsilon,\beta}$$

$$= \varepsilon(2\partial_x^4 u_{\varepsilon,\beta} + E(\partial_x u_{\varepsilon,\beta})^2 + Fu_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta})\partial_x^2 u_{\varepsilon,\beta}$$

$$+\varepsilon(\varepsilon G^2\partial_t u_{\varepsilon,\beta}-\varepsilon\beta H^2\partial_t\partial_x^2 u_{\varepsilon,\beta})\partial_x^2 u_{\varepsilon,\beta}. \tag{39}$$

Since

$$\int_{\mathbb{R}}\left(2\partial_x^4 u_{\varepsilon,\beta}+E(\partial_x u_{\varepsilon,\beta})^2+Fu_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}\right)\partial_t u_{\varepsilon,\beta}dx$$

$$=\frac{d}{dt}\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+E\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2\partial_t u_{\varepsilon,\beta}dx+F\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta}dx,$$

$$\int_{\mathbb{R}}\left(\varepsilon G^2\partial_t u_{\varepsilon,\beta}-\varepsilon\beta H^2\partial_t\partial_x^2 u_{\varepsilon,\beta}\right)\partial_t u_{\varepsilon,\beta}dx$$

$$=\varepsilon G^2\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\varepsilon\beta H^2\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$a\int_{\mathbb{R}}\left(2\partial_x^4 u_{\varepsilon,\beta}+E(\partial_x u_{\varepsilon,\beta})^2+Fu_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}\right)u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx$$

$$=-2a\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2\partial_x^3 u_{\varepsilon}-2a\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}dx+a\left(E-F\right)\int_{\mathbb{R}}u_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^3dx$$

$$=5a\int_{\mathbb{R}}\partial_x u_{\varepsilon,\beta}(\partial_x^2 u_{\varepsilon,\beta})^2dx+a\left(E-F\right)\int_{\mathbb{R}}u_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^3dx,$$

$$a\int_{\mathbb{R}}\left(\varepsilon G^2\partial_t u_{\varepsilon,\beta}-\varepsilon\beta H^2\partial_t\partial_x^2 u_{\varepsilon,\beta}\right)u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx$$

$$=a\varepsilon G^2\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta}dx-a\varepsilon\beta H^2\int_{\mathbb{R}}u_{\varepsilon}\partial_x u_{\varepsilon}\partial_t\partial_x^2 u_{\varepsilon,\beta}dx,$$

$$\alpha\int_{\mathbb{R}}\left(2\partial_x^4 u_{\varepsilon,\beta}+E(\partial_x u_{\varepsilon,\beta})^2+Fu_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}\right)\partial_x^3 u_{\varepsilon,\beta}dx$$

$$=-\alpha\left(2E+\frac{F}{2}\right)\int_{\mathbb{R}}u_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^3dx,$$

$$\alpha\int_{\mathbb{R}}\left(\varepsilon G^2\partial_t u_{\varepsilon,\beta}-\varepsilon\beta H^2\partial_t\partial_x^2 u_{\varepsilon,\beta}\right)\partial_x^3 u_{\varepsilon,\beta}dx$$

$$=\varepsilon\alpha G^2\int_{\mathbb{R}}\partial_t u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}dx-\varepsilon\beta\alpha H^2\int_{\mathbb{R}}\partial_t\partial_x^2 u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}dx$$

$$\beta^2\int_{\mathbb{R}}\left(2\partial_x^4 u_{\varepsilon,\beta}+E(\partial_x u_{\varepsilon,\beta})^2+Fu_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}\right)\partial_t\partial_x^4 u_{\varepsilon,\beta}dx$$

$$=\beta^2\frac{d}{dt}\left\|\partial_x^4 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-\beta^2\left(2E+F\right)\int_{\mathbb{R}}\partial_x u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}\partial_t\partial_x^3 u_{\varepsilon,\beta}dx$$

$$-\beta^2F\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}\partial_t\partial_x^3 u_{\varepsilon,\beta}dx,$$

$$\beta^2\int_{\mathbb{R}}\left(\varepsilon G^2\partial_t u_{\varepsilon,\beta}-\varepsilon\beta H^2\partial_t\partial_x^2 u_{\varepsilon,\beta}\right)\partial_t\partial_x^4 u_{\varepsilon,\beta}dx$$

$$= \varepsilon\beta^2 G^2 \left\| \partial_t \partial_x^2 u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \varepsilon\beta^3 H^2 \left\| \partial_t \partial_x^3 u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})},$$

$$= \varepsilon \int_{\mathbb{R}} \left(2\partial_x^4 u_{\varepsilon,\beta} + E(\partial_x u_{\varepsilon,\beta})^2 + F u_{\varepsilon,\beta} \partial_x^2 u_{\varepsilon,\beta} \right) \partial_x^2 u_{\varepsilon,\beta} dx$$

$$= -2\varepsilon \left\| \partial_x^3 u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \varepsilon F \int_{\mathbb{R}} u_{\varepsilon,\beta} (\partial_x^2 u_{\varepsilon,\beta})^2 dx,$$

$$\varepsilon \int_{\mathbb{R}} \left(\varepsilon G^2 \partial_t u_{\varepsilon,\beta} - \varepsilon\beta H^2 \partial_t \partial_x^2 u_{\varepsilon,\beta} \right) \partial_x^2 u_{\varepsilon,\beta} dx$$

