Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Study of Multiobjective Fractional Variational Formulation Using η-Approximated Method

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Received: 27 May 2024; **Revised:** 12 August 2024; **Accepted:** 16 August 2024

Abstract: This paper investigates the idea of the η-approximated technique for converting the nonlinear and nonconvex multiobjective fractional variational dual problems (MFP) and (MFD) with inequality constraints to the linear and convex counterparts of the problems, namely, $(\text{MFP})_\eta$ and $(\text{MFD})_\eta$, respectively. Weak, strong, and converse duality theorems are obtained for the original as well as the η-approximated dual pair under invexity for weak Pareto as well as Pareto solutions. Furthermore, the connection between the original and modified problems has also been established. A suitable numerical example is constructed to bolster the research paper.

***Keywords*:** variational problem, multiobjective, Mond-Weir dual, (weak) Pareto solution

MSC: 90C29, 90C30, 90C32, 90C46

1. Introduction

A wide range of mathematical theories that have developed via the study of variational principles, optimum control, and linear and nonlinear systems with the goal of minimizing the function are reflected in variational analysis. Utilizing the concepts of variational calculus, vector-valued variational programming seeks to determine the minimum or maximum of a vector-valued function that arises in optimization problems. Applications of variational calculus in optimization include aeronautical design and space management structures [1, 2], optimization in orbit transfer [3], engineering [4], economics [5], and so on. A great deal of effort has been made by several researchers to establish the foundation for optimality-that is, results that are both necessary and sufficient. Assuming the function to be convex, Mond and Hanson [6] worked on the dual model of variational problems and derived various duality results. After that, Mond et al. [7] demonstrated the duality theorems by extending the findings of Mond and Hanson [6] by employing an invex function rather than a convex one. Making use of a generalized invex function, Mond and Husain [8] developed the KKT-sufficiency criterion to identify optimality and duality relations for optimization problems. In addition to establishing the relationship between variational and multiobjective programming problems under the convexity assumption, Bector and Husain [9] constructed dual models for multiobjective variational programs and derived duality theorems. By creating a multiobjective variational problem, Nahak and Nanda [10] widened the work of Bector and Husain [9] from convex to invex function and obtained duality results. The duality and optimality conditions for variational problems under ρ-(η, θ)-B-type-I and the generalized

DOI: https://doi.org/10.37256/cm.5420245031

ρ-(η, θ)-B-type-I functions were examined by Nahak and Behera [11]. For the vector variational problem, Khazafi and Rueda [12] proposed the mixed dual model, derived optimality criteria, and several dual results. Assuming the function to be (β, ρ)-type I, Khazafi et al. [13] focused on the optimality as well as duality conditions for vector optimization formulations. Using a parametric approach, Stancu-Minasian and Mititelu [14] developed a dual model for the fractional problem and discussed the necessary conditions. Appropriate duality results under the (ρ, b)-quasiinvex function are derived based on the dual model.

In the course of the aforementioned research, Antczak developed a novel strategy known as the η-approximated method for determining the optimal answer. The relation between the original vector problem and the modified problem under generalized invexity was described by Antczak in [15]. He also came up with an η-approximated approach to solve the differentiable vector-valued problem. Additionally, Antczak [16] worked on the dual model by using the η-approximation approach to transform the problem and dual. From this, they were able to establish many duality relations for the η-approximated problem and dual pair. Subsequently, Antczak [17] enlarged on the findings of [16] by investigating the relationship between the optimal solution of the original optimization problem and the related η-approximated problem, making use of the concept of r-invexity as a tool. After that, Antczak and Michalak [18] worked on the η-approximated technique for nonconvex vector variational formulations and developed a relationship for the Pareto solution between the problem generated with the η-approximated method and the vector variational problem. The optimality and saddle-point criteria for a nonconvex variational optimization problem were examined by Jayswal et al. [19]. Additionally, they demonstrated the relationship between the saddle point of the modified variational problem and the optimal solution to the variational programming problem by modifying the objective function using an η-approximated technique. Jha et al. [20] just proposed a η-approximated approach for using the (p, r)-invex function to solve nonconvex variational problems, and they demonstrated multiple duality results under the Mond-Weir dual model. Jha et al. [21] subsequently employed the η-approximation approach to frame the multi-time vector variational formulations and obtained the equivalency results based on the multi-time problem in conjunction with the η-approximation problem.

Motivated by the aforementioned studies, we have investigated multiobjective fractional variational problems through the η-approximated method. This article's development is summarized as follows: Basic notations that are utilized throughout the article's sequel are provided in Section 2. Further, we construct a multiobjective fractional variational problem and recall the optimality criteria. In Section 3, we construct the Mond-Weir dual model, introduce the η-approximated method, and formulate the modified variational problem and modified dual problem by modifying both the objective function and constraints using the η-approximated approach. Further, we derive several (weak, strong, and converse) duality theorems. Additionally, we formulate a numerical example of a nonlinear nonconvex fractional variational problem with its dual, and using the η-approximation method, we construct the modified problem with its dual pair, through which we can validate that using the η-approximation method, the nonlinear nonconvex fractional problem may be reduced to linear, convex, and nonfractional problems. Finally, Section 4 summarizes the accomplished work in the form of conclusions.

2. Preliminaries

The following equalities and inequalities are set out in this section and will be used throughout the article. For any $\varkappa^1 = (\varkappa_1^1, \varkappa_2^1, \ldots, \varkappa_n^1)$, $\varkappa^2 = (\varkappa_1^2, \varkappa_2^2, \ldots, \varkappa_n^2)$ in $\mathbb{R}^n$, we have

(i) $\varkappa^1 = \varkappa^2 \Leftrightarrow \varkappa_i^1 = \varkappa_i^2, \ \forall i = 1, \ldots, n$;

(ii) $\varkappa^1 > \varkappa^2 \Leftrightarrow \varkappa_i^1 > \varkappa_i^2, \ \forall i = 1, \ldots, n$;

(iii) $\varkappa^1 \geqq \varkappa^2 \Leftrightarrow \varkappa_i^1 \geqq \varkappa_i^2, \ \forall i = 1, \ldots, n$;

(iv) $\varkappa^1 \geq \varkappa^2 \Leftrightarrow \varkappa^1 \geqq \varkappa^2, \varkappa^1 \neq \varkappa^2$.

In the sequel to the paper, consider $\mathscr{I} = [\tau_1, \tau_2]$ as a closed interval, and suppose X represents the space of continuously differentiable functions $\varsigma\colon \mathscr{I} \to \mathbb{R}^n$ having the norm $\|\varsigma\| = \|\varsigma\|_\infty + \|D\varsigma\|_\infty$. Let the differential operator D be specified as

$$\eth = D\varsigma \Leftrightarrow \varsigma(\theta) = \varsigma(\tau_1) + \int_{\tau_1}^{\theta} \eth(s)ds,$$

where $\varsigma(\tau_1)$ is boundary value. Therefore, $D \equiv \frac{d}{d\theta}$ excludes points where the functions are not continuous.

Let the multiobjective fractional variational problem be constructed as:

$$\text{(MFP)} \quad \min \quad \phi(\varsigma) = \left(\frac{\int_{\tau_1}^{\tau_2} \varphi_1(\theta, \varsigma(\theta), \dot{\varsigma}(\theta))d\theta}{\int_{\tau_1}^{\tau_2} \psi_1(\theta, \varsigma(\theta), \dot{\varsigma}(\theta))d\theta}, \ldots, \frac{\int_{\tau_1}^{\tau_2} \varphi_k(\theta, \varsigma(\theta), \dot{\varsigma}(\theta))d\theta}{\int_{\tau_1}^{\tau_2} \psi_k(\theta, \varsigma(\theta), \dot{\varsigma}(\theta))d\theta} \right),$$

$$\text{subject to} \quad h(\theta, \varsigma(\theta), \dot{\varsigma}(\theta)) \leqq 0, \ \theta \in \mathscr{I},$$

$$\varsigma(\tau_1) = \alpha, \quad \varsigma(\tau_2) = \beta,$$

where

(i) the functions φ: $\mathscr{I} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$, ψ: $\mathscr{I} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ and h: $\mathscr{I} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ possess continuous differentiability;

(ii) $\int_{\tau_1}^{\tau_2} \varphi_i(\theta, \varsigma(\theta), \dot{\varsigma}(\theta))d\theta \geqq 0$, $\int_{\tau_1}^{\tau_2} \psi_i(\theta, \varsigma(\theta), \dot{\varsigma}(\theta))d\theta > 0$, $i \in \mathscr{K} = \{1, \ldots, k\}$.

Further, the set of feasible solutions is represented by

$$\Upsilon: = \{\varsigma \in X: \varsigma(\tau_1) = \alpha, \ \varsigma(\tau_2) = \beta \text{ and } h(\theta, \varsigma, \dot{\varsigma}) \leqq 0, \ \theta \in \mathscr{I}\}.$$

For ς: $\mathscr{I} \to \mathbb{R}^n$, $\dot{\varsigma}(\theta)$ denotes the derivative of ς with regard to θ. Moreover, φ_{ς} and $\varphi_{\dot{\varsigma}}$ denote $k \times n$ Jacobian representations of first-order partial derivatives of $\varphi(\theta, \varsigma(\theta), \dot{\varsigma}(\theta))$ with regard to ς and $\dot{\varsigma}$ respectively, i.e., $\varphi_{1\varsigma}, \ldots, \varphi_{k\varsigma}$ and $\varphi_{1\dot{\varsigma}}, \ldots, \varphi_{k\dot{\varsigma}}$ with

$$\varphi_{i_{\varsigma}} = \left(\frac{\partial \varphi_i}{\partial \varsigma_1}, \frac{\partial \varphi_i}{\partial \varsigma_2}, \ldots, \frac{\partial \varphi_i}{\partial \varsigma_n} \right)^T \text{ and } \varphi_{i_{\dot{\varsigma}}} = \left(\frac{\partial \varphi_i}{\partial \dot{\varsigma}_1}, \frac{\partial \varphi_i}{\partial \dot{\varsigma}_2}, \ldots, \frac{\partial \varphi_i}{\partial \dot{\varsigma}_n} \right)^T, \ \forall i \in \mathscr{K},$$

where the superscript T represents the transpose operator. Similarly, ψ_{ς}, h_{ς} and $\psi_{\dot{\varsigma}}$, $h_{\dot{\varsigma}}$ denote $k \times n$ and $m \times n$ Jacobian representation of first-order partial derivatives of the function $\psi(\theta, \varsigma(\theta), \dot{\varsigma}(\theta))$ and $h(\theta, \varsigma(\theta), \dot{\varsigma}(\theta))$ with regard to ς and $\dot{\varsigma}$ respectively.

Special cases

(i) If we consider the nonfractional single objective function instead of the multiobjective fractional problem (MFP), then it reduces to the problem discussed in Jha et al. [20].

(ii) Additionally, in (i), if we consider the static case, then we get the problem discussed in Antczak [22].

Definition 1 A point $\widehat{\varsigma} \in \Upsilon$ is known as a Pareto solution (efficient solution) to (MFP) provided there is no point $\varsigma \in \Upsilon$ satisfying

$$\frac{\int_{\tau_1}^{\tau_2} \varphi_i(\theta, \varsigma(\theta), \dot{\varsigma}(\theta))d\theta}{\int_{\tau_1}^{\tau_2} \psi_i(\theta, \varsigma(\theta), \dot{\varsigma}(\theta))d\theta} \leqq \frac{\int_{\tau_1}^{\tau_2} \varphi_i(\theta, \widehat{\varsigma}(\theta), \dot{\widehat{\varsigma}}(\theta))d\theta}{\int_{\tau_1}^{\tau_2} \psi_i(\theta, \widehat{\varsigma}(\theta), \dot{\widehat{\varsigma}}(\theta))d\theta}; \ \forall i \in \mathscr{K},$$

with at least one strict inequality.

Definition 2 A point $\widehat{\varsigma} \in \Upsilon$ is known as a weak Pareto solution (weak efficient solution) to MFP, provided there is no point $\varsigma \in \Upsilon$ satisfying

$$\frac{\int_{\tau_1}^{\tau_2} \varphi_i(\theta, \varsigma(\theta), \dot{\varsigma}(\theta))d\theta}{\int_{\tau_1}^{\tau_2} \psi_i(\theta, \varsigma(\theta), \dot{\varsigma}(\theta))d\theta} < \frac{\int_{\tau_1}^{\tau_2} \varphi_i(\theta, \widehat{\varsigma}(\theta), \dot{\widehat{\varsigma}}(\theta))d\theta}{\int_{\tau_1}^{\tau_2} \psi_i(\theta, \widehat{\varsigma}(\theta), \dot{\widehat{\varsigma}}(\theta))d\theta}; \ \forall i \in \mathscr{K}.$$

As for convenience, we write $\varphi_i(\theta, \varsigma(\theta), \dot{\varsigma}(\theta))$ shortly by $\varphi_i(\theta, \varsigma, \dot{\varsigma})(i \in \mathscr{K})$. Moreover, let us consider η: $\mathscr{I} \times X \times X \to \mathbb{R}^n$ as a multi-valued differentiable function with the condition $\eta(\theta, \varsigma, \varsigma) = 0$, for all $\varsigma(\theta) \in X$.

Next, let us generalize the definition of invexity given by Jayswal et al. [19] from scalar-valued to vector-valued functions.

