Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Bi-Skew Lie (Jordan) Product on Factor Von Neumann Algebras

Mohd Arif Raza[1*], Huda Eid Almehmadi[2]

[1] Department of Mathematics, College of Science & Arts-Rabigh, King Abdulaziz University, Jeddah, Saudi Arabia
[2] Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
E-mail: arifraza03@gmail.com

Received: 26 May 2024; **Revised:** 9 August 2024; **Accepted:** 12 August 2024

Abstract: In this manuscript, we characterize the bijective mapping Λ: $\mathscr{U} \to \mathscr{V}$ between two von Neumann algebras $\mathscr{U}$ and $\mathscr{V}$ with dim $\geqslant 1$. The mapping Λ is to satisfy the following condition on the mixed bi-skew Lie (Jordan) product $\Lambda([\mathscr{D}, \mathscr{E}]_\bullet \diamond \mathscr{G}) = [\Lambda(\mathscr{D}), \Lambda(\mathscr{E})]_\bullet \diamond \Lambda(\mathscr{G})$, for all $\mathscr{D}$, $\mathscr{E}$, $\mathscr{G} \in \mathscr{U}$, where the bi-skew Lie (Jordan) product is defined as $[\mathscr{D}, \mathscr{E}]_\bullet = \mathscr{D}\mathscr{E}^* - \mathscr{E}\mathscr{D}^*$, and $\mathscr{D} \diamond \mathscr{E} = \mathscr{D}\mathscr{E}^* + \mathscr{E}\mathscr{D}^*$. Specifically, we elaborate on the properties of the mapping Λ, establishing that it is additive, a $*$-somorphism, and linear.

Keywords: von Neumann algebra, bi-skew Lie (Jordan) product, $*$-isomorphism

MSC: 46J10, 47B48, 46L10

1. Brief historical development and motivation

The main objective of this research is to investigate maps that preserve the bi-skew Lie (Jordan) mixed product on factor von Neumann algebras. More specifically, we aim to study maps between factor von Neumann algebras that satisfy certain conditions related to the bi-skew Lie (Jordan) mixed product. These maps, known as bi-skew Lie (Jordan) isomorphisms, provide a way to preserve the algebraic structure of factor von Neumann algebras under the bi-skew Lie (Jordan) mixed product. These products plays a crucial role in the theory of factor von Neumann algebras, $*$-algebras, operator algebra, C^*-algebras, etc. Understanding the properties of this product is essential for developing a deeper understanding of factor von Neumann algebras and their applications. The study of maps preserving the bi-skew Lie(Jordan) product on factor von Neumann algebras is a relatively recent development in operator algebra, von Neumann algebras, $*$-algebras, C^*-algebras, etc. The bi-skew Lie (Jordan) product is considered a generalization of the skew Lie (Jordan) product, which was originally studied by [1–5] in the context of von Neumann algebra, operator algebra, C^*-algebras, etc.

A map Λ: $\mathsf{R} \to \mathsf{R}'$ preserves product or is multiplicative if $\Lambda(\mathscr{D}\mathscr{E}) = \Lambda(\mathscr{D})\,\Lambda(\mathscr{E})$, for every $\mathscr{D}, \mathscr{E} \in \mathsf{R}$, where R and R' are rings. One of the most important fields of research is the study of non-linear maps. In [6], Martindale discussed how multiplicative mapping turns to additive. This has prompted various researchers to focus on developing mappings on rings or algebras that preserve the Jordan product. $\mathscr{D} \circ \mathscr{E} = \mathscr{D}\mathscr{E} + \mathscr{E}\mathscr{D}$ (for instance, see [7–12]) or $[\mathscr{D}, \mathscr{E}] = \mathscr{D}\mathscr{E} - \mathscr{E}\mathscr{D}$ (for instance, see [13]). The foregoing observations demonstrate that the Jordan/Lie product combination is suitable for determining the algebraic structure.

DOI: https://doi.org/10.37256/cm.5420245028

In recent times, a great deal of effort has been put into investigating skew products, such as $[\mathscr{D},\mathscr{E}]_\bullet = \mathscr{D}\mathscr{E}^* - \mathscr{E}\mathscr{D}^*$, and $\mathscr{D}\diamond\mathscr{E} = \mathscr{D}\mathscr{E}^* + \mathscr{E}\mathscr{D}^*$, for all $\mathscr{D},\mathscr{E}\in \mathrm{R}$, where R is a $*$-ring (see, for example, [14–21]). An obvious objective is to find out if the mapping that preserves the new products on rings or algebras is additive. Cui and Li studied the nonlinear bijective mapping on factor von Neumann algebras in [22], preserving the existence of the skew Lie product $[\mathscr{D},\mathscr{E}]_* = \mathscr{D}\mathscr{E} - \mathscr{E}\mathscr{D}^*$. They proved that this mapping is in fact an isomorphism of the $*$-ring. The nonlinear bijective mapping that preserves the skew Jordan product $\mathscr{D}\diamond\mathscr{E} = \mathscr{D}\mathscr{E} + \mathscr{E}\mathscr{D}^*$ was discussed by Li et al. in [16]. Indeed, they proved that the previously described mapping on factor von Neumann algebras is also an isomorphism between $*$- rings. Furthermore, they explicitly showed that is either a unitary isomorphism or a conjugate unitary isomorphism if type $\mathscr{I}$ factors exist for the von Neumann algebras $\mathscr{U}$ and $\mathscr{V}$. These investigations produced a plethora of distinct approaches to various algebraic structures such as operator algebras, von Neumann algebras, Banach algebras, and so on, yielding a range of intriguing conclusions about nonadditive mappings (see [6, 8, 9, 13, 15, 22–24]) and its references include a variety of significantly different approaches for dealing with nonlinear maps. Motivated by prior works, the key objective of this article is to elaborate the characteristics of the map Λ such as additivity, $*$-isomorphism, and linearity, where Λ satisfies $\Lambda\colon \mathscr{U}\to\mathscr{V}$ $\Lambda([\mathscr{D},\mathscr{E}]_\bullet\diamond\mathscr{G}) = [\Lambda(\mathscr{D}),\Lambda(\mathscr{E})]_\bullet\diamond\Lambda(\mathscr{G})$ for all $\mathscr{D},\mathscr{E},\mathscr{G}\in\mathscr{U}$. We shall present various constructive proofs for clarification on the properties of the map Λ.

2. Key content and results

Just before providing the appropriate results, let's keep in mind that a von Neumann algebra $\mathscr{U}$ on a Hilbert space $\mathscr{H}$ containing the identity operator $\mathscr{I}$, is a weakly closed, self-adjoint algebra of operators. The set $\mathfrak{X}_{\mathscr{U}} = \{\mathscr{B}\in\mathscr{U} \mid \mathscr{B}\mathscr{D} = \mathscr{D}\mathscr{B}$ for all $\mathscr{D}\in\mathscr{U}\}$ is known for the center of $\mathscr{U}$. The intersection of all central projections $\mathscr{G}$ such that $\mathscr{G}\mathscr{U}=\mathscr{U}$ represents the central carrier of $\mathscr{D}$, for $\mathscr{D}\in\mathscr{U}$ and mentioned by $\overline{\mathscr{U}}$. We indicate the core of $\mathscr{U}$ as $\underline{\mathscr{U}} = \sup\{\mathscr{S}\in\mathfrak{X}_{\mathscr{U}} \mid \mathscr{S}=\mathscr{U}^*, \mathscr{U}\leq\mathscr{D}\}$ for any self-adjoint operator $\mathscr{D}\in\mathscr{U}$. $\mathscr{G}$ is referred to as a core-free projection if $\mathscr{G}$ is a projection and $\mathscr{G}=0$. One obviously has $\mathscr{D}-\underline{\mathscr{D}}\geq\mathscr{S}\geq 0$. Additionally, if $\mathscr{S}\in\mathfrak{X}_{\mathscr{U}}$ and $\mathscr{D}-\underline{\mathscr{D}}\geq\mathscr{S}\geq 0$, then $\mathscr{S}=0$. It is obvious that $\underline{\mathscr{G}}$ is the greatest central projection $\leq\mathscr{G}$, if $\mathscr{G}$ is a projection. In this case, $\overline{\mathscr{I}-\mathscr{G}}$ stands for the central carrier of $\mathscr{I}-\mathscr{G}$, and it is simple to demonstrate that $\underline{\mathscr{G}}=0$ if and only if $\overline{\mathscr{I}-\mathscr{G}}=\mathscr{I}$. The argument in support of our theorem necessitates specific lemmas/facts, and we will regularly employ numerous essential characteristics of von Neumann algebra.

3. Additivity of mappings that preserve the bi-skew Lie (Jordan) product

The additivity property of maps has significant implications for the structure of von Neumann algebras. A bi-skew Lie (Jordan) isomorphism, which preserves the bi-skew Lie (Jordan) product and is additive, establishes an isomorphism between the original von Neumann algebra and another von Neumann algebra, preserving the bi-skew Lie (Jordan) product and the additivity of maps.

