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Research Article

Stable Truncated Trigonometric Moment Problems

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Abstract: Let $\gamma = \{\gamma_k\}_{-2n\leq k\leq 2n},\ \gamma_k = \overline{\gamma_{-k}}$ be an one dimensional complex sequence of degree at most $2n$. In the present paper we give a necessary condition such that γ admits on $z\bar{z} = 1$ an atomic representing measure with a finite number of atoms. The necessary condition is expressed in terms of "stability" of the Riesz linear non-negative functional, $(z^k + (1-z\bar{z}) \cap C_{2n}[z,\ \bar{z}]) \to \gamma_k$, associated to the given sequence. We also give a necessary and sufficient condition such that the extended sequence $\widehat{\gamma} = \{\widehat{\gamma}_j\}_{j\in Z},\ \widehat{\gamma}_j = \widehat{\overline{\gamma_{-j}}},\ \widehat{\gamma}_k = \gamma_k,\ -2n \leq k \leq 2n$ to admit on $z\bar{z} = 1$ an unique atomic representing measure with a finite number of atoms. The "stability" condition of the introduced Riesz functional is an adaption of the concept "dimension stability" by Vasilescu introduced for solving Hamburger moment problems in [5]. In section 3 of the present paper, we apply the main existence theorem for determining representing measures with 1, 2, 3 atoms, according to the rank of the moment matrix. The representing measures of the data of the quadratic moment problem have the support in the unit circle.

Keywords: full and truncated trigonometric moment problems, liniar unital functional $L_{2n} : C^1_{2n}[z,\ \bar{z}]/I_{2n} \to C$ positive on squares, unitary operators, dimensional stability, representing measure

MSC: 47A57, 44A60, 15A57

1. Introduction

The moment problem is one of the most interesting subject in mathematics. It appears as a distinct problem in functional analysis; it gives information about the continuous medium using discrete data, the moment sequence. The information are given by setting a duality between the space of moments and spaces of functions belonging to different domains of science. The duality keeps the "positivity" of both associated terms. The advantage is that the discrete data are sampled and are input in applications of modeling and simulation, in science and engineering with relevant applied mathematics and computational approaches.Domains in which the moment sequence is associated with functions representing wave signals are of great interests. For example seismology, transmissions, to quote only few of them, e.g. [1].

Remarkable papers on truncated or full real or complex moment problems are those: by Curto and Fialkow [2], by Putinar [1, 3], by Vasilescu [5]. Complex truncated moment problems are also found in [4]. The problem of finding atomic representing measures for truncated real Hamburger and Stieltjes moment problems are solved in [2]. The finite moment sequence $\gamma \subset R$ is associated with the matrix $M_n(\gamma)$ (the moment matrix). The condition on the existence of the

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representing measure for γ is expressed in terms of positivity and extension property of $M_n(\gamma)$ to $M_{n+1}(\gamma)$, $M_{n+1}(\gamma)$ with the same rank as $M_n(\gamma)$, "(flat extension)". By an operator approach, the problem of finding representing atomic measures for Hamburger truncated moments, in conditions of stability of the Riesz functional induced by the assignment $t^k \to \gamma_k$, associated with the given moment sequence $\gamma \subset R$, in [5] is solved.

Given a finite multi-sequence, $\gamma = \{\gamma_{ij}\}_{i,\ j \in N^d,\ |i|+|j| \le 2n} \subset C$, subject on $\gamma_{00} > 0$, $\gamma_{ij} = \overline{\gamma_{ji}}$, the truncated d dimensional complex moment problem for γ entails determining necessary and sufficient conditions for the existing of a positive Borel measure μ on C^d such that:

$$\gamma_{ij} = \int_{C^d} \overline{z}^i z^j d\mu(z,\ \overline{z}),\ 0 \le |i| + |j| \le 2n. \tag{1}$$

The measure μ is called a representing measure for γ.

In the present paper, whenever $n \ge 1$, given a finite complex sequence $\gamma = \{\gamma_k\}_{-2n \le k \le 2n} \subset C$, subject on $\gamma_k = \overline{\gamma_{-k}}$, we prove that if γ is non-negative definite and satisfies a "stability condition", γ has an unique positive atomic representing measure with support in the unit circle. The finite data γ are the values of a linear non-negative functional introduced by the assignment $\dfrac{C_n[z,\ \overline{z}]}{(1 - z\overline{z}) \cap C_n[z,\ \overline{z}]} \ni \widehat{z^\alpha \overline{z}^\beta} \to \gamma_{\alpha-\beta} \in C$, (a kind of Riesz functional). The stability condition of the data γ expressed as a stability condition of the associated Riesz functional (Definition 2.1), in the present paper, is an adaption of the concept "dimensionally stable" by Vasilescu introduced in [5], (Definition 2.2). Such a non-negative functional is always completely defined by the moments $\widehat{z}^k \to \gamma_k,\ -2n \le k \le 2n$. The spaces on which the Riesz functional acts represents one of the novelty of the present paper. This technical essence assures the representing measure of the given data, in case of existence, to have the support in the unit circle.