Definition 3 A vector-valued function $\int_{\tau_1}^{\tau_2} h(\theta, \varsigma, \dot{\varsigma})d\theta$ is said to be invex (strictly invex) at a point $\widehat{\varsigma} \in X$ in connection with η provided

$$\int_{\tau_1}^{\tau_2} h(\theta, \varsigma, \dot{\varsigma})d\theta - \int_{\tau_1}^{\tau_2} h(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})d\theta$$

$$\geqq (>) \int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \varsigma, \widehat{\varsigma})^T h_{\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) + \left(\frac{d}{d\theta}\eta(\theta, \varsigma, \widehat{\varsigma}) \right)^T h_{\dot{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) \right\} d\theta; \ \forall \varsigma \in X.$$

Consequently, we work out the optimality criteria for (MFP), which can be seen in Antczak [15], Stancu-Minasian and Mititelu [14].

Theorem 1 Let the feasible point $\widehat{\varsigma} \in \Upsilon$ be a (weak) Pareto solution to (MFP) and satisfies the Slater's constraint qualification. Then a vector $\delta \in \mathbb{R}^k$ and a smooth piecewise function ρ: $\mathscr{I} \to \mathbb{R}^m$ exist and satisfy the following conditions:

$$\sum_{i=1}^{k} \delta_i [\aleph_i(\widehat{\varsigma})\varphi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})] + (\rho(\theta))^T h_{\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})$$

$$= \frac{d}{d\theta}\left(\sum_{i=1}^{k} \delta_i [\aleph_i(\widehat{\varsigma})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})] + (\rho(\theta))^T h_{\dot{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) \right), \tag{1}$$

$$(\rho(\theta))^T h(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) = 0, \tag{2}$$

$$\delta > 0,\ \ \rho(\theta) \geqq 0\ (\theta \in \mathscr{I}),\ \ \sum_{i=1}^{k} \delta_i = 1, \tag{3}$$

where $\Phi_i(\widehat{\varsigma})$ stands for the numerator and $\aleph_i(\widehat{\varsigma})$ for the denominator of the i-th component of the objective function $\phi(\widehat{\varsigma})$.

3. Dual formulation

This section outlines the formulation of the Mond-Weir dual model of (MFP) and discusses duality relations by modifying the original problem (MFP) and its dual model with the η-approximated method.

The dual model of the problem (MFP) is constructed as

$$\text{(MFD)}\quad \max\quad \phi(\eth) = \left(\frac{\int_{\tau_1}^{\tau_2} \varphi_1(\theta, \eth(\theta), \dot{\eth}(\theta))d\theta}{\int_{\tau_1}^{\tau_2} \psi_1(\theta, \eth(\theta), \dot{\eth}(\theta))d\theta}, \ldots, \frac{\int_{\tau_1}^{\tau_2} \varphi_k(\theta, \eth(\theta), \dot{\eth}(\theta))d\theta}{\int_{\tau_1}^{\tau_2} \psi_k(\theta, \eth(\theta), \dot{\eth}(\theta))d\theta} \right),$$

$$\text{subject to}\quad \eth(\tau_1) = \alpha, \quad \eth(\tau_2) = \beta, \tag{4}$$

$$\sum_{i=1}^{k} \delta_i [\aleph_i(\eth)\varphi_{i_\varsigma}(\theta, \eth, \dot{\eth}) - \Phi_i(\eth)\psi_{i_\varsigma}(\theta, \eth, \dot{\eth})] + (\rho(\theta))^T h_\varsigma(\theta, \eth, \dot{\eth})$$

$$= \frac{d}{d\theta}\left(\sum_{i=1}^{k} \delta_i [\aleph_i(\eth)\varphi_{i_{\dot{\varsigma}}}(\theta, \eth, \dot{\eth}) - \Phi_i(\eth)\psi_{i_{\dot{\varsigma}}}(\theta, \eth, \dot{\eth})] + (\rho(\theta))^T h_{\dot{\varsigma}}(\theta, \eth, \dot{\eth}) \right), \tag{5}$$

$$\int_{\tau_1}^{\tau_2} (\rho(\theta))^T h(\theta, \eth, \dot{\eth})d\theta \geqq 0, \tag{6}$$

$$\delta > 0,\ \ \rho(\theta) \geqq 0\ (t \in \mathscr{I}),\ \ \sum_{i=1}^{k} \delta_i = 1. \tag{7}$$

The set containing all feasible solutions to (MFD) is labelled as Ω. Therefore, for any $(\eth, \delta, \rho(\theta)) \in \Omega$, let us construct the modified problem and its dual model using the η-approximated method as follows:

$$(\mathrm{MFP})_{\eta} \quad \min \quad \int_{\tau_1}^{\tau_2} \left\{ \aleph_1(\widehat{\eth})\varphi_1(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_1(\widehat{\eth})\psi_1(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) \right\} d\theta$$

$$+ \int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \varsigma, \widehat{\eth}) \left\{ \aleph_1(\widehat{\eth})\varphi_{1_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_1(\widehat{\eth})\psi_{1_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) \right\} \right.$$

$$\left. + \left(\frac{d}{d\theta} \eta(\theta, \varsigma, \widehat{\eth}) \right) \left\{ \aleph_1(\widehat{\eth})\varphi_{1_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_1(\widehat{\eth})\psi_{1_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) \right\} \right\} d\theta$$

$$, \ldots, \int_{\tau_1}^{\tau_2} \left\{ \aleph_k(\widehat{\eth})\varphi_k(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_k(\widehat{\eth})\psi_k(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) \right\} d\theta$$

$$+ \int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \varsigma, \widehat{\eth}) \left\{ \aleph_k(\widehat{\eth})\varphi_{k_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_k(\widehat{\eth}\psi_{k_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) \right\} \right.$$

$$\left. + \left(\frac{d}{d\theta} \eta(\theta, \varsigma, \widehat{\eth}) \right) \left\{ \aleph_k(\widehat{\eth})\varphi_{k_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_k(\widehat{\eth}\psi_{k_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) \right\} \right\} d\theta$$

$$\text{subject to} \qquad \varsigma(\tau_1) = \alpha, \quad \varsigma(\tau_2) = \beta, \tag{8}$$

$$\int_{\tau_1}^{\tau_2} \rho(\theta)^T h(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) d\theta + \int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \varsigma, \widehat{\eth}) \rho(\theta)^T h_{\varsigma}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) \right.$$

$$\left. + \left(\frac{d}{d\theta} \eta(\theta, \varsigma, \widehat{\eth}) \right) \rho(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) \right\} d\theta \leqq 0, \;\; t \in \mathscr{I}. \tag{9}$$

$$(\mathrm{MFD})_{\eta} \; \max \int_{\tau_1}^{\tau_2} \left\{ \aleph_1(\widehat{\eth})\varphi_1(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_1(\widehat{\eth})\psi_1(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) \right\} d\theta$$

$$+ \int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \eth, \widehat{\eth}) \left\{ \aleph_1(\widehat{\eth})\varphi_{1_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_1(\widehat{\eth})\psi_{1_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) \right\} \right.$$

$$\left. + \left(\frac{d}{d\theta} \eta(\theta, \eth, \widehat{\eth}) \right) \left\{ \aleph_1(\widehat{\eth})\varphi_{1_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_1(\widehat{\eth})\psi_{1_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) \right\} \right\} d\theta$$

$$, \ldots, \int_{\tau_1}^{\tau_2} \left\{ \aleph_k(\widehat{ð})\varphi_k(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_k(\widehat{ð})\psi_k(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right\} d\theta$$

$$+ \int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, ð, \widehat{ð}) \left\{ \aleph_k(\widehat{ð})\varphi_{k_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_k(\widehat{ð})\psi_{k_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right\} \right.$$

$$\left. + \left(\frac{d}{d\theta}\eta(\theta, ð, \widehat{ð}) \right) \left\{ \aleph_k(\widehat{ð})\varphi_{k_{\dot{\varsigma}}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_k(\widehat{ð})\psi_{k_{\dot{\varsigma}}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right\} \right\} d\theta$$

$$\text{subject to} \quad ð(\tau_1) = \alpha, \quad ð(\tau_2) = \beta, \tag{10}$$

$$\sum_{i=1}^{k} \delta_i [\aleph_i(\widehat{ð})\varphi_{i_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð})\psi_{i_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}})] + (\rho(\theta))^T h_\varsigma(\theta, \widehat{ð}, \dot{\widehat{ð}})$$

$$= \frac{d}{d\theta}\left(\sum_{i=1}^{k} \delta_i [\aleph_i(\widehat{ð})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{ð}, \dot{\widehat{ð}})] + (\rho(\theta))^T h_{\dot{\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right), \tag{11}$$

$$\int_{\tau_1}^{\tau_2} \rho(\theta)^T h(\theta, \widehat{ð}, \dot{\widehat{ð}}) d\theta + \int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, ð, \widehat{ð}) \rho(\theta)^T h_\varsigma(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right.$$

$$\left. + \left(\frac{d}{d\theta}\eta(\theta, ð, \widehat{ð}) \right) \rho(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right\} d\theta \geqq 0, \tag{12}$$

$$\delta > 0, \ \rho(\theta) \geqq 0 \ (\theta \in \mathscr{I}), \ \sum_{i=1}^{k} \delta_i = 1. \tag{13}$$

Remark 1 The set containing all feasible solutions to $(\text{MFP})_\eta$ and $(\text{MFD})_\eta$ is the same as the original problem (MFP) and its dual model (MFD).

Example 1 Let X and ς: $\mathscr{I} \rightarrow \mathscr{I}$, $\mathscr{I} = [0, 1]$ be the sets of continuously differentiable functions. The pair of primal and dual models for the multiobjective variational problem are characterized by:

$$\text{(MFP1)} \quad \min \quad \phi(\varsigma) = \left(\frac{\int_0^1 \left(\theta\varsigma^2(\theta) + \sin\varsigma(\theta) - 1 \right) d\theta}{\int_0^1 \left(\varsigma^4(\theta) + \varsigma(\theta) + 1 \right) d\theta}, \ \frac{\int_0^1 \left(\theta\varsigma^2(\theta) - 2\cos\varsigma(\theta) + 1 \right) d\theta}{\int_0^1 \left(\varsigma^2(\theta) + \varsigma(\theta) + 1 \right) d\theta} \right)$$

subject to $h(\theta, \varsigma, \dot{\varsigma}) = \varsigma^2(\theta) - \varsigma(\theta) \leqq 0,$

$\varsigma(0) = \varsigma(1) = 0.$

Let $\Upsilon := \{\varsigma \in X\colon \varsigma(0) = \varsigma(1) = 0, \text{ and } 0 \leqq \varsigma(\theta) \leqq 1, \text{ where } \theta \in \mathscr{I}\}$ stands for the set of feasible solutions. The function $\eta\colon \mathscr{I} \times X \times X \to \mathbb{R}$ is specified by

$$\eta(\theta, \varsigma, \widehat{\varsigma}) = \varsigma(\theta) + \widehat{\varsigma}(\theta). \tag{14}$$

Let $\rho(\theta) = \frac{3}{2},\ \delta_1 = \frac{1}{2},\ \delta_2 = \frac{1}{2}$ and consider $\widehat{\varsigma} = 0$.

Now the dual of (MFP1) is constructed by utilizing the expression given in (4)-(7).

The dual model:

$$\text{(MFD1)}\ \max \quad \phi(\eth) = \left(\frac{\int_0^1 \left(\theta\eth^2(\theta) + \sin\eth(\theta) - 1\right) d\theta}{\int_0^1 \left(\eth^4(\theta) + \eth(\theta) + 1\right) d\theta},\ \frac{\int_0^1 \left(\theta\eth^2(\theta) - 2\cos\eth(\theta) + 1\right) d\theta}{\int_0^1 \left(\eth^2(\theta) + \eth(\theta) + 1\right) d\theta} \right)$$

subject to $\eth(0) = \eth(1) = 0,$

$$\left(\eth^4(\theta) + \eth(\theta) + 1\right)\left(2t\eth(\theta) + \cos\eth(\theta)\right) - \left(\theta\eth^2(\theta) + \sin\eth(\theta) - 1\right)\left(4\eth^3(\theta) + 1\right)$$

$$+ \left(\eth^2(\theta) + \eth(\theta) + 1\right)\left(2t\eth(\theta) + 2\sin\eth(\theta)\right) - \left(\theta\eth^2(\theta) - 2\cos\eth(\theta) + 1\right)\left(2\eth(\theta) + 1\right)$$

$$+ 2\rho(\theta)(2\eth(\theta) - 1) = 0,$$

$$\int_0^1 \rho(\theta)^T (\eth^2(\theta) - \eth(\theta)) \geqq 0,$$

$$\rho(\theta)^T \geqq 0,\ t \in \mathscr{I}.$$

Next, let us construct the modified problem and its dual pairs for the feasible point $(\widehat{\eth}, \delta_1, \delta_2, \rho(\theta)) = \left(0, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right)$ as follows:

$$(\text{MFP1})_\eta \quad \min \left(\int_0^1 2\varsigma(\theta) d\theta, \int_0^1 \varsigma(\theta) d\theta \right)$$

$$\text{subject to} \quad \varsigma(0) = \varsigma(1) = 0,$$

$$\int_0^1 -\varsigma(\theta)\rho(\theta)^T d\theta \leqq 0.$$

$$(\text{MFD1})_\eta \quad \max \left(\int_0^1 2ð(\theta) d\theta, \int_0^1 ð(\theta) d\theta \right)$$

$$\text{subject to} \quad ð(0) = ð(1) = 0,$$

$$3 - 2\rho(\theta) = 0,$$

$$\int_0^1 -ð(\theta)\rho(\theta)^T d\theta \geqq 0.$$

$$\rho(\theta^T) \geqq 0.$$

Clearly, $\widehat{\varsigma}(\theta) = 0$ and $(\widehat{ð}, \delta_1, \delta_2, \rho(\theta)^T) = \left(0, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right)$ are feasible solutions to the modified problem and its dual, respectively.

