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Research Article

Efficient Collocation Algorithm for High-Order Boundary Value Problems via Novel Exponential-Type Chebyshev Polynomials

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Abstract: This paper presents an innovative collocation algorithm designed to effectively handle a specific class of boundary value problems with high-order characteristics. The approach involves utilizing a novel variant of exponential-type Chebyshev polynomials that meet all the necessary equation conditions. A key aspect of the algorithm is the transformation of both linear and nonlinear forms of the equations, along with their respective boundary conditions, into systems of algebraic equations. By solving these systems, a unique iterative technique is employed that significantly reduces the computational time required for solving these types of equations. To validate the effectiveness of the algorithm, numerous experiments using various examples with differing orders and types are conducted. The proposed technique is compared against other similar methods. The results obtained demonstrate the exceptional accuracy of the proposed approach and its potential for extension to other models in the future. Additionally, a comprehensive and detailed error analysis of the proposed method is developed, further confirming its robustness and precision in practical applications.

Keywords: exponential chebyshev polynomials, nonlinear higher-order, collocation method, convergence analysis

MSC: 65L10, 65L60

1. Introduction

Over the years, differential equations (DE) played a vital role in simulating physical phenomena in science and engineering. Different forms of differential equations were used to simulate these various phenomena, whether in the form of ordinary differential equations (ODE) or partial differential equations (PDE). The concept of change dates back to

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the era of Newton and Leibniz in the 1860s, with small changes in dependent variables by independent variables treated as ordinary numbers and magnitude, which constituted an ordinary differential equation. For more history on ordinary differential equations, see [1]. The use of these types had wide importance in different areas of science and engineering. For example, in the areas of chemical science, there was a lot of research involving the application of differential equations to simulate chemical reactive flow, as in [2]. Additionally, solving problems related to the world population, the swinging pendulum in fluid, and many others were examples from our daily lives that could be simulated through differential equations [3]. High-order differential equations, which included high derivatives, were found in different areas of science. For example, when heating an infinite horizontal layer of fluid from below with the action of rotation, Chandrasekhar [4] proved that instability occurred. This could be simulated through high-order differential equations. Also, in the modeling of viscoelastic or inelastic flows and deformation of beams, a fourth-order boundary value problem was used to model the effect of these flows or deformations [5]. These were some of the basic models that used differential equations to study their dynamics. The reader might find others in [6–9] and references therein.

Given the importance of applications involving differential equations, especially high-order ones, researchers have consistently sought suitable numerical and analytical techniques to address these problems. Among these methods, the collocation and spectral methods have gained prominence due to their ability to provide accurate solutions. These techniques have various forms of bases, each with its advantages and drawbacks. For example, the Laguerre tau and collocation methods have been used to solve high-order differential equations [10]. Bernoulli polynomials, along with shifted Chebyshev points, were employed to address fourth-sixth, and eighth-order problems by Gamel et al. [11]. The same basis was used to simulate linear equations with applications in electronic oscillations [12]. Astrophysics also saw applications of the collocation method, such as the successful solution of the Ambartsumian equation, a delay differential equation used to model the variation of light in the Milky Way using a shifted form of Bernoulli polynomials [13]. Additionally, Hassani et al. [14] employed generalized Bernoulli-Laguerre polynomials to simulate a coupled fractional nonlinear system of variable order. The method has even been extended to epidemic models, where Avazzadeh et al. [15] adapted a collocation approach based on generalized Laguerre polynomials to solve a fractional-order tuberculosis disease model, providing deeper insights into the dynamics of the disease. Izadi et al. [16] also expanded the definition of generalized Bessel polynomials to enhance simulations of Troesch's problem, while Avazzadeh et al. [17] studied the same model using a different approach. Moreover, a transcendental Bernstein series approach was used by Hassani et al. [18] for solving variable-order space-time fractional telegraph equations. Other models included the Hunter-Saxton equation [19, 20], nonlinear second-order Lane-Emden pantograph delay models [21], squeezing flow problems [22], the Rosenau-Hyman equation [23], fourth-order Sturm-Liouville problems [24], and porous fin simulation [25], with references therein. Further information about using collocation methods can be found in [26–29].

In this study, we focus on the application of a novel technique based on exponential Chebyshev polynomials for solving high-order boundary value problems of odd order on some infinite domains. The presented technique was adapted to solve several examples on various domains, including $[-\infty, \infty]$, the finite interval $[-1, 1]$, and the semi-infinite interval [30–32]. The method presented in this paper offered significant advantages, including the ability to efficiently handle high-order odd boundary value problems by transforming both linear and nonlinear forms of equations, along with boundary conditions, into algebraic systems. The use of exponential-type Chebyshev polynomials enhanced accuracy and facilitated a unique iterative technique that significantly reduced computational time. However, the method also had limitations. It was sensitive to initial guesses, which could impact convergence and robustness. Its applicability was mainly restricted to specific problem classes, limiting its generalizability. Additionally, the method might have struggled with highly nonlinear or complex boundary conditions, and the study did not address the convergence criteria, which introduced uncertainty regarding its performance across different scenarios. Several works were done on using this technique and other similar ones to solve different problems. For example, Protasov et al. in [33] presented the solution of some linear systems using exponential Chebyshev polynomials. Also, Baranoski et al. in [34] applied the exponential Chebyshev inequality to find the nondeterministic computation of some factors. A hybrid method of the exponential Chebyshev was adapted for simulating the time-fractional coupled Burgers equations defined on the semi-infinite domain in [35]. Vigo-Aguiar and Ramos [36] employed a family of implicit methods mainly based on the Chebyshev interpolation technique for solving second-order differential equations. Additionally, a similar method for solving an initial value problem of second order with a variable step size was introduced in [37]. The utilization of the fourth-order Runge-Kutta method with the aid of Chebyshev approximations for a stiff initial-value problem was elaborated in [38]. Other applications of Chebyshev polynomials for solving different applicable problems included the Basset equation [39], nonlinear fractional PDEs with delay [40], the convection-diffusion equation [41–44] with references therein. To the best of our knowledge, only a few papers dealt with the use of this method, which drove us to investigate this technique further.

The third and fifth order linear and nonlinear equations that take the following general form are considered in this work:

$$y^{(\delta)}(x)+\sum_{k=1}^{\delta-1} c_k(x)y^{(k)}(x)=\Theta(x,\ y^r(x)),\ x\in[a,\ b], \tag{1}$$

with initial and boundary conditions

$$\begin{cases} y^{(j)}(a)=\mu_j,\ j=0,\ 1,\ \dots,\ (\delta-1)/2, \\ \\ y^{(m)}(b)=\alpha_m,\ m=0,\ 1,\ \dots,\ (\delta-3)/2, \end{cases} \tag{2}$$

where δ can either be odd or even, $n\geq 1$, $y^{(k)}(x)=d^k(y(x))/dx^{(k)}$, and $\Theta(x,\ y^r(x))$ is a generalization to linear term, when $r=1$, and nonlinear terms which shall be defined in later sections. The function $y(x)$ is the unknown function, the known functions $c_k(x)$ is well defined and bounded on the interval $[a,\ b]$. In Eq. (2), μ_j and α_m are real constants. The novelty of the paper lies within the following points:

(1) The paper introduces a novel collocation algorithm specifically designed to address high-order odd boundary value problems, which are not commonly covered by existing methods.

(2) A unique variant of exponential-type Chebyshev polynomials is utilized, fulfilling all necessary equation conditions and enhancing the flexibility and accuracy of the approach.

(3) The proposed method involves the transformation of both linear and nonlinear forms of equations, as well as their boundary conditions, into systems of algebraic equations, streamlining the solution process.

(4) An innovative iterative technique is employed that significantly reduces computational time, offering a more efficient solution process compared to traditional methods.

(5) The algorithm's effectiveness is validated through numerous experiments involving various examples with differing orders and types, demonstrating its robustness across different scenarios.

(6) A complete and detailed error analysis of the proposed method is provided, confirming its precision and reliability in practical applications.

(7) The results indicate the proposed approach's potential for extension to other models, opening avenues for future research and application in various fields.