The present paper presents also a full trigonometric moment problem. Given a non-negative sequence $\gamma = \{\gamma_k\}_{k \in Z}$, $\gamma_k = \overline{\gamma_{-k}}$, γ admits an unique atomic representing measure on the unit circle if and only if the unique extension of the Riesz functional $\dfrac{C[z,\ \overline{z}]}{(1 - z\overline{z})} \ni \widehat{z^k} \to \gamma_k \in C$, (a kind of classical Riesz functional) is "stable". In both cases of the truncated and full moment problem solved in this paper, the construction of the spaces on which the Riesz functional acts, $\dfrac{C_n[z,\ \overline{z}]}{(1 - z\overline{z}) \cap C_n[z,\ \overline{z}]}$, respectively $\dfrac{C[z,\ \overline{z}]}{(1 - z\overline{z})}$, together with the standard form of the moment matrix $M(\gamma)$ make results in Proposition 3.1 in [2] to be obvious.

In section 3 of the present paper the main theorem in section 2 is applied for the data γ of the quadratic moment problem. By applying the given theorem to find representing measures with 1, 2, 3 atoms for the quadratic moment data, another direct proof of the same result in [2] is obtained. As a novelty of the paper, the study of the existence of representing measures on $z\overline{z} = 1$ for γ proves that: the introduced invariants "rank of the moment matrices" in [2] and "stable dimension of the Hilbert spaces constructed with help of the moment data" in [5] are the same.

The conclusions of the study of the quadratic moment problem in 3 are stated in the equivalent assertions of theorem 3.1. One of the equivalences is the common value of the main invariants in [2], respectively in [5].

The structure of the present paper is:

Section 1. Introduction.

Sub-section 1.1. Preliminary Notions.

Section 2. Stability of the L_{2n} functional. The unique extension of L_{2n} to a functional on $\dfrac{C[z,\ \overline{z}]}{(1 - z\overline{z})}$.

Section 3. Application. The quadratic moment problem on $z\overline{z} = 1$.

Section 4. Conclusions.

1.1 *Preliminary notions*

Let $n \ge 1$ and $d = 1$ be fixed. We denote with $C_n^1[z,\ \overline{z}] := C_n[z,\ \overline{z}]$ the space of polynomials with complex coefficients in the indeterminates $z,\ \overline{z} \in C$ with total degree at most n. For $i,\ j \in N$, $z^i = z \cdot\ \ldots\ \cdot z$ (i-times), $\overline{z}^j = \overline{z} \cdot\ \ldots\ \cdot \overline{z}$ (j-times), the total degree of $z^i \overline{z}^j$ is $i + j$. We have: $C_n[z,\ \overline{z}] = \{z^i \overline{z}^j,\ i + j \le n,\ i,\ j \in N\}$. Let $I := (1 - z\overline{z})$ be the ideal generated by $(1-z\bar{z})$ in $C[z, \bar{z}]$, the algebra of polynomials in indeterminates $z, \bar{z}$, of any total degree. For $k \in \{1, ..., n\}$ let also $I_k = (1-z\bar{z}) \cap C_k[z, \bar{z}]$ be a subspace in $C_k[z, \bar{z}]$. We consider the quotients spaces $C_k[z, \bar{z}]/I_k$, $0 \le k \le n$; when $k = 0, 1$, let $I_k = C_k[z, \bar{z}] \cap (1-z\bar{z}) = \{0\}$, and the quotients $C_k[z, \bar{z}]/I_k = C_k[z, \bar{z}]$. For the same integers n, d, let $L_{2\cdot n} : C_{2\cdot n}[z, \bar{z}]/I_{2\cdot n} \to C$ be a linear functional satisfying the conditions:

$$L_{2\cdot n}(\widehat{\bar{f}}) = \overline{L_{2\cdot n}(\widehat{f})}, \ \forall \ \widehat{f} \in C_{2\cdot n}[z, \bar{z}]/I_{2n}. \tag{2}$$