$$= -\frac{d}{dt} \left(\frac{\varepsilon^2 G^2}{2} \left\| \partial_x u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon^2 \beta H^2}{2} \left\| \partial_x^2 u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \right),$$

an integration on $\mathbb{R}$ of (39) gives

$$\frac{d}{dt} \left(\left\| \partial_x^2 u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \beta^2 \left\| \partial_x^4 u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \right)$$

$$+ E \int_{\mathbb{R}} (\partial_x u_{\varepsilon,\beta})^2 \partial_t u_{\varepsilon,\beta} dx + F \int_{\mathbb{R}} u_{\varepsilon,\beta} \partial_x^2 u_{\varepsilon,\beta} \partial_t u_{\varepsilon,\beta} dx$$

$$+ \frac{d}{dt} \left(\frac{\varepsilon^2 G^2}{2} \left\| \partial_x u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon^2 \beta H^2}{2} \left\| \partial_x^2 u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \right)$$

$$+ \varepsilon G^2 \left\| \partial_t u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \varepsilon\beta H^2 \left\| \partial_t \partial_x^2 u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})}$$

$$+ \varepsilon\beta^2 G^2 \left\| \partial_t \partial_x^2 u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \varepsilon\beta^3 H^2 \left\| \partial_t \partial_x^2 u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + 2\varepsilon \left\| \partial_x^3 u_{\varepsilon,\beta}(t,\cdot) \right\|^2_{L^2(\mathbb{R})}$$

$$= \left(2\alpha E + \frac{\alpha F}{2} - 5a \right) \int_{\mathbb{R}} \partial_x u_{\varepsilon,\beta} (\partial_x^2 u_{\varepsilon,\beta})^2 dx - a(E - F) \int_{\mathbb{R}} u_{\varepsilon,\beta} (\partial_x u_{\varepsilon,\beta})^3 dx$$

$$- a\varepsilon G^2 \int_{\mathbb{R}} u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \partial_t u_{\varepsilon,\beta} dx + a\varepsilon\beta H^2 \int_{\mathbb{R}} u_{\varepsilon} \partial_x u_{\varepsilon} \partial_t \partial_x^2 u_{\varepsilon,\beta} dx$$

$$- \varepsilon\alpha G^2 \int_{\mathbb{R}} \partial_t u_{\varepsilon,\beta} \partial_x^3 u_{\varepsilon,\beta} dx - \varepsilon\beta\alpha H^2 \int_{\mathbb{R}} \partial_t \partial_x^2 u_{\varepsilon,\beta} \partial_x^3 u_{\varepsilon,\beta} dx$$

$$+ \beta^2 (2E + F) \int_{\mathbb{R}} \partial_x u_{\varepsilon,\beta} \partial_x^2 u_{\varepsilon,\beta} \partial_t \partial_x^3 u_{\varepsilon,\beta} dx + \beta^2 F \int_{\mathbb{R}} u_{\varepsilon,\beta} \partial_x^3 u_{\varepsilon,\beta} \partial_t \partial_x^3 u_{\varepsilon,\beta} dx$$

$$+ \varepsilon F \int_{\mathbb{R}} u_{\varepsilon,\beta} (\partial_x^2 u_{\varepsilon,\beta})^2 dx. \tag{40}$$

Observe that

$$E \int_{\mathbb{R}} (\partial_x u_{\varepsilon,\beta})^2 \partial_t u_{\varepsilon,\beta} dx + F \int_{\mathbb{R}} u_{\varepsilon,\beta} \partial_x^2 u_{\varepsilon,\beta} \partial_t u_{\varepsilon,\beta} dx$$

$$= (E - F) \int_{\mathbb{R}} (\partial_x u_{\varepsilon,\beta})^2 \partial_t u_{\varepsilon,\beta} dx - F \int_{\mathbb{R}} u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \partial_t \partial_x u_{\varepsilon,\beta} dx$$

$$=\left(E-F\right)\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2\partial_t u_{\varepsilon,\beta}dx-\frac{F}{2}\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_t((\partial_x u_{\varepsilon,\beta})^2)dx.$$

Consequently, by (40),

$$\frac{d}{dt}\left(\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^4 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)$$

$$+\left(E-F\right)\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2\partial_t u_{\varepsilon,\beta}dx-\frac{F}{2}\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_t((\partial_x u_{\varepsilon,\beta})^2)dx$$

$$+\frac{d}{dt}\left(\frac{\varepsilon^2G^2}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\varepsilon^2\beta H^2}{2}\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)$$

$$+\varepsilon G^2\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\varepsilon\beta H^2\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+\varepsilon\beta^2G^2\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\varepsilon\beta^3H^2\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+2\varepsilon\left\|\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$=\left(2\alpha E+\frac{\alpha F}{2}-5a\right)\int_{\mathbb{R}}\partial_x u_{\varepsilon,\beta}(\partial_x^2 u_{\varepsilon,\beta})^2dx-a\left(E-F\right)\int_{\mathbb{R}}u_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^3dx$$

$$+a\varepsilon G^2\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta}dx+a\varepsilon\beta H^2\int_{\mathbb{R}}u_{\varepsilon}\partial_x u_{\varepsilon}\partial_t\partial_x^2 u_{\varepsilon,\beta}dx$$

$$-\varepsilon\alpha G^2\int_{\mathbb{R}}\partial_t u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}dx-\varepsilon\beta\alpha H^2\int_{\mathbb{R}}\partial_t\partial_x^2 u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}dx$$

$$-\varepsilon\alpha G^2\int_{\mathbb{R}}\partial_t u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}dx-\varepsilon\beta\alpha H^2\int_{\mathbb{R}}\partial_t\partial_x^2 u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}dx$$

$$+\beta^2\left(2E+F\right)\int_{\mathbb{R}}\partial_x u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}\partial_t\partial_x^3 u_{\varepsilon,\beta}dx+\beta^2F\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}\partial_t\partial_x^3 u_{\varepsilon,\beta}dx$$

$$+\varepsilon F\int_{\mathbb{R}}u_{\varepsilon,\beta}(\partial_x^2 u_{\varepsilon,\beta})^2dx. \tag{41}$$