Theorem 1 Let $\mathscr{U}$ and $\mathscr{V}$ are two factor von Neumann algebras with dim $\geqslant 1$. If $\Lambda\colon\mathscr{U}\to\mathscr{V}$ is an unital bijective map satisfies

$$\Lambda([\mathscr{D},\mathscr{E}]_\bullet\diamond\mathscr{G}) = [\Lambda(\mathscr{D}),\Lambda(\mathscr{E})]_\bullet\diamond\Lambda(\mathscr{G}), \tag{1}$$

for all $\mathscr{D},\mathscr{E},\mathscr{G}\in\mathscr{U}$, where $[\mathscr{D},\mathscr{E}]_\bullet = \mathscr{D}\mathscr{E}^*-\mathscr{E}\mathscr{D}^*$ and $\mathscr{D}\diamond\mathscr{E} = \mathscr{D}\mathscr{E}^*+\mathscr{E}\mathscr{D}^*$, then Λ is additive.

For the establishment of the above theorem, we need to employ numerous lemmas.

Lemma 1 If $\Lambda\colon\mathscr{U}\to\mathscr{V}$ preserves the bi-skew products on $\mathscr{U}$ into $\mathscr{V}$, then $\Lambda(0)=0$.

Proof. Since we have $\Lambda([\mathscr{D},\mathscr{E}]_\bullet\diamond\mathscr{G}) = [\Lambda(\mathscr{D}),\Lambda(\mathscr{E})]_\bullet\diamond\Lambda(\mathscr{G})$, for all $\mathscr{D},\mathscr{E},\mathscr{G}\in\mathscr{U}$. Suppose $\mathscr{D}=\mathscr{E}=1_{\mathscr{U}}$, then we get $\Lambda(0)=\Lambda([I,I]_\bullet\diamond\mathscr{G})$. Since Λ is unital, therefore we have $\Lambda(0)=[I,I]_\bullet\diamond\Lambda(\mathscr{G}) = 0\Lambda(\mathscr{G})^*+\Lambda(\mathscr{G})0=0$. Hence $\Lambda(0)=0$. □

Lemma 2 If Λ: $\mathscr{U} \to \mathscr{V}$ preserves the bi-skew products on $\mathscr{U}$, $\mathscr{V}$ and suppose $\Lambda(T) = \Lambda(\mathscr{D}) + \Lambda(\mathscr{E})$, for every $T, \mathscr{D}, \mathscr{E} \in \mathscr{U}$, then

$$\Lambda([X_1, T]_\bullet \diamond X_2) = \Lambda([X_1, \mathscr{D}]_\bullet \diamond X_2) + \Lambda([X_1, \mathscr{E}]_\bullet \diamond X_2), \text{ for all } X_1, X_2 \in \mathscr{U}.$$

Proof. In view of (1) and for any $X_1, X_2, T \in \mathscr{U}$, we have

$$\Lambda([X_1, T]_\bullet \diamond X_2) = [\Lambda(X_1), \Lambda(T)]_\bullet \diamond \Lambda(X_2).$$

By using $\Lambda(T) = \Lambda(\mathscr{D}) + \Lambda(\mathscr{E})$ in the above equation, we see that

$$[\Lambda(X_1), \Lambda(\mathscr{D}) + \Lambda(\mathscr{E})]_\bullet \diamond \Lambda(X_2) = [\Lambda(X_1), \Lambda(\mathscr{D})]_\bullet \diamond \Lambda(X_2) + [\Lambda(X_1), \Lambda(\mathscr{E})]_\bullet \diamond \Lambda(X_2).$$

Therefore, we get

$$\Lambda([X_1, T]_\bullet \diamond X_2) = \Lambda([X_1, \mathscr{D}]_\bullet \diamond X_2) + \Lambda([X_1, \mathscr{D}]_\bullet \diamond X_2).$$

□

Suppose there is a nontrivial projection $P_1 \in \mathscr{U}$ and $P_2 = 1 - P_1$. Let $\mathscr{U}_{kl} = P_k \mathscr{U} P_l$ where $k, l \in \{1, 2\}$. Therefore, we get an algebra $\mathscr{U} = \sum_{k, l=1}^{2} \mathscr{U}_{kl}$. We can write $\mathscr{D} = \mathscr{D}_{11} + \mathscr{D}_{12} + \mathscr{D}_{21} + \mathscr{D}_{22}$ for all $\mathscr{D} \in \mathscr{U}$. If we write $\mathscr{D}_{kl}$, it mean that $\mathscr{D}_{kl} \in \mathscr{U}_{kl}$.

Lemma 3 If Λ: $\mathscr{U} \to \mathscr{V}$ preserves the bi-skew products on $\mathscr{U}$ into $\mathscr{V}$, then

$$\Lambda(\mathscr{D}_{11} + \mathscr{E}_{12}) = \Lambda(\mathscr{D}_{11}) + \Lambda(\mathscr{E}_{12}), \text{ for all } \mathscr{D}_{11} \in \mathscr{U}_{11}, \mathscr{E}_{12} \in \mathscr{U}_{12}.$$

Proof. By the surjectivity of Λ, we find an element $T = T_{11} + T_{12} + T_{21} + T_{22}$ such that

$$\Lambda(T) = \Lambda(\mathscr{D}_{11}) + \Lambda(\mathscr{E}_{12}) \tag{2}$$

We need to show $T = \mathscr{D}_{11} + \mathscr{E}_{12}$.

By using Lemma 2 on equation (2) for $X_1 = P_1$ and $X_2 = P_2$, then

$$\Lambda([P_1, T]_\bullet \diamond P_2) = \Lambda([P_1, \mathscr{D}_{11}]_\bullet \diamond P_2) + \Lambda([P_1, \mathscr{E}_{12}]_\bullet \diamond P_2).$$

So,

$$\Lambda([P_1, T_{11} + T_{12} + T_{21} + T_{22}]_\bullet \diamond P_2) = \Lambda((\mathscr{D}_{11}^* - \mathscr{D}_{11}) \diamond P_2) + \Lambda(0 \diamond P_2).$$

Thus,

$$\Lambda((T_{11}^* + T_{21}^* - T_{11} - T_{21}) \diamond P_2) = 0.$$

Hence, we get

$$\Lambda(T_{21}^* + T_{21}) = 0.$$

Since Λ is an injective map, then we have

$$T_{21}^* + T_{21} = 0.$$

We obtain $T_{21} = 0$, by multiplying the previous equation from the left side by P_2.
Also, if we apply Lemma 2 on equation (2) for $X_1 = P_2$ and $X_2 = P_1$, then we get

$$\Lambda([P_2, T]_\bullet \diamond P_1) = \Lambda([P_2, \mathscr{D}_{11}]_\bullet \diamond P_1) + \Lambda([P_2, \mathscr{E}_{12}]_\bullet \diamond P_1).$$

Thus, we have

$$\Lambda(T_{12}^* + T_{12}) = \Lambda(\mathscr{E}_{12}^* + \mathscr{E}_{12}).$$

Since Λ is an injective map, we get

$$T_{12}^* + T_{12} = \mathscr{E}_{12}^* + \mathscr{E}_{12}.$$

Upon multiplying the previous equation by P_1 from the left side, we obtain $T_{12} = \mathscr{E}_{12}$.
Again, if we use Lemma 2 on equation (2) for $X_1 = X_{12}$ and $X_2 = P_2$, hence we get

$$\Lambda([X_{12}, T]_\bullet \diamond P_2) = \Lambda([X_{12}, \mathscr{D}_{11}]_\bullet \diamond P_2) + \Lambda([X_{12}, \mathscr{E}_{12}]_\bullet \diamond P_2).$$

Therefore, we get

$$\Lambda(X_{12}T_{22}^* + T_{22}X_{12}^*) = 0.$$

By the injectivity of Λ, then we have

$$X_{12}T_{22}^* + T_{22}X_{12}^* = 0.$$

Upon multiplying the previous equation by P_1 from the left side, we obtain $X_{12}T_{22}^* = 0$, for all $X_{12} \in \mathscr{U}_{12}$. Since $\mathscr{U}$ prime, so $T_{22} = 0$.