$$L_{2\cdot n}(\widehat{|f|}^2) \ge 0, \ \forall \ \widehat{f} \in C_n[z, \bar{z}]/I_n. \tag{3}$$

$$L_{2\cdot n}(\widehat{1}) = 1. \tag{4}$$

If $L_{2\cdot n}(\widehat{\bar{f}}) = \overline{L_{2\cdot n}(\widehat{f})}$, $\widehat{f} \in C_{2\cdot n}[z, \bar{z}]/I_{2\cdot n}$, we have $L_{2\cdot n}(\widehat{z^i\bar{z}^j}) = L_{2\cdot n}(\widehat{|z^i|^2\bar{z}^{j-i}}) = \overline{L_{2\cdot n}(\widehat{|z^i|^2 z^{j-i}})}$. Consequently, $\overline{L_{2\cdot n}(\widehat{z^{j-i}})} = L_{2\cdot n}(\widehat{\bar{z}^{j-i}})$ with $-2n \le j-i \le 2n$. Such a functional is uniquely determined by the complex moments $L_{2\cdot n}(\widehat{z^p})$, $p \in \{-2n, ..., 2n\}$. With aid of $L_{2\cdot n}$ we introduced on $C_k[z, \bar{z}]/I_k$ the pre-Hilbert space product:

$$< \widehat{f}, \widehat{g} >= L_{2\cdot n}(\widehat{f\bar{g}}), \ \widehat{f}, \widehat{g} \in C_k[z, \bar{z}]/I_k, \ k \in \{0, 1, ..., n\}. \tag{5}$$

From (5) and $L'_{2\cdot n}$s properties, the following assertions are true:
(i) $<\alpha_1\widehat{f}_1 + \alpha_2\widehat{f}_2, \widehat{g}> = \alpha_1 < \widehat{f}_1, \widehat{g}> + \alpha_2 < \widehat{f}_2, \widehat{g}>$, $\widehat{f}_1, \widehat{f}_2, \widehat{g} \in C_k[z, \bar{z}]/I_k$.
(ii) $<\widehat{f}, \widehat{g}> = \overline{<\widehat{g}, \widehat{f}>}$, $\widehat{f}, \widehat{g} \in C_k[z, \bar{z}]/I_k$.
(iii) $<\widehat{f}, \widehat{f}> \ge 0$, $\forall \ \widehat{f} \in C_k[z, \bar{z}]/I_k$, $k \in \{0, 1, ..., n\}$.
If the introduced pre-Hilbert space product has (iii), the Cauchy-Schwarz inequality holds:

$$| < \widehat{f}, \widehat{g} > | \le < \widehat{f}, \widehat{f} >^{\frac{1}{2}} < \widehat{g}, \widehat{g} >^{\frac{1}{2}}, \ \forall \widehat{f}, \widehat{g} \in C_k[z, \bar{z}]/I_k, \ k \in \{0, 1, ..., n\}. \tag{6}$$

Remark 1.1 If, in place of the functional $L_{2\cdot n} : C_{2\cdot n}[z, \bar{z}]/I_{2\cdot n} \to C$ with (2-4), we give its values $L_{2\cdot n}(\widehat{z^k}) = \gamma_k$, $k \in \{-2n, ..., 2n\}$, $\overline{\gamma_k} = \gamma_{-k}$, we ask for a representing measure of γ_k, i.e.

$$\gamma_k = \int_{z, |(z\bar{z}=1)} z^k \mathrm{d}\mu(z), -2n \le k \le 2n. \tag{7}$$

We obtain, in this way, an authentic truncated trigonometric moment problem.

2. Stability of the L_{2n} functional. The unique extension of L_{2n} to a functional on $C[z, \bar{z}]/(1-z\bar{z})$

In this section, we present an extension theorem of a linear stable functional with given properties, $L_{2\cdot n}$ on $C_{2\cdot n}[z, \bar{z}]/I_{2\cdot n}$, $n \ge 1$, to a linear functional with given properties L on $C[z, \bar{z}]/(1-z\bar{z})$ and give some applications of it in solving truncated and full trigonometric moment problems on T_1.

If $0 \le k \le l \le n$, we have $C_k[z, \bar{z}] \subseteq C_l[z, \bar{z}]$, the subspaces $I_k \subseteq I_l \subseteq I_n$ and the quotients $C_k[z, \bar{z}]/I_k \subseteq C_l[z, \bar{z}]/I_l \subseteq C_n[z, \bar{z}]/I_n$ are vector subspaces in $C_n[z, \bar{z}]/I_n$.

