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Research Article

An Approach to Solve Gas Dynamic Equation by Fuzzy Hilfer Fractional Differential Equation

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Abstract: In this work, Adomian decomposition method (ADM) based on Hilfer derivative is proposed to solve nonlinear fuzzy fractional gas dynamic equation. We reform the provided fractional gas dynamic equation into fuzzy Hilfer fractional differential equation. Subsequently with description of Hilfer fractional derivative and suitable initial conditions under fuzzy sense, we obtain approximate solution to the given problem. This method provides an analytical solution in the form of infinite power series. The behavior of the solution is illustrated using graphical representation.

***Keywords*:** Fractional gas differential equation, Riemann-Liouville fractional derivative, Caputo fractional derivative, Hilfer fractional derivative, Adomian decomposition method

MSC: 34A08, 34K37, 35R11

1. Introduction

Fractional calculus is implemented constantly due its virtue in variety of domains, such as physics, biology, signal processing, economics, financial markets, etc. An essential application of fractional calculus lies in fractional partial differential equations (FPDEs), which serve as effective models for numerous natural phenomena. Different definitions of fractional derivatives and integrals have been proposed in the literature. Amongst them, the most prominent involve the fractional definitions by Caputo and Riemann-Liouville. Hilfer extended the concept of the Riemann-Liouville fractional derivative recently. This extension, called the Hilfer derivative, leads to a wider range of steady states and provides more flexibility in initial condition setting by introducing new parameters. Researchers have been prompted to enhance theoretical structures and applied methods in this domain owing to the increased use of FPDEs in the field of science and engineering.

Agarwal et al. [1] presented the solution approach for fuzzy fractional differential equations (FFDEs), marking the initial instance in literature where a fractional differential equation integrating uncertainty has been addressed. In multivariable functions with fuzzy values, Long et al. [2] introduced the concepts of Caputo gH-partial and fuzzy fractional integral and reported two new results about the existence of two distinct kinds of gH-weak solutions to these problems. Using multiple fractional power series (MFPS) formulation, Alaroud et al. [3] approximated solutions for a nonlinear

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time-fractional gas dynamics equation (FGDE) by applying the Laplace residual power series (LRPS) technique. Authors also used limit principles to discover the unknown coefficients of Laplace fractional series expansion (LFSE) for the new equation in Laplace space. Using the fuzzy Adomian decomposition method (FADM) Saeed et al. [4] examined mathematical solutions of nonlinear fuzzy fractional partial differential equations (FFPDEs) under Caputo derivative. They evaluated FADMs consistency and performance to produce series solutions wherein persistence depends on the fuzzy fractional derivative. Askari et al. [5] showed the convergence of a sequence with distinct forms of differentiability, towards the precise solution within the Caputo derivative. The objective of Padmapriya et al. [6] is to explore the solutions of Hilfer fractional differential equations in a fuzzy sense using the ADM. Additionally they illustrated the feasibility of obtaining numerical solutions for fuzzy Hilfer fractional differential equations under different conditions. With Schauder fixed-point theorem and method of upper and lower solutions Malahi et al. [7] obtained sufficient conditions for a fractional differential equation. In [8, 9], authors investigated the existence and approximate controllability for Hilfer fractional differential systems using fixed point theorem. These requirements are important for ensuring the existence of a positive solution for the given problem. Several authors have proposed solution for the nonlinear fractional gas dynamics equation (FGDE) using the combination of homotopy perturbation Sumudu transform method (HPSTM) and the Adomian decomposition method (ADM), natural transform and homotopy perturbation method to obtain exact solution of the problems. These methods are highly potent and efficient in scientific and engineering domains, offering a refined approach in the field of numerical techniques in solving different kinds of linear and nonlinear fractional differential equations [10, 11]. Olaniyi [12] have examined time-fractional non-linear gas dynamic equations for both homogeneous and non-homogeneous system and calculated approximate solutions for these equations in the form of series using q-homotopy analysis method (q-HAM). By evaluating a Caputo fractional proportional type nonlinear boundary value issue [13] linked with Caputo fractional proportional type slit-strips and Riemann-Stieltjes integral boundary conditions, solution were extracted for a particular example. Orthogonal collocation on finite elements (OCFE) [14] with quadratic B-spline basis functions were applied to the space fractional diffusion equation. Quadratic B-spline's main advantages are their superior interpolation skills and easy adaption to problems on irregular grids. A study on coronavirus transmission models was carried out by Srilekha at el. [15] using Simulink, a model-based design system that promotes modeling and at the system level design. The Caputo fractional derivative was used as a guide for creating the model and Simulink to forecast and simulate the COVID-19 pandemic. Selvam et al. [16] used the ψ-Caputo fractional derivative to study the observability of linear and non-linear fractional dynamical systems. The observability Grammian matrix, controlled by the Mittag-Leffler function, was employed for the linear case. Banach's fixed point theorem was applied in the nonlinear case to provide sufficient criteria for the observability of fractional dynamical systems.

A dynamic structure for analyzing systems with memory impacts is offered by the fuzzy Hilfer fractional derivative. This is crucial in gas dynamics, as previous conditions have an impact on present behavior. It successfully integrates fractional-order dynamics, providing benefits in analytical and numerical solutions, resulting in forms that are easier to handle, more accurate and stable simulations. Gas dynamics has practical applications in environmental science, combustion research, aeronautical engineering, and other fields. To maximize efficiency, enhance security, and reduce environmental impact, effective gas flow simulation is essential. This research offers a chance to develop new approaches in gas dynamics and may change our current knowledge and modeling abilities for complex flow phenomena. In order to integrate theoretical understanding with real-world applications, research into FFDEs integrating the Riemann-Liouville derivative provides potential for recent advances in gas dynamics. These equations may find application for the study of gas dynamics to describe non-local or memory-dependent effects in the behavior of gas flows resulting in more precise and predicting real-world phenomenon models. The functioning of a system at certain times is predicted via conventional differential equations to rely exclusively on its current state and the external inputs available at that point. On the other hand, the system's historical history may impact its behavior in everyday circumstances, such as gas dynamics, such memory-dependent behavior can be modeled mathematically using fractional derivatives. These equations provide an additional context that conventional differential equations are unable to adequately represent for describing complex phenomena in gas dynamics.