We search E, F such that

$$E-F=-\frac{F}{2},\quad 2\alpha E+\frac{\alpha F}{2}-5a=0. \tag{42}$$

Since

$$\left(E,F\right)=\left(\frac{5a}{3\alpha},\frac{10a}{3\alpha}\right)$$

is the unique solution of (42), it follows from (41) that

$$
\frac{d}{dt}\left(\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^4 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)
$$

$$
-\frac{5a}{3\alpha}\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2\partial_t u_{\varepsilon,\beta}dx-\frac{5a}{3\alpha}\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_t((\partial_x u_{\varepsilon,\beta})^2)dx
$$

$$
+\frac{d}{dt}\left(\frac{\varepsilon^2 G^2}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\varepsilon^2\beta H^2}{2}\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)
$$

$$
+\varepsilon G^2\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\varepsilon\beta H^2\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}
$$

$$
+\varepsilon\beta^2 G^2\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\varepsilon\beta^3 H^2\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}
$$

$$
+2\varepsilon\left\|\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}
$$

$$
=\frac{5a^2}{3\alpha}\int_{\mathbb{R}}u_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^3dx-a\varepsilon G^2\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta}dx
$$

$$
+a\varepsilon\beta H^2\int_{\mathbb{R}}u_{\varepsilon}\partial_x u_{\varepsilon,\beta}\partial_t\partial_x^2 u_{\varepsilon,\beta}dx-\varepsilon\alpha G^2\int_{\mathbb{R}}\partial_t u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}dx
$$

$$
+\varepsilon\beta\alpha H^2\int_{\mathbb{R}}\partial_t\partial_x^2 u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}dx-\varepsilon\alpha G^2\int_{\mathbb{R}}\partial_t u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}dx
$$

$$
-\varepsilon\beta\alpha H^2\int_{\mathbb{R}}\partial_t\partial_x^2 u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}dx+\frac{20a\beta^2}{3\alpha}\int_{\mathbb{R}}\partial_x u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}\partial_t\partial_x^3 u_{\varepsilon,\beta}dx
$$

$$
+\frac{10a\beta^2}{3\alpha}\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}\partial_t\partial_x^3 u_{\varepsilon,\beta}dx+\frac{10a\varepsilon}{3\alpha}\int_{\mathbb{R}}u_{\varepsilon,\beta}(\partial_x^2 u_{\varepsilon,\beta})^2dx,
$$

that is

$$
\frac{d}{dt}\left(\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-\frac{5a}{3\alpha}\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2 u_{\varepsilon,\beta}dx+\beta^2\left\|\partial_x^4 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)
$$

$$
+\frac{d}{dt}\left(\frac{\varepsilon^2 G^2}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\varepsilon^2\beta H^2}{2}\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)
$$

$$
+\varepsilon G^2\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\varepsilon\beta H^2\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}
$$

$$
+\varepsilon\beta^2 G^2\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\varepsilon\beta^3 H^2\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon\left\|\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}
$$

$$
=\frac{5a^2}{3\alpha}\int_{\mathbb{R}}u_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^3dx-a\varepsilon G^2\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta}dx
$$

$$+a\varepsilon\beta H^2\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_t\partial_x^2 u_{\varepsilon,\beta}dx-\varepsilon\alpha G^2\int_{\mathbb{R}}\partial_t u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}dx$$

$$+\varepsilon\beta\alpha H^2\int_{\mathbb{R}}\partial_t\partial_x^2 u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}dx-\varepsilon\alpha G^2\int_{\mathbb{R}}\partial_t u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}dx$$

$$-\varepsilon\beta\alpha H^2\int_{\mathbb{R}}\partial_t\partial_x^2 u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}dx+\frac{20a\beta^2}{3\alpha}\int_{\mathbb{R}}\partial_x u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}\partial_t\partial_x^3 u_{\varepsilon,\beta}dx$$

$$+\frac{10a\beta^2}{3\alpha}\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}\partial_t\partial_x^3 u_{\varepsilon,\beta}dx+\frac{10a\varepsilon}{3\alpha}\int_{\mathbb{R}} u_{\varepsilon,\beta}(\partial_x^2 u_{\varepsilon,\beta})^2\,dx. \tag{43}$$