2. Preliminaries

2.1 *Exponential chebyshev functions*

To solve the differential equations, Chebyshev polynomials method is the most important special function matrix method, which is widely used in numerical analysis. The Chebyshev functions of the first kind polynomials $T_n(t)$ is get from the recurrence relation [45–47]

$$T_0(t)=1,\ T_1(t)=t,\ T_{n+1}(t)=2tT_n(t)-T_{n-1}(t),\ n\geq 1. \tag{3}$$

In these relation we use basic rational Chebyshev functions as [48],

$$R_n(t) = T_n\left(\frac{t-1}{t+1}\right). \tag{4}$$

So, exponential Chebyshev functions (ECFs) is defined by

$$E_n(t) = T_n\left(\frac{e^t-1}{e^t+1}\right), \tag{5}$$

and where recurrence relation for the ECFs is

$$E_0(t) = 1,\ E_1(t) = \left(\frac{e^t-1}{e^t+1}\right),$$

$$E_{n+1}(t) = 2\left(\frac{e^t-1}{e^t+1}\right)E_n(t) - E_{n-1}(t),\ n \geq 1.$$

At final stage, let us state the orthogonality relation for this set of ECFs. In [49, 50], the authors proved that

$$\int_{-\infty}^{+\infty} E_n(t)\ E_m(t)\ w(t)dt = \begin{cases} \pi, \text{ if } n = m = 0, \\ \frac{\pi}{2}, \text{ if } n = m \neq 0, \\ 0, \text{ if } n \neq m, \end{cases} \tag{6}$$

where the associated weight function is $w(t) := \dfrac{e^{t/2}}{e^t+1}$.

2.2 *Function expansion with ECFs: Convergence analysis*

A (square-integrable) function $f(t)$ is defined over any interval $[a,\ b] \subseteq \Omega := (-\infty,\ +\infty)$ may be expanded in terms of exponential second Chebyshev function in such a way that

$$f(t) = \sum_{i=0}^{\infty} a_i\ E_i(t), \tag{7}$$

where we seek for the unknown coefficients a_i for $i \geq 0$. Upon the orthogonality condition (6), we may obtain the subsequent representations for the coefficients a_i in the forms

$$a_0 = \frac{1}{\pi}\int_\Omega f(t)\ w(t)dt,\ a_i = \frac{2}{\pi}\int_\Omega E_i(t)\ f(t)\ w(t)dt,\ i \geq 1.$$

In the above equation (8) the summation is restricted to N where $N < \infty$ so that the function $f(t)$ is approximated by $f_N(t)$ as following form

$$f(t) \cong f_N(t) := \sum_{i=0}^{N} a_i\ E_i(t) = \mathbf{E}(t)\ \mathbf{A}, \tag{8}$$

where

$$\mathbf{E}(t) = [E_0(t),\ E_1(t),\ \ldots,\ E_N(t)],\ \mathbf{A} = [a_0,\ a_1,\ a_2,\ \ldots,\ a_N]^T.$$

Consequently, let us define the space V_N spanned consists of these $(N+1)$ ECFs as follows

$$V_N := \text{Span}\Big\{E_0(t),\ E_1(t),\ \ldots,\ E_N(t)\Big\} \subseteq V := L_{2,\ w}([a,\ b]).$$

We next define the inner product and the associated norm for elements in the aforesaid spaces. We have

$$(f,\ g)_w := \int_a^b f(t)\ g(t)\, w(t) dt,\ \|f\|_w : = (f,\ f)^{1/2}.$$

We now assume that a function $f \in V$ to be approximated by another function $g_N \in V_N$. Let us recall that the element $g_* \in V_N$ is called to be the closest (best) approximation to $f \in V$ if the subsequent relation holds true:

$$\|f - g_*\|_w \le \|f - g\|_w,\ \forall g \in V.$$

To ensure the existence of g_* we use the fact that V_N is a closed and complete subspace of V. For a proof and detailed information we may refer to cf. [47, Thm. I1].

In the sequel, the aim would be to measure the difference between $f(t)$ and $f_N(t)$ when we subtract the corresponding series forms given above. So, we define the error $E_N(t) := f(t) - f_N(t) = \sum_{i=N+1}^{\infty} a_i\ E_i(t)$. To derive an upper bound in the weighted L_2 norm, we need the subsequent result.

Lemma 1 Let a given function f is $(N+1)$ times continuously differentiable on $[a,\ b]$, where $a < b$. Also, assume that $F_N(t)$ shows the associated interpolating function at $(N+1)$ zeros of Chebyshev function $T_{N+1}(t)$ on $[a,\ b]$. Then, the subsequent error estimate holds true for any $t \in [a,\ b]$

$$\max_{t \in [a,\ b]} |f(t) - F_N(t)| \le \frac{2(b-a)^{N+1}}{4^{N+1}\ (N+1)!} \|f\|_\infty, \tag{9}$$

where $\|f\|_\infty : = \sup_{t \in [a,\ b]} |f^{(N+1)}(t)|$. See [51], for a proof.

Theorem 1 Let suppose that the hypotheses of Lemma 2.1 admitted. If $f_N(t) := E(t)\ A$ signifies the finest (best) approximation to $f(t)$ out of space V_N, then one gets the subsequent error bound

$$\|E_N(t)\|_w \le \frac{2\sqrt{\pi}(b-a)^{N+1}}{4^{N+1}\ (N+1)!}\|f\|_\infty.$$

Proof. We start the proof by using the fact that the error norm can be written as

$$\|E_N(t)\|_w^2 = \|f(t)-f_N(t)\|_w^2 = \int_a^b |f(t)-f_N(t)|^2\ w(t)dt.$$

Since f_N stands for the best approximation, we get

$$\|E_N(t)\|_w = \|f(t)-f_N(t)\|_w \le \|f(t)-g(t)\|_w,\ \forall g \in V_N.$$

The previous inequality holds true if we replace $g(t)$ by $F_N(t)$. Together with relation (9), it follows that

$$\|E_N(t)\|_w^2 \le \int_a^b |f(t)-F_N(t)|^2\ w(t)dt \le \Big[\frac{2(b-a)^{N+1}}{4^{N+1}\ (N+1)!}\|f\|_\infty\Big]^2 \int_a^b w(t)dt.$$

On the other hand, it is not a difficult job to show that

$$\int_a^b w(t)dt \le \int_\Omega w(t)dt = 2\tan^{-1}\left(e^{t/2}\right)\Big|_{-\infty}^{+\infty} = \pi.$$

By preforming the square root we have accomplished the proof. □

2.3 *Operational matrix of differentiation*

The Chebyshev polynomials of first kind $T_n(t)$ that can be expressed in terms of t^n by using different formula [51]. We have

$$T_n(t) = \sum_{j=0}^{[n/2]} (-1)^j 2^{n-2m-1} \frac{n-m}{m} \binom{n-m}{m} t^{n-2m},\ 2m \le n.$$

Similarly, we can find the relation for Chebyshev of the second kind in the form

$$U_n(t) = \sum_{m=0}^{[n/2]} (-1)^m \binom{n-m}{m}\ (2t)^{n-2m},\ n > 0.$$

Utilizing the exponential expression from the recurrence relation $U_t = (\dfrac{e^t-1}{e^t+1})$ this expression replace in the place of $2t$, in the term of power U_t. For similar relations, the reader can refer to [46]

$$E_n(t) = \sum_{m=0}^{[n/2]} (-1)^m 2^{n-2m} \binom{n-m}{m} \; (U(t))^{n-2m}.$$

By using this above relation, we form new expression for different values if n is even,

$$E_{2l}^{(k)}(t) = \sum_{m=0}^{l} (-1)^m 2^{2l-2m} \binom{2l-m}{m} (U(t))^{2l-2m}. \tag{10}$$

Then, if n is odd, then we use

$$E_{2i+1}^{(k)}(t) = \sum_{m=0}^{i} (-1)^m 2^{2i+1-2m} \binom{2i+1-m}{m} m \;\; (U(t))^{2i+1-2m}. \tag{11}$$

Based on the aforementioned relations, we can derive a new general matrix from the (ECFs) as below:

$$\mathbf{E}(t) = \mathbf{U}(t)\mathbf{D}^T, \tag{12}$$

where $\mathbf{E}(t)$ and $\mathbf{U}(t)$ are matrices of this form

$$\mathbf{E}(t) = [E_0(t)E_1(t), \cdots, E_N(t)], \; \mathbf{U}(t) = [U^0(t) \; U^1(t) \; \cdots \; U^n(t)],$$

and

$$U^0(t) = 1, \; U^1(t) = \left(\frac{e^t - 1}{e^t + 1}\right),$$

$$U^2(t) = \left(\frac{e^t - 1}{e^t + 1}\right)^2 \cdots \quad U^n(t) = \left(\frac{e^t - 1}{e^t + 1}\right)^n,$$

and $\mathbf{D}$ is a lower triangular constant matrix of size $(n+1) \times (n+1)$ is

$$\mathbf{D} = \begin{bmatrix} \binom{0}{0} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 2\binom{2}{0} & 0 & 0 & \cdots & 0 \\ (-1)2^0\binom{1}{1} & 0 & 2^2\binom{2}{0} & 0 & \cdots & 0 \\ 0 & (-1)2^1\binom{2}{1} & 0 & 2^3\binom{3}{0} & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & (-1)^i 2^0 \binom{i}{i} & 0 & (-1)^{i-1}2^2\binom{i+1}{i-1} & 0 & 2^{2i}\binom{2i+1}{0} \end{bmatrix}.$$

In this case, we are going to use the last row for odd values of $n = 2l+1$, and for even values $n = 2l$, it will replace the last row of the matrix **D**.