The virtue of Adomian decomposition method in different types of equations has made several scholars interested in it. This method was initially developed by George Adomian from the 1970s to 1990s. It stands out as an excellent

technique as it can be directly applied to integral and differential equations with variable or constant coefficients regardless of whether they are homogeneous or non-homogeneous, linear or nonlinear. Moreover the solution is obtained in terms of fast converging power series with estimable terms, from the provided initial condition of the given problem.

2. Preliminaries

Some basic concepts and results for fuzzy Hilfer fractional derivative are provided in this section see [6, 7].

Definition 1 [6, 7]

The Riemann-Liouville fractional integration of order α is defined as

$$(S_{0+}^{\alpha}x)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds,\ \alpha>0,\ t>0.$$

Definition 2 [6, 7]

The fuzzy Hilfer fractional derivative $\chi_{0+}^{\alpha,\beta}$ of a function x with order $0<\alpha<1$ and type $0\le\beta\le 1$ is defined as follows

$$\chi_{0+}^{\alpha,\beta}x(t) = S_{0+}^{\beta(1-\alpha)}\chi S_{0+}^{(1-\beta)(1-\alpha)}x(t),$$

where $\chi = \frac{d}{dt}$. Then

$$[(\chi_{0+}^{\alpha,\beta}x)(t)]^r = [(\chi_{0+}^{\alpha,\beta}\underline{x_r})(t),\ (\chi_{0+}^{\alpha,\beta}\overline{x_r})(t)],$$

$$[(\chi_{0+}^{\alpha,\beta}x)(t)]^r = [(S_{0+}^{\beta(1-\alpha)}\chi S_{0+}^{(1-\beta)(1-\alpha)}\underline{x_r})(t),\ (S_{0+}^{\beta(1-\alpha)}\chi S_{0+}^{(1-\beta)(1-\alpha)}\overline{x_r})(t)].$$

Remark 1 [6, 7]

(i) The Hilfer derivative $\chi_{0+}^{\alpha,\beta}$ can be expressed in the following form

$$\chi_{0+}^{\alpha,\beta} = S_{0+}^{\beta(1-\alpha)}\chi S_{0+}^{(1-\beta)(1-\alpha)} = S_{0+}^{\beta(1-\alpha)}\chi_{0+}^{\gamma},$$

where $\gamma=\alpha+\beta-\alpha\beta$.

(ii) Between the Riemann-Liouville and Caputo fractional derivatives, the Hilfer fractional derivative $\chi_{0+}^{\alpha,\beta}$ is employed as an interpolator since

$$\chi_{0+}^{\alpha,\beta} = \begin{cases} \chi S^{1-\alpha} =^{RL} \chi_{0+}^{\alpha}, & \text{if} \quad \beta=0, \\ S_{0+}^{1-\alpha}\chi =^{C} \chi_{0+}^{\alpha}, & \text{if} \quad \beta=1. \end{cases}$$

Theorem 1 [7]

Suppose that $x \in \rho_\gamma[0,\ 1]$, where $0 < \alpha < 1$, $0 \leq \gamma < 1$ and ρ_γ is the weighted spaces of continuous functions on the interval [0, 1],

$$\chi_{0+}^{\alpha} S_{0+}^{\alpha} x(t) = x(t),\ \forall\, t \in (0,\ 1].$$

Further, if $x \in \rho_\gamma[0,\ 1]$ and $S_{0+}^{(1-\beta)(1-\alpha)} x \in \rho_\gamma^1[0,\ 1]$, then

$$\chi_{0+}^{\alpha,\ \beta} S_{0+}^{\alpha} x(t) = x(t),\ \text{ for a.e } t \in (0,\ 1].$$

Theorem 2 [6, 7]

Consider $\alpha,\ \beta \geq 0$ and $x \in \rho_\gamma^1[0,\ 1]$. Then

$$S_{0+}^{\alpha} S_{0+}^{\beta} x(t) = S_{0+}^{\alpha+\beta} x(t).$$

Lemma 1 [6, 7]

Consider $\alpha \geq 0$ and $\sigma > 0$. Then

$$S_{0+}^{\alpha} t^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\alpha+\sigma)} t^{\alpha+\sigma-1},\ \ t > 0,$$

and

$$\chi_{0+}^{\alpha} t^{\sigma-1} = 0,\ \ 0 < \alpha < 1.$$

Lemma 2 [6, 7]

If $x \in \rho_\gamma[0,\ 1]$ and $S_{0+}^{1-\alpha} x \in \rho_\gamma^1[0,\ 1]$ for $0 < \alpha < 1$ and $0 \leq \gamma < 1$, then

- $(S_{0+}^{\alpha} \chi_{0+}^{\alpha} x)(t) = x(t) \ominus \dfrac{S_{0+}^{1-\alpha} x(0)}{\Gamma(\sigma)} t^{\sigma-1}$, for x is (1)-differentiable (gH differentiable)
- $(S_{0+}^{\alpha} \chi_{0+}^{\alpha} x)(t) = (-1)\dfrac{S_{0+}^{1-\alpha} x(0)}{\Gamma(\sigma)} t^{\sigma-1} \ominus (-1)x(t)$, for x is (2)-differential (gH differentiable)

where $\ominus$ is fuzzy subtraction.