Note: In Example 1, the objective functions of the primal and its dual are nonconvex and nonlinear, which can be easily verified. Moreover, we observe that after modifying the considered problem and its dual pair using the η-approximated method, the objective function reduces to convex as well as linear. Also, the fractional problem simplifies to nonfractional ones. Hence, one can conclude that in some cases, the modified fractional variational problem is simpler than the original fractional variational problem.

Next, let us derive the duality theorems for the original problem (MFP) and the modified problem $(\text{MFP})_\eta$ with the help of its dual models (MFD) and $(\text{MFD})_\eta$, respectively.

Proposition 1 (Weak duality for modified problems under the weak Pareto solution) Let ς be the feasible solution to the problem $(\text{MFP})_\eta$ and $(ð, \delta, \rho(\theta))$ be the feasible solution to the dual $(\text{MFD})_\eta$. Then the following cannot hold:

$$\int_{\tau_1}^{\tau_2} \left\{ \aleph_i(\widehat{ð})\varphi_i(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð})\psi_i(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right\} d\theta$$

$$+ \int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \varsigma, \widehat{ð}) \left\{ \aleph_i(\widehat{ð})\varphi_{i_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð})\psi_{i_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right\} \right.$$

$$+\left(\frac{d}{d\theta}\eta(\theta, \varsigma, \widehat{\eth})\right)\left\{\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right\}\Bigg\} d\theta$$

$$< \int_{\tau_1}^{\tau_2} \left\{\aleph_i(\widehat{\eth})\varphi_i(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_i(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right\} d\theta$$

$$+\int_{\tau_1}^{\tau_2} \Bigg\{\eta(\theta, \eth, \widehat{\eth}) \left\{\aleph_i(\widehat{\eth})\varphi_{i_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right\}$$

$$+\left(\frac{d}{d\theta}\eta(\theta, \eth, \widehat{\eth})\right)\left\{\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right\}\Bigg\} d\theta, \forall i \in \mathscr{K}.$$

Proof. Let us assume the opposite of the result that the above inequality holds. Hence, we get

$$\int_{\tau_1}^{\tau_2} \Bigg\{\eta(\theta, \varsigma, \widehat{\eth}) \left\{\aleph_i(\widehat{\eth})\varphi_{i_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right\}$$

$$+\left(\frac{d}{d\theta}\eta(\theta, \varsigma, \widehat{\eth})\right)\left\{\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right\}\Bigg\} d\theta$$

$$< \int_{\tau_1}^{\tau_2} \Bigg\{\eta(\theta, \eth, \widehat{\eth}) \left\{\aleph_i(\widehat{\eth})\varphi_{i_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right\}$$

$$+\left(\frac{d}{d\theta}\eta(\theta, \eth, \widehat{\eth})\right)\left\{\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right\}\Bigg\} d\theta, \forall i \in \mathscr{K}.$$

Multiplying both sides of the above inequality with the Lagrange multiplier $\delta > 0$ and summing up from $i = \{1, \ldots, k\}$, we obtain

$$\int_{\tau_1}^{\tau_2} \Bigg\{\eta(\theta, \varsigma, \widehat{\eth})\left\{\sum_{i=1}^{k} \delta_i [\aleph_i(\widehat{\eth})\varphi_{i_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})]\right\}$$

$$+\left(\frac{d}{d\theta}\eta(\theta, \varsigma, \widehat{\eth})\right)\left\{\sum_{i=1}^{k} \delta_i [\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})]\right\}\Bigg\} d\theta$$

$$< \int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \eth, \widehat{\eth}) \left\{ \sum_{i=1}^{k} \delta_i [\aleph_i(\widehat{\eth})\varphi_{i_\varsigma}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_\varsigma}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})] \right\} \right.$$

$$\left. + \left(\frac{d}{d\theta}\eta(\theta, \eth, \widehat{\eth}) \right) \left\{ \sum_{i=1}^{k} \delta_i [\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})] \right\} \right\} d\theta. \tag{15}$$

On the other hand, since (ς) and ($\eth$, δ, $\rho(\theta)$) are feasible solutions to (MFP)η and (MFD)η, respectively. Therefore, one can have

$$\int_{\tau_1}^{\tau_2} \rho(\theta)^T h(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) d\theta + \int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \varsigma, \widehat{\eth}) \rho(\theta)^T h_\varsigma(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) \right.$$

$$\left. + \left(\frac{d}{d\theta}\eta(\theta, \varsigma, \widehat{\eth}) \right) \rho(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) \right\} d\theta \leqq 0, \tag{16}$$

and

$$\int_{\tau_1}^{\tau_2} \rho(\theta)^T h(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) d\theta + \int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \eth, \widehat{\eth}) \rho(\theta)^T h_\varsigma(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) \right.$$

$$\left. + \left(\frac{d}{d\theta}\eta(\theta, \eth, \widehat{\eth}) \right) \rho(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) \right\} d\theta \geqq 0. \tag{17}$$

By combining the inequality (16) with the inequality (17), one can obtain

$$\int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \varsigma, \widehat{\eth}) \rho(\theta)^T h_\varsigma(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) + \left(\frac{d}{d\theta}\eta(\theta, \varsigma, \widehat{\eth}) \right) \rho(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) \right\} d\theta$$

$$\leqq \int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \eth, \widehat{\eth}) \rho(\theta)^T h_\varsigma(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) + \left(\frac{d}{d\theta}\eta(\theta, \eth, \widehat{\eth}) \right) \rho(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) \right\} d\theta. \tag{18}$$

Further, multiplying both sides of equation (11) with $\eta(\theta, \eth, \widehat{\eth})$ and integrating within the limit τ_1 to τ_2, we get

$$\int_{\tau_1}^{\tau_2} \eta(\theta, \eth, \widehat{\eth}) \left\{ \sum_{i=1}^{k} \delta_i [\aleph_i(\widehat{\eth})\varphi_{i_\varsigma}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_\varsigma}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})] \right.$$

$$+(\rho(\theta))^T h_{\varsigma}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\Bigg\}d\theta$$

$$=\int_{\tau_1}^{\tau_2} \eta(\theta, \eth, \widehat{\eth})\Bigg\{\frac{d}{d\theta}\Bigg(\sum_{i=1}^{k}\delta_i\,[\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})]$$

$$+(\rho(\theta))^T h_{\dot{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\Bigg)\Bigg\}d\theta.$$

With the help of integration by parts in the above equation, we get

$$\int_{\tau_1}^{\tau_2} \eta(\theta, \eth, \widehat{\eth})\Bigg\{\sum_{i=1}^{k}\delta_i\,[\aleph_i(\widehat{\eth})\varphi_{i_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})]$$

$$+(\rho(\theta))^T h_{\varsigma}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\Bigg\}d\theta$$

$$=\Bigg[\eta(\theta, \eth, \widehat{\eth})\Bigg\{\sum_{i=1}^{k}\delta_i\,[\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})]$$

$$+(\rho(\theta))^T h_{\dot{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\Bigg\}\Bigg]_{\tau_1}^{\tau_2}$$

$$-\int_{\tau_1}^{\tau_2}\left(\frac{d}{d\theta}\eta(\theta, \eth, \widehat{\eth})\right)\Bigg\{\sum_{i=1}^{k}\delta_i\,[\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})]$$

$$+(\rho(\theta))^T h_{\dot{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\Bigg\}d\theta. \tag{19}$$

Equation (19) together with (10) and $\eta(\theta, \widehat{\eth}, \widehat{\eth}) = 0$ gives

$$\int_{\tau_1}^{\tau_2} \eta(\theta, \eth, \widehat{\eth})\Bigg\{\sum_{i=1}^{k}\delta_i\,[\aleph_i(\widehat{\eth})\varphi_{i_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})]$$

$$+(\rho(\theta))^T h_{\varsigma}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\Bigg\}d\theta$$

$$= -\int_{\tau_1}^{\tau_2} \left(\frac{d}{d\theta}\eta(\theta, \eth, \widehat{\eth})\right)\left\{ \sum_{i=1}^{k} \delta_i \left[\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right]\right.$$

$$\left. + (\rho(\theta))^T h_{\dot{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right\} d\theta. \tag{20}$$

The equation (20) can be rephrased as

$$\int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \eth, \widehat{\eth})\left\{ \sum_{i=1}^{k} \delta_i \left[\aleph_i(\widehat{\eth})\varphi_{i_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right]\right\}\right.$$

$$\left. + \left(\frac{d}{d\theta}\eta(\theta, \eth, \widehat{\eth})\right)\left\{ \sum_{i=1}^{k} \delta_i \left[\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right]\right\}\right\} d\theta$$

$$= -\int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \eth, \widehat{\eth})(\rho(\theta))^T h_{\varsigma}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right.$$

$$\left. + \left(\frac{d}{d\theta}\eta(\theta, \eth, \widehat{\eth})\right)(\rho(\theta))^T h_{\dot{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right\} d\theta. \tag{21}$$

Similarily, for the feasible point $(\varsigma, \delta, \rho(\theta))$ in $(\text{MFD})_{\eta}$, one can have

$$\int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \varsigma, \widehat{\eth})\left\{ \sum_{i=1}^{k} \delta_i \left[\aleph_i(\widehat{\eth})\varphi_{i_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right]\right\}\right.$$

$$\left. + \left(\frac{d}{d\theta}\eta(\theta, \varsigma, \widehat{\eth})\right)\left\{ \sum_{i=1}^{k} \delta_i \left[\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right]\right\}\right\} d\theta$$

$$= -\int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \varsigma, \widehat{\eth})(\rho(\theta))^T h_{\varsigma}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right.$$

$$\left. + \left(\frac{d}{d\theta}\eta(\theta, \varsigma, \widehat{\eth})\right)(\rho(\theta))^T h_{\dot{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right\} d\theta. \tag{22}$$

Inequality (18) together with the equations (21) and (22) gives

$$\int_{\tau_1}^{\tau_2}\left\{\eta(\theta,\varsigma,\widehat{\eth})\left\{\sum_{i=1}^{k}\delta_i[\aleph_i(\widehat{\eth})\varphi_{i_\varsigma}(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i(\widehat{\eth})\psi_{i_\varsigma}(\theta,\widehat{\eth},\dot{\widehat{\eth}})]\right\}\right.$$

$$\left.+\left(\frac{d}{d\theta}\eta(\theta,\varsigma,\widehat{\eth})\right)\left\{\sum_{i=1}^{k}\delta_i[\aleph_i(\widehat{\eth})\varphi_{i_{\dot\varsigma}}(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i(\widehat{\eth})\psi_{i_{\dot\varsigma}}(\theta,\widehat{\eth},\dot{\widehat{\eth}})]\right\}\right\}d\theta$$

$$\geqq\int_{\tau_1}^{\tau_2}\left\{\eta(\theta,\eth,\widehat{\eth})\left\{\sum_{i=1}^{k}\delta_i[\aleph_i(\widehat{\eth})\varphi_{i_\varsigma}(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i(\widehat{\eth})\psi_{i_\varsigma}(\theta,\widehat{\eth},\dot{\widehat{\eth}})]\right\}\right.$$

$$\left.+\left(\frac{d}{d\theta}\eta(\theta,\eth,\widehat{\eth})\right)\left\{\sum_{i=1}^{k}\delta_i[\aleph_i(\widehat{\eth})\varphi_{i_{\dot\varsigma}}(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i(\widehat{\eth})\psi_{i_{\dot\varsigma}}(\theta,\widehat{\eth},\dot{\widehat{\eth}})]\right\}\right\}d\theta,$$

which opposes the inequality (15). Thus, the proof is complete.

Proposition 2 (Weak duality for modified problems under the Pareto solution) Let ς be the feasible solution to the problem $(\text{MFP})_\eta$ and $(\eth,\delta,\rho(\theta))$ be the feasible solution to the dual $(\text{MFD})_\eta$. Then the following cannot hold:

$$\int_{\tau_1}^{\tau_2}\left\{\aleph_i(\widehat{\eth})\varphi_i(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i(\widehat{\eth})\psi_i(\theta,\widehat{\eth},\dot{\widehat{\eth}})\right\}d\theta$$

$$+\int_{\tau_1}^{\tau_2}\left\{\eta(\theta,\varsigma,\widehat{\eth})\left\{\aleph_i(\widehat{\eth})\varphi_{i_\varsigma}(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i(\widehat{\eth})\psi_{i_\varsigma}(\theta,\widehat{\eth},\dot{\widehat{\eth}})\right\}\right.$$

$$\left.+\left(\frac{d}{d\theta}\eta(\theta,\varsigma,\widehat{\eth})\right)\left\{\aleph_i(\widehat{\eth})\varphi_{i_{\dot\varsigma}}(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i(\widehat{\eth})\psi_{i_{\dot\varsigma}}(\theta,\widehat{\eth},\dot{\widehat{\eth}})\right\}\right\}d\theta$$

$$\leqq\int_{\tau_1}^{\tau_2}\left\{\aleph_i(\widehat{\eth})\varphi_i(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i(\widehat{\eth})\psi_i(\theta,\widehat{\eth},\dot{\widehat{\eth}})\right\}d\theta$$

$$+\int_{\tau_1}^{\tau_2}\left\{\eta(\theta,\eth,\widehat{\eth})\left\{\aleph_i(\widehat{\eth})\varphi_{i_\varsigma}(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i(\widehat{\eth})\psi_{i_\varsigma}(\theta,\widehat{\eth},\dot{\widehat{\eth}})\right\}\right.$$

$$\left.+\left(\frac{d}{d\theta}\eta(\theta,\eth,\widehat{\eth})\right)\left\{\aleph_i(\widehat{\eth})\varphi_{i_{\dot\varsigma}}(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i(\widehat{\eth})\psi_{i_{\dot\varsigma}}(\theta,\widehat{\eth},\dot{\widehat{\eth}})\right\}\right\}d\theta,\ \forall i\in\mathscr{K},$$

with at least one strict inequality.