Finally, if we use Lemma 2 on equation (2) $X_1 = X_{21}$ and $X_2 = P_1$, then we get

$$\Lambda([X_{21}, T]_\bullet \diamond P_1) = \Lambda([X_{21}, \mathscr{D}_{11}]_\bullet \diamond P_1) + \Lambda([X_{21}, \mathscr{E}_{12}]_\bullet \diamond P_1).$$

Hence,

$$\Lambda(X_{21}T_{11}^* + T_{11}X_{21}^*) = \Lambda(X_{21}\mathscr{D}_{11}^* + \mathscr{D}_{11}X_{21}^*).$$

Since Λ is an injective map, then

$$X_{21}T_{11}^* + T_{11}X_{21}^* = X_{21}\mathscr{D}_{11}^* + \mathscr{D}_{11}X_{21}^*.$$

Upon multiplying the previous equation by P_2 from the left side, we obtain $X_{21}T_{11}^* = X_{21}\mathscr{D}_{11}^*$, for all $X_{21} \in \mathscr{U}_{21}$. So, we obtain $T_{11} = \mathscr{D}_{11}$. Therefore $T = \mathscr{D}_{11} + \mathscr{E}_{12}$. □

Lemma 4 If Λ: $\mathscr{U} \to \mathscr{V}$ preserves the bi-skew products on $\mathscr{U}$ into $\mathscr{V}$, then we have

$$\Lambda(\mathscr{D}_{11} + \mathscr{E}_{12} + \mathscr{G}_{21}) = \Lambda(\mathscr{D}_{11}) + \Lambda(\mathscr{E}_{12}) + \Lambda(\mathscr{G}_{21}),$$

for all $\mathscr{D}_{11} \in \mathscr{U}_{11}$, $\mathscr{E}_{12} \in \mathscr{U}_{12}$ and $\mathscr{G}_{21} \in \mathscr{U}_{21}$.

Proof. By the surjective map of Λ, we have $T = T_{11} + T_{12} + T_{21} + T_{22}$ such that

$$\Lambda(T) = \Lambda(\mathscr{D}_{11}) + \Lambda(\mathscr{E}_{12}) + \Lambda(\mathscr{G}_{21}). \tag{3}$$

We need to show $T = \mathscr{D}_{11} + \mathscr{E}_{12} + \Lambda(\mathscr{G}_{21})$. By using Lemma 2 on equation (3) for $X_1 = P_1$ and $X_1 = P_2$, we have

$$\Lambda([P_1, T]_\bullet \diamond P_2) = \Lambda([P_1, \mathscr{D}_{11}]_\bullet \diamond P_2) + \Lambda([P_1, \mathscr{E}_{12}]_\bullet \diamond P_2) + \Lambda([P_1, \mathscr{G}_{21}]_\bullet \diamond P_2).$$

We have

$$\Lambda(T_{21}^* + T_{21}) = \Lambda(\mathscr{G}_{21}^* + \mathscr{G}_{21}).$$

Since Λ is an injective map, then we have

$$T_{21}^* + T_{21} = \mathscr{G}_{21}^* + \mathscr{G}_{21}.$$

So we gat $T_{21}=0$. Similarly, if we use Lemma 2 on equation (3) for $X_1=P_2$ and $X_1=P_1$, thus we get

$$\Lambda(T_{12}^*+T_{12})=\Lambda(\mathscr{E}_{12}^*+\mathscr{E}_{12}).$$

By the injective of Λ, we obtain

$$T_{12}^*+T_{12}=\mathscr{E}_{12}^*+\mathscr{E}_{12}.$$

Upon multiplying the previous equation by P_1 from the left side, we obtain $T_{12}=\mathscr{E}_{12}$.
Also, if we use Lemma 2 on equation (3) for $X_1=X_{12}$ and $X_2=P_2$, then we get

$$\Lambda([X_{12},T]_\bullet\diamond P_2)=\Lambda([X_{12},\mathscr{D}_{11}]_\bullet\diamond P_2)+\Lambda([X_{12},\mathscr{E}_{12}]_\bullet\diamond P_2)+\Lambda([X_{12},\mathscr{G}_{21}]_\bullet\diamond P_2).$$

So, we get

$$\Lambda(X_{12}T_{22}^*+T_{22}X_{12}^*)=0.$$

Since Λ is an injective map, then we find

$$X_{12}T_{22}^*+T_{22}X_{12}^*=0.$$

Upon multiplying the previous equation by P_1 from the left side, we obtain $X_{12}T_{22}^*=0$, for all $X_{12}\in\mathscr{U}_{12}$. Since $\mathscr{U}$ is prime. Then, we get $T_{22}=0$.

Finally, by using Lemma 2 on equation (3) $X_1=X_{21}$ and $X_2=P_1$, as similar proof above shows $T_{11}=\mathscr{D}_{11}$. So, $T=\mathscr{D}_{11}+\mathscr{E}_{12}+\mathscr{G}_{21}$. □

Lemma 5 If Λ: $\mathscr{U}\to\mathscr{V}$ preserves the bi-skew products on $\mathscr{U}$ into $\mathscr{V}$, then we have

$$\Lambda(\mathscr{D}_{11}+\mathscr{E}_{12}+\mathscr{G}_{21}+\mathscr{D}_{22})=\Lambda(\mathscr{D}_{11})+\Lambda(\mathscr{E}_{12})+\Lambda(\mathscr{G}_{21})+\Lambda(\mathscr{D}_{22}),$$

for all $\mathscr{D}_{11}\in\mathscr{U}_{11}$, $\mathscr{E}_{12}\in\mathscr{U}_{12}$, $\mathscr{G}_{21}\in\mathscr{U}_{21}$ and $\mathscr{D}_{22}\in\mathscr{U}_{22}$.

Proof. Assume $T=T_{11}+T_{12}+T_{21}+T_{22}$ such that

$$\Lambda(T)=\Lambda(\mathscr{D}_{11})+\Lambda(\mathscr{E}_{12})+\Lambda(\mathscr{G}_{21})+\Lambda(\mathscr{D}_{22}) \tag{4}$$

By applying Lemma 2 on equation (4) for $X_1=P_1$ and $X_2=P_2$, and the similar proof as in Lemma 4, we obtain $T_{21}=0$.

Similarly, if we use Lemma 2 on equation (4) for $X_1=P_2$ and $X_2=P_1$, thus we get $T_{12}=\mathscr{E}_{12}$. Now, if we apply Lemma 2 on equation (4) for $X_1=X_{12}$ and $X_2=P_2$, hence we have

$$\Lambda([X_{12}, T]_\bullet \diamond P_2) = \Lambda([X_{12}, \mathcal{D}_{11}]_\bullet \diamond P_2) + \Lambda([X_{12}, \mathcal{E}_{12}]_\bullet \diamond P_2)$$

$$+ \Lambda([X_{12}, \mathcal{G}_{21}]_\bullet \diamond P_2) + \Lambda([X_{12}, \mathcal{D}_{22}]_\bullet \diamond P_2).$$

Therefore, we have

$$\Lambda(X_{12}T_{22}^* + T_{22}X_{12}^*) = 0.$$

Since Λ is an injective map, then we find

$$X_{12}T_{22}^* + T_{22}X_{12}^* = X_{12}\mathcal{D}_{22}^* + \mathcal{D}_{22}X_{12}^*.$$

Upon multiplying the previous equation by P_1 from the left side, we obtain $X_{12}T_{22}^* = X_{12}\mathcal{D}_{22}^*$, for all $X_{12} \in \mathcal{U}_{12}$. Since $\mathcal{U}$ is prime, we get $T_{22} = \mathcal{D}_{22}$.

Also, if we apply Lemma 2 on equation (4) $X_1 = X_{21}$ and $X_2 = P_1$, as similar proof above shows $T_{11} = \mathcal{D}_{11}$. So, $T = \mathcal{D}_{11} + \mathcal{E}_{12} + \mathcal{G}_{21} + \mathcal{D}_{22}$. □

Lemma 6 If Λ: $\mathcal{U} \to \mathcal{V}$ preserves the bi-skew products on $\mathcal{U}$ into $\mathcal{V}$, then we have

$$\Lambda(\mathcal{D}_{kl} + \mathcal{E}_{kl}) = \Lambda(\mathcal{D}_{kl}) + \Lambda(\mathcal{E}_{kl}),$$

with $1 \leq k \neq l \leq 2$, for all $\mathcal{D}_{kl}, \mathcal{E}_{kl} \in \mathcal{U}_{kl}$.

Proof. Assume that $\mathcal{D}_{kl} = a_{kl}^* + a_{kl}$ and $\mathcal{E}_{kl} = b_{kl}^* + b_{kl}$, where $a_{kl}, b_{kl} \in \mathcal{U}_{kl}$. By using the following identities and the previous Lemmas, we get

$$\left[P_k + a_{kl}^* + a_{kl}, \frac{-iI}{2}\right]_\bullet \diamond i(P_l + b_{kl}^* + b_{kl}) = \mathcal{D}_{kl} + \mathcal{E}_{kl} + \mathcal{E}_{kl}^*\mathcal{D}_{kl} + \mathcal{D}_{kl}^*\mathcal{E}_{kl}.$$

By applying Λ on the above identities, then we get

$$\Lambda\left(\left[P_k + a_{kl}^* + a_{kl}, \frac{-iI}{2}\right]_\bullet \diamond i(P_l + b_{kl}^* + b_{kl})\right) = \Lambda(\mathcal{D}_{kl} + \mathcal{E}_{kl}) + \Lambda(\mathcal{E}_{kl}^*\mathcal{D}_{kl} + \mathcal{D}_{kl}^*\mathcal{E}_{kl}). \tag{5}$$