Let $<, >$ be the pre-Hilbert product in (5). When $0 \le k \le n$, if we take $T_k = \{\widehat{f} \in C_k[z, \bar{z}]/I_k, <\widehat{f}, \widehat{f}>= L_{2n}(\widehat{|f|^2}) = 0\}$, from Cauchy-Schwarz inequality $T_k \subset C_k[z, \bar{z}]/I_k$, are all vector subspaces in the mentioned vector spaces. If $C_k[z, \bar{z}]/I_k$ are finite dimensional, the quotients $H_k = \dfrac{C_k[z, \bar{z}]/I_k}{T_k}$, $0 \le k \le n$, are all finite dimensional Hilbert spaces with the scalar product given by:

$$<\widehat{f}+T_k, \widehat{g}+T_k>_k=<\widehat{f}, \widehat{g}>= L_{2n}(\widehat{f\bar{g}}), \ \widehat{f}, \widehat{g} \in C_k[z, \bar{z}/I_k, \ 0 \le k \le n. \tag{8}$$

Now, if l is another integer, with, $0 \le k \le l \le n$, since $C_k[z, \bar{z}]/I_k \subset C_l[z, \bar{z}]/I_l$, and $T_k \subset T_l$, we have natural maps $J_{k,\,l}: \ H_k \to H_l, \ J_{k,\,l}(\widehat{f}+T_k) = (\widehat{f}+T_l)$. If $\left\|\widehat{f}+T_k\right\|_k =<\widehat{f}+T_k, \ \widehat{f}+T_k>_k^{\frac{1}{2}}=<\widehat{f}+T_l, \ \widehat{f}+T_l>_l^{\frac{1}{2}}= L_{2n}(\widehat{f\bar{f}})^{\frac{1}{2}} = \left\|J_{k,\,l}(\widehat{f}+T_k)\right\|_l$; $\left\|(\widehat{f}+T_1)\right\|_1 =<\widehat{f}+T_1, \ \widehat{f}+T_1>_1^{\frac{1}{2}}= \left\|J_{0,\,1}(\widehat{f}+T_0)\right\|_1 = \left\|(\widehat{f}+T_0)\right\|_0$, $\left\|(\widehat{f}+T_k)\right\|_k =<\widehat{f}+T_k, \ \widehat{f}+T_k>_k^{\frac{1}{2}}= L_{2n}(\widehat{f\bar{f}})^{\frac{1}{2}} = \left\|J_{j,\,k}(\widehat{f}+T_j)\right\|_k = \left\|(\widehat{f}+T_j)\right\|_j$, $j = 0, \ 1$ all $J_{k,\,l}, \ J_{j,\,k}$ with $j \le k$ are isometries. We have $J_{k,\,k}, \ J_{1,\,1}, \ J_{0,\,0}$ the identity maps of $H_k, \ H_1$, respectively H_0.

For a given linear functional $L_{2\cdot n}: \ C_{2\cdot n}[z, \bar{z}]/I_{2n} \to C$, with properties (2-4), the Hilbert spaces $\{H_k\}_{k=0}^{n}$ will be referred as the Hilbert spaces built via $L_{2\cdot n}$, the maps $J_{k,\,l}: \ H_k \to H_l, \ 0 \le k \le l \le n$, as the associated isometries. When $l = k+1, \ 0 \le k \le (n-1)$, we write J_k instead $J_{k,\,k+1}$.

Definition 2.1 Let $L_{2\cdot n}: \ C_{2\cdot n}[z, \bar{z}]/I_{2\cdot n}, \to C, \ n \ge 1$, be a linear functional with properties (2-4), $\{H_k\}_{k=0}^{n}$, the Hilbert spaces built via $L_{2\cdot n}$, $<, >_k, \ k \in \{0, \ ..., \ n\}$ the scalar products in H_k, $J_k: \ H_k \to H_{k+1}, \ 0 \le k \le n-1$, the associated isometries. If for some $k \in \{0, \ ..., \ n-1\}$ one has $J_k(H_k) = H_{k+1}$ we say that L_{2n} is dimensionally stable at k.

Remark 2.2 i) It is immediate that the functional $L_{2\cdot n}$ is stable at $(n-1)$ if and only if $C_n[z, \bar{z}]/I_n = C_{n-1}[z, \bar{z}]/I_{n-1} + T_n, \ n \ge 1$.