Proof. The proof of this proposition follows from Proposition 1.

Theorem 2 (Weak duality for original problems under the weak Pareto solution) Let $\widehat{\varsigma}$ be the feasible solution to the problem (MFP) and $(\widehat{\eth}, \widehat{\delta}, \widehat{\rho}(\theta))$ be the feasible solution to the dual (MFD). Further, suppose that $\int_{\tau_1}^{\tau_2} \{\aleph_i(\varsigma)\varphi_i(\theta, \varsigma, \dot{\varsigma}) - \Phi_i(\varsigma)\psi_i(\theta, \varsigma, \dot{\varsigma})\}d\theta$ $(i \in \mathscr{K})$ and $\int_{\tau_1}^{\tau_2} \widehat{\rho}(\theta)^T h(\theta, \varsigma, \dot{\varsigma})d\theta$ are invex functions at a feasible point $\widehat{\eth}$ on X with regard to η. Then the following inequality cannot hold:

$$\frac{\int_{\tau_1}^{\tau_2} \varphi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})d\theta}{\int_{\tau_1}^{\tau_2} \psi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})d\theta} < \frac{\int_{\tau_1}^{\tau_2} \varphi_i(\theta, \widehat{\eth}, \dot{\widehat{\eth}})d\theta}{\int_{\tau_1}^{\tau_2} \psi_i(\theta, \widehat{\eth}, \dot{\widehat{\eth}})d\theta}, \forall i \in \mathscr{K}.$$

Proof. Let us assume the contrary of the result that

$$\frac{\int_{\tau_1}^{\tau_2} \varphi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})d\theta}{\int_{\tau_1}^{\tau_2} \psi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})d\theta} < \frac{\int_{\tau_1}^{\tau_2} \varphi_i(\theta, \widehat{\eth}, \dot{\widehat{\eth}})d\theta}{\int_{\tau_1}^{\tau_2} \psi_i(\theta, \widehat{\eth}, \dot{\widehat{\eth}})d\theta}, \forall i \in \mathscr{K},$$

holds. That is,

$$\int_{\tau_1}^{\tau_2} \left\{\aleph_i(\widehat{\varsigma})\varphi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right\} d\theta$$

$$< \int_{\tau_1}^{\tau_2} \left\{\aleph_i(\widehat{\eth})\varphi_i(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_i(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right\} d\theta, \ \forall i \in \mathscr{K}. \tag{23}$$

On the other hand, we will show that $\widehat{\varsigma}$ and $(\widehat{\eth}, \widehat{\delta}, \widehat{\rho}(\theta))$ are the sets of feasible solutions to $(\text{MFP})_\eta$ and $(\text{MFD})_\eta$, respectively. Since $\widehat{\varsigma}$ is a feasible solution to (MFP). Hence, we have

$$h(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) \leqq 0.$$

Using the condition $\widehat{\rho}(\theta)^T \geqq 0$ in the above inequality, we get

$$\int_{\tau_1}^{\tau_2} \widehat{\rho}(\theta)^T h(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\, d\theta \leqq 0. \tag{24}$$

Since the function $\int_{\tau_1}^{\tau_2} \widehat{\rho}(\theta)^T h(\theta, \varsigma, \dot{\varsigma})d\theta$ is invex at a point $\widehat{\eth}$ on X with regard to η. So,

$$\int_{\tau_1}^{\tau_2} \widehat{\rho}(\theta)^T h(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})d\theta \geqq \int_{\tau_1}^{\tau_2} \widehat{\rho}(\theta)^T h(\theta, \widehat{ð}, \dot{\widehat{ð}})d\theta$$

$$+\int_{\tau_1}^{\tau_2} \left\{\eta(\theta, \widehat{\varsigma}, \widehat{ð})\widehat{\rho}(\theta)^T h_{\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}}) + \left(\frac{d}{d\theta}\eta(\theta, \widehat{\varsigma}, \widehat{ð})\right)\widehat{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}})\right\}d\theta. \tag{25}$$

According to inequality (24), the inequality (25) reduces to

$$\int_{\tau_1}^{\tau_2} \widehat{\rho}(\theta)^T h(\theta, \widehat{ð}, \dot{\widehat{ð}})d\theta + \int_{\tau_1}^{\tau_2} \left\{\eta(\theta, \widehat{\varsigma}, \widehat{ð})\widehat{\rho}(\theta)^T h_{\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}})\right.$$

$$\left. + \left(\frac{d}{d\theta}\eta(\theta, \widehat{\varsigma}, \widehat{ð})\right)\widehat{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}})\right\}d\theta \leqq 0,$$

which confirms that $\widehat{\varsigma}$ is a feasible solution to $(\text{MFP})_{\eta}$. Moreover, since $(\widehat{ð}, \widehat{\delta}, \widehat{\rho}(\theta))$ is a feasible solution to the dual (MFD). Therefore, from inequality (6), one can obtain

$$\int_{\tau_1}^{\tau_2} (\rho(\theta))^T h(\theta, \widehat{ð}, \dot{\widehat{ð}})d\theta \geqq 0.$$

Using the condition $\eta(\theta, \widehat{ð}, \widehat{ð}) = 0,$ the above inequality can be written as

$$\int_{\tau_1}^{\tau_2} \widehat{\rho}(\theta)^T h(\theta, \widehat{ð}, \dot{\widehat{ð}})d\theta + \int_{\tau_1}^{\tau_2} \left\{\eta(\theta, \widehat{ð}, \widehat{ð})\widehat{\rho}(\theta)^T h_{\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}})\right.$$

$$\left. + \left(\frac{d}{d\theta}\eta(\theta, \widehat{ð}, \widehat{ð})\right)\widehat{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}})\right\}d\theta \geqq 0,$$

which is equivalent to (12). Hence, $(\widehat{ð}, \widehat{\delta}, \widehat{\rho}(\theta))$ is a feasible solution to the dual $(\text{MFD})_{\eta}$. The function $\int_{\tau_1}^{\tau_2} \{\aleph_i(\varsigma)\varphi_i(\theta, \varsigma, \dot{\varsigma}) - \Phi_i(\varsigma)\psi_i(\theta, \varsigma, \dot{\varsigma})\}d\theta$ $(i \in \mathscr{K})$ is invex at a point $\widehat{ð}$ on X with redard to η yields

$$\int_{\tau_1}^{\tau_2} \{\aleph_i(\widehat{\varsigma})\varphi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\}d\theta$$

$$\geqq \int_{\tau_1}^{\tau_2} \{\aleph_i(\widehat{\eth})\varphi_i(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_i(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\} d\theta$$

$$+ \int_{\tau_1}^{\tau_2} \Big\{ \eta(\theta, \widehat{\varsigma}, \widehat{\eth}) \{\aleph_i(\widehat{\eth})\varphi_{i_\varsigma}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_\varsigma}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\}$$

$$+ \left(\frac{d}{d\theta} \eta(\theta, \widehat{\varsigma}, \widehat{\eth}) \right) \{\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\} \Big\} d\theta, \ \forall i \in \mathscr{K}.$$

The above inequality, together with Proposition 1, gives

$$\int_{\tau_1}^{\tau_2} \{\aleph_i(\widehat{\varsigma})\varphi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\} d\theta$$

$$\geqq \int_{\tau_1}^{\tau_2} \Big\{\aleph_i(\widehat{\eth})\varphi_i(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_i(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\Big\} d\theta, \ \forall i \in \mathscr{K},$$

which opposes the inequality (23). Thus, the proof is finalized.

Theorem 3 (Weak duality for original problems under the Pareto solution) Let $\widehat{\varsigma}$ be the feasible solution to the problem (MFP) and $(\widehat{\eth}, \widehat{\delta}, \widehat{\rho}(\theta))$ be the feasible solution to the dual (MFD). Further, suppose that $\int_{\tau_1}^{\tau_2} \{\aleph_i(\varsigma)\varphi_i(\theta, \varsigma, \dot{\varsigma}) - \Phi_i(\varsigma)\psi_i(\theta, \varsigma, \dot{\varsigma})\} d\theta$ $(i \in \mathscr{K})$ is strictly invex, and $\int_{\tau_1}^{\tau_2} \widehat{\rho}(\theta)^T h(\theta, \varsigma, \dot{\varsigma}) d\theta$ is an invex function at a feasible point $\widehat{\eth}$ on X with regard to η. Then the following inequality cannot hold:

$$\frac{\int_{\tau_1}^{\tau_2} \varphi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) d\theta}{\int_{\tau_1}^{\tau_2} \psi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) d\theta} \leqq \frac{\int_{\tau_1}^{\tau_2} \varphi_i(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) d\theta}{\int_{\tau_1}^{\tau_2} \psi_i(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) d\theta}, \ \forall i \in \mathscr{K},$$

with at least one strict inequality.

Proof. The proof of this theorem follows from Theorem 2.

Theorem 4 If the feasible point $\widehat{\varsigma}$ is a weak Pareto solution to (MFP), then it is also a weak Pareto solution to $(\text{MFP})_\eta$.

Proof. Since the feasible point $\widehat{\varsigma}$ is a weak Pareto solution to (MFP). Therefore, from Theorem 1, one can conclude that a vector $\delta > 0$ and $\rho(\theta)\colon \mathscr{I} \to \mathbb{R}^m$ exist and satisfy

$$\sum_{i=1}^{k}\delta_i[\aleph_i(\widehat{\varsigma})\varphi_{i_\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})] + \rho(\theta)^T h_\varsigma(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})$$

$$= \frac{d}{d\theta}\left(\sum_{i=1}^{k}\delta_i[\aleph_i(\widehat{\varsigma})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})] + \rho(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right), \tag{26}$$

$$\rho(\theta)^T h(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) = 0, \tag{27}$$

$$\delta > 0, \ \rho(\theta) \geqq 0 \ (\theta \in \mathscr{I}), \ \sum_{i=1}^{k}\delta_i = 1. \tag{28}$$

Let us suppose that $\widehat{\varsigma}$ is not a weak Pareto solution to $(\mathrm{MFP})_\eta$. Then $\widetilde{\varsigma} \in \Upsilon$ exists, so that

$$\int_{\tau_1}^{\tau_2} \left\{\aleph_i(\widehat{\varsigma})\varphi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right\} d\theta$$

$$+ \int_{\tau_1}^{\tau_2} \Bigg\{\eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma}) \left\{\aleph_i(\widehat{\varsigma})\varphi_{i_\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right\}$$

$$+ \left(\frac{d}{d\theta}\eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma})\right) \left\{\aleph_i(\widehat{\varsigma})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right\} \Bigg\} d\theta$$

$$< \int_{\tau_1}^{\tau_2} \left\{\aleph_i(\widehat{\varsigma})\varphi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right\} d\theta$$

$$+ \int_{\tau_1}^{\tau_2} \Bigg\{\eta(\theta, \widehat{\varsigma}, \widehat{\varsigma}) \left\{\aleph_i(\widehat{\varsigma})\varphi_{i_\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right\}$$

$$+ \left(\frac{d}{d\theta}\eta(\theta, \widehat{\varsigma}, \widehat{\varsigma})\right) \left\{\aleph_i(\widehat{\varsigma})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right\} \Bigg\} d\theta, \ \forall i \in \mathscr{K}.$$

Applying the condition $\eta(\theta, \widehat{\varsigma}, \widehat{\varsigma}) = 0,$ in the above inequality, we get

$$\int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma}) \left\{ \aleph_i(\widehat{\varsigma}) \varphi_{i_\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma}) \psi_{i_\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) \right\} \right.$$

$$\left. + \left(\frac{d}{d\theta} \eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma}) \right) \left\{ \aleph_i(\widehat{\varsigma}) \varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma}) \psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) \right\} \right\} d\theta < 0, \ \forall i \in \mathscr{K}.$$