Also, we have

$$\Lambda\left(\left[P_k + a_{kl}^* + a_{kl}, \frac{-iI}{2}\right]_\bullet \diamond (iP_l + ib_{kl}^* + ib_{kl})\right) = \left[\Lambda(P_k + a_{kl}^* + a_{kl}), \Lambda\left(\frac{-iI}{2}\right)\right]_\bullet$$

$$\diamond \Lambda(iP_l + ib_{kl}^* + ib_{kl}).$$

Thus, we have

$$\Lambda\left(\left[P_k+a_{kl}^*+a_{kl},\, \frac{-iI}{2}\right]_{\bullet}\diamond(iP_l+ib_{kl}^*+ib_{kl})\right)$$

$$=\left[\Lambda(P_k)+\Lambda(a_{kl}^*)+\Lambda(a_{kl}),\, \Lambda\left(\frac{-iI}{2}\right)\right]_{\bullet}\diamond(\Lambda(iP_l)+\Lambda(ib_{kl}^*)+\Lambda(ib_{kl}))$$

$$=\left[\Lambda(P_k),\, \Lambda(\frac{-iI}{2})\right]_{\bullet}\diamond\Lambda(iP_l)+\left[\Lambda(P_k),\, \Lambda\left(\frac{-iI}{2}\right)\right]_{\bullet}\diamond\Lambda(ib_{kl})$$

$$+\left[\Lambda(P_k),\, \Lambda\left(\frac{-iI}{2}\right)\right]_{\bullet}\diamond\Lambda(ib_{kl}^*)+\left[\Lambda(a_{kl}),\, \Lambda\left(\frac{-iI}{2}\right)\right]_{\bullet}\diamond\Lambda(iP_l)$$

$$+\left[\Lambda(a_{kl}),\, \Lambda\left(\frac{-iI}{2}\right)\right]_{\bullet}\diamond\Lambda(ib_{kl})+\left[\Lambda(a_{kl}),\, \Lambda\left(\frac{-iI}{2}\right)\right]_{\bullet}\diamond\Lambda(ib_{kl}^*)$$

$$+\left[\Lambda(a_{kl}^*),\, \Lambda\left(\frac{-iI}{2}\right)\right]_{\bullet}\diamond\Lambda(iP_l)+\left[\Lambda(a_{kl}^*),\, \Lambda\left(\frac{-iI}{2}\right)\right]_{\bullet}\diamond\Lambda(ib_{kl})$$

$$+\left[\Lambda(a_{kl}^*),\, \Lambda\left(\frac{-iI}{2}\right)\right]_{\bullet}\diamond\Lambda(ib_{kl}^*).$$

$$=\Lambda\left(\left[P_k,\, \frac{-iI}{2}\right]_{\bullet}\diamond iP_l\right)+\Lambda\left(\left[P_k,\, \frac{-iI}{2}\right]_{\bullet}\diamond ib_{kl}\right)+\Lambda\left(\left[P_k,\, \frac{-iI}{2}\right]_{\bullet}\diamond ib_{kl}^*\right)$$

$$+\ \Lambda\left(\left[a_{kl},\, \frac{-iI}{2}\right]_{\bullet}\diamond iP_l\right)+\Lambda\left(\left[a_{kl},\, \frac{-iI}{2}\right]_{\bullet}\diamond ib_{kl}\right)+\Lambda\left(\left[a_{kl},\, \frac{-iI}{2}\right]_{\bullet}\diamond ib_{kl}^*\right)$$

$$+\ \Lambda\left(\left[a_{kl}^*,\, \frac{-iI}{2}\right]_{\bullet}\diamond iP_l\right)+\Lambda\left(\left[a_{kl}^*,\, \frac{-iI}{2}\right]_{\bullet}\diamond ib_{kl}\right)+\Lambda\left(\left[a_{kl}^*,\, \frac{-iI}{2}\right]_{\bullet}\diamond ib_{kl}^*\right).$$

Hence, we get

$$\Lambda\left(\left[P_k+a_{kl}^*+a_{kl},\, \frac{-iI}{2}\right]_{\bullet}\diamond(iP_l+ib_{kl}^*+ib_{kl})\right)=\Lambda(\mathscr{D}_{kl})+\Lambda(\mathscr{E}_{kl})+\Lambda(\mathscr{E}_{kl}^*\mathscr{D}_{kl}+\mathscr{D}_{kl}^*\mathscr{E}_{kl}). \tag{6}$$

Then, by equations (5) and (6) we have

$$\Lambda(\mathscr{D}_{kl}+\mathscr{E}_{kl})=\Lambda(\mathscr{D}_{kl})+\Lambda(\mathscr{E}_{kl}),\ \text{ for all } \mathscr{D}_{kl},\mathscr{E}_{kl}\in\mathscr{U}_{kl}.$$

□

Lemma 7 If Λ: $\mathscr{U} \to \mathscr{V}$ preserves the bi-skew products on $\mathscr{U}$ into $\mathscr{V}$, then we have

$$\Lambda(\mathscr{D}_{kk} + \mathscr{E}_{kk}) = \Lambda(\mathscr{D}_{kk}) + \Lambda(\mathscr{E}_{kk}), \text{ with } k \in \{l, 2\}, \text{ for all } \mathscr{D}_{kk}, \mathscr{E}_{kk} \in \mathscr{U}_{kk}.$$

Proof. Since Λ is a surjective map, then we have $T = T_{kk} + T_{kl} + T_{lk} + T_{ll}$ where $1 \leq k \neq l \leq 2$ such that

$$\Lambda(T) = \Lambda(\mathscr{D}_{kk}) + \Lambda(\mathscr{E}_{kk}). \tag{7}$$

By applying Lemma 2 on equation (7) for $X_1 = P_k$ and $X_2 = P_l$, thus we have

$$\Lambda([P_k, T]_\bullet \diamond P_l) = \Lambda([P_k, \mathscr{D}_{kk}]_\bullet \diamond P_l) + \Lambda([P_k, \mathscr{E}_{kk}]_\bullet \diamond P_l).$$

Then, we have

$$\Lambda(T_{lk}^* + T_{lk}) = 0.$$

By the injective of Λ, we have

$$T_{lk}^* + T_{lk} = 0.$$

Multiplying the previous equation by P_l from the left side yields $T_{lk} = 0$.
Also, by using Lemma 2 on equation (7) for $X_1 = P_l$ and $X_2 = P_k$, thus we have

$$\Lambda(T_{kl}^* + T_{kl}) = 0.$$

Since Λ is an injective map, we obtain

$$T_{kl}^* + T_{kl} = 0.$$

By multiplying the previous equation on the left side by P_k, we get $T_{kl} = 0$. Also, if we use Lemma 2 on equation (7) for $X_1 = X_{kl}$ and $X_2 = P_l$, then we have

$$\Lambda\left([X_{kl}, T]_\bullet \diamond P_l\right) = \Lambda\left([X_{kl}, \mathscr{D}_{kk}]_\bullet \diamond P_l\right) + \Lambda\left([X_{kl}, \mathscr{E}_{kk}]_\bullet \diamond P_l\right).$$

Hence, we have

$$\Lambda(X_{kl} T_{ll}^* + T_{ll} X_{kl}^*) = 0.$$

By the injective of Λ, thus we find

$$X_{kl}T_{ll}^{*}+T_{ll}X_{kl}^{*}=0.$$

Multiplying the preceding equation from the left side by P_k yields $X_{kl}T_{ll}^{*}=0,$ where $X_{kl}\in\mathcal{U}_{kl}$. Since $\mathcal{U}$ is prime, we get $T_{ll}=0.$ Hence, we have $T=T_{kk}$. Finally, by using Lemma 2 on equation (7) for $X_1=X_{lk}$ and $X_2=P_k$, then we have

$$\Lambda\left(X_{lk}T_{kk}^{*}+T_{kk}X_{lk}^{*}\right)=\Lambda\left(X_{lk}\mathcal{D}_{kk}^{*}+\mathcal{D}_{kk}X_{lk}^{*}\right)+\Lambda\left(X_{lk}\mathcal{E}_{kk}^{*}+\mathcal{E}_{kk}X_{lk}^{*}\right).$$

By using Lemma 6, we find

$$\Lambda\left(X_{lk}T_{kk}^{*}+T_{kk}X_{lk}^{*}\right)=\Lambda\left(X_{lk}\mathcal{D}_{kk}^{*}+\mathcal{D}_{kk}X_{lk}^{*}+X_{lk}\mathcal{E}_{kk}^{*}+\mathcal{E}_{kk}X_{lk}^{*}\right).$$

Since Λ is an injective, then

$$X_{lk}T_{kk}^{*}+T_{kk}X_{lk}^{*}=X_{lk}\mathcal{D}_{kk}^{*}+\mathcal{D}_{kk}X_{lk}^{*}+X_{lk}\mathcal{E}_{kk}^{*}+\mathcal{E}_{kk}X_{lk}^{*}.$$