Let $L: C[z, \bar{z}]/(1-z\bar{z})$ be a linear, functional with properties (2-4). Similar constructions may be done for the functional L. Thus, let $\{H_j\}_{j=0}^{\infty}$ be the Hilbert spaces constructed via L, for all $k, \ l \in N, \ J_{k,\,l}, \ 0 \le k < l$ be the associated isometries.

We say that L is dimensionally stable at k if there exists integers $k, \ l$ with $0 \le k < l$, such that $L|_{C_{2\cdot l}[z, \ \bar{z}]/I_{2\cdot l}}$ is stable at k. The number $\mathbf{s} = \mathbf{sd}(\mathbf{L}) = \dim\mathbf{H_k} = \dim\mathbf{H_{k+1}}$ is called the stable dimension of L.

The same constructions and same invariant "stable dimension" was primary introduced in [5] (Remark 2.1, Definition 2.2).

We consider the operators $M_z, \ M_{\bar{z}}: \ C_{n-1}[z, \bar{z}]/I_{n-1} \to C_n[z, \bar{z}]/I_n$, defined by $M_z(\widehat{f}) = \widehat{zf}$, $M_{\bar{z}}(\widehat{f}) = (\widehat{\bar{z}f})$.

Remark 2.3 i) The operators $M_z, \ M_{\bar{z}}: \ C_{n-1}[z,\bar{z}]/I_{n-1} \to C_n[z, \bar{z}]/I_n$ are correctly defined.

Indeed, if $zf \in I_n = I \cap C_n[z, \bar{z}]$, we have $zf \in C_n[z, \bar{z}], \ zf \in I$. Consequently $f \in C_{n-1}[z, \bar{z}], \ f \in I$. That is $\widehat{f} \in C_{n-1}[z, \bar{z}]/I_{n-1}$, implying that $M_z: \ C_{n-1}[z, \bar{z}]/I_{n-1} \to C_n[z, \bar{z}]/I_n$ is correctly defined. The same for $M_{\bar{z}}: \ C_{n-1}[z, \bar{z}]/I_{n-1} \to C_n[z, \bar{z}]/I_n$.

Remark 2.4 Let $<, >$ the pre-Hilbert space product on $C_n[z, \bar{z}]/I_n, C_{n-1}[z, \bar{z}]/I_{n-1}$, as in (5), we have:

1.1^0 $<M_z\widehat{f}, \ M_z\widehat{f}>=<\widehat{f}, \ \widehat{f}>, <M_{\bar{z}}\widehat{f}, \ M_{\bar{z}}\widehat{f}>=<\widehat{f}, \ \widehat{f}>= L_n(\widehat{|f|^2})$, for all $\widehat{f} \in C_{n-1}[z, \bar{z}]/I_{n-1}$.

1.2^0 $<M_z\widehat{f}, \ \widehat{q}>=<\widehat{f}, \ M_{\bar{z}}\widehat{q}>, <M_{\bar{z}}\widehat{f}, \ \widehat{q}>=<\widehat{f}, \ M_z\widehat{q}>, \ \widehat{f}, \ \widehat{q} \in C_{n-1}[z, \bar{z}]/I_{n-1}$.

Proposition 2.5 Let $<, >$ be the pre-Hilbert product on $C_{n-1}[z, \bar{z}]/I_{n-1}$, respectively on $C_n[z, \bar{z}]/I_n$ as in (5) and let $\{H_j\}_{j=0}^{n}$ be the Hilbert spaces built via $L_{2\cdot n}$. The linear operators $M_z: \ H_{n-1} \to H_n, \ M_{\bar{z}}: \ H_{n-1} \to H_n, \ M_z(\widehat{f}+T_{n-1}) = z\widehat{f}+T_n, \ M_{\bar{z}}(\widehat{f}+T_{n-1}) = \bar{z}\widehat{f}+T_n$, are correctly defined, injective and $\|M_z\| = 1, \ \|M_{\bar{z}}\| = 1$ whenever $\widehat{f}+T_{n-1} \in H_{n-1}$.