3. Adomian decomposition method

Examine the following differential equation of fractional order that involves the Hilfer derivative, where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$,

$$\chi_{0+}^{\alpha,\ \beta} v(\xi,\ t,\ \zeta) = \lambda^2 v(\xi,\ t,\ \zeta) + \lambda\, v(\xi,\ t,\ \zeta) + f(\xi,\ t,\ \zeta), \tag{1}$$

with initial condition

$$S_{0+}^{1-\psi}\nu(\xi,\,0,\,\zeta)=b,\quad \psi=\alpha+\beta-\alpha\beta. \tag{2}$$

Operating S_{0+}^{η} on both sides of Eqn. (1)

$$S_{0+}^{\eta}\chi_{0+}^{\alpha,\,\beta}\nu(\xi,\,t,\,\zeta)=S_{0+}^{\eta}\lambda^2\nu(\xi,\,t,\,\zeta)+S_{0+}^{\eta}\lambda\nu(\xi,\,t,\,\zeta)+S_{0+}^{\eta}f(\xi,\,t,\,\zeta).$$

We have,

$$\nu(\xi,\,t,\,\zeta)=\frac{bt^{\psi-1}}{\Gamma(\psi)}+\lambda^2S_{0+}^{\eta}\nu(\xi,\,t,\,\zeta)+\lambda S_{0+}^{\eta}\nu(\xi,\,t,\,\zeta)+S_{0+}^{\eta}f(\xi,\,t,\,\zeta). \tag{3}$$

According to the ADM, the solution $\nu(\xi,\,t,\,\zeta)$ consists of various components, including

$$\nu(\xi,\,t,\,\zeta)=\sum_{n=0}^{\infty}\nu_n(\xi,\,t,\,\zeta). \tag{4}$$

Combining both sides of equation (3) with the decomposition series (4), yields

$$\sum_{n=0}^{\infty}\nu_n(\xi,\,t,\,\zeta)=\frac{bt^{\psi-1}}{\Gamma(\psi)}+S_{0+}^{\eta}f(\xi,\,t,\,\zeta)+\lambda^2S_{0+}^{\eta}\sum_{n=0}^{\infty}\nu_n(\xi,\,t,\,\zeta)+\lambda S_{0+}^{\eta}\sum_{n=0}^{\infty}\nu_n(\xi,\,t,\,\zeta).$$

We can obtain the following recursive relation using this equation

$$\begin{cases}\nu_0(\xi,\,t,\,\zeta)=\dfrac{bt^{\psi-1}}{\Gamma(\psi)}+S_{0+}^{\eta}f(\xi,\,t,\,\zeta),\\[2ex] \nu_{k+1}(\xi,\,t,\,\zeta)=\lambda^2S_{0+}^{\eta}\nu_k(\xi,\,t,\,\zeta)+\lambda S_{0+}^{\eta}\nu_k(\xi,\,t,\,\zeta),\ \ k\geq 0.\end{cases}$$

Continuing this process, we have

$$\nu_n(\xi,\,t,\,\zeta)=\lambda^n(\lambda+1)^n\left[\frac{bt^{n\eta+\psi-1}}{\Gamma(n\eta+\psi)}+S_{0+}^{n\eta+\eta}f(\xi,\,t,\,\zeta)\right].$$

Then the solution $\nu(\xi,\,t,\,\zeta)$ can be written by Eqn. (4) as follows

$$\nu(\xi,\,t,\,\zeta)=b\sum_{n=0}^{\infty}\frac{\lambda^n(\lambda+1)^nt^{n\eta+\psi-1}}{\Gamma(n\eta+\psi)}+\int_0^t\sum_{n=0}^{\infty}\frac{\lambda^n(\lambda+1)^n(t-s)^{n\eta+\eta-1}}{\Gamma(n\eta+\eta)}f(s)ds,$$

$$v(\xi,\ t,\ \zeta) = bt^{\psi-1}\sum_{n=0}^{\infty}\frac{[\lambda(\lambda+1)t^{\eta}]^n}{\Gamma(n\eta+\psi)} + \int_0^t (t-s)^{\eta-1}\sum_{n=0}^{\infty}\frac{[\lambda(\lambda+1)(t-s)^{\eta}]^n}{\Gamma(n\eta+\eta)}f(s)ds.$$

Thus the initial problem of (1) and (2) can be solved as follows:

$$v(\xi,\ t,\ \zeta) = bt^{\psi-1}E_{\eta,\ \psi}[\lambda(\lambda+1)t^{\eta}] + \int_0^t (t-s)^{\eta-1}E_{\eta,\ \eta}[\lambda(\lambda+1)(t-s)^{\eta}]f(s)ds. \tag{5}$$

Remark 2
1. If $\beta = 0$, then $\psi = \eta$ and

$$v(\xi,\ t,\ \zeta) = bt^{\eta-1}E_{\eta,\ \eta}[\lambda(\lambda+1)t^{\eta}] + \int_0^t (t-s)^{\eta-1}E_{\eta,\ \eta}[\lambda(\lambda+1)(t-s)^{\eta}]f(s)ds, \tag{6}$$

is the solution to the fractional differential equation involving the Riemann-Liouville derivative shown below,

$$\begin{cases} {}^{RL}\chi_{0+}^{\alpha}v(\xi,\ t,\ \zeta) = \lambda^2 v(\xi,\ t,\ \zeta) + \lambda v(\xi,\ t,\ \zeta) + f(\xi,\ t,\ \zeta), \\ \\ S_{0+}^{1-\alpha}v(\xi,\ 0,\ \zeta) = b. \end{cases}$$