Then, $X_{lk}(T_{kk}^{*}-\mathcal{D}_{kk}^{*}-\mathcal{E}_{kk}^{*})=(-T_{kk}+\mathcal{D}_{kk}+\mathcal{E}_{kk})X_{lk}^{*},$ for all $X_{lk}\in\mathcal{U}_{lk}$. Multiplying the preceding equation from the left side by P_l yields $X_{lk}(T_{kk}^{*}-\mathcal{D}_{kk}^{*}-\mathcal{E}_{kk}^{*})=0,$ where $X_{lk}\in\mathcal{U}_{lk}$. Hence by the properties of factor von Numann algebra $\mathcal{U}$ and the involution map, we get $T_{kk}=\mathcal{D}_{kk}+\mathcal{E}_{kk}$. □

Proof of Theorem 1 Let $\mathcal{D},\mathcal{E}\in\mathcal{U}$ and write $\mathcal{D}=\sum_{i,\,j=1}^{2}\mathcal{D}_{ij},\ \mathcal{E}=\sum_{i,\,j=1}^{2}\mathcal{E}_{ij}$. Then, by using Lemmas 5-7, we have

$$
\begin{aligned}
\Lambda(\mathscr{D}+\mathscr{E}) =&\Lambda\left(\sum_{i,\, j=1}^{2} \mathscr{D}_{ij}+\sum_{i,\, j=1}^{2} \mathscr{E}_{ij}\right) \\
=&\Lambda\left(\sum_{i,\, j=1}^{2} (\mathscr{D}_{ij}+\mathscr{E}_{ij})\right) \\
=&\sum_{i,\, j=1}^{2} \Lambda(\mathscr{D}_{ij}+\mathscr{E}_{ij}) \\
=&\sum_{i,\, j=1}^{2} \Lambda(\mathscr{D}_{ij})+\Lambda(\mathscr{E}_{ij}) \\
=&\Lambda\left(\sum_{i,\, j=1}^{2} \mathscr{D}_{ij}\right)+\Lambda\left(\sum_{i,\, j=1}^{2} \mathscr{E}_{ij}\right) \\
=&\Lambda(\mathscr{D})+\Lambda(\mathscr{E}).
\end{aligned}
$$

Thus, Λ is additive.

4. Main results

Theorem 2 Let $\mathscr{U}$ and $\mathscr{V}$ be two factor von Neumann algebras with dim $\geqslant 1$. If Λ: $\mathscr{U} \to \mathscr{V}$ is an unital bijective map satisfies

$$\Lambda([\mathscr{D}, \mathscr{E}]_{\bullet} \diamond \mathscr{G}) = [\Lambda(\mathscr{D}), \Lambda(\mathscr{E})]_{\bullet} \diamond \Lambda(\mathscr{G}),$$

for all $\mathscr{D}, \mathscr{E}, \mathscr{G} \in \mathscr{U}$, where $[\mathscr{D}, \mathscr{E}]_{\bullet} = \mathscr{D}\mathscr{E}^* - \mathscr{E}\mathscr{D}^*$ and $\mathscr{D} \diamond \mathscr{E} = \mathscr{D}\mathscr{E}^* + \mathscr{E}\mathscr{D}^*$, then Λ is linear.

To prove the previously mentioned theorem, we have to employ several lemmas. We begin with isomorphisms and then go on to linearity.

Lemma 8 Let Λ: $\mathscr{U} \to \mathscr{V}$ be a bijective map that preserves the bi-skew product on a factor von Neumann algebras $\mathscr{U}$ into a factor von Neumann algebra $\mathscr{V}$. Then we have

$$\Lambda(-\mathscr{D}) = -\Lambda(\mathscr{D}),$$

Proof. By Theorem 1, we have Λ is an additive, then

$$0 = \Lambda(\mathscr{D} - \mathscr{D}) = \Lambda(\mathscr{D}) + \Lambda(-\mathscr{D}), \text{ for all } \mathscr{D} \in \mathscr{U}.$$

Hence, we get

$$\Lambda(-\mathscr{D}) = -\Lambda(\mathscr{D}).$$

□

Lemma 9 If Λ: $\mathscr{U} \to \mathscr{V}$ preserves the bi-skew products on $\mathscr{U}$ into $\mathscr{V}$ and $\mathscr{D}^* = -\mathscr{D}$, then

$$\Lambda(\mathscr{D})^* = -\Lambda(\mathscr{D}), \text{ for all } \mathscr{D} \in \mathscr{U}.$$

Proof. Since we have

$$0 = [\mathscr{D}, I]_{\bullet} \diamond I, \text{ for all } \mathscr{D} \in \mathscr{U}.$$

If we apply Λ on the above identities, then we get

$$0 = \Lambda([\mathscr{D}, I]_{\bullet} \diamond I) = \Lambda((\mathscr{D} - \mathscr{D}^*) \diamond I) = \Lambda(\mathscr{D}) - \Lambda(\mathscr{D}^*) + \Lambda(\mathscr{D})^* - \Lambda(\mathscr{D}^*)^*.$$

Assume that $\mathscr{D}$ is an anti-self-adjoint (i.e) $\mathscr{D}^* = -\mathscr{D}$. Therefore, we get

$$\Lambda(\mathscr{D}) + \Lambda(\mathscr{D}) + \Lambda(\mathscr{D})^* + \Lambda(\mathscr{D})^* = 0.$$

Thus, we have

$$\Lambda(\mathscr{D})^* = -\Lambda(\mathscr{D}).$$

□

Lemma 10 If Λ: $\mathscr{U} \to \mathscr{V}$ preserves the bi-skew products on $\mathscr{U}$ into $\mathscr{V}$ and $\mathscr{D}^* = -\mathscr{D}$, then

$$\Lambda(iI)^* = -\Lambda(iI) \quad \text{and} \quad \Lambda(iI)^2 = -I, \text{ for all } \mathscr{D} \in \mathscr{U}.$$

Proof. By Lemma 9, we can easy to see $\Lambda(iI)^* = -\Lambda(iI)$. Furthermore, since we have

$$[iI, I]_{\bullet} \diamond iI = 4I.$$

Now, apply Λ in the above identity, one can see that

$$
\begin{aligned}
4\Lambda(I) =& [\Lambda(iI), \Lambda(I)]_{\bullet} \diamond \Lambda(iI) \\
=& \Lambda(iI)\Lambda(iI)^* - \Lambda(iI)^*\Lambda(iI) + \Lambda(iI)\Lambda(iI)^* + \Lambda(iI)\Lambda(iI).
\end{aligned}
$$

Since we know that $\Lambda(iI)^* = -\Lambda(iI)$ and Λ is unital, so we get

$$
\Lambda(iI)^2 = -I.
$$

□

Lemma 11 If Λ: $\mathscr{U} \to \mathscr{V}$ preserves the bi-skew products on $\mathscr{U}$ into $\mathscr{V}$ and $\mathscr{D}^* = \mathscr{D}$, then $\Lambda(\mathscr{D})^* = \Lambda(\mathscr{D})$.

Proof. Suppose $\mathscr{D}^* = \mathscr{D} \in \mathscr{U}$. Then, we have

$$
\mathscr{D} = \left[\frac{iI}{2}, \mathscr{D}\right]_{\bullet} \diamond \frac{iI}{2}.
$$

By applying Λ on the above identities, then we get

$$
\begin{aligned}
\Lambda(\mathscr{D}) =& \left[\Lambda\left(\frac{iI}{2}\right), \Lambda(\mathscr{D})\right]_{\bullet} \diamond \Lambda\left(\frac{iI}{2}\right) \\
\Lambda(\mathscr{D}) =& \Lambda\left(\frac{iI}{2}\right)\Lambda(\mathscr{D})^*\Lambda\left(\frac{iI}{2}\right)^* - \Lambda(\mathscr{D})\Lambda\left(\frac{iI}{2}\right)^*\Lambda\left(\frac{iI}{2}\right)^* \\
&+ \Lambda\left(\frac{iI}{2}\right)\Lambda(\mathscr{D})\Lambda\left(\frac{iI}{2}\right)^* - \Lambda\left(\frac{iI}{2}\right)\Lambda\left(\frac{iI}{2}\right)\Lambda(\mathscr{D})^*.
\end{aligned}
$$

Now, taking involution from both sides, then we have

$$
\begin{aligned}
\Lambda(\mathscr{D})^* =& \Lambda\left(\frac{iI}{2}\right)\Lambda(\mathscr{D})\Lambda\left(\frac{iI}{2}\right)^* - \Lambda\left(\frac{iI}{2}\right)\Lambda\left(\frac{iI}{2}\right)\Lambda(\mathscr{D})^* \\
&+ \Lambda\left(\frac{iI}{2}\right)\Lambda(\mathscr{D})^*\Lambda\left(\frac{iI}{2}\right)^* - \Lambda(\mathscr{D})\Lambda\left(\frac{iI}{2}\right)^*\Lambda\left(\frac{iI}{2}\right)^*
\end{aligned}
$$

Thus, we get $\Lambda(\mathscr{D})^* = \Lambda(\mathscr{D})$. □

Lemma 12 If Λ: $\mathscr{U} \to \mathscr{V}$ preserves the bi-skew products on $\mathscr{U}$ into $\mathscr{V}$, then $\Lambda(iI)$ is a conjugate central.