Since $0<\varepsilon,\beta<1$, due to (17), (18), (9) and the Young inequality,

$$\begin{aligned}\left|\frac{5a^2}{3\alpha}\right|\int_{\mathbb{R}}|u_{\varepsilon,\beta}||\partial_x u_{\varepsilon,\beta}|^3\,dx &\le \left|\frac{5a^2}{3\alpha}\right|\left\|u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}|^3 dx\\
&\le C_0\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le C_0\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}\\
&\le C_0+\frac{1}{2}\left\|\partial_x u_{\varepsilon,\beta}\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\\
&\le C_0\left(1+\left\|\partial_x u_{\varepsilon,\beta}\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right),\end{aligned}$$

$$\begin{aligned}|a|\varepsilon G^2\int_{\mathbb{R}}|u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}||\partial_t u_{\varepsilon,\beta}|\,dx &= 2\varepsilon G^2\int_{\mathbb{R}}\left|\frac{\sqrt{3}au_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}}{2}\right|\left|\frac{\partial_t u_{\varepsilon,\beta}}{\sqrt{3}}\right|dx\\
&\le \frac{3\varepsilon a^2G^2}{4}\int_{\mathbb{R}} u^2_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^2\,dx+\frac{\varepsilon G^2}{3}\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{3\varepsilon a^2G^2}{4}\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\varepsilon G^2}{3}\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le C_0G^2+\frac{\varepsilon G^2}{3}\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\end{aligned}$$

$$\begin{aligned}|a|\varepsilon\beta H^2\int_{\mathbb{R}}|u_\varepsilon\partial_x u_\varepsilon||\partial_t\partial_x^2 u_{\varepsilon,\beta}|\,dx &= 2\varepsilon H^2\int_{\mathbb{R}}\left|\frac{\sqrt{3}au_\varepsilon\partial_x u_\varepsilon}{2}\right|\left|\frac{\beta_t\partial_x^2 u_{\varepsilon,\beta}}{\sqrt{3}}\right|\\
&\le \frac{\varepsilon 3a^2H^2}{4}\int_{\mathbb{R}} u^2_\varepsilon(\partial_x u_\varepsilon)^2\,dx+\frac{\varepsilon\beta^2H^2}{3}\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{\varepsilon 3a^2H^2}{4}\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\varepsilon\beta^2H^2}{3}\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\end{aligned}$$

$$\leq C_0 H^2 + \frac{\varepsilon\beta^2 H^2}{3}\left\|\partial_t \partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$\varepsilon|\alpha|G^2\int_{\mathbb{R}}\partial_t u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}dx \leq \frac{\varepsilon G^2}{2}\left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon\alpha^2 G^2}{2}\left\|\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$\varepsilon\beta|\alpha|H^2\int_{\mathbb{R}}|\partial_t\partial_x^2 u_{\varepsilon,\beta}||\partial_x^3 u_{\varepsilon,\beta}|dx = \varepsilon H^2\int_{\mathbb{R}}\left|\beta\partial_t\partial_x^2 u_{\varepsilon,\beta}\right|\left|\alpha\partial_x^3 u_{\varepsilon,\beta}\right|dx$$

$$\leq \frac{\varepsilon\beta^2 H^2}{2}\left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon\alpha^2 H^2}{2}\left\|\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$\left|\frac{20a\beta^2}{3\alpha}\right|\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}||\partial_t\partial_x^3 u_{\varepsilon,\beta}|dx$$

$$=\int_{\mathbb{R}}\left|\frac{20a\beta^{\frac{1}{2}}\partial_x u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}}{3\alpha H\sqrt{\varepsilon}}\right|\left|\beta^{\frac{3}{2}}\sqrt{\varepsilon}H\partial_t\partial_x^3 u_{\varepsilon,\beta}\right|dx$$

$$\leq \frac{200a^2\beta}{9\alpha^2 H^2\varepsilon}\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2(\partial_x^2 u_{\varepsilon,\beta})^2 dx + \frac{\varepsilon\beta^3 H^2}{2}\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq \frac{200a^2\beta}{9\alpha^2 H^2\varepsilon}\left\|\partial_x u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon\beta^3 H^2}{2}\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq \frac{200a^2 D^2\varepsilon^4}{9\alpha^2 H^2\varepsilon}\left\|\partial_x u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon\beta^3 H^2}{2}\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq \frac{200a^2 D^2\varepsilon^3}{9\alpha^2 H^2}\left\|\partial_x u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon\beta^3 H^2}{2}\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq \frac{200a^2 D^2\varepsilon}{9\alpha^2 H^2}\left\|\partial_x u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon\beta^3 H^2}{2}\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$\left|\frac{10a\beta^2}{3\alpha}\right|\int_{\mathbb{R}}|u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}||\partial_t\partial_x^3 u_{\varepsilon,\beta}|dx = 2\int_{\mathbb{R}}\left|\frac{5\sqrt{3}a\beta^{\frac{1}{2}}u_{\varepsilon,\beta}\partial_x^3 u_{\varepsilon,\beta}}{3\alpha H\sqrt{\varepsilon}}\right|\left|\frac{\beta^{\frac{3}{2}}\sqrt{\varepsilon}H\partial_t\partial_x^3 u_{\varepsilon,\beta}}{\sqrt{3}}\right|dx$$

$$\leq \frac{75a^2\beta}{9\alpha^2 H^2\varepsilon}\int_{\mathbb{R}}u^2_{\varepsilon,\beta}(\partial_x^3 u_{\varepsilon,\beta})^2 dx + \frac{\varepsilon\beta^3 H^2}{3}\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq \frac{75a^2\beta}{9\alpha^2 H^2\varepsilon}\left\|u_{\varepsilon,\beta}\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon\beta^3 H^2}{3}\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq \frac{C_0 D^2 \varepsilon^4}{H^2 \varepsilon} \left\|\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon\beta^3 H^2}{3} \left\|\partial_t \partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq \frac{C_0 D^2 \varepsilon^3}{H^2} \left\|\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon\beta^3 H^2}{3} \left\|\partial_t \partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq \frac{C_0 D^2 \varepsilon}{H^2} \left\|\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon\beta^3 H^2}{3} \left\|\partial_t \partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$\left|\frac{10a}{3\alpha}\right| \varepsilon \int_{\mathbb{R}} |u_{\varepsilon,\beta}| (\partial_x^2 u_{\varepsilon,\beta})^2 dx \leq \left|\frac{10a}{3\alpha}\right| \varepsilon \left\|u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})} \left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq C_0 \varepsilon \left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