2. If $\beta = 1$, then $\psi = 1$ and

$$v(\xi,\ t,\ \zeta) = bE_{\eta,\ 1}[\lambda(\lambda+1)t^{\eta}] + \int_0^t (t-s)^{\eta-1}E_{\eta,\ \eta}[\lambda(\lambda+1)(t-s)^{\eta}]f(s)ds, \tag{7}$$

is the solution to the fractional differential equation involving the Caputo derivative shown below,

$$\begin{cases} {}^{C}\chi_{0+}^{\alpha}v(\xi,\ t,\ \zeta) = \lambda^2 v(\xi,\ t,\ \zeta) + \lambda v(\xi,\ t,\ \zeta) + f(\xi,\ t,\ \zeta), \\ \\ v(\xi,\ 0,\ \zeta) = b. \end{cases}$$

Lemma 3 [6]
Let $x_i(t),\ i = 0,\ 1,\ 2,\ ...,\ n$, be fuzzy continuous functions. Then

$$\chi_{0+}^{\alpha,\ \beta}\sum_{i=0}^{n}x_i(t) = \sum_{i=0}^{n}\chi_{0+}^{\alpha,\ \beta}x_i(t).$$

4. Convergence of Adomian decomposition method

Many scholars have been examining the convergence problem for ADM with the Caputo fractional derivative [6]. We show the convergence for the Adomian series involving the Hilfer fractional derivative in this section via the reference [6].

Consider the following FFDE with Hilfer derivative of order $0 < \alpha < 1, 0 \leq \beta \leq 1$

$$\chi_{0+}^{\alpha,\beta} x(t) = L_x(t) + N_x(t), \tag{8}$$

with initial condition

$$S_{0+}^{1-\gamma} x(0) = x_0, \ \gamma = \alpha + \beta - \alpha\beta, \ x_0 \in \mathbb{E}. \tag{9}$$

Here $\mathbb{E}$ is the set of all fuzzy numbers, and L and N refer to the linear and nonlinear operators respectively.

In the view of the ADM, the solution $x(t)$ is decomposed into

$$x(t) = \sum_{n=0}^{\infty} x_n(t)$$

and decomposing the nonlinear term N as

$$N_x(t) = \sum_{n=0}^{\infty} A_n(t).$$

Adomian polynomials are denoted by A_n and are provided by

$$A_n = \frac{1}{n} \frac{d^n}{d\lambda^n} \left[N\left(\sum_{j=0}^{\infty} \lambda^j x_j \right) \right]_{\lambda=0}.$$

According to Lemma 3 the solution of $x(t) = \sum_{n=0}^{\infty} x_n(t)$ is uniformly convergent by ADM.

4.1 *Algorithm for numerical analysis of nonlinear fuzzy fractional gas dynamic equations*

Step 1: Initialization

• Begin by defining the parameters α, β, λ, and the initial conditions for the function $v(\xi, t, \zeta)$.

• Set the spatial and temporal domains within the specified ranges, and initialize the function $v(\xi, t, \zeta)$ based on its initial condition.

Step 2: Decomposition of the solution

• Decompose the function $v(\xi, t, \zeta)$ into an infinite series using the Adomian decomposition method (ADM).

• Initialize the series components $v_n^r(\xi, t, \zeta)$ for $n = 0, 1, 2, \ldots$.

Step 3: Application of the ADM

• Iteratively compute each component $v_n^r(\xi, t, \zeta)$ by applying the fractional differential equation, utilizing the initial conditions and series decomposition for recursive calculation.

Step 4: Numerical integration

• For each time step, use numerical integration techniques, such as the Grunwald-Letnikov or trapezoidal rule, to compute the necessary fractional derivatives and integrals.

• Ensure that the fractional orders α and β are correctly handled during these computations.

Step 5: Update and iteration

• For each spatial point ξ within the domain, update the function $v(\xi, t, \zeta)$ for the current time step.

• Store the results after each iteration for subsequent analysis, and repeat the process until the final time step is reached.

5. Numerical example

Consider the following nonlinear fuzzy fractional gas dynamic equation [4]

$$\begin{cases} \chi_t^{\psi} v(\xi, t, \zeta) + v(\xi, t, \zeta)\omega(\xi, t, \zeta) - v(\xi, t, \zeta) + v^2(\xi, t, \zeta) = 0, \quad 0 \le \xi, \ t \le 1, \\ v(\xi, 0, \zeta) = [\zeta, 3 - 2\zeta]e^{-\xi}. \end{cases}$$

Consider Eqn. (10) as Fuzzy Hilfer fractional differential equation

$$\chi_{0+}^{\alpha, \beta} v(\xi, t, \zeta) = -\lambda^2 \odot [v^2(\xi, t, \zeta) + v(\xi, t, \zeta)\omega(\xi, t, \zeta)] + \lambda \odot v(\xi, t, \zeta), \quad 0 \le \alpha \le 1, \ 0 \le \beta \le 1, \tag{10}$$

with initial condition

$$S_{0+}^{1-\psi} v(\xi, 0, \zeta) = [\zeta, 3 - 2\zeta]e^{-\xi}, \tag{11}$$

where $[v(\xi, t, \zeta)]^r = [\underline{v_r}(\xi, t, \zeta), \overline{v_r}(\xi, t, \zeta)]$ and $\odot$ represents fuzzy multiplication.