Proof. By Lemma 11, we have $\Lambda(\mathscr{D})^* = \Lambda(\mathscr{D})$ for any self adjoint $\mathscr{D} \in \mathscr{U}$. We want to show $\Lambda(iI) \in \mathscr{Z}(\mathscr{V})$, where $\mathscr{Z}(\mathscr{V})$ is the center of $\mathscr{V}$. Since we have

$$0=\Lambda([iI, I]_{\bullet}\diamond\mathcal{D})=[\Lambda(iI), I]_{\bullet}\diamond\Lambda(\mathcal{D})=2\Lambda(iI)\Lambda(\mathcal{D})-2\Lambda(\mathcal{D})\Lambda(iI).$$

So, we get $\Lambda(iI)\Lambda(\mathcal{D})=\Lambda(\mathcal{D})\Lambda(iI)$. Let $\Lambda(\mathcal{D})=\mathcal{E}$ is an arbitrary element. we can write any element $\mathcal{E}\in\mathcal{V}$ as $\mathcal{E}=\mathcal{E}_1+i\mathcal{E}_2$, where $\mathcal{E}_1=\frac{\mathcal{E}+\mathcal{E}^*}{2}$ and $\mathcal{E}_1=\frac{\mathcal{E}-\mathcal{E}^*}{2i}$. Thus, $\Lambda(iI)\mathcal{E}=\mathcal{E}\Lambda(iI)$, for each $\mathcal{E}\in\mathcal{V}$. Hence $\Lambda(iI)\in\mathcal{Z}(\mathcal{V})$. □

Lemma 13 If Λ: $\mathcal{U}\to\mathcal{V}$ preserves the bi-skew products on $\mathcal{U}$ into $\mathcal{V}$, then $\Lambda(\mathcal{D}^*)=\Lambda(\mathcal{D})^*$, for all $\mathcal{D}\in\mathcal{U}$.

Proof. Since we have

$$\Lambda([\mathcal{D}, iI]_{\bullet}\diamond iI)=[\Lambda(\mathcal{D}), \Lambda(iI)]_{\bullet}\diamond\Lambda(iI).$$

By using Lemma 8, 10 and 12 we get

$$-\Lambda(2\mathcal{D}+2\mathcal{D}^*)=-2\Lambda(\mathcal{D})-2\Lambda(\mathcal{D})^*.$$

Since Λ is an additive map, then

$$-2\Lambda(\mathcal{D})-2\Lambda(\mathcal{D}^*)=-2\Lambda(\mathcal{D})-2\Lambda(\mathcal{D})^*.$$

Therefore, $\Lambda(\mathcal{D}^*)=\Lambda(\mathcal{D})^*$. □

Lemma 14 If Λ: $\mathcal{U}\to\mathcal{V}$ preserves the bi-skew products on $\mathcal{U}$ into $\mathcal{V}$ and $\Lambda(\mathcal{D})^*=\Lambda(\mathcal{D})$, then $\mathcal{D}^*=\mathcal{D}$.

Proof. Let $\mathcal{E}\in\mathcal{V}$ be a self adjoint, i.e. $\mathcal{E}^*=\mathcal{E}$. Since Λ is a surjective map, then there is $\mathcal{D}\in\mathcal{U}$ such that $\mathcal{E}=\Lambda(\mathcal{D})$. Thus,

$$\begin{aligned}2\Lambda(\mathcal{D})=2\mathcal{E}=&\Lambda(iI)\mathcal{E}^*\Lambda\left(\frac{iI}{2}\right)^*-\mathcal{E}\Lambda(iI)^*\Lambda\left(\frac{iI}{2}\right)^*\\&+\Lambda(iI)\mathcal{E}\Lambda\left(\frac{iI}{2}\right)^*-\Lambda(iI)\Lambda\left(\frac{iI}{2}\right)\mathcal{E}^*.\end{aligned}$$

By using Lemmas 10 and 12, we get

$$2\Lambda(\mathcal{D})=2\mathcal{E}=\mathcal{E}^*+\mathcal{E}=\Lambda(\mathcal{D})^*+\Lambda(\mathcal{D}).$$

So, by using 13, then we get

$$2\Lambda(\mathcal{D})=\Lambda(\mathcal{D}^*)+\Lambda(\mathcal{D})=\Lambda(\mathcal{D}^*+\mathcal{D}).$$

Since Λ is an injective and additive map, then we have $\mathcal{D}=\frac{\mathcal{D}^*+\mathcal{D}}{2}=\mathcal{D}^*$. □

Lemma 15 If Λ: $\mathscr{U} \to \mathscr{V}$ preserves the bi-skew products on $\mathscr{U}$ into $\mathscr{V}$, then

$$\Lambda(\mathscr{D}_1 + i\mathscr{D}_2) = \Lambda(\mathscr{D}_1) + \Lambda(iI)\Lambda(\mathscr{D}_2), \text{ where } \mathscr{D}_1 = \frac{\mathscr{D} + \mathscr{D}^*}{2}, \mathscr{D}_2 = \frac{\mathscr{D} - \mathscr{D}^*}{2i}.$$

Proof. For any $\mathscr{D} = \mathscr{D}_1 + i\mathscr{D}_2 \in \mathscr{U}$, we suppose that $\Lambda(\mathscr{D}_1 + i\mathscr{D}_2) = \mathscr{E}_1 + i\mathscr{E}_2$, where $\mathscr{E}_1 = \frac{\mathscr{E}+\mathscr{E}^*}{2}$ and $\mathscr{E}_2 = \frac{\mathscr{E}-\mathscr{E}^*}{2i}$. Since we know $\mathscr{D}_1^* = \mathscr{D}_1$, then $\Lambda(\mathscr{D}_1)^* = \Lambda(\mathscr{D}_1)$. So, we have

$$\begin{aligned}
\Lambda(4\mathscr{D}_1) =& \Lambda([iI, \mathscr{D}_1 + i\mathscr{D}_2]_\bullet \diamond iI) \\
=& [\Lambda(iI), \Lambda(\mathscr{D}_1 + i\mathscr{D}_2)]_\bullet \diamond \Lambda(iI)) \\
=& 4\Lambda(\mathscr{D}_1) \\
=& 4\mathscr{E}_1.
\end{aligned}$$

Now, we want to show $\Lambda(i\mathscr{D}_2) = i\mathscr{E}_2$. We see that

$$\begin{aligned}
\Lambda(4\mathscr{D}_2) =& \Lambda([\mathscr{D}_1 + i\mathscr{D}_2, I]_\bullet \diamond iI) \\
=& [\Lambda(\mathscr{D}_1 + i\mathscr{D}_2), I]_\bullet \diamond \Lambda(iI) \\
=& [\Lambda(\mathscr{D}_1) + \Lambda(i\mathscr{D}_2), I]_\bullet \diamond \Lambda(iI)) \\
=& [\Lambda(\mathscr{D}_1), I]_\bullet \diamond \Lambda(iI)) + [\Lambda(i\mathscr{D}_2), I]_\bullet \diamond \Lambda(iI) \\
=& [\Lambda(\mathscr{D}_1), I]_\bullet \diamond \Lambda(iI)) + [\Lambda(i\mathscr{D}_2), I]_\bullet \diamond \Lambda(iI) \\
=& (\Lambda(i\mathscr{D}_2) - \Lambda(i\mathscr{D}_2))^* \diamond \Lambda(iI).
\end{aligned}$$

By using Lemmas 9 and 12 so we get

$$\Lambda(4\mathscr{D}_2) = -4\Lambda(iI)\Lambda(i\mathscr{D}_2).$$

If we multiple the above equation by $\Lambda(iI)$, then we get $\Lambda(i\mathscr{D}_2) = \Lambda(iI)\Lambda(\mathscr{D}_2)$. Also, we have $\Lambda(i\mathscr{D}_2) = \Lambda\left(\frac{\mathscr{D}-\mathscr{D}^*}{2}\right)$. Since Λ is an additive map, then we have

$$\Lambda(i\mathscr{D}_2) = \frac{\Lambda(\mathscr{D}) - \Lambda(\mathscr{D})^*}{2}) = i\mathscr{E}_2.$$

Hence, we have

$$\begin{aligned}\Lambda(\mathscr{D}_1 + i\mathscr{D}_2) =& \Lambda(\mathscr{D}_1) + \Lambda(i\mathscr{D}_2) \\ =& \Lambda(\mathscr{D}_1) + \Lambda(iI)\Lambda(\mathscr{D}_2) \\ =& \mathscr{E}_1 + i\mathscr{E}_2.\end{aligned}$$

□

Lemma 16 If Λ: $\mathscr{U} \to \mathscr{V}$ preserves the bi-skew products on $\mathscr{U}$ into $\mathscr{V}$, then

$$\Lambda(\mathscr{D}_1\mathscr{D}_2) = \Lambda(\mathscr{D}_1)\Lambda(\mathscr{D}_2), \text{ for all } \mathscr{D}_1, \mathscr{D}_2 \in \mathscr{U}.$$