Multiplying both sides of the above inequality by $\delta > 0$ and then suming up from $i = 1, \ldots, k$ with at least one strict inequality, we get

$$\int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma}) \left\{ \sum_{i=1}^{k} \delta_i [\aleph_i(\widehat{\varsigma}) \varphi_{i_\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma}) \psi_{i_\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})] \right\} \right.$$

$$\left. + \left(\frac{d}{d\theta} \eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma}) \right) \left\{ \sum_{i=1}^{k} \delta_i [\aleph_i(\widehat{\varsigma}) \varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma}) \psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})] \right\} \right\} d\theta$$

$$< 0. \tag{29}$$

Since $\widetilde{\varsigma}$ is a feasible solution to the modified problem $(\text{MFP})_\eta$. Therefore, one can have

$$\int_{\tau_1}^{\tau_2} \rho(\theta)^T h(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) d\theta + \int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma}) \rho(\theta)^T h_\varsigma(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) \right.$$

$$\left. + \left(\frac{d}{d\theta} \eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma}) \right) \rho(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) \right\} d\theta \leqq 0.$$

Equation (27), together with the above inequality, yields

$$\int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma}) \rho(\theta)^T h_\varsigma(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) + \left(\frac{d}{d\theta} \eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma}) \right) \rho(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) \right\} d\theta \leqq 0. \tag{30}$$

On the flip side, multiplying the equation (26) with $\eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma})$ and integrating within the limits τ_1 and τ_2, we get

$$\int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma}) \left(\sum_{i=1}^{k} \delta_i [\aleph_i(\widehat{\varsigma}) \varphi_{i_\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma}) \psi_{i_\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})] \right. \right.$$

$$+\rho(\theta)^T h_{\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\Bigg)\Bigg\}d\theta$$

$$=\int_{\tau_1}^{\tau_2} \eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma})\Bigg\{\frac{d}{d\theta}\Bigg(\sum_{i=1}^{k}\delta_i\,[\aleph_i(\widehat{\varsigma})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})-\Phi_i\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})]$$

$$+\rho(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\Bigg)\Bigg\}d\theta.$$

Integrating by parts and using the condition $\eta(\theta, \widehat{\varsigma}, \widehat{\varsigma})=0$ in the above equation, we get

$$\int_{\tau_1}^{\tau_2} \eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma})\Bigg\{\sum_{i=1}^{k}\delta_i\,[\aleph_i(\widehat{\varsigma})\varphi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})-\Phi_i(\widehat{\varsigma})\psi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})]$$

$$+\rho(\theta)^T h_{\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\Bigg\}d\theta$$

$$=-\int_{\tau_1}^{\tau_2}\left(\frac{d}{d\theta}\eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma})\right)\Bigg\{\sum_{i=1}^{k}\delta_i\,[\aleph_i(\widehat{\varsigma})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})-\Phi_i(\widehat{\varsigma})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})]$$

$$+\rho(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\Bigg\}d\theta,$$

which can be rephrased as

$$\int_{\tau_1}^{\tau_2}\Bigg\{\eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma})\Bigg\{\sum_{i=1}^{k}\delta_i\,[\aleph_i(\widehat{\varsigma})\varphi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})-\Phi_i(\widehat{\varsigma})\psi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})]\Bigg\}$$

$$+\left(\frac{d}{d\theta}\eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma})\right)\Bigg\{\sum_{i=1}^{k}\delta_i\,[\aleph_i(\widehat{\varsigma})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})-\Phi_i(\widehat{\varsigma})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})]\Bigg\}\Bigg\}d\theta$$

$$=-\int_{\tau_1}^{\tau_2}\Bigg\{\eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma})\rho(\theta)^T h_{\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})+\left(\frac{d}{d\theta}\eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma})\right)\rho(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\Bigg\}d\theta.$$

Using inequality (30), the above inequality reduces to

$$\int_{\tau_1}^{\tau_2}\left\{\eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma})\left\{\sum_{i=1}^{k}\delta_i[\aleph_i(\widehat{\varsigma})\varphi_{i_\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})]\right\}\right.$$

$$\left. + \left(\frac{d}{d\theta}\eta(\theta, \widetilde{\varsigma}, \widehat{\varsigma})\right)\left\{\sum_{i=1}^{k}\delta_i[\aleph_i(\widehat{\varsigma})\varphi_{i_{\dot\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_{\dot\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})]\right\}\right\}d\theta \geqq 0,$$

which contradicts (29). Thus, the proof is finalized.

Theorem 5 If the feasible point $\widehat{\varsigma}$ is a Pareto solution to (MFP), then it is also a Pareto solution to $(\mathrm{MFP})_\eta$.

Proof. The proof of this theorem follows from Theorem 4.

Proposition 3 (Strong duality for modified problems under the weak Pareto solution) Let the feasible point $\widehat{\varsigma}$ be a weak Pareto solution to $(\mathrm{MFP})_\eta$. Then the Lagrange multiplier $\widehat{\delta} > 0$ and a smooth piecewise function $\widehat{\rho}\colon \mathscr{I} \to \mathbb{R}^m_+$ exist, so that $(\widehat{\varsigma}, \widehat{\delta}, \widehat{\rho})$ is a weak Pareto solution to the modified dual problem $(\mathrm{MFD})_\eta$.

Proof. Since the feasible point $\widehat{\varsigma}$ is a weak Pareto solution to $(\mathrm{MFP})_\eta$. Therefore, there exists the Lagrange multiplier $\widehat{\delta} > 0$ and a smooth piecewise function $\widehat{\rho}\colon \mathscr{I} \to \mathbb{R}^m_+$ satisfying (26)-(28). Using equations (26) and (27), along with the condition $\eta(\theta, \widehat{\varsigma}, \widehat{\varsigma}) = 0$, we obtain

$$\sum_{i=1}^{k}\widehat{\delta}_i[\aleph_i(\widehat{\varsigma})\varphi_{i_\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})] + \widehat{\rho}(\theta)^T h_\varsigma(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})$$

$$= \frac{d}{d\theta}\left(\sum_{i=1}^{k}\widehat{\delta}_i[\aleph_i(\widehat{\varsigma})\varphi_{i_{\dot\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i\psi_{i_{\dot\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})] + \widehat{\rho}(\theta)^T h_{\dot\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right),$$

$$\int_{\tau_1}^{\tau_2}\widehat{\rho}(\theta)^T h(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})d\theta + \int_{\tau_1}^{\tau_2}\eta(\theta, \widehat{\varsigma}, \widehat{\varsigma})\widehat{\rho}(\theta)^T h_\varsigma(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})$$

$$+ \left(\frac{d}{d\theta}\eta(\theta, \widehat{\varsigma}, \widehat{\varsigma})\right)\widehat{\rho}(\theta)^T h_{\dot\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})d\theta = 0,$$

which validates the feasibility of $(\widehat{\varsigma}, \widehat{\delta}, \widehat{\rho})$ to $(\mathrm{MFD})_\eta$. Now, let us assume to the contrary that $(\widehat{\varsigma}, \widehat{\delta}, \widehat{\rho})$ is not a weak Pareto solution to $(\mathrm{MFD})_\eta$. Then $(\widehat{\eth}, \widetilde{\delta}, \widehat{\rho}) \in \Omega$ exists and satisfying

$$\int_{\tau_1}^{\tau_2}\left\{\aleph_i(\widehat{\varsigma})\varphi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right\}d\theta$$

$$+ \int_{\tau_1}^{\tau_2}\left\{\eta(\theta, \widehat{\eth}, \widehat{\varsigma})\left\{\aleph_i(\widehat{\varsigma})\varphi_{i_\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right\}\right.$$

$$+\left(\frac{d}{d\theta}\eta(\theta, \widehat{\eth}, \widehat{\varsigma})\right)\left\{\aleph_i(\widehat{\varsigma})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right\}\Bigg\}d\theta$$

$$> \int_{\tau_1}^{\tau_2} \left\{\aleph_i(\widehat{\varsigma})\varphi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_i(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right\} d\theta$$

$$+\int_{\tau_1}^{\tau_2} \Bigg\{\eta(\theta, \widehat{\varsigma}, \widehat{\varsigma})\left\{\aleph_i(\widehat{\varsigma})\varphi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right\}$$

$$+\left(\frac{d}{d\theta}\eta(\theta, \widehat{\varsigma}, \widehat{\varsigma})\right)\left\{\aleph_i(\widehat{\varsigma})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right\}\Bigg\}d\theta, \ \forall i \in \mathscr{K}.$$

With the help of the condition $\eta(\theta, \widehat{\varsigma}, \widehat{\varsigma}) = 0$, the inequality mentioned above becomes

$$\int_{\tau_1}^{\tau_2} \Bigg\{\eta(\theta, \widehat{\eth}, \widehat{\varsigma})\left\{\aleph_i(\widehat{\varsigma})\varphi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right\}$$

$$+\left(\frac{d}{d\theta}\eta(\theta, \widehat{\eth}, \widehat{\varsigma})\right)\left\{\aleph_i(\widehat{\varsigma})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right\}\Bigg\}d\theta > 0, \ \forall i \in \mathscr{K}.$$

Multiplying both sides with $\widehat{\delta} > 0$ and adding from $i = 1, \ldots, k,$ we get

$$\int_{\tau_1}^{\tau_2} \Bigg\{\eta(\theta, \widehat{\eth}, \widehat{\varsigma})\Bigg\{\sum_{i=1}^{k}\widehat{\delta}_i[\aleph_i(\widehat{\varsigma})\varphi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})]\Bigg\}$$

$$+\left(\frac{d}{d\theta}\eta(\theta, \widehat{\eth}, \widehat{\varsigma})\right)\Bigg\{\sum_{i=1}^{k}\widehat{\delta}_i[\aleph_i(\widehat{\varsigma})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})]\Bigg\}\Bigg\}d\theta$$

$$> 0. \tag{31}$$

As $(\widehat{\varsigma}, \widehat{\delta}, \widehat{\rho})$ is a feasible point to $(\mathrm{MFD})_{\eta}$, therefore, using (11) we get

$$\sum_{i=1}^{k}\widehat{\delta}_i[\aleph_i(\widehat{\varsigma})\varphi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}}) - \Phi_i(\widehat{\varsigma})\psi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})] + \widehat{\rho}(\theta)^T h_{\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})$$

$$=\frac{d}{d\theta}\left(\sum_{i=1}^{k}\widehat{\delta}_i[\aleph_i(\widehat{\varsigma})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})-\Phi_i(\widehat{\varsigma})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})]+\widehat{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right).$$

Multiplying the above equation with $\eta(\theta, \widehat{\eth}, \widehat{\varsigma})$ and integrating within limits τ_1 and τ_2, we get

$$\int_{\tau_1}^{\tau_2}\eta(\theta, \widehat{\eth}, \widehat{\varsigma})\left\{\sum_{i=1}^{k}\widehat{\delta}_i[\aleph_i(\widehat{\varsigma})\varphi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})-\Phi_i(\widehat{\varsigma})\psi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})]\right.$$

$$\left.+\widehat{\rho}(\theta)^T h_{\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right\}d\theta$$

$$=\int_{\tau_1}^{\tau_2}\eta(\theta, \widehat{\eth}, \widehat{\varsigma})\left\{\frac{d}{d\theta}\left(\sum_{i=1}^{k}\widehat{\delta}_i[\aleph_i(\widehat{\varsigma})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})-\Phi_i(\widehat{\varsigma})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})]\right.\right.$$

$$\left.\left.+\widehat{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right)\right\}d\theta.$$

Using integration by parts and the condition $\eta(\theta, \widehat{\varsigma}, \widehat{\varsigma})=0$, the above equation gives

$$\int_{\tau_1}^{\tau_2}\eta(\theta, \widehat{\eth}, \widehat{\varsigma})\left\{\sum_{i=1}^{k}\widehat{\delta}_i[\aleph_i(\widehat{\varsigma})\varphi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})-\Phi_i(\widehat{\varsigma})\psi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})]\right.$$

$$\left.+\widehat{\rho}(\theta)^T h_{\varsigma}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right\}d\theta$$

$$=-\int_{\tau_1}^{\tau_2}\left(\frac{d}{d\theta}\eta(\theta, \widehat{\eth}, \widehat{\varsigma})\right)\left\{\sum_{i=1}^{k}\widehat{\delta}_i[\aleph_i(\widehat{\varsigma})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})-\Phi_i(\widehat{\varsigma})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})]\right.$$

$$\left.+\widehat{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})\right\}d\theta,$$

which can be rephrased as

$$\int_{\tau_1}^{\tau_2}\left\{\eta(\theta, \widehat{\eth}, \widehat{\varsigma})\left\{\sum_{i=1}^{k}\widehat{\delta}_i[\aleph_i(\widehat{\varsigma})\varphi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})-\Phi_i(\widehat{\varsigma})\psi_{i_{\varsigma}}(\theta, \widehat{\varsigma}, \dot{\widehat{\varsigma}})]\right\}\right.$$

$$+\left(\frac{d}{d\theta}\eta(\theta, \hat{\eth}, \hat{\varsigma})\right)\left\{\sum_{i=1}^{k}\hat{\delta}_i[\aleph_i(\hat{\varsigma})\varphi_{i_{\dot{\varsigma}}}(\theta, \hat{\varsigma}, \dot{\hat{\varsigma}}) - \Phi_i(\hat{\varsigma})\psi_{i_{\dot{\varsigma}}}(\theta, \hat{\varsigma}, \dot{\hat{\varsigma}})]\right\}\Bigg\}d\theta$$

$$= -\int_{\tau_1}^{\tau_2}\left\{\eta(\theta, \hat{\eth}, \hat{\varsigma})\hat{\rho}(\theta)^T h_{\varsigma}(\theta, \hat{\varsigma}, \dot{\hat{\varsigma}}) + \left(\frac{d}{d\theta}\eta(\theta, \hat{\eth}, \hat{\varsigma})\right)\hat{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta, \hat{\varsigma}, \dot{\hat{\varsigma}})\right\}d\theta.$$