It follows from (43) that

$$\frac{d}{dt}\left(\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \frac{5a}{3\alpha} \int_{\mathbb{R}} (\partial_x u_{\varepsilon,\beta})^2 u_{\varepsilon,\beta} dx + \beta^2 \left\|\partial_x^4 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \right)$$

$$+ \frac{d}{dt}\left(\frac{\varepsilon^2 G^2}{2} \left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon^2 \beta H^2}{2} \left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \right)$$

$$+ \frac{\varepsilon G^2}{6} \left\|\partial_t u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \varepsilon\beta H^2 \left\|\partial_t \partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+ \frac{\varepsilon\beta^2 G^2}{6} \left\|\partial_t \partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon\beta^3 H^2}{6} \left\|\partial_t \partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+ \left(2 - \frac{\alpha^2 G^2}{2} - \frac{\alpha^2 H^2}{2} - \frac{C_0 D^2}{H^2} \right) \varepsilon \left\|\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq C_0 \left(1 + \left\|\partial_x u_{\varepsilon,\beta}\right\|^2_{L^\infty((0,T)\times\mathbb{R})} \right) + C_0 G^2 + C_0 H^2$$

$$+ \frac{200 a^2 D^2 \varepsilon}{9\alpha^2 H^2} \left\|\partial_x u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0 \varepsilon \left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \tag{44}$$

We search G, H such that

$$2 - \frac{\alpha^2 G^2}{2} - \frac{\alpha^2 H^2}{2} - \frac{C_0 D^2}{H^2} > 0. \tag{45}$$

Choosing

$$G^2 = \frac{2}{\alpha^2} \tag{46}$$

(45) reads

$$1 - \frac{\alpha^2 H^2}{2} - \frac{C_0 D^2}{H^2} > 0.$$

Therefore, we search H such that

$$\alpha^2 H^4 - 2H^2 + 2C_0 D^2 < 0. \tag{47}$$

Thanks to (10), there exists $0 < H_1^2 < H_2^2$, such that choosing

$$H_1^2 < H^2 < H_2^2, \tag{48}$$

(47) is verified.

Therefore, by (10), (44), (46), (47) and (48), we have

$$\begin{aligned}
&\frac{d}{dt}\left(\left\| \partial_x^2 u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 - \frac{5a}{3\alpha}\int_{\mathbb{R}} (\partial_x u_{\varepsilon,\beta})^2 u_{\varepsilon,\beta} dx + \beta^2 \left\| \partial_x^4 u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\
&+ \frac{d}{dt}\left(\frac{\varepsilon^2 G^2}{2} \left\| \partial_x u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\varepsilon^2 \beta H^2}{2} \left\| \partial_x^2 u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\
&+ \frac{\varepsilon G^2}{6} \left\| \partial_t u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + \varepsilon\beta H^2 \left\| \partial_t \partial_x^2 u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&+ \frac{\varepsilon\beta^2 G^2}{6} \left\| \partial_t \partial_x^2 u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\varepsilon\beta^3 H^2}{6} \left\| \partial_t \partial_x^3 u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + K_3^2 \varepsilon \left\| \partial_x^3 u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&\leq C_0 \left(1 + \left\| \partial_x u_{\varepsilon,\beta} \right\|_{L^\infty((0,T)\times\mathbb{R})}^2 \right) + C_0 \varepsilon \left\| \partial_x^2 u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&+ \frac{200 a^2 D^2 \varepsilon}{9\alpha^2 H^2} \left\| \partial_x u_\varepsilon \right\|_{L^\infty((0,T)\times\mathbb{R})}^2 \left\| \partial_x^2 u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2,
\end{aligned} \tag{49}$$

where K_3^2 is an appropriate positive constant.

It follows from (5), (18) and an integration on $(0, t)$ that

$$\begin{aligned}
&\left\| \partial_x^2 u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 - \frac{5a}{3\alpha}\int_{\mathbb{R}} (\partial_x u_{\varepsilon,\beta})^2 u_{\varepsilon,\beta} dx + \beta^2 \left\| \partial_x^4 u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&+ \frac{\varepsilon^2 G^2}{2} \left\| \partial_x u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\varepsilon^2 \beta H^2}{2} \left\| \partial_x^2 u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&+ \frac{\varepsilon G^2}{6} \int_0^t \left\| \partial_t u_{\varepsilon,\beta}(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 ds + \varepsilon\beta H^2 \int_0^t \left\| \partial_t \partial_x^2 u_{\varepsilon,\beta}(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 ds
\end{aligned}$$

$$+\frac{\varepsilon\beta^2 G^2}{6}\int_0^t \left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(s,\cdot)\right\|_{L^2(\mathbb{R})}^2 ds+\frac{\varepsilon\beta^3 H^2}{6}\int_0^t \left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(s,\cdot)\right\|_{L^2(\mathbb{R})}^2 ds$$

$$+K_3^2\varepsilon\int_0^t \left\|\partial_x^3 u_{\varepsilon,\beta}(s,\cdot)\right\|_{L^2(\mathbb{R})}^2 ds$$

$$\leq C_0+C_0\left(1+\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}^2\right)t+C_0\varepsilon\int_0^t \left\|\partial_x^2 u_{\varepsilon,\beta}(s,\cdot)\right\|_{L^2(\mathbb{R})}^2 ds$$

$$+\frac{200a^2D^2\varepsilon}{9\alpha^2H^2}\left\|\partial_x u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})}^2\int_0^t \left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|_{L^2(\mathbb{R})}^2 ds$$

$$\leq C(T)\left(1+\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}^2\right).$$