Assume $\lambda = 1$ then using (1)-differentiable we have

$$\begin{cases} \chi_{0+}^{\alpha, \beta} \underline{v_r}(\xi, t, \zeta) = -\underline{v_r}^2(\xi, t, \zeta) - \underline{v_r}(\xi, t, \zeta)\underline{\omega_r}(\xi, t, \zeta)] + \underline{v_r}(\xi, t, \zeta), \\ \chi_{0+}^{\alpha, \beta} \overline{v_r}(\xi, t, \zeta) = -\overline{v_r}^2(\xi, t, \zeta) - \overline{v_r}(\xi, t, \zeta)\overline{\omega_r}(\xi, t, \zeta)] + \overline{v_r}(\xi, t, \zeta), \\ \underline{v_r}(\xi, 0, \zeta) = \zeta e^{-\xi}, \\ \overline{v_r}(\xi, 0, \zeta) = (3 - 2\zeta)e^{-\xi}. \end{cases} \tag{12}$$

Operating S_{0+}^{η} on both sides of Eqn. (13), we have

$$\begin{cases} \underline{\nu_r}(\xi,\, t,\, \zeta) = \zeta e^{-\xi}\left(\dfrac{t^{\xi-1}}{\Gamma(\psi)}\right) - S^{\eta}_{0+}\underline{\nu_r}^{2}(\xi,\, t,\, \zeta) - S^{\eta}_{0+}\left[\underline{\nu_r}(\xi,\, t,\, \zeta)\underline{\omega_r}(\xi,\, t,\, \zeta)\right] + S^{\eta}_{0+}\underline{\nu_r}(\xi,\, t,\, \zeta), \\ \\ \overline{\nu_r}(\xi,\, t,\, \zeta) = (3-2\zeta)e^{-\xi}\left(\dfrac{t^{\xi-1}}{\Gamma(\psi)}\right) - S^{\eta}_{0+}\overline{\nu_r}^{2}(\xi,\, t,\, \zeta) - S^{\eta}_{0+}\left[\overline{\nu_r}(\xi,\, t,\, \zeta)\overline{\omega_r}(\xi,\, t,\, \zeta)\right] + S^{\eta}_{0+}\overline{\nu_r}(\xi,\, t,\, \zeta). \end{cases} \tag{13}$$

The following is the decomposition of the solutions $\overline{\nu_r}(\xi,\, t,\, \zeta)$ and $\underline{\nu_r}(\xi,\, t,\, \zeta)$ into infinite series from the perspective of the ADM.

$$\underline{\nu_r}(\xi,\, t,\, \zeta) = \sum_{n=0}^{\infty}\underline{\nu_{nr}}(\xi,\, t,\, \zeta) \text{ and } \overline{\nu_r}(\xi,\, t,\, \zeta) = \sum_{n=0}^{\infty}\overline{\nu_{nr}}(\xi,\, t,\, \zeta). \tag{14}$$

Substituting the decomposition series (15) into both sides of (14) yields

$$\begin{cases} \displaystyle\sum_{n=0}^{\infty}\underline{\nu_{nr}}(\xi,\, t,\, \zeta) = \zeta e^{-\xi}\left(\frac{t^{\xi-1}}{\Gamma(\psi)}\right) - S^{\eta}_{0+}\sum_{n=0}^{\infty}\underline{\nu_{nr}}(\xi,\, t,\, \zeta) - S^{\eta}_{0+}\sum_{n=0}^{\infty}\left[\underline{\nu_{nr}}(\xi,\, t,\, \zeta)\underline{\omega_{nr}}(\xi,\, t,\, \zeta)\right] \\ \qquad\qquad + S^{\eta}_{0+}\displaystyle\sum_{n=0}^{\infty}\underline{\nu_{nr}}(\xi,\, t,\, \zeta), \\ \\ \displaystyle\sum_{n=0}^{\infty}\overline{\nu_{nr}}(\xi,\, t,\, \zeta) = (3-2\zeta)e^{-\xi}\left(\frac{t^{\xi-1}}{\Gamma(\psi)}\right) - S^{\eta}_{0+}\sum_{n=0}^{\infty}\overline{\nu_{nr}}(\xi,\, t,\, \zeta) - S^{\eta}_{0+}\sum_{n=0}^{\infty}\left[\overline{\nu_{nr}}(\xi,\, t,\, \zeta)\overline{\omega_{nr}}(\xi,\, t,\, \zeta)\right] \\ \qquad\qquad + S^{\eta}_{0+}\displaystyle\sum_{n=0}^{\infty}\overline{\nu_{nr}}(\xi,\, t,\, \zeta). \end{cases} \tag{15}$$

Hence the solutions $\overline{\nu_r}(\xi,\, t,\, \zeta)$ and $\underline{\nu_r}(\xi,\, t,\, \zeta)$ are obtained as follows

$$\underline{\nu_r}(\xi,\, t,\, \zeta) = \zeta e^{-\xi}t^{\psi-1}\sum_{n=0}^{\infty}\frac{(t^{\eta})^{n}}{\Gamma(n\eta+\psi)},$$

$$\underline{\nu_r}(\xi,\, t,\, \zeta) = \zeta e^{-\xi}t^{\psi-1}E_{\eta,\,\psi}(t^{\eta}),$$

$$\overline{\nu_r}(\xi,\, t,\, \zeta) = (3-2\zeta)e^{-\xi}t^{\psi-1}\sum_{n=0}^{\infty}\frac{(t^{\eta})^{n}}{\Gamma(n\eta+\psi)},$$

$$\overline{\nu_r}(\xi,\, t,\, \zeta) = (3-2\zeta)e^{-\xi}t^{\psi-1}E_{\eta,\,\psi}(t^{\eta}).$$

If $\beta = 0$, then

$$\begin{cases} \underline{v_r}(\xi,\, t,\, \zeta) = \zeta e^{-\xi} t^{\eta-1} E_{\eta,\,\eta}(t^{\eta}), \\ \\ \overline{v_r}(\xi,\, t,\, \zeta) = (3-2\zeta) e^{-\xi} t^{\eta-1} E_{\eta,\,\eta}(t^{\eta}), \end{cases} \tag{16}$$

is the solution of following FFDE involving the Riemann-Liouville derivative

$$\begin{cases} {}^{RL}\chi_{0+}^{\alpha} v(\xi,\, t,\, \zeta) = \lambda^2 \odot [v^2(\xi,\, t,\, \zeta) + v(\xi,\, t,\, \zeta)\omega(\xi,\, t,\, \zeta)] + \lambda \odot v(\xi,\, t,\, \zeta), \quad 0 < \alpha < 1, \\ \\ S_{0+}^{1-\alpha} v(\xi,\, 0,\, \zeta) = [\zeta,\, 3-2\zeta] e^{-\xi}. \end{cases} \tag{17}$$