2.4 *Derivatives of exponential functions*

Now, from (11) we can obtain the k^{th} order derivatives of the matrix $\mathbf{E}(t)$ as follow,

$$\mathbf{E}(t) = \mathbf{U}(t)\mathbf{D}^T,$$

$$\mathbf{E}^{(1)}(t) = \mathbf{U}^{(1)}(t)\mathbf{D}^T,$$

$$\mathbf{E}^{(2)}(t) = \mathbf{U}^{(2)}(t)\mathbf{D}^T,$$

$$\vdots$$

similarly we find the k^{th} order derivatives of $\mathbf{E}(t)$ as

$$\mathbf{E}^{(k)}(t) = \mathbf{U}^{(k)}(t)\mathbf{D}^T. \tag{13}$$

Equation (12) represents the new operational matrix of the derivatives of exponential Chebyshev functions.

3. Fundamental matrix relations

By the Eq. (7) if $y(t)$ is approximated by truncated series in terms of ECFs as

$$y_N(t) \cong \sum_{i=0}^{N} a_i E_i(t), \tag{14}$$

then, we take $y(t)$ and its derivative $y^{(k)}$ can be represented in the form of matrix as

$$[y(t)] = \mathbf{E}(t)\mathbf{A}, \Rightarrow \mathbf{Y} = \mathbf{U}(t)\mathbf{D}^T\mathbf{A}, \tag{15}$$

and

$$[y^{(k)}(t)] = \mathbf{Y}^{(k)} = \mathbf{E}^{(k)}(t)\mathbf{A},\ k = 0,\ 1,\ 2,\ \ldots,\ m \leq N, \tag{16}$$

where

$$\mathbf{E}^{(k)}(t) = [(E_0(t))^k,\ (E_1(t))^k,\ \ldots,\ (E_N(t))^k],\ \mathbf{A} = [a_0,\ a_1,\ a_2,\ \ldots,\ a_N]^T.$$

Here, matrix $\mathbf{E}(t)$ is the ECFs matrix and $\mathbf{A}$ matrix is a scalar matrix and $a_0,\ a_1,\ a_2,\ \ldots,\ a_N$ are coefficients to be determined in Eq. (13). Now, we replace (12) in (15) to get

$$\mathbf{Y}^{(k)} = \mathbf{U}^{(k)}(t)\mathbf{D}^T\mathbf{A}. \tag{17}$$

To adapt the previously mentioned matrix relation and for more clarity, we shall assign the right-hand side of Eq. (1) as $\Theta(x,\ y^r(x)) = f(x) - \sum_{r=1}^{n} q_r(x)y^r(x)$ where $q_r(x)$, and $f(x)$ are well defined and bounded on the interval $[a,\ b]$. The main equation in (1) will take the following form

$$y^{(\delta)}(x) + \sum_{k=1}^{\delta-1} c_k(x)y^{(k)}(x) + \sum_{r=1}^{n} q_r(x)y^r(x) = f(x),\ x \in [a,\ b]. \tag{18}$$

The form of high order linear non homogeneous differential equations with variable coefficients in finite domains is by using general Chebyshev collocation points in the following form

$$x_i = \frac{b-a}{2}\cos\left(\frac{(2i-1)\pi}{2N}\right) + \frac{b+a}{2},\ i = 1,\ 2,\ \ldots,\ N. \tag{19}$$

Thus we obtain the following

$$y^{(\delta)}(x_i) + \sum_{k=1}^{\delta-1} c_k(x_i)y^{(k)}(x_i) + \sum_{r=1}^{n} q_r(x_i)y^r(x_i) = f(x_i). \tag{20}$$

The matrix form to Eq. (19) can be written as the next one,

$$\mathbf{Y}^{(\delta)} + \sum_{k=1}^{\delta-1} \mathbf{C}_k\mathbf{Y}^{(k)} + \sum_{r=1}^{n} \mathbf{Q}_r\mathbf{Y}^r = \mathbf{F},\ n \geq 1, \tag{21}$$

where:

$$\mathbf{F} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix},\ \mathbf{C}_k = \begin{bmatrix} c_k(x_0) & 0 & \cdots & 0 \\ 0 & c_k(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_k(x_N) \end{bmatrix},$$

$$\mathbf{Q}_r = \begin{bmatrix} q_r(x_0) & 0 & \cdots & 0 \\ 0 & q_r(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_r(x_N) \end{bmatrix}.$$

Next, we shall study the two cases of linear and nonlinear type of Eq. (19). Firstly, the study of the linear case is as follows.

3.1 *Linear cases*

To deal with the linear case problems, we set $n = 1$ in Eq. (20) to obtain the linear matrix form as:

$$\mathbf{Y}^{(\delta)} + \sum_{k=1}^{\delta-1} \mathbf{C}_k \mathbf{Y}^{(k)} + \mathbf{Q}_1 \mathbf{Y} = \mathbf{F}, \tag{22}$$

where the matrix equation may now be formed by replacing the relations (16), and (14) in Eq. (21):

$$\left\{ \mathbf{U}^{(\delta)} \mathbf{D}^T + \sum_{k=1}^{\delta-1} \mathbf{C}_k \mathbf{U}^{(k)} \mathbf{D}^T + \mathbf{Q}_1 \mathbf{U} \mathbf{D}^T \right\} \mathbf{A} = \mathbf{F}. \tag{23}$$

Eq. (22) can be written in the canonical form as:

$$\mathbf{\Omega} \mathbf{A} = \mathbf{F} \;\Rightarrow\; [\mathbf{\Omega};\, \mathbf{F}], \tag{24}$$

where,

$$\mathbf{\Omega} = \mathbf{U}^{(\delta)} \mathbf{D}^T + \sum_{k=1}^{\delta-1} \mathbf{C}_k \mathbf{U}^{(k)} \mathbf{D}^T + \mathbf{Q}_1 \mathbf{U} \mathbf{D}^T.$$

Eq. (23) indicates a system of $(N+1)$ linear algebraic equations with unknown coefficients matrix $\mathbf{A}$. To complete the solution by finding $\mathbf{A}$, the boundary conditions in Eq. (2) must be presented in matrix form and then inserted into (23). We generate the following matrix form by utilizing relation in (16) at $x_i = x_0$ and $x_i = x_N$:

$$\left\{ \mathbf{U}^{(j)}(x_0) \mathbf{D}^T \right\} \mathbf{A} = \mu_j, \; j = 0,\, 1,\, \ldots,\, (\delta-1)/2,$$

$$\left\{ \mathbf{U}^{(m)}(x_N) \mathbf{D}^T \right\} \mathbf{A} = \alpha_m, \; m = 0,\, 1,\, \ldots,\, (\delta-3)/2,$$

or equivalent to

$$
\begin{cases}
\Phi_j \mathbf{A} = [\mu_j] \Rightarrow \ [\Phi_j;\ \mu_j],\ j = 0,\ 1,\ \ldots,\ (\delta - 1)/2, \\
\\
\Psi_m \mathbf{A} = [\alpha_m] \Rightarrow \ [\Psi_m;\ \alpha_m],\ m = 0,\ 1,\ \ldots,\ (\delta - 3)/2,
\end{cases}
\tag{25}
$$

where

$$
\Phi_j = \mathbf{U}^{(j)}(x_0)\mathbf{D}^T = [\phi_{j,\,0}\ \phi_{j,\,1}\ \phi_{j,\,2} \cdots \phi_{j,\,N}],
$$

$$
\Psi_m = \mathbf{U}^{(m)}(x_N)\mathbf{D}^T = [\psi_{m,\,0}\ \psi_{m,\,1}\ \psi_{m,\,2} \cdots \psi_{m,\,N}].
$$