With the help of inequality (31), the above inequality reduces to

$$\int_{\tau_1}^{\tau_2}\left\{\eta(\theta, \hat{\eth}, \hat{\varsigma})\hat{\rho}(\theta)^T h_{\varsigma}(\theta, \hat{\varsigma}, \dot{\hat{\varsigma}}) + \left(\frac{d}{d\theta}\eta(\theta, \hat{\eth}, \hat{\varsigma})\right)\hat{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta, \hat{\varsigma}, \dot{\hat{\varsigma}})\right\}d\theta < 0. \tag{32}$$

As $(\hat{\varsigma}, \hat{\delta}, \hat{\rho})$ is a feasible point to $(\text{MFD})_\eta$, therefore, using (12) we get

$$\int_{\tau_1}^{\tau_2}\hat{\rho}(\theta)^T h(\theta, \hat{\varsigma}, \dot{\hat{\varsigma}})d\theta + \int_{\tau_1}^{\tau_2}\Bigg\{\eta(\theta, \hat{\eth}, \hat{\varsigma})\hat{\rho}(\theta)^T h_{\varsigma}(\theta, \hat{\varsigma}, \dot{\hat{\varsigma}})$$

$$+\left(\frac{d}{d\theta}\eta(\theta, \hat{\eth}, \hat{\varsigma})\right)\hat{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta, \hat{\varsigma}, \dot{\hat{\varsigma}})\Bigg\}d\theta \geqq 0.$$

Implementing equation (27) to the above inequality, we obtain

$$\int_{\tau_1}^{\tau_2}\left\{\eta(\theta, \hat{\eth}, \hat{\varsigma})\hat{\rho}(\theta)^T h_{\varsigma}(\theta, \hat{\varsigma}, \dot{\hat{\varsigma}}) + \left(\frac{d}{d\theta}\eta(\theta, \hat{\eth}, \hat{\varsigma})\right)\hat{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta, \hat{\varsigma}, \dot{\hat{\varsigma}})\right\}d\theta \geqq 0,$$

which contradicts the inequality (32). Hence, the feasible point $(\hat{\eth}, \hat{\delta}, \hat{\rho})$ is a weak Pareto solution to $(\text{MFD})_\eta$.

Proposition 4 (Strong duality for modified problems under the Pareto solution) Let the feasible point $\hat{\varsigma}$ be a Pareto solution to $(\text{MFP})_\eta$. Then the Lagrange multiplier $\hat{\delta} > 0$ and a smooth piecewise function $\hat{\rho}\colon \mathscr{I} \to \mathbb{R}^m_+$ exist, so that $(\hat{\varsigma}, \hat{\delta}, \hat{\rho})$ is a Pareto solution to the modified dual problem $(\text{MFD})_\eta$.

Proof. The proof of this proposition follows from Proposition 3.

Theorem 6 (Strong duality for original problems under the weak Pareto solution) Let the feasible point $\hat{\varsigma}$ be a weak Pareto solution to (MFP) and satisfy all the assumptions of Theorem 2. Then the feasible point $(\hat{\varsigma}, \hat{\delta}, \hat{\rho})$ will become a weak Pareto solution to (MFD) and the extremal value will be the same.

Proof. Since the feasible point $\hat{\varsigma}$ is a weak Pareto solution to (MFP). Therefore, by Theorem 4, $\hat{\varsigma}$ is a weak Pareto solution to $(\text{MFP})_\eta$. Moreover, using Theorem 3, one can conclude that $(\hat{\varsigma}, \hat{\delta}, \hat{\rho})$ is a weak Pareto solution to $(\text{MFD})_\eta$ satisfying

$$\sum_{i=1}^{k}\widehat{\delta}_i[\aleph_i(\widehat{\varsigma})\varphi_{i_\varsigma}(\theta,\widehat{\varsigma},\dot{\widehat{\varsigma}})-\Phi_i(\widehat{\varsigma})\psi_{i_\varsigma}(\theta,\widehat{\varsigma},\dot{\widehat{\varsigma}})]+\widehat{\rho}(\theta)^T h_\varsigma(\theta,\widehat{\varsigma},\dot{\widehat{\varsigma}})$$

$$=\frac{d}{d\theta}\left(\sum_{i=1}^{k}\widehat{\delta}_i[\aleph_i(\widehat{\varsigma})\varphi_{i_{\dot{\varsigma}}}(\theta,\widehat{\varsigma},\dot{\widehat{\varsigma}})-\Phi_i\psi_{i_{\dot{\varsigma}}}(\theta,\widehat{\varsigma},\dot{\widehat{\varsigma}})]+\widehat{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta,\widehat{\varsigma},\dot{\widehat{\varsigma}})\right),$$

$$\int_{\tau_1}^{\tau_2}\widehat{\rho}(\theta)^T h(\theta,\widehat{\varsigma},\dot{\widehat{\varsigma}})d\theta+\int_{\tau_1}^{\tau_2}\eta(\theta,\widehat{\varsigma},\widehat{\varsigma})\widehat{\rho}(\theta)^T h_\varsigma(\theta,\widehat{\varsigma},\dot{\widehat{\varsigma}})$$

$$+\left(\frac{d}{d\theta}\eta(\theta,\widehat{\varsigma},\widehat{\varsigma})\right)\widehat{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta,\widehat{\varsigma},\dot{\widehat{\varsigma}})d\theta\geqq 0,$$

$$\widehat{\delta}>0,\ \widehat{\rho}(\theta)\geqq 0,\ \theta\in\mathscr{I},$$

which validates that $(\widehat{\varsigma},\widehat{\delta},\widehat{\rho})$ is a feasible solution to (MFD) and also satisfies all the assumptions of Theorem 2. Hence, $(\widehat{\varsigma},\widehat{\delta},\widehat{\rho})$ becomes a weak Pareto solution to (MFD) and gives the same extremal values as (MFP).

Theorem 7 (Strong duality for original problems under the Pareto solution) Let the feasible point $\widehat{\varsigma}$ be a Pareto solution to (MFP) and satisfy all the assumptions of Theorem 3. Then the feasible point $(\widehat{\varsigma},\widehat{\delta},\widehat{\rho})$ will become a Pareto solution to (MFD) and the extremal value will be the same.

Proof. The proof of this theorem follows from Theorem 6.

Proposition 5 (Converse duality for modified problems under weak Pareto solution) If $(\widehat{\eth},\widehat{\delta},\widehat{\rho})$ is a weak Pareto solution to the modified dual $(\text{MFD})_\eta$. Then $\widehat{\eth}$ is a weak Pareto solution to $(\text{MFP})_\eta$.

Proof. Let us assume that the feasible point $\widehat{\eth}$ is not a weak Pareto solution to $(\text{MFP})_\eta$. Then, there exists $\varsigma\in\Upsilon$ satisfying

$$\int_{\tau_1}^{\tau_2}\left\{\aleph_i(\widehat{\eth})\varphi_i(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i(\widehat{\eth})\psi_i(\theta,\widehat{\eth},\dot{\widehat{\eth}})\right\}d\theta$$

$$+\int_{\tau_1}^{\tau_2}\left\{\eta(\theta,\varsigma,\widehat{\eth})\left\{\aleph_i(\widehat{\eth})\varphi_{i_\varsigma}(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i(\widehat{\eth})\psi_{i_\varsigma}(\theta,\widehat{\eth},\dot{\widehat{\eth}})\right\}\right.$$

$$\left.+\left(\frac{d}{d\theta}\eta(\theta,\varsigma,\widehat{\eth})\right)\left\{\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i(\widehat{\eth})\psi_{i_{\dot{\varsigma}}}(\theta,\widehat{\eth},\dot{\widehat{\eth}})\right\}\right\}d\theta$$

$$<\int_{\tau_1}^{\tau_2}\left\{\aleph_i(\widehat{\eth})\varphi_i(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i(\widehat{\eth})\psi_i(\theta,\widehat{\eth},\dot{\widehat{\eth}})\right\}d\theta$$

$$+\int_{\tau_1}^{\tau_2}\left\{\eta(\theta,\widehat{ð},ð)\left\{\aleph_i(\widehat{ð})\varphi_{i_\varsigma}(\theta,\widehat{ð},\dot{\widehat{ð}})-\Phi_i(\widehat{ð})\psi_{i_\varsigma}(\theta,\widehat{ð},\dot{\widehat{ð}})\right\}\right.$$

$$\left.+\left(\frac{d}{d\theta}\eta(\theta,\widehat{ð},\widehat{ð})\right)\left\{\aleph_i(\widehat{ð})\varphi_{i_{\dot\varsigma}}(\theta,\widehat{ð},\dot{\widehat{ð}})-\Phi_i(\widehat{ð})\psi_{i_{\dot\varsigma}}(\theta,\widehat{ð},\dot{\widehat{ð}})\right\}\right\}d\theta,\ \forall i\in\mathscr{K}.$$

Using the condition $\eta(\theta,\widehat{ð},\widehat{ð})=0,$ the above relation can be written as

$$\int_{\tau_1}^{\tau_2}\left\{\eta(\theta,\varsigma,\widehat{ð})\left\{\aleph_i(\widehat{ð})\varphi_{i_\varsigma}(\theta,\widehat{ð},\dot{\widehat{ð}})-\Phi_i(\widehat{ð})\psi_{i_\varsigma}(\theta,\widehat{ð},\dot{\widehat{ð}})\right\}\right.$$

$$\left.+\left(\frac{d}{d\theta}\eta(\theta,\varsigma,\widehat{ð})\right)\left\{\aleph_i(\widehat{ð})\varphi_{i_{\dot\varsigma}}(\theta,\widehat{ð},\dot{\widehat{ð}})-\Phi_i(\widehat{ð})\psi_{i_{\dot\varsigma}}(\theta,\widehat{ð},\dot{\widehat{ð}})\right\}\right\}d\theta<0,\ \ \forall i\in\mathscr{K}.$$

Multiplying with $\widehat{\delta}>0$ and summing up, we get

$$\int_{\tau_1}^{\tau_2}\left\{\eta(\theta,\varsigma,\widehat{ð})\sum_{i=1}^{k}\widehat{\delta}_i\left\{\aleph_i(\widehat{ð})\varphi_{i_\varsigma}(\theta,\widehat{ð},\dot{\widehat{ð}})-\Phi_i(\widehat{ð})\psi_{i_\varsigma}(\theta,\widehat{ð},\dot{\widehat{ð}})\right\}\right.$$

$$\left.+\left(\frac{d}{d\theta}\eta(\theta,\varsigma,\widehat{ð})\right)\sum_{i=1}^{k}\widehat{\delta}_i\left\{\aleph_i(\widehat{ð})\varphi_{i_{\dot\varsigma}}(\theta,\widehat{ð},\dot{\widehat{ð}})-\Phi_i(\widehat{ð})\psi_{i_{\dot\varsigma}}(\theta,\widehat{ð},\dot{\widehat{ð}})\right\}\right\}d\theta$$

$$<0. \tag{33}$$

Since $(\widehat{ð},\widehat{\delta},\widehat{\rho})$ is a feasible solution to $(\mathrm{MFD})_\eta,$ therefore from (12), we obtain

$$\int_{\tau_1}^{\tau_2}\widehat{\rho}(\theta)^T h(\theta,\widehat{ð},\dot{\widehat{ð}})d\theta+\int_{\tau_1}^{\tau_2}\left\{\eta(\theta,\widehat{ð},\widehat{ð})\widehat{\rho}(\theta)^T h_\varsigma(\theta,\widehat{ð},\dot{\widehat{ð}})\right.$$

$$\left.+\left(\frac{d}{d\theta}\eta(\theta,\widehat{ð},\widehat{ð})\right)\widehat{\rho}(\theta)^T h_{\dot\varsigma}(\theta,\widehat{ð},\dot{\widehat{ð}})\right\}d\theta\geqq 0.$$

Using the condition $\eta(\theta,\widehat{ð},\widehat{ð})=0,$ the inequality mentioned above simplifies to

$$\int_{\tau_1}^{\tau_2} \widehat{\rho}(\theta)^T h(\theta, \widehat{ð}, \dot{\widehat{ð}}) d\theta \geqq 0. \tag{34}$$

As $\widehat{ð}$ is a feasible solution to $(\text{MFP})_\eta$, one can have

$$\int_{\tau_1}^{\tau_2} \widehat{\rho}(\theta)^T h(\theta, \widehat{ð}, \dot{\widehat{ð}}) d\theta + \int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \varsigma, \widehat{ð}) \widehat{\rho}(\theta)^T h_\varsigma(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right.$$

$$\left. + \left(\frac{d}{d\theta} \eta(\theta, \varsigma, \widehat{ð}) \right) \widehat{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right\} d\theta \leqq 0.$$

Using inequality (34), the above relation reduces to

$$\int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \varsigma, \widehat{ð}) \widehat{\rho}(\theta)^T h_\varsigma(\theta, \widehat{ð}, \dot{\widehat{ð}}) + \left(\frac{d}{d\theta} \eta(\theta, \varsigma, \widehat{ð}) \right) \widehat{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right\} d\theta \leqq 0. \tag{35}$$