Consider the following FFDE with Riemann-Liouville derivative

$$\begin{cases} {}^{RL}\chi_{0+}^{\alpha} v(\xi,\, t,\, \zeta) = \lambda^2 \odot [v^2(\xi,\, t,\, \zeta) + v(\xi,\, t,\, \zeta)\omega(\xi,\, t,\, \zeta)] + \lambda \odot v(\xi,\, t,\, \zeta), \quad 0 < \alpha < 1, \\ \\ S_{0+}^{1-\alpha} v(\xi,\, 0,\, \zeta) = v_0. \end{cases} \tag{18}$$

The exact solution of Eqn. (18) obtained by Laplace transform is as follows

$$\begin{cases} \underline{v_r}(\xi,\, t,\, \zeta) = \underline{v}_0 t^{\eta-1} E_{\eta,\,\eta}[\lambda(\lambda+1)t^{\eta}], \\ \\ \overline{v_r}(\xi,\, t,\, \zeta) = \overline{v}_0 t^{\eta-1} E_{\eta,\,\eta}[\lambda(\lambda+1)t^{\eta}]. \end{cases} \tag{19}$$

By comparing equations (18) and (19), we have $v_0 = [\zeta,\, 3-2\zeta] e^{-\xi}$.
If $\beta = 1$, then

$$\begin{cases} \underline{v_r}(\xi,\, t,\, \zeta) = \zeta e^{-\xi} E_{\eta,\,1}(t^{\eta}), \\ \\ \overline{v_r}(\xi,\, t,\, \zeta) = (3-2\zeta) e^{-\xi} E_{\eta,\,1}(t^{\eta}). \end{cases} \tag{20}$$

The result (21) is the exact solution of the following Caputo FFDE

$$\begin{cases} {}^{C}\chi_{0+}^{\alpha} v(\xi,\, t,\, \zeta) = \lambda^2 \odot [v^2(\xi,\, t,\, \zeta) + v(\xi,\, t,\, \zeta)\omega(\xi,\, t,\, \zeta)] + \lambda \odot v(\xi,\, t,\, \zeta), \quad 0 < \alpha < 1, \\ \\ S_{0+}^{1-\alpha} v(\xi,\, 0,\, \zeta) = v_0. \end{cases}$$