Proof. If $\widehat{f} \in T_{n-1}$, it follows from remark 2.4 that $<M_z(\widehat{f}+T_{n-1}), \ M_z(\widehat{f}+T_{n-1})>_n=<\widehat{f}, \ \widehat{f}>_{(n-1)}= 0$; the same $<M_{\bar{z}}(\widehat{f}+T_{n-1}), \ M_{\bar{z}}(\widehat{f}+T_{n-1})>_n=<\widehat{f}, \ \widehat{f}>_{(n-1)}= 0$, implying $M_z: \ H_{n-1} \to H_n, \ M_{\bar{z}}: \ H_{n-1} \to H_n$ are correctly defined. If $0 =<M_z(\widehat{f}+T_{n-1}), \ M_z(\widehat{f}+T_{n-1})>_n=<M_{\bar{z}}(\widehat{f}+T_{n-1}), \ M_{\bar{z}}(\widehat{f}+T_{n-1})>_n=<\widehat{f}, \ \widehat{f}>_{(n-1)}$, it implies $\widehat{f} \in T_{n-1}$. That is: M_z and $M_{\bar{z}}$ are injective. Whenever $\|M_z(\widehat{g}+T_{n-1})\|_n =<M_z(\widehat{g}+T_{n-1}), \ M_z(\widehat{g}+T_{n-1})>_n^{\frac{1}{2}}=<\widehat{g}+T_{n-1}, \ \widehat{g}+T_{n-1}>_{n-1}^{\frac{1}{2}}=$

$\|\widehat{g}+T_{n-1}\|_{(n-1)}$, we have $\||M_z\|=1$. The same, $\|M_{\bar{z}}(g+T_{n-1})\|_n =< M_{\bar{z}}(\widehat{g}+T_{n-1}),\ M_{\bar{z}}(\widehat{g}+T_{n-1} >_n^{\frac{1}{2}} =< \widehat{g}+T_{n-1},\ \widehat{g}+ T_{n-1}) >_{n-1}^{\frac{1}{2}} = \|\widehat{g}+T_{n-1}\|_{n-1}$. That is $\|M_{\bar{z}}\|=1$. □

Lemma 2.6 Let $L_{2\cdot n}:\ C_{2\cdot n}[z,\ \bar{z}]/I_{2\cdot n}\to C$ be a linear functional with properties (2-4), stable at $(n-1)$, $<,>$ be the pre-Hilbert product introduced with help of $L_{2\cdot n}$. We have $M_zT_n\cap(C_n[z,\ \bar{z}]/I_n)\subset T_n$, $M_{\bar{z}}T_n\cap(C_n[z,\ \bar{z}]/I_n)\subset T_n$.

Proof. Let be $M_z\widehat{f},\ M_{\bar{z}}\widehat{f}\in C_n[z,\ \bar{z}]/I_n$ and $\widehat{f}\in T_n$. If $L_{2\cdot n}$ is stable at $(n-1)$, there exists $\widehat{r},\ \widehat{q}\in C_{n-1}[z,\ \bar{z}]/I_{n-1}$ such that $(M_z\widehat{f}-\widehat{q})\in T_n,\ (M_{\bar{z}}\widehat{f}-\widehat{r})\in T_n$.

Using remark 2.4 and proposition 2.5, we have $< M_z\widehat{f},\ M_z\widehat{f} >_n =< M_z\widehat{f},\ M_z\widehat{f}-\widehat{q} >_n + < M_z\widehat{f},\ \widehat{q} >_n =< M_z\widehat{f},\ M_z\widehat{f}-\widehat{q} >_n + < \widehat{f},\ M_{\bar{z}}\widehat{q} >_n = 0$. Moreover, $< M_{\bar{z}}\widehat{f},\ M_{\bar{z}}\widehat{f} >_n =< M_{\bar{z}}\widehat{f},\ M_{\bar{z}}\widehat{f}-\widehat{r} >_n + < M_{\bar{z}}\widehat{f},\ \widehat{r} >_n =< M_{\bar{z}}\widehat{f},\ M_{\bar{z}}\widehat{f}-\widehat{r} >_n + < \widehat{f},\ M_z\widehat{r} >_n = 0$.

Therefore, when $\widehat{f}\in T_n$ we have also $M_z\widehat{f}\in T_n$ and $M_{\bar{z}}\widehat{f}\in T_n$, implying $M_zT_n\cap(C_n[z,\ \bar{z}]/I_n)\subset T_n$ and $M_{\bar{z}}T_n\cap(C_n[z,\ \bar{z}]/I_n)\subset T_n$. □

Remark 2.7 Let $L_{2\cdot n}:\ C_{2\cdot n}[z,\ \bar{z}]/I_{2\cdot n}\to C$ be a linear functional with properties (2-4), stable at $(n-1)$, $<,>_n$ the scalar product constructed via it as in (8) and $J_{n-1,\ n}:\ H_{n-1}\to H_n$ the associated isometry. The operators $M_z:\ H_{n-1}\to H_n$, $M_{\bar{z}}:\ H_{n-1}\to H_n$, defined by: $M_z(\widehat{f}+T_{n-1})=\widehat{zf}+T_n$, $M_{\bar{z}}(\widehat{f}+T_{n-1})=\widehat{\bar{z}f}+T_n$, are bijective.