Adding the inequalities (33) and (35), we get

$$\int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, \varsigma, \widehat{ð}) \left(\sum_{i=1}^{k} \widehat{\delta}_i \left\{ \aleph_i(\widehat{ð}) \varphi_{i_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð}) \psi_{i_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right\} \right. \right.$$

$$\left. + \widehat{\rho}(\theta)^T h_\varsigma(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right)$$

$$+ \left(\frac{d}{d\theta} \eta(\theta, \varsigma, \widehat{ð}) \right) \left(\sum_{i=1}^{k} \widehat{\delta}_i \left\{ \aleph_i(\widehat{ð}) \varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð}) \psi_{i_{\dot{\varsigma}}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right\} \right.$$

$$\left. \left. + \widehat{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right) \right\} d\theta$$

$$< 0. \tag{36}$$

As $(\widehat{ð}, \widehat{\delta}, \widehat{\rho})$ is a feasible solution to $(\text{MFD})_\eta$, so,

$$\sum_{i=1}^{k}\widehat{\delta}_i[\aleph_i(\widehat{\eth})\varphi_{i_\varsigma}(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i(\widehat{\eth})\psi_{i_\varsigma}(\theta,\widehat{\eth},\dot{\widehat{\eth}})]+\widehat{\rho}(\theta)^T h_\varsigma(\theta,\widehat{\eth},\dot{\widehat{\eth}})$$

$$=\frac{d}{d\theta}\left(\sum_{i=1}^{k}\widehat{\delta}_i[\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i\psi_{i_{\dot{\varsigma}}}(\theta,\widehat{\eth},\dot{\widehat{\eth}})]+\widehat{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta,\widehat{\eth},\dot{\widehat{\eth}})\right).$$

Multiplying the above equation with $\eta(\theta,\varsigma,\widehat{\eth})$ and integrating within limits τ_1 and τ_2, we get

$$\int_{\tau_1}^{\tau_2}\eta(\theta,\varsigma,\widehat{\eth})\left\{\sum_{i=1}^{k}\widehat{\delta}_i[\aleph_i(\widehat{\eth})\varphi_{i_\varsigma}(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i(\widehat{\eth})\psi_{i_\varsigma}(\theta,\widehat{\eth},\dot{\widehat{\eth}})]\right.$$

$$\left.+\widehat{\rho}(\theta)^T h_\varsigma(\theta,\widehat{\eth},\dot{\widehat{\eth}})\right\}d\theta$$

$$=\int_{\tau_1}^{\tau_2}\eta(\theta,\varsigma,\widehat{\eth})\left\{\frac{d}{d\theta}\left(\sum_{i=1}^{k}\widehat{\delta}_i[\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i(\widehat{\varsigma})\psi_{i_{\dot{\varsigma}}}(\theta,\widehat{\eth},\dot{\widehat{\eth}})]\right.\right.$$

$$\left.\left.+\widehat{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta,\widehat{\eth},\dot{\widehat{\eth}})\right)\right\}d\theta.$$

Using integration by parts and the condition $\eta(\theta,\widehat{\eth},\widehat{\eth})=0$, the above equation gives

$$\int_{\tau_1}^{\tau_2}\eta(\theta,\varsigma,\widehat{\eth})\left\{\sum_{i=1}^{k}\widehat{\delta}_i[\aleph_i(\widehat{\eth})\varphi_{i_\varsigma}(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i(\widehat{\eth})\psi_{i_\varsigma}(\theta,\widehat{\eth},\dot{\widehat{\eth}})]\right.$$

$$\left.+\widehat{\rho}(\theta)^T h_\varsigma(\theta,\widehat{\eth},\dot{\widehat{\eth}})\right\}d\theta$$

$$=-\int_{\tau_1}^{\tau_2}\left(\frac{d}{d\theta}\eta(\theta,\varsigma,\widehat{\eth})\right)\left\{\sum_{i=1}^{k}\widehat{\delta}_i[\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta,\widehat{\eth},\dot{\widehat{\eth}})-\Phi_i(\widehat{\eth})\psi_{i_{\dot{\varsigma}}}(\theta,\widehat{\eth},\dot{\widehat{\eth}})]\right.$$

$$\left.+\widehat{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta,\widehat{\eth},\dot{\widehat{\eth}})\right\}d\theta,$$

which can be written as

$$\int_{\tau_1}^{\tau_2} \eta(\theta, \varsigma, \widehat{ð})\left\{\sum_{i=1}^{k} \widehat{\delta}_i [\aleph_i(\widehat{ð})\varphi_{i_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð})\psi_{i_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}})]\right.$$

$$\left. + \widehat{\rho}(\theta)^T h_\varsigma(\theta, \widehat{ð}, \dot{\widehat{ð}})\right\} d\theta$$

$$+ \int_{\tau_1}^{\tau_2} \left(\frac{d}{d\theta}\eta(\theta, \varsigma, \widehat{ð})\right)\left\{\sum_{i=1}^{k} \widehat{\delta}_i [\aleph_i(\widehat{ð})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{ð}, \dot{\widehat{ð}})]\right.$$

$$\left. + \widehat{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}})\right\} d\theta$$

$$=0,$$

which contradicts (36). Hence, the feasible point $\widehat{ð}$ becomes a weak Pareto solution to $(\text{MFD})_\eta$.

Proposition 6 (Converse duality for modified problems under Pareto solution) If $(\widehat{ð}, \widehat{\delta}, \widehat{\rho})$ is a Pareto solution to the modified dual $(\text{MFD})_\eta$. Then $\widehat{ð}$ is a Pareto solution to $(\text{MFP})_\eta$.

Proof. The proof of this proposition follows from Proposition 5.

Theorem 8 (Converse duality for original problems under the weak Pareto solution) Let the feasible point $(\widehat{ð}, \widehat{\delta}, \widehat{\rho})$ be a weak Pareto solution to the dual (MFD) and satisfy the condition $h(\theta, \widehat{ð}, \dot{\widehat{ð}}) = 0$. Moreover, suppose $\int_{\tau_1}^{\tau_2} \{\aleph_i(\varsigma)\varphi_i(\theta, \varsigma, \dot{\varsigma}) - \Phi_i(\varsigma)\psi_i(\theta, \varsigma, \dot{\varsigma})\} d\theta,\ \forall i \in \mathscr{I}$ and $\int_{\tau_1}^{\tau_2} \widehat{\rho}(\theta) h(\theta, \varsigma, \dot{\varsigma}) d\theta$ are invex functions at a point $\widehat{ð}$ on X with regard to η. Then $\widehat{ð}$ is a weak Pareto solution to (MFP).

Proof. First of all, we need to verify that the feasible point $(\widehat{ð}, \widehat{\delta}, \widehat{\rho})$ is a weak Pareto solution to $(\text{MFD})_\eta$. Let us assume to the contrary that $(\widehat{ð}, \widehat{\delta}, \widehat{\rho})$ is not a weak Pareto solution to $(\text{MFD})_\eta$, then $(\bar{ð}, \bar{\delta}, \bar{\rho}) \in \Omega$ exist and satisfy

$$\int_{\tau_1}^{\tau_2} \left\{\aleph_i(\widehat{ð})\varphi_i(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð})\psi_i(\theta, \widehat{ð}, \dot{\widehat{ð}})\right\} d\theta$$

$$+ \int_{\tau_1}^{\tau_2} \left\{\eta(\theta, \widehat{ð}, \widehat{ð}) \left\{\aleph_i(\widehat{ð})\varphi_{i_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð})\psi_{i_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}})\right\}\right.$$

$$\left. + \left(\frac{d}{d\theta}\eta(\theta, \widehat{ð}, \widehat{ð})\right) \left\{\aleph_i(\widehat{ð})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{ð}, \dot{\widehat{ð}})\right\}\right\} d\theta$$

$$< \int_{\tau_1}^{\tau_2} \left\{\aleph_i(\widehat{ð})\varphi_i(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð})\psi_i(\theta, \widehat{ð}, \dot{\widehat{ð}})\right\} d\theta$$

$$+\int_{\tau_1}^{\tau_2}\left\{\eta(\theta, ð, \widehat{ð})\left\{\aleph_i(\widehat{ð})\varphi_{i_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}})-\Phi_i(\widehat{ð})\psi_{i_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}})\right\}\right.$$

$$\left.+\left(\frac{d}{d\theta}\eta(\theta, ð, \widehat{ð})\right)\left\{\aleph_i(\widehat{ð})\varphi_{i_{\dot\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}})-\Phi_i(\widehat{ð})\psi_{i_{\dot\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}})\right\}\right\}d\theta, \forall i\in\mathscr{K}.$$

Using $\eta(\theta, \widehat{ð}, \widehat{ð})=0,$ the above relation reduces to

$$\int_{\tau_1}^{\tau_2}\left\{\eta(\theta, ð, \widehat{ð})\left\{\aleph_i(\widehat{ð})\varphi_{i_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}})-\Phi_i(\widehat{ð})\psi_{i_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}})\right\}\right.$$

$$\left.+\left(\frac{d}{d\theta}\eta(\theta, ð, \widehat{ð})\right)\left\{\aleph_i(\widehat{ð})\varphi_{i_{\dot\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}})-\Phi_i(\widehat{ð})\psi_{i_{\dot\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}})\right\}\right\}d\theta>0,\ \forall i\in\mathscr{K}.$$

Multiplying with $\bar{\delta}>0$ and summing up, we get

$$\int_{\tau_1}^{\tau_2}\left\{\eta(\theta, ð, \widehat{ð})\sum_{i=1}^{k}\bar{\delta}_i\left\{\aleph_i(\widehat{ð})\varphi_{i_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}})-\Phi_i(\widehat{ð})\psi_{i_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}})\right\}\right.$$

$$\left.+\left(\frac{d}{d\theta}\eta(\theta, ð, \widehat{ð})\right)\sum_{i=1}^{k}\bar{\delta}_i\left\{\aleph_i(\widehat{ð})\varphi_{i_{\dot\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}})-\Phi_i(\widehat{ð})\psi_{i_{\dot\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}})\right\}\right\}d\theta$$

$$>0. \tag{37}$$

As $(ð, \bar{\delta}, \bar{\rho})$ is a feasible solution to $(\text{MFD})_\eta$. Therefore, we have

$$\sum_{i=1}^{k}\bar{\delta}_i[\aleph_i(\widehat{ð})\varphi_{i_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}})-\Phi_i(\widehat{ð})\psi_{i_\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}})]+\bar{\rho}(\theta)^T h_\varsigma(\theta, \widehat{ð}, \dot{\widehat{ð}})$$

$$=\frac{d}{d\theta}\left(\sum_{i=1}^{k}\bar{\delta}_i[\aleph_i(\widehat{ð})\varphi_{i_{\dot\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}})-\Phi_i\psi_{i_{\dot\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}})]+\bar{\rho}(\theta)^T h_{\dot\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}})\right).$$

Multiplying the above equation with $\eta(\theta, ð, \widehat{ð})$ and integrating within limits τ_1 and $\tau_2,$ we get

$$\int_{\tau_1}^{\tau_2} \eta(\theta, \eth, \widehat{\eth})\left\{\sum_{i=1}^{k} \bar{\delta}_i [\aleph_i(\widehat{\eth})\varphi_{i_\varsigma}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_\varsigma}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})]\right.$$

$$\left. + \bar{\rho}(\theta)^T h_\varsigma(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right\} d\theta$$

$$= \int_{\tau_1}^{\tau_2} \eta(\theta, \eth, \widehat{\eth})\left\{\frac{d}{d\theta}\left(\sum_{i=1}^{k} \bar{\delta}_i [\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\varsigma})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})]\right.\right.$$

$$\left.\left. + \bar{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right)\right\} d\theta.$$

Using integration by parts and the condition $\eta(\theta, \widehat{\eth}, \widehat{\eth}) = 0,$ the above equation gives

$$\int_{\tau_1}^{\tau_2} \eta(\theta, \eth, \widehat{\eth})\left\{\sum_{i=1}^{k} \bar{\delta}_i [\aleph_i(\widehat{\eth})\varphi_{i_\varsigma}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_\varsigma}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})]\right.$$

$$\left. + \bar{\rho}(\theta)^T h_\varsigma(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right\} d\theta$$

$$= -\int_{\tau_1}^{\tau_2} \left(\frac{d}{d\theta}\eta(\theta, \eth, \widehat{\eth})\right)\left\{\sum_{i=1}^{k} \bar{\delta}_i [\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})]\right.$$

$$\left. + \bar{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right\} d\theta,$$

which can be expressed as

$$\int_{\tau_1}^{\tau_2} \left\{\eta(\theta, \eth, \widehat{\eth})\left\{\sum_{i=1}^{k} \bar{\delta}_i [\aleph_i(\widehat{\eth})\varphi_{i_\varsigma}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_\varsigma}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})]\right.\right.$$

$$\left.\left. + \left(\frac{d}{d\theta}\eta(\theta, \eth, \widehat{\eth})\right)\left\{\sum_{i=1}^{k} \bar{\delta}_i [\aleph_i(\widehat{\eth})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) - \Phi_i(\widehat{\eth})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})]\right\}\right\} d\theta$$

$$= -\int_{\tau_1}^{\tau_2} \left\{\eta(\theta, \eth, \widehat{\eth})\bar{\rho}(\theta)^T h_\varsigma(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) + \left(\frac{d}{d\theta}\eta(\theta, \eth, \widehat{\eth})\right)\bar{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{\eth}, \dot{\widehat{\eth}})\right\} d\theta,$$

which, along with the inequality (37) yields

$$\int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, ð, \widehat{ð}) \bar{\rho}(\theta)^T h_{\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}}) + \left(\frac{d}{d\theta} \eta(\theta, ð, \widehat{ð}) \right) \bar{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right\} d\theta < 0.$$