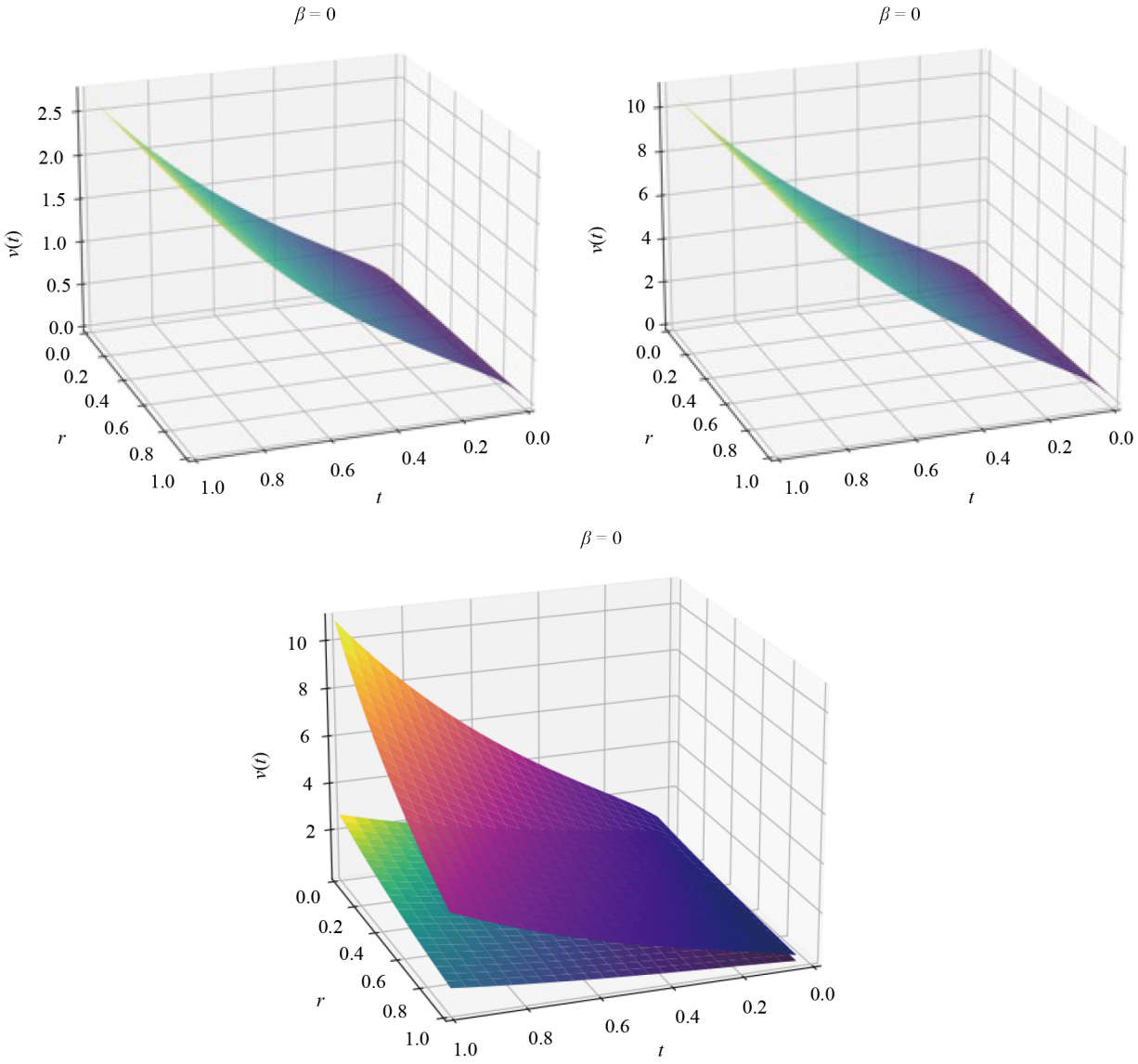

Figure 1. Approximate solution with $\eta = 0.5$ and $\beta = 0$

The approximate solutions of the numerical analysis at $\eta = 0.5$ for various values of β are displayed in Figures 1, 2, and 3. Table 1 presents the approximate solutions of $\underline{v_r}$ and $\overline{v_r}$ for different values of r and β at $t = 1$. It is evident that, for $\beta = 0$ and $\beta = 1$, the numerical analysis solution corresponds exactly with the FFDE exact solution with the Riemann-Liouville derivative and with the FFDE exact solution with the Caputo derivative.

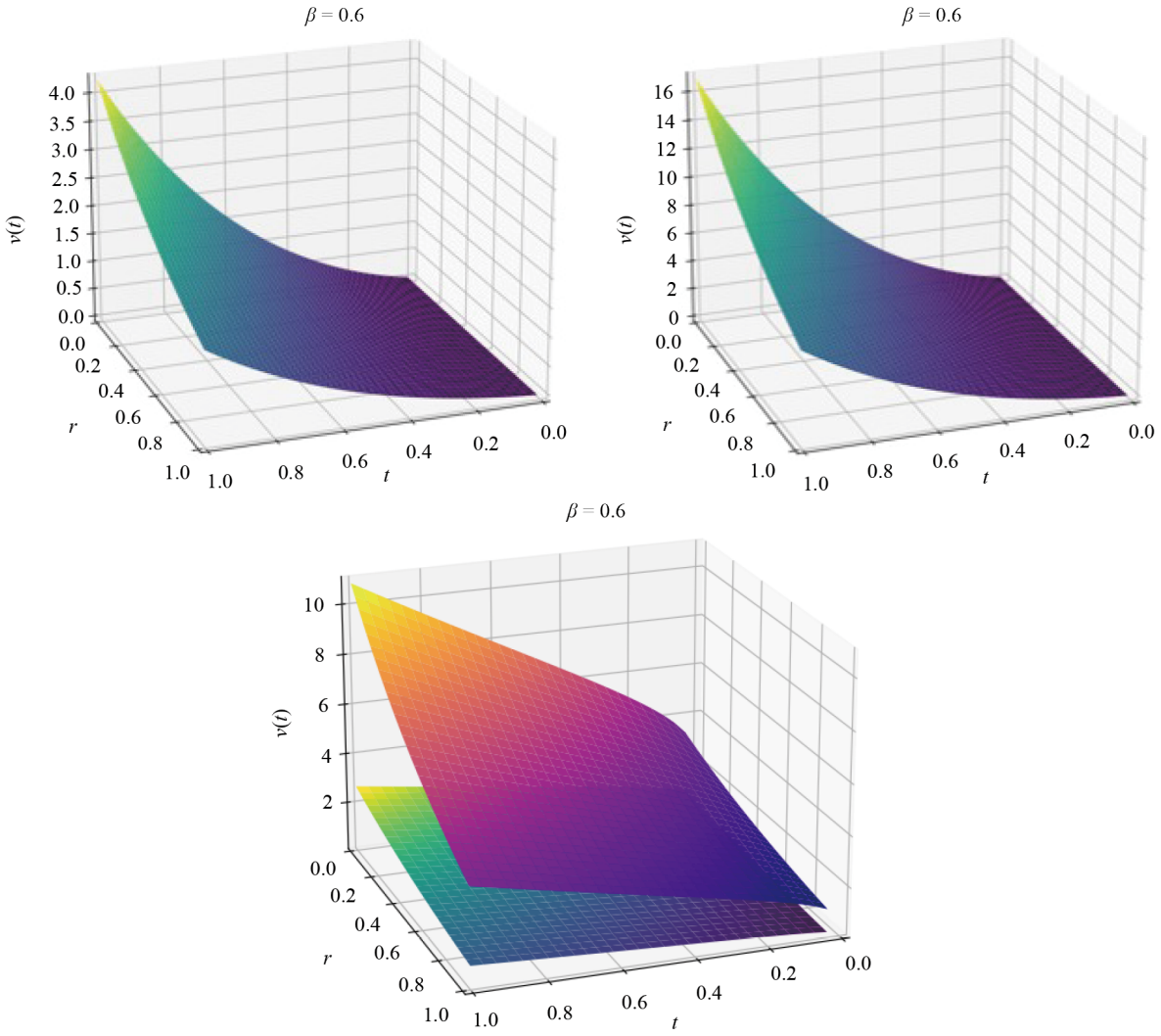

Figure 2. Approximate solution with $\eta = 0.5$ and $\beta = 0.6$

Table 1. Numerical results

	$\beta = 0$		$\beta = 0.6$		$\beta = 1$	
r	$\underline{v_r}$	$\overline{v_r}$	$\underline{v_r}$	$\overline{v_r}$	$\underline{v_r}$	$\overline{v_r}$
0	0.281363	5.158599	0.815361	5.060214	0.192210	5.158599
0.1	0.426318	4.825493	0.928703	4.831367	0.426318	4.825493
0.2	0.692072	4.503855	0.966194	4.445072	0.559411	4.192918
0.3	0.972971	4.192918	0.974065	4.233204	0.899974	3.891882
0.4	1.048230	3.891882	1.090795	3.894736	1.048230	3.316024
0.5	1.375979	3.599894	1.235263	3.583330	1.289650	3.039228
0.6	1.755496	3.316024	1.277345	3.404944	1.755496	2.501726
0.7	1.968364	3.039228	1.317666	3.132700	1.968364	2.237642
0.8	2.198684	2.768289	2.026657	2.764598	2.320893	1.973442
0.9	2.448082	2.501726	2.362277	2.543553	2.580470	1.426752
1	2.718282	2.237642	2.889703	2.173400	2.718282	1.125453