Proof. Indeed, if $L_{2\cdot n}$ is stable at $n-1$, $\dim H_{n-1}=\dim H_n$, consequently the injective operators $M_z,\ M_{\bar{z}}$ are surjective and bijective too. Setting $J:=J_{n-1,\ n}$ we may consider on the Hilbert space H_n, the linear operators $A=M_zJ^{-1},\ B=M_{\bar{z}}J^{-1},\ A,\ B:\ H_n\to H_n$. Note that, in condition of stability of $<,>_n$ at $n-1$, $A,\ B$ are correctly defined, bounded and bijective. □

Proposition 2.8 The linear operators $A:H_n\to H_n,\ A=M_zJ^{-1}$, $B:H_n\to H_n,\ B=M_{\bar{z}}J^{-1}$ are unitary.

Proof. As in Remark 2.7, $A,\ B:H_n\to H_n,\ A=M_zJ^{-1},\ B=M_{\bar{z}}J^{-1}$ are correctly defined, linear, bounded, bijective operators on H_n. Let us prove that $A,\ B$ are unitary ones. Indeed, if J is a bijective isometry, M_z as in Proposition 2.5, we have $\left\|A(\widehat{f}+T_n)\right\|_n = \left\|M_z(J^{-1}(\widehat{f}+T_n))\right\|_n = \left\|J^{-1}(\widehat{f}+T_n)\right\|_{n-1} = \left\|\widehat{f}+T_n\right\|_n$ for all $(\widehat{f}+T_n)\in H_n$. With $\left\|A(\widehat{f}+T_n)\right\|_n^2 =< A(\widehat{f}+T_n),\ A(\widehat{f}+T_n) >_n =< A^*A(\widehat{f}+T_n),\ (\widehat{f}+T_n) >_n =< (\widehat{f}+T_n),\ (\widehat{f}+T_n) >_n$ for all $(\widehat{f}+T_n)\in H_n$; using the polarization relation for $< A^*A(x+T_n),\ (y+T_n) >_n$, we obtain $A^*A(\widehat{f}+T_n)=\widehat{f}+T_n$ for all $(\widehat{f}+T_n)\in H_n$, that is $A^*A=Id_{H_n}$. Consequently we have $A^*=A^*\circ Id_{H_n}=A^*\circ(A\circ A^{-1})=(A^*A)\circ A^{-1}=A^{-1}$; thus A is unitary. The same happens for B. □

Proposition 2.9 Let $L_{2n}:\ C_{2n}[z,\ \bar{z}]/I_{2n}\to C$ be a linear functional with properties (2-4), stable at $(n-1)$, $<,>_n$, the scalar product constructed via it in (8), $\{H_l\}_{l=0}^{l=n}$ be the Hilbert spaces built via L_{2n} and A,B be the unitary operator in 2.8 . We have $A=B^{-1}$.

Proof. Whenever $A=M_zJ^{-1},\ B=M_{\bar{z}}J^{-1}$ are the linear, bounded, bijective operators in remark 2.7, for $\widehat{f}+T_n\in H_n$, we have $A^{-1}(\widehat{f}+T_n)=JM_z^{-1}(\widehat{f}+T_n)=J(\widehat{g}+T_{n-1})$, with property $(z\widehat{g}-\widehat{f})\in T_n$. We have also $B(\widehat{f}+T_n)=M_{\bar{z}}J^{-1}(\widehat{f}+T_n)=M_{\bar{z}}(\widehat{h}+T_{n-1})=\bar{z}\widehat{h}+T_n$ with property $(\widehat{f}-\widehat{h})\in T_n$. Applying lemma 2.6 for both $(\widehat{zg}-\widehat{f}),(\widehat{f}-\widehat{h})\in T_n$, we obtain:

$$(\bar{z}z\widehat{g}-\bar{z}\widehat{f})\in T_n. \tag{9}$$

Respectively,

$$(\bar{z}\widehat{f}-\bar{z}\widehat{h})\in T_n. \tag{10}$$

From (9), (10) and that T_n is a vector space, we get $(\widehat{g}-\bar{z}\widehat{f}+\bar{z}\widehat{f}-\bar{z}\widehat{h})=(\widehat{g}-\bar{z}\widehat{h})\in T_n$ with result: $A^{-1}=B$ on H_n. □

Proposition 2.10 Let $n\geq 1$, $L_{2\cdot n}:C_{2\cdot n}[z,\ \bar{z}]/I_{2\cdot n}\to C$ a linear functional with properties (2-4), stable at $(n-1)$, $A,\ B$ the operator in proposition 2.8. We have:

(i) $A^k(1+T_n)=\widehat{z^k}+T_n$ when $0\leq k\leq n$.