The new solution augmented matrix is obtained by substituting the row matrices in (24) with the $(j + m = \delta)$ rows in the augmented matrix in (23), and will take the following form:

$$
\overline{\mathbf{\Omega}}\mathbf{A} = \mathbb{F} \ \Rightarrow \ [\overline{\mathbf{\Omega}};\ \mathbb{F}],
$$

$$
[\overline{\mathbf{\Omega}}; \mathbb{F}] = \begin{bmatrix}
\omega_{0,\,0} & \omega_{0,\,1} & \omega_{0,\,2} & \cdots & \omega_{0,\,N} & ; & f(x_0) \\
\omega_{1,\,0} & \omega_{1,\,1} & \omega_{1,\,2} & \cdots & \omega_{1,\,N} & ; & f(x_1) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\omega_{N-\delta,\,0} & \omega_{N-\delta,\,1} & \omega_{N-\delta,\,2} & \cdots & \omega_{N-\delta,\,N} & ; & f(x_{N-\delta}) \\
\\
\phi_{0,\,0} & \phi_{0,\,1} & \phi_{0,\,2} & \cdots & \phi_{0,\,N} & ; & \mu_0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\phi_{j,\,0} & \phi_{j,\,1} & \phi_{j,\,2} & \cdots & \phi_{j,\,N} & ; & \mu_j \\
\psi_{0,\,0} & \psi_{0,\,1} & \psi_{0,\,2} & \cdots & \psi_{0,\,N} & ; & \alpha_0 \\
\psi_{m,\,0} & \psi_{m,\,1} & \psi_{m,\,2} & \cdots & \psi_{m,\,N} & ; & \alpha_m
\end{bmatrix}.
\tag{26}
$$

The preceding system is a linear algebraic system composed of $(N + 1)$ equations, and by solving it from (25), the unknown exponential Chebyshev coefficients matrix $\mathbf{A}$ is found. Upon finding the coefficient matrix $\mathbf{A}$, we substitute it using (14), eventually, the exponential Chebyshev polynomial approximation solution is found.

3.2 *Nonlinear cases*

Recalling again Eq. (20), by setting $n \geq 2$ the nonlinear case of the next form will be appear as:

$$
\mathbf{Y}^{(\delta)} + \sum_{k=1}^{\delta-1} \mathbf{C}_k \mathbf{Y}^{(k)} + \sum_{r=2}^{n} \mathbf{Q}_r \mathbf{Y}^r = \mathbf{F},\ n \geq 2.
\tag{27}
$$

To show the procedure that we deal with the nonlinear case in Eq. (26), it can be formulated as the following form:

$$\mathbf{Y}^{(\delta)}+\sum_{k=1}^{\delta-1}\mathbf{C}_k\mathbf{Y}^{(k)}+\left(\sum_{r=2}^{n}\mathbf{Q}_r\mathbb{Y}^{r-1}\right)\mathbf{Y}=\mathbf{F},\ n\geq 2, \tag{28}$$

where

$$\mathbb{Y}^{\xi}=\begin{bmatrix} y^{\xi}(x_0) & 0 & \cdots & 0 \\ 0 & y^{\xi}(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y^{\xi}(x_N) \end{bmatrix},\ \xi\geq 1,$$

$$\mathbb{Y}=\mathbb{U}\mathbb{D}^T\mathbb{A},\ \mathbb{U}=\begin{bmatrix} \mathbf{U}(x_0) & 0 & \cdots & 0 \\ 0 & \mathbf{U}(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{U}(x_N) \end{bmatrix},$$

$$\mathbb{D}^T=\begin{bmatrix} \mathbf{D}^T & 0 & \cdots & 0 \\ 0 & \mathbf{D}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{D}^T \end{bmatrix},\ \mathbb{A}=\begin{bmatrix} \mathbf{A} & 0 & \cdots & 0 \\ 0 & \mathbf{A} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A} \end{bmatrix}. \tag{29}$$

Now, by using the relations (16), (14), and (28) in Eq. (27), the matrix equation can be written as:

$$\left\{\mathbf{U}^{(\delta)}\mathbf{D}^T+\sum_{k=1}^{\delta-1}\mathbf{C}_k\mathbf{U}^{(k)}\mathbf{D}^T+\sum_{r=1}^{n}\mathbf{Q}_r(\mathbb{U}\mathbb{D}^T\mathbb{A})^{r-1}\mathbf{U}\mathbf{D}^T\right\}\mathbf{A}=\mathbf{F}. \tag{30}$$

Eq. (29) can be written in the canonical form as:

$$\mho\mathbf{A}=\mathbf{F}\ \Rightarrow\ [\mho\mathbf{A};\ \mathbf{F}], \tag{31}$$

where,

$$\mho\mathbf{A}=\mathbf{U}^{(\delta)}\mathbf{D}^T\mathbf{A}+\sum_{k=1}^{\delta-1}\mathbf{C}_k\mathbf{U}^{(k)}\mathbf{D}^T\mathbf{A}+\sum_{r=1}^{n}\mathbf{Q}_r(\mathbb{U}\mathbb{D}^T\mathbb{A})^{r-1}\mathbf{U}\mathbf{D}^T\mathbf{A}.$$

Then, after substituting the row matrices of boundary conditions in relation (24) with the $(j+m=\delta)$ rows in the augmented matrix (30), the resulting solution augmented matrix will be

$$\overline{\mho}\mathbf{A}=\mathbb{F}\ \Rightarrow\ [\overline{\mho}\mathbf{A};\ \mathbb{F}],$$

$$
[\overline{\mho}\mathbf{A};\ \mathbb{F}] = \begin{bmatrix}
\rho_{0,\,0} & \rho_{0,\,1} & \rho_{0,\,2} & \cdots & \rho_{0,\,N} & ; & f(x_0) \\
\rho_{1,\,0} & \rho_{1,\,1} & \rho_{1,\,2} & \cdots & \rho_{1,\,N} & ; & f(x_1) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\rho_{N-\delta,\,0} & \rho_{N-\delta,\,1} & \rho_{N-\delta,\,2} & \cdots & \rho_{N-\delta,\,N} & ; & f(x_{N-\delta}) \\
\phi_{0,\,0}a_0 & \phi_{0,\,1}a_1 & \phi_{0,\,2}a_2 & \cdots & \phi_{0,\,N}a_N & ; & \mu_0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\phi_{j,\,0}a_0 & \phi_{j,\,1}a_1 & \phi_{j,\,2}a_2 & \cdots & \phi_{j,\,N}a_N & ; & \mu_j \\
\psi_{0,\,0}a_0 & \psi_{0,\,1}a_1 & \psi_{0,\,2}a_2 & \cdots & \psi_{0,\,N}a_N & ; & \alpha_0 \\
\psi_{m,\,0}a_0 & \psi_{m,\,1}a_1 & \psi_{m,\,2}a_2 & \cdots & \psi_{m,\,N}a_N & ; & \alpha_m
\end{bmatrix},
$$

The previous resulting system will be a nonlinear one consists of $(N+1)$ equations in $(N+1)$ unknowns, the exponential Chebyshev coefficients (a_i), we used the Newton technique to solve it. After finding the coefficients, we replace them into (14), eventually the Exponential Chebyshev polynomial approximation solution is found.

4. Residual error function

This part allows us to evaluate the suggested method's precision in terms of the residual error function. Since the truncated Exponential Chebyshev series in Eq. (13) is thought to represent a reasonable solution of Eq. (17), the resulting equation must satisfy Eq. (17) when the approximate solution $y_N(x)$ and its derivatives are substituted, with the assistance of the collocation points specified in Eq. (18). The estimated solution's residual error function can be determined using the next formula:

$$
\Re(x) = \left| y_N^{(\delta)}(x) + \sum_{k=1}^{\delta-1} c_k(x) y_N^{(k)}(x) + \sum_{r\geq 1}^{n} q_r(x) y_N^r(x) - f(x) \right| \cong 0,\ x \in [a,\ b]. \tag{32}
$$

Calculating the regular absolute error and the residual error shown by Eq. (31) together will guarantee the efficacy of the suggested method.

5. Numerical results

In this section, the third and fifth order linear and nonlinear equations that take the following general form are considered in this work: To show the efficiency of the proposed technique, we compare the numerical results with the exact solution and some methods used to solve the same problem. The formulas for the absolute error and the maximum absolute error will be:

- The absolute error ε_N of the solution is defined by

$$
\varepsilon_N(x) = |y(x) - y_N(x)|,\ x \in [a,\ b].
$$

- The maximum error $\varepsilon_{Max.}$ of the solution is defined by

$$\varepsilon_{Max.} = \max_{1\le i\le M} \varepsilon_N(x_i)|,$$

where M in the number of used points in $[a,\ b]$.