Consequently, by (17) and (18), we get

$$\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+\beta^2\left\|\partial_x^4 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+\frac{\varepsilon^2G^2}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+\frac{\varepsilon^2\beta H^2}{2}\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2$$

$$+\frac{\varepsilon G^2}{6}\int_0^t \left\|\partial_t u_{\varepsilon,\beta}(s,\cdot)\right\|_{L^2(\mathbb{R})}^2 ds+\varepsilon\beta H^2\int_0^t \left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(s,\cdot)\right\|_{L^2(\mathbb{R})}^2 ds+\frac{\varepsilon\beta^2G^2}{6}\int_0^t \left\|\partial_t\partial_x^2 u_{\varepsilon,\beta}(s,\cdot)\right\|_{L^2(\mathbb{R})}^2 ds$$

$$+\frac{\varepsilon\beta^3H^2}{6}\int_0^t \left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(s,\cdot)\right\|_{L^2(\mathbb{R})}^2 ds+K_3^2\varepsilon\int_0^t \left\|\partial_x^3 u_{\varepsilon,\beta}(s,\cdot)\right\|_{L^2(\mathbb{R})}^2 ds$$

$$\leq C(T)\left(1+\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}^2\right)+\frac{5a}{3\alpha}\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2 u_{\varepsilon,\beta}dx$$

$$\leq C(T)\left(1+\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}^2\right)+\left|\frac{5a}{3\alpha}\right|\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2\,|u_{\varepsilon,\beta}|\,dx$$

$$\leq C(T)\left(1+\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}^2\right)+\left|\frac{5a}{3\alpha}\right|\left\|u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2$$

$$\leq C(T)\left(1+\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}^2\right). \tag{50}$$

We prove (37). Thanks to (16), (18) and (50),

$$(\partial_x u_{\varepsilon,\beta}(t,x))^2\leq C(T)\sqrt{\left(1+\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}^2\right)}.$$

Hence, we have

$$\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}^4-C(T)\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}^2-C(T)\leq 0,$$

which gives (37).

Finally, (37) and (50) give (38).

Lemma 2.4 Fix $T > 0$. There exists a constant $C(T) > 0$, independent on β and ε, such that

$$\frac{\beta^2}{6}\left\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^4\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq C(T), \tag{51}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Multiplying (2) by $-2\beta^2\partial_t\partial_x^2 u_{\varepsilon,\beta}$, we have that

$$\begin{aligned}&-2\beta^2\partial_t\partial_x^2 u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta} - 2\beta^4\partial_t\partial_x^2 u_{\varepsilon,\beta}\partial_t\partial_x^3 u_{\varepsilon,\beta}\\ &= 2\beta^2\partial_t\partial_x^2 u_{\varepsilon,\beta} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta} - 2\beta^2\alpha\partial_x^3 u_{\varepsilon,\beta}\partial_t\partial_x^2 u_{\varepsilon,\beta} - 2\varepsilon\beta^2\partial_x^2 u_{\varepsilon,\beta}\partial_t\partial_x^2 u_{\varepsilon,\beta}.\end{aligned} \tag{52}$$

Observe that

$$\begin{aligned}&-2\beta^2\partial_t\partial_x^2 u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta}dx - 2\beta^4\int_{\mathbb{R}}\partial_t\partial_x^2 u_{\varepsilon,\beta}\partial_t\partial_x^3 u_{\varepsilon,\beta}dx = 2\beta^2\left\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^4\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ &2\beta^2\int_{\mathbb{R}}\partial_t\partial_x^2 u_{\varepsilon,\beta} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx = -2\beta^2\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2\partial_t\partial_x u_{\varepsilon,\beta}dx - 2\beta^2\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}\partial_t\partial_x u_{\varepsilon,\beta}dx,\\ &-2\beta^2\alpha\int_{\mathbb{R}}\partial_x^3 u_{\varepsilon,\beta}\partial_t\partial_x^2 u_{\varepsilon,\beta}dx = 2\beta^2\alpha\int_{\mathbb{R}}\partial_x^4 u_{\varepsilon,\beta}\partial_t\partial_x u_{\varepsilon,\beta}dx,\\ &-2\varepsilon\beta^2\int_{\mathbb{R}}\partial_x^2 u_{\varepsilon,\beta}\partial_t\partial_x^2 u_{\varepsilon,\beta}dx = 2\varepsilon\beta^2\int_{\mathbb{R}}\partial_x u_{\varepsilon,\beta}\partial_t\partial_x^3 u_{\varepsilon,\beta}dx.\end{aligned} \tag{53}$$