Proof. First we want to show that $\Lambda(i\mathscr{D}) = \Lambda(iI)\Lambda(\mathscr{D})$. Let $\mathscr{D} = \mathscr{D}_1 + i\mathscr{D}_2 \in \mathscr{U}$, then we have

$$\Lambda(i\mathscr{D}) = \Lambda(i\mathscr{D}_1 - \mathscr{D}_2).$$

By using the additive of Λ and Lemma 8, then

$$\begin{aligned}\Lambda(i\mathscr{D}) =& \Lambda(i\mathscr{D}_1) - \Lambda(\mathscr{D}_2) \\ =& \Lambda(iI)\Lambda(\mathscr{D}_1) - \Lambda(\mathscr{D}_2) \\ =& \Lambda(iI)\left[\Lambda(\mathscr{D}_1) + \Lambda(iI)\Lambda(\mathscr{D}_2)\right] \\ =& \Lambda(iI)[\Lambda(\mathscr{D}_1 + i\mathscr{D}_2)] \\ =& \Lambda(iI)\Lambda(\mathscr{D}).\end{aligned}$$

Now, we need to prove $\Lambda(\mathscr{D}_1\mathscr{D}_2) = \Lambda(\mathscr{D}_1)\Lambda(\mathscr{D}_2)$, for all $\mathscr{D}_1, \mathscr{D}_2 \in \mathscr{U}$. We have

$$\Lambda([iI, \mathscr{D}_2] \diamond i\mathscr{D}_1) = \Lambda(\mathscr{D}_2^*\mathscr{D}_1^*) + \Lambda(\mathscr{D}_2\mathscr{D}_1^*) + \Lambda(\mathscr{D}_1\mathscr{D}_2) + \Lambda(\mathscr{D}_1\mathscr{D}_2^*).$$

Also, we have

$$\Lambda([iI,\mathscr{D}_2]\diamond i\mathscr{D}_1) = [\Lambda(iI),\Lambda(\mathscr{D}_2)]\diamond\Lambda(i\mathscr{D}_1)$$

$$=\Lambda(\mathscr{D}_2)^*\Lambda(\mathscr{D}_1)^*+\Lambda(\mathscr{D}_2)\Lambda(\mathscr{D}_1)^*+\Lambda(\mathscr{D}_1)\Lambda(\mathscr{D}_2)+\Lambda(\mathscr{D}_1)\Lambda(\mathscr{D}_2)^*.$$

So, we get

$$\Lambda(\mathscr{D}_2^*\mathscr{D}_1^*)+\Lambda(\mathscr{D}_2\mathscr{D}_1^*)+\Lambda(\mathscr{D}_1\mathscr{D}_2)+\Lambda(\mathscr{D}_1\mathscr{D}_2^*)$$
$$=\Lambda(\mathscr{D}_2)^*\Lambda(\mathscr{D}_1)^*+\Lambda(\mathscr{D}_2)\Lambda(\mathscr{D}_1)^*+\Lambda(\mathscr{D}_1)\Lambda(\mathscr{D}_2)+\Lambda(\mathscr{D}_1)\Lambda(\mathscr{D}_2)^*. \tag{8}$$

Similarly, if we use the following identity, we get

$$\Lambda([I,\mathscr{D}_2]\diamond\mathscr{D}_1)=\Lambda(\mathscr{D}_2^*\mathscr{D}_1^*)-\Lambda(\mathscr{D}_2\mathscr{D}_1^*)+\Lambda(\mathscr{D}_1\mathscr{D}_2)-\Lambda(\mathscr{D}_1\mathscr{D}_2^*).$$

On the other hand,

$$\Lambda([I,\mathscr{D}_2]\diamond\mathscr{D}_1)=\Lambda(\mathscr{D}_2)^*\Lambda(\mathscr{D}_1)^*-\Lambda(\mathscr{D}_2)\Lambda(\mathscr{D}_1)^*+\Lambda(\mathscr{D}_1)\Lambda(\mathscr{D}_2)-\Lambda(\mathscr{D}_1)\Lambda(\mathscr{D}_2)^*.$$

Then, we get

$$\Lambda(\mathscr{D}_2^*\mathscr{D}_1^*)-\Lambda(\mathscr{D}_2\mathscr{D}_1^*)+\Lambda(\mathscr{D}_1\mathscr{D}_2)-\Lambda(\mathscr{D}_1\mathscr{D}_2^*)$$
$$=\Lambda(\mathscr{D}_2)^*\Lambda(\mathscr{D}_1)^*-\Lambda(\mathscr{D}_2)\Lambda(\mathscr{D}_1)^*+\Lambda(\mathscr{D}_1)\Lambda(\mathscr{D}_2)-\Lambda(\mathscr{D}_1)\Lambda(\mathscr{D}_2)^*. \tag{9}$$

Now, if we add equations (8) and (9), we get

$$\Lambda(\mathscr{D}_2^*\mathscr{D}_1^*)+\Lambda(\mathscr{D}_1\mathscr{D}_2)=\Lambda(\mathscr{D}_2)^*\Lambda(\mathscr{D}_1)^*+\Lambda(\mathscr{D}_1)\Lambda(\mathscr{D}_2).$$

So, we get

$$\Lambda(\mathscr{D}_1\mathscr{D}_2^*+\mathscr{D}_1\mathscr{D}_2)=(\Lambda(\mathscr{D}_2)\Lambda(\mathscr{D}_1))^*+\Lambda(\mathscr{D}_1)\Lambda(\mathscr{D}_2).$$

Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be a self-adjoint, then we get

$$\Lambda(\mathscr{D}_1\mathscr{D}_2)=\Lambda(\mathscr{D}_1)\Lambda(\mathscr{D}_2).$$

Let $\Lambda(\mathscr{D}_1) = \mathscr{G}$ and $\Lambda(\mathscr{D}_2) = \mathscr{D}$ are arbitrary elements. We can write any elements $\mathscr{G}, \mathscr{D} \in \mathscr{V}$ as $\mathscr{G} = \mathscr{G}_1 + i\mathscr{G}_2$, where $\mathscr{G}_1 = \frac{\mathscr{G}+\mathscr{G}^*}{2}$, $\mathscr{G}_2 = \frac{\mathscr{G}-\mathscr{G}^*}{2i}$, and $\mathscr{D} = \mathscr{D}_1 + i\mathscr{D}_2$, where $\mathscr{D}_1 = \frac{\mathscr{D}+\mathscr{D}^*}{2}$ and $\mathscr{D}_2 = \frac{\mathscr{D}-\mathscr{D}^*}{2i}$. Hence, $\Lambda(\mathscr{D}_1\mathscr{D}_2) = \Lambda(\mathscr{D}_1)\Lambda(\mathscr{D}_2)$, for each $\mathscr{D}_1, \mathscr{D}_2 \in \mathscr{U}$. □

From now on, the following Lemmas demonstrate that Λ is linear.

Lemma 17 If Λ: $\mathscr{U} \to \mathscr{V}$ preserves the bi-skew products on $\mathscr{U}$ into $\mathscr{V}$ and q is a rational number, then $\Lambda(qI) = qI$, for any $q \in \mathbb{Q}$.

Proof. Since q is a rational number, then there exist two integers r_1, $r_2 \in \mathbb{Z}$, where $r_2 \neq 0$ such that $q = \frac{r_1}{r_2}$. Since Λ is an additive and unital, we get

$$\Lambda(qI) = \Lambda\left(\frac{r_1}{r_2}I\right) = r_1\Lambda\left(\frac{1}{r_2}I\right) = \frac{r_1}{r_2}\Lambda(I) = \frac{r_1}{r_2}I.$$

□

Lemma 18 If Λ: $\mathscr{U} \to \mathscr{V}$ preserves the bi-skew products on $\mathscr{U}$ into $\mathscr{V}$, then Λ preserves positive element.

Proof. Let $\mathscr{D}$ is a positive element in $\mathscr{U}$, then there exist unique self-adjoint element $\mathscr{E}$ such that $\mathscr{D} = \mathscr{E}^2$. Here we want to show $\Lambda(\mathscr{E}^2) = \Lambda(\mathscr{E})^2$. Since we have

$$4\Lambda(\mathscr{D}) = \Lambda(4\mathscr{E}^2) = \Lambda([i\mathscr{E}, \mathscr{D}]_\bullet \diamond iI) = [\Lambda(i\mathscr{E}), \Lambda(\mathscr{D})]_\bullet \diamond \Lambda(iI).$$

By using Lemmas 10, 12, and 16, we get

$$4\Lambda(\mathscr{D}) = \Lambda(4\mathscr{E}^2) = 4\Lambda(\mathscr{E})^2.$$

Also, since we know Λ is a multiplicative, then it is clear that $\Lambda(\mathscr{D}) = \Lambda(\mathscr{E}\mathscr{E}) = \Lambda(\mathscr{E})^2$. □

Lemma 19 If Λ: $\mathscr{U} \to \mathscr{V}$ preserves the bi-skew products on $\mathscr{U}$ into $\mathscr{V}$ and q is a real number, then we have $\Lambda(qI) = qI$, for any $q \in \mathbb{R}$.