It is important to note that the examples selected for this study were carefully chosen to thoroughly evaluate the performance and versatility of the proposed collocation algorithm across a diverse range of scenarios. Each example represents a distinct type of boundary value problem, with varying orders and characteristics, thereby enabling us to assess the algorithm's applicability and effectiveness in different contexts. Additionally, we selected examples from the literature to facilitate comparison with existing techniques, which helps in assessing the relative performance of our method. The first example is as follows:

Example 1 Consider the linear third order differential equation by setting $\delta = 3$, $c_k(x) = 0$, and $q_r(x) = 0$ except that $q_1(x) = -x$ in (1) to give

$$y'''(x) - xy(x) = f(x),\ x \in (0,\ 1),$$

with, the boundary conditions are

$$\begin{cases} y(0) = 0,\ y'(0) = 1, \\ y(1) = 0. \end{cases}$$

Here, $f(x) = (x^3 - 2x^2 - 5x - 3)\exp(x)$ so that the exact solution will be $y(x) = x(1-x)\exp(x)$.
For $N = 4$, using (18), the Chebyshev collocation points includes the following points:

$$\{x_0 = 0,\ x_1 = 0.146,\ x_2 = 0.5,\ x_3 = 0.854,\ x_4 = 1.0\}.$$

From equation (22) we get fundamental matrix equation of the problem as:

$$\left\{\mathbf{U}^{(3)}(\mathbf{D}^T) + \mathbf{Q}_1\mathbf{U}(\mathbf{D}^T)\right\}\mathbf{A} = \mathbf{F},$$

where

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 4 & 0 & 0 \\ 0 & -4 & 0 & 8 & 0 \\ 1 & 0 & -12 & 0 & 16 \end{bmatrix},\ \mathbf{Q}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -0.146 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & -0.854 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$
\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0.0731 & 0.0053 & 0.004 & 0.00002 \\ 1 & 0.2449 & 0.0599 & 0.0147 & 0.0036 \\ 1 & 0.4026 & 0.1621 & 0.0653 & 0.0263 \\ 1 & 0.4621 & 0.2136 & 0.0987 & 0.0456 \end{bmatrix},
$$

$$
\mathbf{U}^{(3)} = \begin{bmatrix} 0 & -0.250 & 0 & 0.75 & 0 \\ 0 & -0.2447 & -0.1442 & 0.7103 & 0.2120 \\ 0 & -0.1927 & -0.4190 & 0.3498 & 0.4821 \\ 0 & -0.1076 & -0.5106 & -0.01233 & 0.2700 \\ 0 & -0.0707 & -0.4940 & -0.2748 & 0.0971 \end{bmatrix},
$$

$$
\mathbf{F} = [-3.000 \quad -4.3668 \quad -9.6862 \quad -19.0256 \quad -24.4645]^T.
$$

By applying the suitable compensation for the provided boundary values, the exponential Chebyshev coefficients can be calculated with the following values:

$$
\mathbf{A} = [-1.254 \quad 0.1667 \quad -1.7419 \quad -0.4167 \quad -0.4870]^T,
$$

then, the approximate solution is obtained as:

$$
y_4(x) = -1.2549E_0(x) + 0.1667E_1(x) - 1.7419E_2(x) - 0.4167E_3(x) - 0.4870E_4(x).
$$

The numerical results in $(0,\ 1)$ for Example 1 are demonstrated in Table 1 for the approximate and exact solution for different values of N. From these results, it can be noticed that the proposed method provides accurate results with few numbers of basis. In addition, Table 2 lists the maximum absolute error for different values of N. Also, Figure 1 demonstrates the behavior of the exact and approximate solution for $N = 18$.

Table 1. Numerical results and error comparison for x_i for Example 1

x_i	$y(x_i)$	$y_{10}(x_i)$	$\varepsilon_{10}(x_i)$	$\varepsilon_{14}(x_i)$	Taylor series [53]
0.1	0.0994653826	0.0994649218	4.60752650e-07	5.0586408673e-10	0.0015490007155
0.2	0.1954244413	0.1954226005	1.84075438e-06	2.0303108083e-09	0.0021358168660
0.3	0.2834703496	0.2834661808	4.16876202e-06	4.5575697105e-09	0.0019739197831
0.4	0.3580379274	0.3580305622	7.36526208e-06	8.1329888379e-09	0.0012844772608
0.5	0.4121803177	0.4121688273	1.14904117e-05	1.2639635505e-08	0.0002965594254
0.6	0.4373085121	0.4372917873	1.67248057e-05	1.8349161190e-08	0.0007507731073
0.7	0.4228880686	0.4228654721	2.25964554e-05	2.4777017216e-08	0.0016035240005
0.8	0.3560865486	0.3560570469	2.95016501e-05	3.2749504997e-08	0.0019837356859
0.9	0.2213642800	0.2213268868	3.73931932e-05	4.1249611976e-08	0.0015748957539

Table 2. Maximum error with different values of N for Example 1

N	8	10	12	14	16	18
Max. error	8.27878408e-04	3.73931932e-05	1.32108602e-06	4.12496120e-08	1.22214899e-09	3.80006582e-11

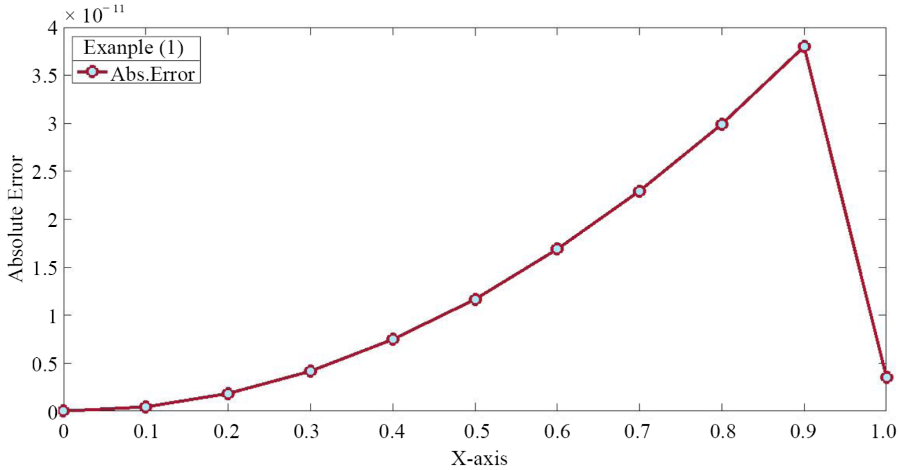

Figure 1. The absolute errors to $N = 18$ with $x \in [0,\ 1]$ for Example 1

Example 2 Consider the non-linear third order differential equation with $\delta = 3,\ c_k(x) = 0$, and $q_r(x) = 0$ except that $q_1(x) = -x,\ q_2(x) = 1$ to get

$$y'''(x) - xy(x) + y^2(x) = f(x),\ x \in (0,\ 1),$$

with the subsequent boundary conditions

$$\begin{cases} y(0) = 0,\ y'(0) = -1, \\ y(1) = 0. \end{cases}$$

Here, $f(x)$ is chosen such that the exact solution is $y(x) = (x-1)\sin(x)$.
By taking $N = 4$, the set of collocation points as before will be:

$$\{x_0 = 0,\ x_1 = 0.146,\ x_2 = 0.5,\ x_3 = 0.854,\ x_4 = 1.0\}. \tag{33}$$

From equation (29) we get fundamental matrix equation of the problem as

$$\left\{\mathbf{U}^{(3)}\mathbf{D}^{T}+\mathbf{Q}_1\mathbf{U}\mathbf{D}^{T}+\mathbf{Q}_2(\mathbb{U}\mathbb{D}^{T}\mathbb{A})\mathbf{U}\mathbf{D}^{T}\right\}\mathbf{A}=\mathbf{F}.$$

The result for the matrices $\mathbf{D}^T$, $\mathbf{U}$, $\mathbf{U}^{(3)}$, and $\mathbf{Q}_1$ all were defined in Example 1; whereas $\mathbb{U}$, $\mathbb{D}^T$, $\mathbb{A}$, were mentioned in Eq. (28), and $\mathbf{Q}_2=\mathbf{I}$. We have

$$\mathbf{F}=[1.0000 \quad 0.4404 \quad -0.82217 \quad -2.0582 \quad -2.52441]^T,$$

$$\mathbf{A}=[1.0119 \quad -0.8333 \quad 1.0180 \quad 0.08333 \quad 0.0061]^T,$$

then, we get the approximate solution as:

$$y_4(x)=1.0119E_0(x)-0.83333E_1(x)+1.0180E_2(x)+0.08333E_3(x)+0.0061E_4(x).$$

Figure 2 demonstrates the exact and approximate solution to the example revealing a good agreement between these solutions. In addition, Table 3 provides the exact, approximate, and absolute error for $N=10,\ 16$. In addition, the residual error for this example at $N=18$ is tabulated which proves the efficiency of the proposed technique. Moreover, the results of maximum absolute errors for diffident values of N are presented in Table 4. It can be noticed from these tables that the method provides accurate and efficient results with few numbers of basis.