Consequently, an integration on $\mathbb{R}$ of (52) gives

$$\begin{aligned}&2\beta^2\left\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^4\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &-2\beta^2\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2\partial_t\partial_x u_{\varepsilon,\beta}dx - 2\beta^2\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}\partial_t\partial_x u_{\varepsilon,\beta}dx\\ &+2\beta^2\alpha\int_{\mathbb{R}}\partial_x^4 u_{\varepsilon,\beta}\partial_t\partial_x u_{\varepsilon,\beta}dx + 2\varepsilon\beta^2\int_{\mathbb{R}}\partial_x u_{\varepsilon,\beta}\partial_t\partial_x^3 u_{\varepsilon,\beta}dx.\end{aligned} \tag{54}$$

Since $0 < \varepsilon, \beta < 1$, thanks to (11), (17), (18), (37), (38) and the Young inequality,

$$\begin{aligned}2\beta^2\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2\,|\,\partial_t\partial_x u_{\varepsilon,\beta}\,|\,dx &\leq \beta^2\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4dx + \beta^2\left\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\leq \left\|\partial_x u_{\varepsilon,\beta}\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2\left\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\leq C(T) + \beta^2\left\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\end{aligned}$$

$$\begin{aligned}2\beta^2\int_{\mathbb{R}}|\,u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}\,||\,\partial_t\partial_x u_{\varepsilon,\beta}\,|\,dx &= \beta^2\int_{\mathbb{R}}|\,2u_{\varepsilon,\beta}\partial_x^2 u_{\varepsilon,\beta}\,||\,\partial_t\partial_x u_{\varepsilon,\beta}\,|\,dx\\ &\leq 2\beta^2\int_{\mathbb{R}} u^2_{\varepsilon,\beta}(\partial_x^2 u_{\varepsilon,\beta})^2\,dx + \frac{\beta^2}{2}\left\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\end{aligned}$$

$$\leq 2\beta^2 \|u_{\varepsilon,\beta}\|^2_{L^\infty((0,T)\times\mathbb{R})} \|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

$$\leq C_0 + \frac{\beta^2}{2}\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$2\beta^2 |\alpha| \int_{\mathbb{R}} |\partial_x^4 u_{\varepsilon,\beta}||\partial_t\partial_x u_{\varepsilon,\beta}| dx = 2\beta^2 \int_{\mathbb{R}} \left|\sqrt{3}\alpha\partial_x^4 u_{\varepsilon,\beta}\right| \left|\frac{\partial_t\partial_x u_{\varepsilon,\beta}}{\sqrt{3}}\right| dx$$

$$\leq 3\alpha^2\beta^2 \|\partial_x^4 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{3}\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

$$\leq C(T) + \frac{\beta^2}{3}\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$2\varepsilon\beta^2 \int_{\mathbb{R}} |\partial_x u_{\varepsilon,\beta}||\partial_t\partial_x^3 u_{\varepsilon,\beta}| dx \leq \varepsilon^2 \|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^4 \|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

$$\leq \|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^4 \|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

$$\leq C_0 + \beta^4 \|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

It follows from (54) that

$$\frac{\beta^2}{6}\|\partial_t\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^4 \|\partial_t\partial_x^3 u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq C(T),$$

which gives (51).

3. Proof of theorem 1.1

In this section, we prove Theorem 1.1. We begin by proving the following result.

Lemma 3.1 Fix $T > 0$ and assume (6),

the sequence $\{u_{\varepsilon,\beta}\}_{\varepsilon,\beta>0}$ is compact in $L^2_{loc}((0,\infty)\times\mathbb{R})$. (55)

Consequently, there exists a subsequence $\{u_{\varepsilon_k,\beta_k}\}_{k\in\mathbb{N}}$ of $\{u_{\varepsilon,\beta}\}_{\varepsilon,\beta>0}$ and $u \in L^2_{loc}((0,\infty)\times\mathbb{R})$ such that, for each compact subset K of $(0,\infty)\times\mathbb{R}$,

$u_{\varepsilon_k,\beta_k} \to u$ in $L^2(K)$ and a.e., (56)

Moreover, u is a solution of (1) and (7) holds.

Proof. We begin by proving (55). To prove (55), we rely on the Aubin-Lions Lemma (see [27-30]). We recall that

$$H^1_{loc}(\mathbb{R}) \hookrightarrow\hookrightarrow L^2_{loc}(\mathbb{R}) \hookrightarrow H^{-1}_{loc}(\mathbb{R}),$$

where the first inclusion is compact and the second is continuous. Owing to the Aubin-Lions Lemma [30], to prove (55), it suffices to show that

$\{u_{\varepsilon,\beta}\}_{\varepsilon,\beta>0}$ is uniformly bounded in $L^2(0, T; H^1_{loc}(\mathbb{R}))$, (57)

$\{\partial_t u_{\varepsilon,\beta}\}_{\varepsilon,\beta>0}$ is uniformly bounded in $L^2(0, T; H^{-1}_{loc}(\mathbb{R})$. (58)

We prove (57). Thanks to (6) and Lemmas 2.1, 2.2 and 2.3,

$$\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{H^2(\mathbb{R})} = \left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T).$$

Therefore,

$\{u_{\varepsilon,\beta}\}_{\varepsilon,\beta>0}$ is uniformly bounded in $L^\infty(0, T; H^2(\mathbb{R}))$,

which gives (57).