Implementing the condition $h(\theta, \widehat{ð}, \dot{\widehat{ð}}) = 0$ in the above inequality, we get

$$\int_{\tau_1}^{\tau_2} \left\{ \bar{\rho}(\theta)^T h(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right\} d\theta + \int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, ð, \widehat{ð}) \bar{\rho}(\theta)^T h_{\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right.$$

$$\left. + \left(\frac{d}{d\theta} \eta(\theta, ð, \widehat{ð}) \right) \bar{\rho}(\theta)^T h_{\dot{\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right\} d\theta < 0,$$

which opposes that $(ð, \bar{\delta}, \bar{\rho})$ is a feasible solution to $(\text{MFD})_{\eta}$. Hence $(\widehat{ð}, \widehat{\delta}, \widehat{\rho})$ is a weak Pareto solution to $(\text{MFD})_{\eta}$, and by Theorem 5, $\widehat{ð}$ is a weak Pareto solution to $(\text{MFP})_{\eta}$. Now, our aim is to show $\widehat{ð}$ is a weak Pareto solution to (MFP). On the contrary, suppose that $\widehat{ð}$ is not a weak Pareto solution to (MFP), thus $ð \in \Upsilon$ exists, such that

$$\int_{\tau_1}^{\tau_2} \{ \aleph_i(ð) \varphi_i(\theta, ð, \dot{ð}) - \Phi_i(ð) \psi_i(\theta, ð, \dot{ð}) \} d\theta$$

$$< \int_{\tau_1}^{\tau_2} \left\{ \aleph_i(\widehat{ð}) \varphi_i(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð}) \psi_i(\theta, \widehat{ð}, \dot{\widehat{ð}}) \right\} d\theta, \ \forall i \in \mathscr{K}. \tag{38}$$

Due to invexity of $\int_{\tau_1}^{\tau_2} \{ \aleph_i(\varsigma) \varphi_i(\theta, \varsigma, \dot{\varsigma}) - \Phi_i(\varsigma) \psi_i(\theta, \varsigma, \dot{\varsigma}) \} d\theta, \ \forall i \in \mathscr{I}$ at $\widehat{ð}$, we have

$$\int_{\tau_1}^{\tau_2} \{ \aleph_i(ð) \varphi_i(\theta, ð, \dot{ð}) - \Phi_i(ð) \psi_i(\theta, ð, \dot{ð}) \} d\theta$$

$$\geqq \int_{\tau_1}^{\tau_2} \{ \aleph_i(\widehat{ð}) \varphi_i(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð}) \psi_i(\theta, \widehat{ð}, \dot{\widehat{ð}}) \} d\theta$$

$$+ \int_{\tau_1}^{\tau_2} \left\{ \eta(\theta, ð, \widehat{ð}) \{ \aleph_i(\widehat{ð}) \varphi_{i_{\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð}) \psi_{i_{\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) \} \right.$$

$$+\left(\frac{d}{d\theta}\eta(\theta, ð, \widehat{ð})\right)\{\aleph_i(\widehat{ð})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{ð}, \dot{\widehat{ð}})\}\Bigg\}d\theta, \ \forall i \in \mathscr{K}.$$

Using (38), the above inequality reduces to

$$\int_{\tau_1}^{\tau_2}\Bigg\{\eta(\theta, ð, \widehat{ð})\{\aleph_i(\widehat{ð})\varphi_{i_{\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð})\psi_{i_{\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}})\}$$

$$+\left(\frac{d}{d\theta}\eta(\theta, ð, \widehat{ð})\right)\{\aleph_i(\widehat{ð})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{ð}, \dot{\widehat{ð}})\}\Bigg\}d\theta < 0, \ \forall i \in \mathscr{K}. \tag{39}$$

Also, $\int_{\tau_1}^{\tau_2}\left\{\widehat{\rho}(\theta))^T h(\theta, \varsigma, \dot{\varsigma})\right\}d\theta$ is invex at $\widehat{ð}$, thus we get

$$\int_{\tau_1}^{\tau_2}\left\{\widehat{\rho}(\theta)^T h(\theta, ð, \dot{ð})\right\}d\theta - \int_{\tau_1}^{\tau_2}\left\{\widehat{\rho}(\theta))^T h(\theta, \widehat{ð}, \dot{\widehat{ð}})\right\}d\theta$$

$$\geqq \int_{\tau_1}^{\tau_2}\Bigg\{\eta(\theta, ð, \widehat{ð})(\widehat{\rho}(\theta)^T h_{\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}}) + \left(\frac{d}{d\theta}\eta(\theta, ð, \widehat{ð})\right)(\widehat{\rho}(\theta))^T h_{\dot{\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}})\Bigg\}d\theta.$$

As $ð \in \Upsilon$ is a feasible solution, so the above expression reduces to be

$$\int_{\tau_1}^{\tau_2}\left\{\widehat{\rho}(\theta))^T h(\theta, \widehat{ð}, \dot{\widehat{ð}})\right\}d\theta + \int_{\tau_1}^{\tau_2}\Bigg\{\eta(\theta, ð, \widehat{ð})(\widehat{\rho}(\theta)^T h_{\varsigma}(\theta, \widehat{ð}, \dot{\widehat{ð}})$$

$$+\left(\frac{d}{d\theta}\eta(\theta, ð, \widehat{ð})\right)(\widehat{\rho}(\theta))^T h_{\dot{\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}})\Bigg\}d\theta \leqq 0,$$

which validates that $ð$ is a feasible solution to $(\text{MFP})_\eta$. Also, since $\widehat{ð}$ is a weak Pareto solution to $(\text{MFP})_\eta$ and $\eta(\theta, \widehat{ð}, \widehat{ð}) = 0$, thus, one can obtain

$$\int_{\tau_1}^{\tau_2}\Bigg\{\eta(\theta, ð, \widehat{ð})\left\{\aleph_i(\widehat{ð})\varphi_{i_{\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð})\psi_{i_{\varsigma}}(\theta, \widehat{ð}, \dot{\widehat{ð}})\right\}$$

$$+\left(\frac{d}{d\theta}\eta(\theta, ð, \widehat{ð})\right)\left\{\aleph_i(\widehat{ð})\varphi_{i_{\dot{\varsigma}}}(\theta, \widehat{ð}, \dot{\widehat{ð}}) - \Phi_i(\widehat{ð})\psi_{i_{\dot{\varsigma}}}(\theta, \widehat{ð}, \dot{\widehat{ð}})\right\}\Bigg\}d\theta > 0, \ \forall i \in \mathscr{K},$$

which contradicts (39). Hence, the feasible point $\widehat{ð}$ is a weak Pareto solution to (MFP).

Theorem 9 (Converse duality for original problems under the Pareto solution) Let the feasible point $(\widehat{\eth}, \widehat{\delta}, \widehat{\rho})$ be a Pareto solution to the dual (MFD) and satisfying the condition $h(\theta, \widehat{\eth}, \dot{\widehat{\eth}}) = 0$. Moreover, suppose $\int_{\tau_1}^{\tau_2} \{\aleph_i(\varsigma)\varphi_i(\theta, \varsigma, \dot{\varsigma}) - \Phi_i(\varsigma)\psi_i(\theta, \varsigma, \dot{\varsigma})\}d\theta, \ \forall i \in \mathscr{I}$ is invex and $\int_{\tau_1}^{\tau_2} \widehat{\rho}(\theta)h(\theta, \varsigma, \dot{\varsigma})d\theta$ is a strictly invex function at a point $\widehat{\eth}$ on X with regard to η. Then $\widehat{\eth}$ is a Pareto solution to (MFP).

Proof. The proof of this theorem follows from Theorem 8.

4. Conclusions

This paper employed the η-approximation method to study the nonlinear nonconvex multiobjective fractional variational problem with inequality constraints. Initially, the objective function and constraints of the original problem (MFP) and its dual (MFD) have been modified to generate the modified η-approximated problem $(\text{MFP})_\eta$ and its dual $(\text{MFD})_\eta$. Weak, strong, and converse duality theorems are established for both the original and modified problems. Furthermore, an appropriate example was envisioned, demonstrating that the nonlinear nonconvex problem could possibly be transformed into a linear and convex optimization problem by employing the η-approximated method.

Acknowledgement

I express my sincere gratitude to Dr. Ashish Kumar Prasad, Dean, Netaji Subhas University, Jamshedpur for his valuable suggestions, motivation and guidance on this research work. I have had the pleasure of working with him during this and other related research work.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Blasi L, Barbato S, Mattei M. A particle swarm approach for flight path optimization in a constrained environment. *Aerospace Science and Technology*. 2013; 26: 128-137.

[2] Khardi S. Aircraft flight path optimization the Hamilton-Jacobi-Bellman considerations. *Applied Mathematical Sciences*. 2012; 6: 1221-1249.

[3] Esmaelzadeh R. Low-thrust orbit transfer optimization using a combined method. *International Journal of Computer Applications*. 2014; 89: 20-24.

[4] Andersson J. *A survey of multiobjective optimization in engineering design*. Linköping, Sweden: Linköping University; 2000.

[5] Martins F, Costa CAV. Multiobjective optimization with economic and environmental objective functions using modified Simulated Annealing. *Computer Aided Chemical Engineering*. 2010; 28: 919-924.

[6] Mond B, Hanson MA. Duality for variational problems. *Journal of Mathematical Analysis and Applications*. 1967; 18: 355-364.

[7] Mond B, Chandra S, Husain I. Duality for variational problems with invexity. *Journal of Mathematical Analysis and Applications*. 1988; 134: 322-328.

[8] Mond B, Husain I. Sufficient optimality criteria and duality for variational problems with generalized invexity. *Journal of the Australian Mathematical Society*. 1989; 31: 108-121. Available from: https://doi.org/10.1017/S0334270000006512.

[9] Bector CR, Husain I. Duality for multiobjective variational problems. *Journal of Mathematical Analysis and Applications*. 1992; 166: 214-229. Available from: https://doi.org/10.1016/0022-247X(92)90337-D.

[10] Nahak C, Nanda S. Duality for multiobjective variational problems with invexity. *Optimization*. 1996; 36: 235-248. Available from: https://doi.org/10.1080/02331939608844181.

[11] Nahak C, Behera N. Optimality conditions and duality for multiobjective variational problems with generalized ρ-(η, θ)-B-type-I functions. *Journal of Control Science and Engineering*. 2011; 2011(1): 497376. Available from: https://doi.org/10.1155/2011/497376.

[12] Khazafi K, Rueda N. Multiobjective variational programming under generalized type I univexity. *Journal of Optimization Theory and Applications*. 2009; 142: 363-376. Available from: https://doi.org/10.1007/s10957-009-9526-3.

[13] Khazafi K, Rueda N, Enflo P. Sufficiency and duality for multiobjective control problems under generalized (B, ρ)-type I functions. *Journal of Global Optimization*. 2010; 46: 111-132. Available from: https://doi.org/10.1007/s10898-009-9412-4.

[14] Stancu-Minasian IM, Mititelu S. Multiobjective fractional variational problems with (ρ, b)-quasiinvexity. *Proceedings of the Romanian Academy, Series A*. 2008; 9: 5-11.

[15] Antczak T. A new approach to multiobjective programming with a modified objective function. *Journal of Global Optimization*. 2003; 27: 485-495. Available from: https://doi.org/10.1023/A:1026080604790.

[16] Antczak T. An η-approximation approach for nonlinear mathematical programming problems involving invex functions. *Numerical Functional Analysis and Optimization*. 2004; 25: 423-438. Available from: https://doi.org/10.1081/NFA-200042183.

[17] Antczak T. A new method of solving nonlinear mathematical programming problems involving r-invex functions. *Journal of Mathematical Analysis and Applications*. 2005; 311: 313-323. Available from: https://doi.org/10.1016/j.jmaa.2005.02.049.

[18] Antczak T, Michalak A. η-approximation method for non-convex multiobjective variational problems. *Numerical Functional Analysis and Optimization*. 2017; 38: 1125-1142. Available from: https://doi.org/10.1080/01630563.2017.1314970.

[19] Jayswal A, Antczak T, Jha S. On equivalence between a variational problem and its modified variational problem with the η-objective function under invexity. *International Transactions in Operational Research*. 2019; 26: 2053-2070. Available from: https://doi.org/10.1111/itor.12377.

[20] Jha S, Das P, Antczak T. Exponential type duality for η-approximated variational problems. *Yugoslav Journal of Operations Research*. 2019; 30: 19-43. Available from: https://doi.org/10.2298/YJOR190415022J.

[21] Jha S, Das P, Bandhyopadhyay S. Multitime multiobjective variational problems via η-approximation method. *Yugoslav Journal of Operations Research*. 2022; 32: 61-86.

[22] Antczak T. Saddle-point criteria in an η-approximation method for nonlinear mathematical programming problems involving invex functions. *Journal of Optimization Theory and Applications*. 2007; 132: 71-87. Available from: https://doi.org/10.1007/s10957-006-9069-9.