Table 3. Numerical results with x_i for Example 2

x_i	$y(x_i)$	$y_{10}(x_i)$	$\varepsilon_{10}(x_i)$	$\varepsilon_{16}(x_i)$	$\mathfrak{R}_{18}(x)$
0.1	-0.0898500750	-0.0898500839	8.90762195e-09	1.551953010e-13	1.914107117e-12
0.2	-0.1589354646	-0.1589355002	3.55855816e-08	6.170064459e-13	3.309036460e-12
0.3	-0.2068641447	-0.2068642253	8.06254604e-08	1.396799342e-12	5.701521347e-12
0.4	-0.2336510054	-0.2336511478	1.42430807e-07	2.467526183e-12	4.435112759e-12
0.5	-0.2397127693	-0.2397129916	2.22267315e-07	3.883809940e-12	1.531291615e-13
0.6	-0.2258569894	-0.2258573132	3.23882733e-07	5.561662242e-12	1.050019844e-11
0.7	-0.1932653062	-0.1932657441	4.37901935e-07	7.633282895e-12	3.645611036e-11
0.8	-0.1434712182	-0.1434717905	5.72304844e-07	9.957562552e-12	8.031827413e-11
0.9	-0.0783326910	-0.0783334177	7.26758635e-07	1.255694159e-11	4.446202639e-10

Table 4. Maximum error with different values of N for Example 2

N	8	10	12	14	16	18
Max. error	1.91523934e-05	7.26758635e-07	2.08160435e-08	5.24184209e-10	1.25569416e-11	2.92404989e-13

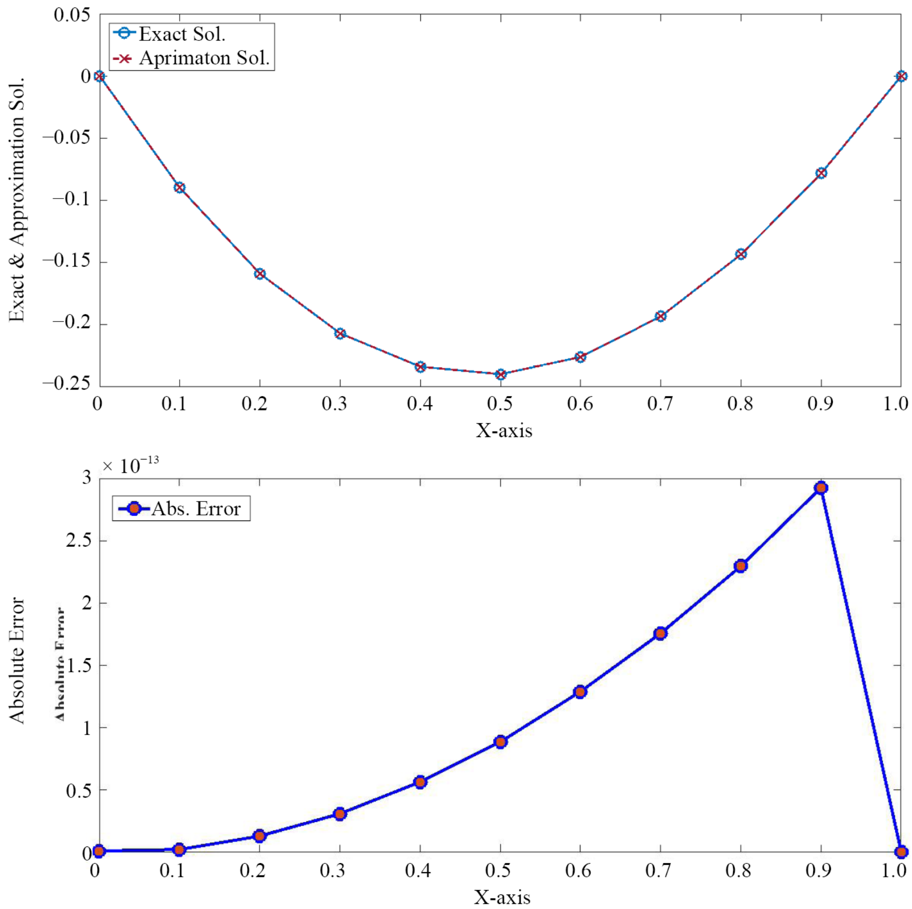

Figure 2. (A) Exact and Approximation solutions, (B) Absolute Errors

Example 3 In this example, a linear fifth–order differential equation will be considered by setting $\delta = 5, c_k(x) = 0$, and $q_r(x) = 0$ except that $q_1(x) = -1$. The resulting equation will takes the following form:

$$y^{(5)}(x) - y(x) = -(15 + 10x)\exp(x),\ x \in (0,\ 1),$$

with the boundary conditions given by

$$\begin{cases} y(0) = 0,\ y'(0) = 1,\ y''(0) = 0, \\ y(1) = 0,\ y'(1) = -\exp(1). \end{cases}$$

Here, the exact solution is $y(x) = x(1-x)\exp(x)$, and by taking $N = 6$, the set of the collocation points will be:

$$\mathbf{X} = \{0, \quad 0.06699, \quad 0.2500, \quad 0.500, \quad 0.7500, \quad 0.93301, \quad 1.000\}.$$

From equation (22), the fundamental matrix equation of the problem will be:

$$\left\{\mathbf{U}^{(5)}(\mathbf{D}^T)+\mathbf{Q}_1\mathbf{U}(\mathbf{D}^T)\right\}\mathbf{A}=\mathbf{F},$$

where

$$\mathbf{D}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 2 & 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 4 & 0 & 0 & 0 & 0\\ 0 & -4 & 0 & 8 & 0 & 0 & 0\\ 1 & 0 & -12 & 0 & 16 & 0 & 0\\ 0 & 6 & 0 & -32 & 0 & 32 & 0\\ 1 & 0 & -24 & 0 & -80 & 0 & 64 \end{bmatrix},\ \mathbf{Q}_1=-\mathbf{I},$$

$$\mathbf{U}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0.03348 & 0.00112 & 3.753e-5 & 1.256e-6 & 4.207e-8 & 1.408e-9\\ 1 & 0.12435 & 0.01546 & 0.00192 & 2.391e-4 & 2.973e-5 & 3.697e-6\\ 1 & 0.24492 & 0.05998 & 0.01469 & 0.00360 & 0.00088 & 0.00022\\ 1 & 0.35835 & 0.12842 & 0.04602 & 0.01649 & 0.00591 & 0.00210\\ 1 & 0.43537 & 0.18954 & 0.08252 & 0.03592 & 0.01564 & 0.00681\\ 1 & 0.46212 & 0.21355 & 0.09869 & 0.04560 & 0.02107 & 0.00974 \end{bmatrix},$$

$$\mathbf{U}^{(5)}=\begin{bmatrix} 0 & 0.5000 & 0 & -3.7500 & 0 & 3.7500 & 0\\ 0 & 0.49524 & 0.28314 & -3.68549 & -0.99537 & 3.60381 & 0.73792\\ 0 & 0.43605 & 0.98451 & -2.89961 & -3.28286 & 1.89451 & 2.06003\\ 0 & 0.27124 & 1.56126 & -0.89544 & 4.29169 & -1.74160 & 0.95633\\ 0 & 0.06996 & 1.56745 & 1.08840 & -2.73668 & -3.69960 & -2.06648\\ 0 & -0.06165 & 1.27799 & 1.97833 & -0.84602 & -3.21467 & -3.42134\\ 0 & -0.10208 & 1.13373 & 2.15029 & 0.16991 & -2.7143 & -3.5260 \end{bmatrix},$$

$$\mathbf{F}=[-15.000\ -16.755\ -22.470\ -32.974\ -47.632\ -61.851\ -67.957]^T,$$

$$\mathbf{A}=[-3.0789\ 0.0559\ -5.0816\ -0.7041\ -2.3114\ -0.1547\ -0.3088]^T,$$

then, the approximate solution will take the next form:

$$y_6(x)=-3.0789E_0(x)+0.0559E_1(x)-5.0816E_2(x)-0.7041E_3(x)+$$

$$-2.3114E_4(x)-0.1547E_5(x)-0.3088E_6(x).$$

Table 5. Numerical results with x_i for Example 3

x_i	$y(x_i)$	$y_{10}(x_i)$	$\varepsilon_{10}(x_i)$	$\varepsilon_{16}(x_i)$
0.1	0.0994653826	0.0994730567	7.6741196886e-06	2.4376844015e-09
0.2	0.1954244413	0.1954783230	5.3881709766e-05	1.7243143191e-08
0.3	0.2834703496	0.2836268449	1.5649535865e-04	5.0630452219e-08
0.4	0.3580379274	0.3583487735	3.1084614293e-04	1.0209147288e-07
0.5	0.4121803177	0.4126701097	4.8979205673e-04	1.6439985251e-07
0.6	0.4373085121	0.4379519980	6.4348596737e-04	2.2361008905e-07
0.7	0.4228880686	0.4235880083	6.9993972535e-04	2.5906011946e-07
0.8	0.3560865486	0.3566656235	5.7907500859e-04	2.4338712901e-07
0.9	0.2213642800	0.2216262062	2.6192622635e-04	1.4304784765e-07