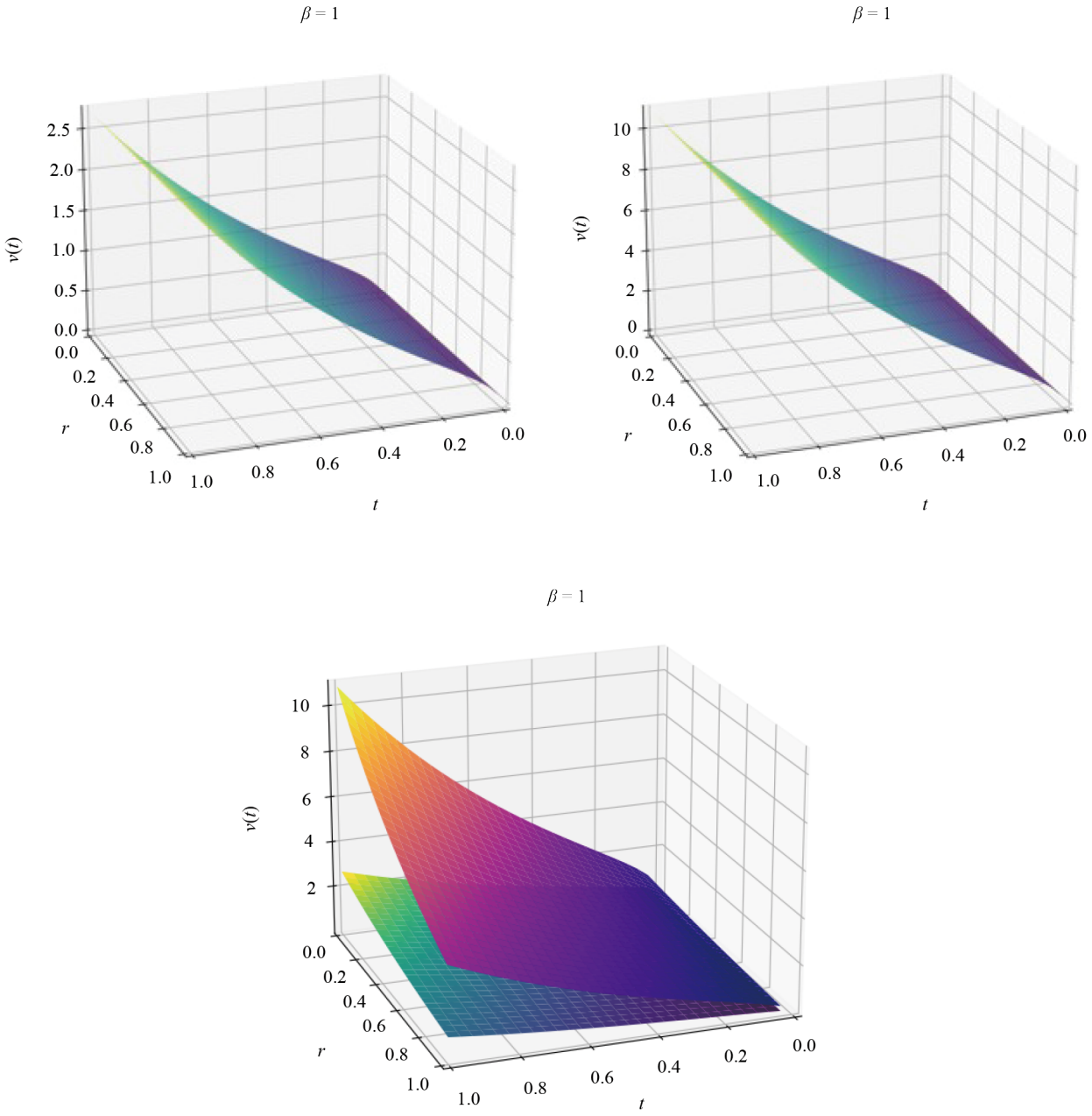

Figure 3. Approximate solution with $\eta = 0.5$ and $\beta = 1$

6. Conclusion

In this paper, the fuzzy sense of Hilfer fractional differential equations is taken into consideration. Additionally, a fuzzy Hilfer fractional differential equation solving numerical technique based on ADM is proposed. The study's results demonstrate that numerical solutions for fuzzy Hilfer fractional differential equations can be developed in various conditions. The results coincide with the Riemann-Liouville FFDE solutions if β goes to 0, and when β approaches 1, the outcome coincides Caputo's FFDE solutions. This study provides a numerical example that demonstrates the efficiency, accuracy, and ease of application of the ADM for solving fuzzy Hilfer fractional differential equations. In addition, we've used visuals to demonstrate how the approximate and ideal solutions are visually identical. To the finest of our

understanding, this is the initial occasion that fuzzy Hilfer fractional have been used to solve the gas dynamic equation within the context of ADM. We aim to apply ADM in the future, to integrate the fuzzy Hilfer fractional derivative into multiscale models, extend its application to real-world issues, and perform comparison studies to assess the benefits of each method.

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Conflict of interest

The authors declare no competing financial interest.

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