Proof. Let $q \in \mathbb{R}$ by taking any two sequences a_n and b_n of rational numbers such that $a_n \leq q \leq b_n$, for every n. If we taking the limit, then we get $lim_{n\to\infty} a_n = lim_{n\to\infty} b_n = q$. Also we have

$$a_n I \leq qI \leq b_n I.$$

By applying Λ on the above then we get

$$a_n I \leq \Lambda(qI) \leq b_n I.$$

Also, if we taking the limit, then we get $\Lambda(qI) = qI$.

Thus in all, we conclude that Λ is $\mathbb{R}-$linear. □

Example 1 Let $\mathscr{U} = \mathscr{M}_n(\mathbb{C})$ and $\mathscr{V} = \mathscr{M}_n(\mathbb{C})$, where $\mathscr{M}_n(\mathbb{C})$ denotes the algebra of $n \times n$ complex matrices. Define the map Λ: $\mathscr{U} \to \mathscr{V}$ by $\Lambda(A) = A^\star$, where $A^\star$ denotes the conjugate transpose of the matrix A. Again, it can be verified that Λ is an unital bijective map and satisfies the condition: $\Lambda([\mathscr{D}, \mathscr{E}] \bullet \diamond \mathscr{G}) = [\Lambda(\mathscr{D}), \Lambda(\mathscr{E})] \bullet \diamond \Lambda(\mathscr{G})$, for all $\mathscr{D}, \mathscr{E}, \mathscr{G} \in \mathscr{U}$. Therefore, according to the theorem, the map Λ is linear.

5. Conclusion

In this manuscript, we have characterized the bijective mapping Λ: $\mathscr{U} \to \mathscr{V}$ between two von Neumann algebras $\mathscr{U}$ and $\mathscr{V}$ with dim $\geqslant 1$. The mapping Λ is shown to satisfy the following condition on the mixed bi-skew Lie (Jordan) product

$$\Lambda([\mathscr{D}, \mathscr{E}]_\bullet \diamond \mathscr{G}) = [\Lambda(\mathscr{D}), \Lambda(\mathscr{E})]_\bullet \diamond \Lambda(\mathscr{G}),$$

for all $\mathscr{D}, \mathscr{E}, \mathscr{G} \in \mathscr{U}$, where the bi-skew Lie (Jordan) product is defined as $[\mathscr{D}, \mathscr{E}]_\bullet = \mathscr{D}\mathscr{E}^* - \mathscr{E}\mathscr{D}^*$, and $\mathscr{D} \diamond \mathscr{E} = \mathscr{D}\mathscr{E}^* + \mathscr{E}\mathscr{D}^*$. Specifically, we have elaborated on the properties of the mapping Λ, establishing that it is additive, a $*$-somorphism, and linear. These results demonstrate the strong structural constraints imposed by the preservation of the mixed bi-skew Lie (Jordan) product under the bijective map Λ.

The approach and results provided in this paper contribute to an enhanced awareness of the relationship between the algebraic structure of von Neumann algebras and the characteristics of bijective maps with defined product identities. This has consequences for the study of operator algebras, quantum mechanics, and other branches of mathematics.

Disclosure statement

The authors declare that they have no conflicts of interest to disclose.

Acknowledgment

Thanking the referees for their remarks and recommendations, which were useful and constructive.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Molnár L. A condition for a subspace of $B(H)$ to be an ideal. *Linear Algebra and Its Applications*. 1996; 235: 229-234. Available from: https://doi.org/10.1016/0024-3795(94)00143-X.

[2] Šemrl P. On Jordan $*$-derivations and an application. *Colloquium Mathematicum*. 1990; 59: 241-251. Available from: http://eudml.org/doc/266544.

[3] Šemrl P. Quadratic functionals and Jordan $*$-derivations. *Studia Mathematica*. 1991; 97: 157-165. Available from: https://eudml.org/doc/219013.

[4] Šemrl P. Quadratic and quasi-quadratic functionals. *Proceedings of the American Mathematical Society*. 1993; 119: 1105-1113. Available from: https://doi.org/10.2307/2159972.

[5] Šemrl P. Jordan $*$-derivations of standard operator algebras. *Proceedings of the American Mathematical Society*. 1994; 120: 515-519. Available from: https://doi.org/10.2307/2159889.

[6] Martindale WSI. When are multiplicative mappings additive? *Proceedings of the American Mathematical Society*. 1969; 21: 695-698. Available from: https://doi.org/10.2307/2036449.

[7] Brešar M. Jodran mappings of semiprime rings. *Journal of Algebra*. 1989; 127: 218-228. Available from: https://doi.org/10.1016/0021-8693(89)90285-8.

[8] Ji P, Liu Z. Additivity of Jordan maps on standard Jordan operator algebras. *Linear Algebra and Its Applications*. 2009; 430(1): 335-343. Available from: https://doi.org/10.1016/j.laa.2008.07.023.

[9] Lu F. Additivity of Jordan maps on standard operator algebras. *Linear Algebra and Its Applications*. 2002; 357: 123-131. Available from: https://doi.org/10.1016/S0024-3795(02)00367-1.

[10] Khan AN. Multiplicative bi-skew Lie triple derivations on factor von Neumann algebras. *Rocky Mountain Journal of Mathematics*. 2021; 51(6): 2103-2114. Available from: https://doi.org/10.1216/rmj.2021.51.2103.

[11] Lu F. Jordan triple maps. *Linear Algebra and Its Applications*. 2003; 375: 311-317. Available from: https://doi.org/10.1016/j.laa.2003.06.004.

[12] Lu F. Jordan maps on associative algebras. *Communications in Algebra*. 2003; 31: 2273-2286. Available from: https://doi.org/10.1081/AGB-120018997.

[13] Bai Z, Du S. Maps preserving products $XY - YX^*$ on von Neumann algebras. *Journal of Mathematical Analysis and Applications*. 2012; 386(1): 103-109.

[14] Alhazmi H, Raza MA, Khan AN, AL-Sobhi T. On mixed nonlinear skew Lie (Jordan) products on von Neumann algebras. *Journal of Mathematics*. 2024; 2024(1): 1407653. Available from: https://doi.org/10.1155/2024/1407653.

[15] Huo D, Zheng B, Xu J, Liu H. Nonlinear mappings preserving Jordan multiple $*$-product on factor von Neumann algebras. *Linear and Multilinear Algebra*. 2015; 63(5): 1026-1036. Available from: https://doi.org/10.1080/03081087.2014.915321.

[16] Li C, Lu F, Fang X. Nonlinear mappings preserving product $XY + YX^*$ on factor von Neumann algebras. *Linear Algebra and Its Applications*. 2013; 438: 2339-2345. Available from: https://doi.org/10.1016/j.laa.2012.10.015.

[17] Darvish V, Razeghi M, Nouri M. Maps preserving the bi-skew Jordan product on factor von Neumann algebras. *Georgian Mathematical Journal*. 2023; 30(6): 845-854. Available from: https://doi.org/10.1515/gmj-2023-2050.

[18] Raza MA, Al-Sobhi T. Non-linear product $LM^* - ML^*$ on prime $*$-algebra. *Korean Journal of Mathematics*. 2023; 31(3): 313-321. Available from: https://doi.org/10.11568/kjm.2023.31.3.313.

[19] Rehman N, Nisar J, Nazim M. Nonlinear mixed Jordan triple $*$-derivations on standard operator algebras. *Filomat*. 2023; 37(10): 3143-3151. Available from: https://doi.org/10.2298/FIL2310143R.

[20] Rehman N, Nisar J, Nazim M. A note on nonlinear mixed Jordan triple derivation on $*$-algebras. *Communications in Algebra*. 2023; 51(4): 1334-1343. Available from: https://doi.org/10.1080/00927872.2022.2134410.

[21] Yang Z, Zhang Y. Nonlinear maps preserving the second mixed Lie triple products on factor von Neumann algebras. *Linear and Multilinear Algebra*. 2020; 68(2): 377-390. Available from: https://doi.org/10.1080/03081087.2018.1506732.

[22] Cui JL, Li CK. Maps preserving product $XY - YX^*$ on factor von Neumann algebras. *Linear Algebra and Its Applications*. 2009; 431: 833-842. Available from: https://doi.org/10.1016/j.laa.2009.03.036.

[23] Chebotar MA, Fong Y, Lee PH. On maps preserving zeros of the polynomial $xy - yx^*$. *Linear Algebra and Its Applications*. 2005; 408: 230-243. Available from: https://doi.org/10.1016/j.laa.2005.06.015.

[24] Raza MA, Al-Sobhi T. Non-linear mappings preserving product $m*n+n*m$ on factor von Neumann algebras. In: *Advances in Ring Theory and Applications, WARA 2022*. Springer Proceedings in Mathematics and Statistics. Springer, Cham; 2024. p.343-351. Available from: https://doi.org/10.1007/978-3-031-50795-3.