Table 6. Maximum error with different values of N for Example 3

N	10	12	14	16	18	20
Max. error	6.999397253e-04	6.536052708e-05	4.540493665e-06	2.590601194e-07	1.285859246e-08	5.907491162e-10

Table 7. Maximum absolute error for some methods to Example 3

The method	Max. error
Present method	5.9075e-10
Sinc-Galerkin method [52]	0.951E-05
Sextic spline method [53]	4.844E-07
Cubic B-spline method [54]	1.41E-05
Finite difference method [55]	1.15E-02
Quartic spline method [56]	7.66E-05

In Table 5 the exact and approximate solutions are tabulated along with the absolute error for $N = 10,\ 16$. Also, Table 6 gives the maximum absolute error for various values of N. A comparison between other techniques for the literature is provided in Table 7. The method is witnessed to perform better than the other methods in terms of the maximum absolute error as introduced in the table. Also, the exact and approximate solutions along with absolute errors are plotted in Figure 3.

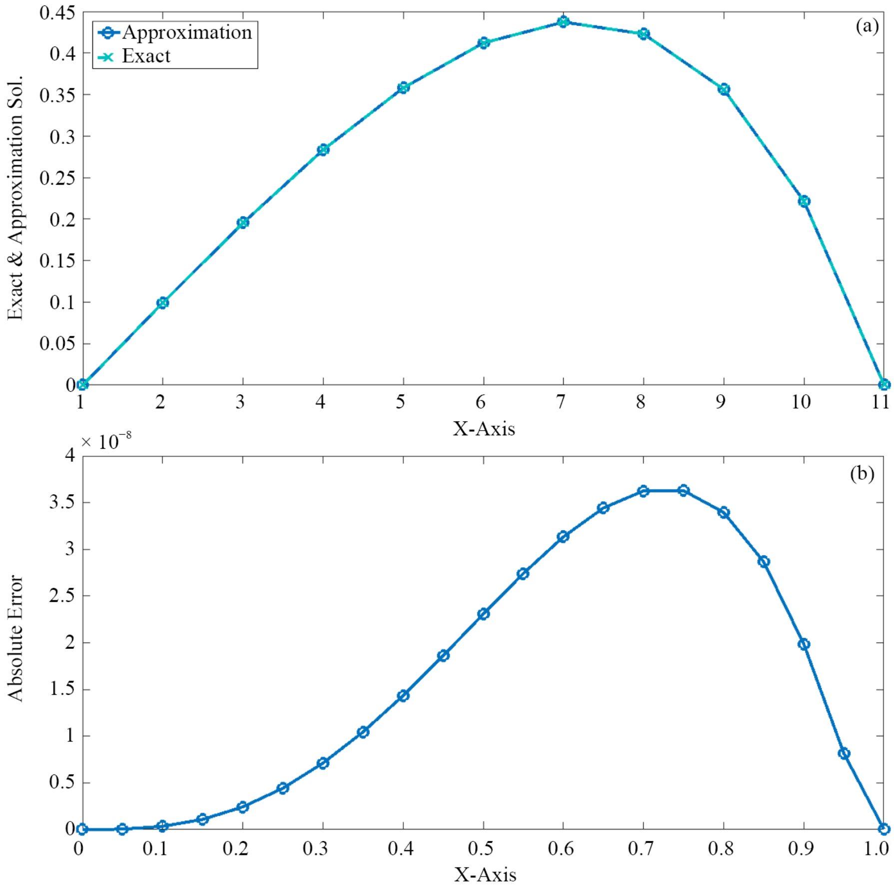

Figure 3. (A) Exact, Approximation solutions, and (B) Absolute Error when $x \in [0, 1]$ where $N = 18$ for Example 3

Example 4 The second example for the fifth order differential equation will be a nonlinear case by setting $\delta = 5$, $c_4(x) = 1$, and $q_r(x) = 0$ except $q_2(x) = \exp(-2x)$ in (1), the obtained equation will be:

$$y^{(5)}(x) + y^{(4)}(x) + \exp(-2x)y^2(x) = 2\exp(x) + 1, \ x \in (0,\ 1).$$

The boundary conditions are

$$\begin{cases} y(0) = y'(0) = y''(0) = 1, \\ y(1) = y'(1) = \exp(1). \end{cases}$$

Here, the theoretical solution is $y(x) = \exp(x)$, and with $N = 6$, the set of collocation points was obtained before in Example 3, and by using equation (29) the fundamental matrix equation of the problem will be

$$\left\{ \mathbf{U}^{(5)}\mathbf{D}^T + \mathbf{C}_4\mathbf{U}^{(4)}\mathbf{D}^T + \mathbf{Q}_2(\mathbb{U}\mathbb{D}^T\mathbb{A})\mathbf{U}\mathbf{D}^T \right\} \mathbf{A} = \mathbf{F},$$

the matrices $\mathbf{D}^T$, $\mathbf{U}$, and $\mathbf{U}^{(5)}$ were defined before in Example (3) whereas $\mathbb{U}$, $\mathbb{D}^T$, $\mathbb{A}$, were defined before in Eq. (28). The matrix $\mathbf{C}_4$ will be equal to matrix $\mathbf{I}$, where

$$\mathbf{Q}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.874612 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.606531 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.367879 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.223130 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.154737 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.135335 \end{bmatrix},$$

$$\mathbf{U}^{(4)} = \begin{bmatrix} 0 & 0 & -1 & 0 & 1.5000 & 0 & 0 \\ 0 & 0.03338 & -0.99049 & -0.24975 & 1.46650 & 0.24792 & 0.02497 \\ 0 & 0.11959 & -0.87211 & -0.86492 & 1.06095 & 0.77650 & 0.30315 \\ 0 & 0.20951 & -0.54248 & -1.35175 & 0.05448 & 0.79409 & 0.75112 \\ 0 & 0.25217 & -0.13992 & -1.31528 & -0.86552 & 0.05306 & 0.61431 \\ 0 & 0.25252 & 0.123310 & -1.02546 & -1.19557 & -0.60245 & 0.09028 \\ 0 & 0.24701 & 0.204168 & -0.88672 & -1.22936 & -0.80177 & -0.14364 \end{bmatrix},$$

$$\mathbf{F} = [3.000\ 3.1385\ 3.5680\ 4.2974\ 5.2340\ 6.0843\ 6.4366]^T,$$

$$\mathbf{A} = [2.23967\ 1.54461\ 1.70567\ 0.31953\ 0.53011\ 0.03148\ 0.06411]^T,$$

then, the approximate solution can be formulated as:

$$y_6(x) = 2.23967E_0(x) + 1.54461E_1(x) + 1.70567E_2(x) + 0.31953E_3(x)$$

$$+ 0.53011E_4(x) + 0.03148E_5(x) + 0.06411E_6(x).$$

Table 8. Numerical results for xi for Example 4

x_i	$y(x_i)$	$y_{10}(x_i)$	$\varepsilon_{10}(x_i)$	$\varepsilon_{18}(x_i)$
0.1	1.1051709181	1.1051699247	9.93341369e-07	1.03832498e-11
0.2	1.2214027582	1.2213959221	6.83604929e-06	7.21074311e-11
0.3	1.3498588076	1.3498393409	1.94666845e-05	2.07839301e-10
0.4	1.4918246976	1.4917867747	3.79229283e-05	4.11666257e-10
0.5	1.6487212707	1.6486626461	5.86245981e-05	6.51812159e-10
0.6	1.8221188004	1.8220432037	7.55966564e-05	8.73101147e-10
0.7	2.0137527075	2.0136719465	8.07609275e-05	9.99176297e-10
0.8	2.2255409285	2.2254752423	6.56861927e-05	9.34473832e-10
0.9	2.4596031111	2.4595738661	2.92450388e-05	5.66207969e-10