We prove (58). We begin by observing that, by (4),

$$\partial_t u_{\varepsilon,\beta} = \partial_x\left(-\frac{a}{2}u^2_{\varepsilon,\beta} - \alpha\partial_x^2 u_{\varepsilon,\beta} - \beta^2\partial_t\partial_x^3 u_{\varepsilon,\beta} + \varepsilon\partial_x u_{\varepsilon,\beta}\right). \tag{59}$$

We have that

$$\frac{a^2}{4}\left\|u^2_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbb{R})} \le C(T). \tag{60}$$

Thanks to (6) and Lemmas 2.1 and 2.2,

$$\frac{a^2}{4}\int_0^T\int_{\mathbb{R}} u^4_{\varepsilon,\beta}dtdx \le \frac{a^2}{4}\left\|u_{\varepsilon,\beta}\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^T\int_{\mathbb{R}} u^2_{\varepsilon,\beta}dtdx$$

$$\le C(T)\int_0^T\int_{\mathbb{R}} u^2_{\varepsilon,\beta}dtdx \le C(T).$$

Observe that, since $0 < \varepsilon < 1$, thanks to (6) and Lemmas 2.1, 2.3 and 2.41

$$\varepsilon^2\left\|\partial_x u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbb{R})}, \alpha^2\left\|\partial_x^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2((0,T)\times\mathbb{R})}, \beta^4\left\|\partial_t\partial_x^3 u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbb{R})} \le C(T). \tag{61}$$

Therefore, by (60) and (61),

$$\left\{\partial_x\left(-\frac{a}{2}u^2_{\varepsilon,\beta} - \alpha\partial_x^2 u_{\varepsilon,\beta} - \beta^2\partial_t\partial_x^3 u_{\varepsilon,\beta} + \varepsilon\partial_x u_{\varepsilon,\beta}\right)\right\}_{\varepsilon,\beta>0} \text{ is bounded in } H^1((0,T)\times\mathbb{R}).$$

Thanks to the Aubin-Lions Lemma, (55) and (56) hold.

Consequently, u is solution of (1) and (7) holds.

Following [32, Theorem 1.1], we prove Theorem 1.1.

Proof of Theorem 1.1 Lemma 3.1 gives the existence of a solution of (1) satisfying (7).

Let u_1 and u_2 be two solutions of (1), which verify (7), that is

$$\begin{cases} \partial_t u_1 + a u_1 \partial_x u_1 + \alpha \partial_x^3 u_1 = 0, & t > 0, x \in \mathbb{R}, \\ u_1(0, x) = u_{1,0}(x), & x \in \mathbb{R}, \end{cases}$$

$$\begin{cases} \partial_t u_2 + a u_2 \partial_x u_2 + \alpha \partial_x^3 u_2 = 0, & t > 0, x \in \mathbb{R}, \\ u_2(0, x) = u_{2,0}(x), & x \in \mathbb{R}. \end{cases}$$

Then, the function

$$\omega = u_1 - u_2 \tag{62}$$

is the solution of the following Cauchy problem:

$$\begin{cases} \partial_t \omega + a(u_1 \partial_x u_1 - u_2 \partial_x u_2) + \alpha \partial_x^3 \omega = 0, & t > 0, x \in \mathbb{R}, \\ \omega(0, x) = u_{1,0}(x) - u_{2,0}(x), & x \in \mathbb{R}. \end{cases} \tag{63}$$

Observe that, thanks to (62)

$$\begin{aligned} u_1 \partial_x u_1 - u_2 \partial_x u_2 &= u_1 \partial_x u_1 - u_1 \partial_x u_2 + u_1 \partial_x u_2 - u_2 \partial_x u_2 \\ &= u_1 \partial_x \omega + \partial_x u_2 \omega, \end{aligned} \tag{64}$$

$$\partial_t \omega = -a u_1 \partial_x \omega - a \partial_x u_2 \omega - \alpha \partial_x^3 \omega. \tag{65}$$

Multiplying (65) by 2ω, an integration on $\mathbb{R}$ gives

$$\begin{aligned} \frac{d}{dt} \|\omega(t, \cdot)\|^2_{L^2(\mathbb{R})} &= 2 \int_{\mathbb{R}} \omega \partial_t \omega dx \\ &= -2a \int_{\mathbb{R}} u_1 \omega \partial_x \omega dx - 2a \int_{\mathbb{R}} \partial_x u_2 \omega^2 dx - 2\alpha \int_{\mathbb{R}} \omega \partial_x^3 \omega dx \\ &= a \int_{\mathbb{R}} \partial_x u_1 \omega^2 dx - 2a \int_{\mathbb{R}} \partial_x u_2 \omega^2 dx + 2\alpha \int_{\mathbb{R}} \partial_x \omega \partial_x^2 \omega dx \\ &\leq |a| \|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})} \|\omega(t, \cdot)\|^2_{L^2(\mathbb{R})} + 2|a| \|\partial_x u_2\|_{L^\infty((0,T)\times\mathbb{R})} \|\omega(t, \cdot)\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{66}$$

Fix $T > 0$. Observe that, since $u_1, u_2 \in H^2(\mathbb{R})$, for every $0 \leq t \leq T$, we have

$$\|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})}, \|\partial_x u_2\|_{L^\infty((0,T)\times\mathbb{R})} \leq C(T). \tag{67}$$

Therefore, by (66) and (67),

$$\frac{d}{dt} \|\omega(t, \cdot)\|^2_{L^2(\mathbb{R})} \leq C(T) \|\omega(t, \cdot)\|^2_{L^2(\mathbb{R})}.$$

The Gronwall Lemma and (62) give (8).

4. Conclusion

By proving several a priori estimates and using the Aubin-Lions Lemma, we show the convergence of the solution of (2) converges to the unique one of (1) as the coefficient β goes to 0.

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