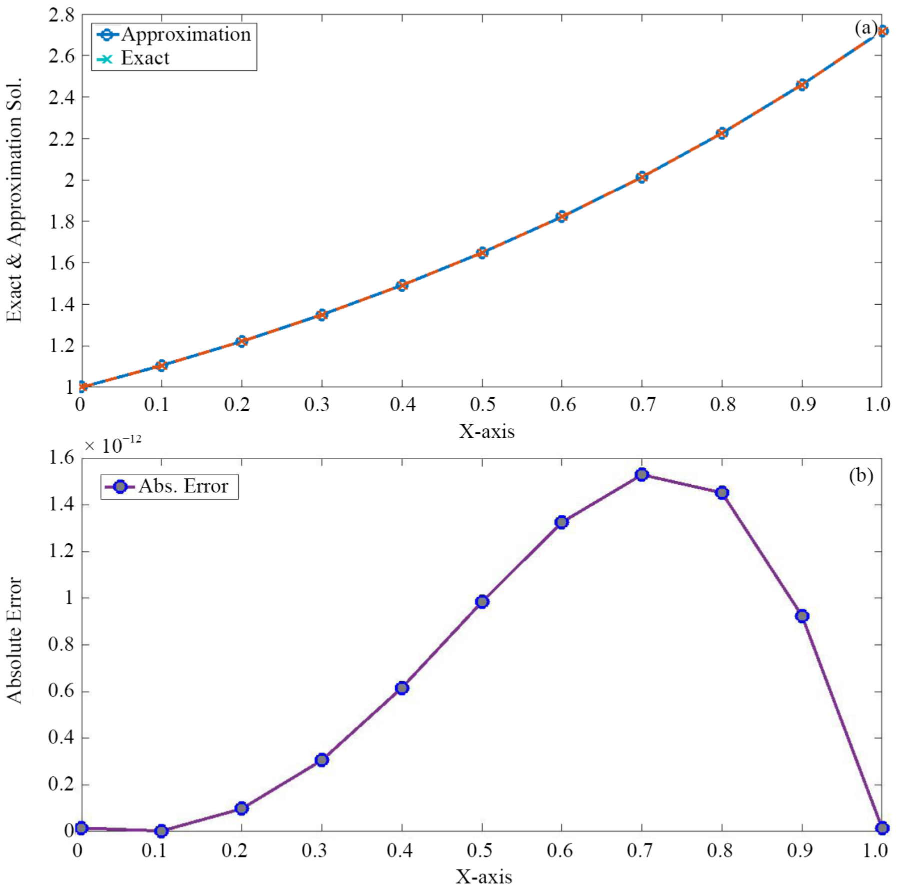

Figure 4. (A) Exact, Approximation solutions, and (B) Absolute Error when $x \in [0,\ 1]$ and $N = 18$ for Example 4

Table 9. Maximum error with different values of N to Example 4

N	12	14	16	18	20	22
Max. error	6.6131662861e-06	4.1368223602e-07	2.1660484073e-08	9.9917629725e-10	4.1923353677e-11	1.5294432387e-12

The approximate and exact solution for this example is graphed in Figure 4 with the absolute error at $N = 18$. The results for this example are also tabulated in Table 8. From this table, it is noticed that the maximum absolute error of the technique in terms of $10e-09$ is better than other methods. These maximum errors are presented in Table 9.

Example 5 To verify the effectiveness for the method, we tackle in this example a fourth order form of the high order BVP of nonlinear form through the next equation;

$$y^{(4)}(x) + y^2(x) = \sin(x) + \sin^2(x),\ x \in (0,\ 1),$$

with the boundary conditions are

$$\begin{cases} y(0)=0,\ y'(0)=1, \\ \\ y(1)=\sin(1),\ y'(1)=\cos(1). \end{cases}$$

This form of equation is a special form of the main equation which demonstrates the ability of the technique to provide accurate solution to even models as well as odd models. Here, the theoretical solutions is $y(x)=\sin(x)$, and by taking $N=6$, the set of collocation points was defined before in Example 3, and by using equation (29) the fundamental matrix equation of the problem will be:

$$\left\{\mathbf{U}^{(4)}\mathbf{D}^T+\mathbf{Q}(\mathbb{U}\mathbb{D}^T\mathbb{A})\mathbf{U}\mathbf{D}^T\right\}\mathbf{A}=\mathbf{F}.$$

All the matrices that appeared in the previous equation were defined before, and $\mathbf{Q}=\mathbf{I}$. The rest of the solution matrices can be found as:

$$\mathbf{F}=[0.000\quad 0.07142\quad 0.30861\quad 0.70927\quad 1.14627\quad 1.44889\quad 1.54954]^T,$$

$$\mathbf{A}=[-0.00871\quad 0.70708\quad -0.02140\quad -0.17971\quad -0.01612\quad -0.02217\quad -0.00343]^T,$$

then, we get the approximate solution as

$$y_6(x)=-0.00871E_0(x)+0.70708E_1(x)-0.02140E_2(x)-0.17971E_3(x)$$

$$-0.01612E_4(x)-0.02217E_5(x)-0.00343E_6(x).$$

Table 10. Numerical results for x_i for Example 5

x_i	$y(x_i)$	$y_{10}(x_i)$	$\varepsilon_{10}(x_i)$	$\varepsilon_{18}(x_i)$	$\mathfrak{R}_{20}(x)$
0.1	0.0998334166	0.0998336397	2.23065463e-07	3.10029779e-13	2.203344897e-16
0.2	0.1986693308	0.1986701170	7.86253144e-07	1.09978693e-12	8.343104154e-14
0.3	0.2955202066	0.2955217373	1.53066400e-06	2.15871765e-12	4.414229275e-13
0.4	0.3894183423	0.3894206394	2.29718157e-06	3.27626815e-12	7.387326861e-13
0.5	0.4794255386	0.4794284648	2.92626285e-06	4.24166257e-12	1.688605841e-14
0.6	0.5646424734	0.5646457334	3.26006021e-06	4.84401407e-12	2.409977642e-12
0.7	0.6442176872	0.6442208248	3.13762856e-06	4.87176965e-12	7.071205268e-12
0.8	0.7173560908	0.7173584882	2.39735261e-06	4.11326528e-12	8.441091644e-12
0.9	0.7833269096	0.7833279525	1.04289051e-06	2.35611530e-12	1.334694997e-10

Table 11. Maximum error with different values of N for Example 5

N	10	12	14	16	18	20
Max. error	3.2600602072e-06	1.4191177267e-07	5.1849752269e-09	1.6698165073e-10	4.8717696543e-12	1.2789769243e-13

In Table 10, the solutions to Eq. (5) along with its boundary conditions are tabulated along with the absolute error of the problem for $N = 10,\ 18$. The residual error at $N = 18$ is also provided form which we conclude that the proposed technique can handle even order models effectively. Maximum absolute error for this example is demonstrated in Table 11.

6. Conclusion

In this article, we have presented an efficient collocation algorithm for handling a specific class of high-order odd boundary value problems. By employing a new version of the exponential-type Chebyshev polynomials that satisfy all the necessary equation conditions, we have successfully transformed the linear and nonlinear forms of the equations, along with their boundary conditions, into systems of algebraic equations. Through the use of a novel iterative technique, we have reduced the computational time required for solving the nonlinear system. Our experimental results, comparing our proposed techniques with other similar methods, have demonstrated the effectiveness and accuracy of our algorithm. The obtained solutions have shown high precision, and the algorithm has proven to be applicable to a wide range of problems, including those of different orders and types. To further validate the reliability of our approach, we have added a detailed section on error analysis, providing a comprehensive evaluation of the accuracy and stability of the proposed method. The presented collocation algorithm opens up new possibilities for efficiently solving complex boundary value problems, and future research can focus on improving the robustness of the iterative technique, extending the method to more generalized and complex problems, incorporating convergence criteria, and applying this algorithm to real-world scenarios in diverse scientific and engineering fields to further validate and expand its practical utility.

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Conflict of interest

The authors declare there is no conflict of interest at any point with reference to research findings.

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