(ii) $A^{-k}(1+T_n)=\widehat{\bar{z}^k}+T_n=B^k(1+T_n)$, for all $0\leq k\leq n$.

(iii) $\mathrm{A}^iA^{-j}(1+T_n)=\widehat{z^i}\widehat{\overline{z}^j}+T_n=A^iB^j(1+T_n)$, for all $0\leq i,\ j,\ i+j\leq n$.

Proof. We prove by induction, that for all , $k\in\{0,\ 1,\ \ldots,\ n\}$, we have $A^k(1+T_n)=\widehat{z^k}+T_n$. For $k=0$, $A^0(1+T_n)=Id_{H_n}(1+T_n)=1+T_n=z^0+T_n$. Assume the assertion is true for some $l\in\{0,\ \ldots,\ n-1\}$ and let us get it for $(l+1)$. We have $A^{l+1}(1+T_n)=A\circ A^l(1+T_n)=A(\widehat{z^l}+T_n)=M_zJ^{-1}(\widehat{z^l}+T_n)$. Because $0\leq l\leq(n-1)$, $\widehat{z^l}+T_{n-1}\in H_{n-1}$ and $J^{-1}(\widehat{z^l}+T_n)=\widehat{z^l}+T_{n-1}$, $A(\widehat{z^l}+T_n)=M_z(\widehat{z^l}+T_{n-1})=\widehat{z^{(l+1)}}+T_n$. These prove (i) whenever $0\leq k\leq n$.

In case $-n\leq k<0$, we have $A^k(1+T_n)=(A^{-1})^{-k}(1+T_n)=B^{-k}(1+T_n)$. As above, by induction, for all $l\in\{0,\ \ldots,\ n\}$, we have $B^l(1+T_n)=\widehat{\overline{z}^l}+T_n=A^{-l}(1+T_n)$. Consequently, we have (i), (ii) for all k, with $-n\leq k\leq n$.

(iii) Let us prove that $A^iA^{-j}(1+T_n)=\widehat{z^i\overline{z}^j}$ for all $0\leq i,\ j,\ i+j\leq n$. The assertion is true for $0\leq i,\ j,\ i+j\leq n-1$ and prove it for $0\leq i,\ j,\ i+j+1\leq n$. We have $A^{i+1}A^{-j}=A(A^iA^{-j}(1+T_n))=A(\widehat{z^i\overline{z}^j}+T_n)=M_zJ^{-1}(\widehat{z^i\overline{z}^j}+T_n)=M_z(\widehat{z^i\overline{z}^j}+T_{n-1})=\widehat{z^{i+1}\overline{z}^j}+T_n$. Thus, we have (iii) for all $0\leq i,\ j,\ i+j\leq n$. □

The next result is a substitute of Theorem 2.6 from [5].

Theorem 2.11 Let $L_{2\cdot n}:\ C_{2\cdot n}[z,\ \overline{z}]/I_{2\cdot n}\to C$ be a linear functional with properties (2-4), stable at $(n-1)$. There exists an unique extension $L:C[z,\ \overline{z}]/I\to C$ of $L_{2\cdot n}$ which is a linear functional with properties (2-4).

Proof. Let $L_{2\cdot n}:\ C_{2\cdot n}[z,\ \overline{z}]/I_{2\cdot n}\to C$ be a linear functional with properties (2-4) stable at $n-1$), $<,>$ be the pre-Hilbert product associated with it, as in (5), A be the unitary operator in 2.7, $\{H_l\}_{l=0}^n$ be the Hilbert spaces built via $L_{2\cdot n}$, $\widehat{f}\in C[z,\ \overline{z}]/(1-z\overline{z})$ arbitrary, $\widehat{f}(z,\ \overline{z})=\sum_{i,\ j}a_{ij}\widehat{z^i\overline{z}^j}:=\sum_{k:\ =i-j\in J\subset Z}f(k)\widehat{z^k}$; when $a_{ij}:=f(k)$. We consider the linear operator $f(A)=\sum_k f(k)A^k$ and define $L:C[z,\ \overline{z}]/(1-z\overline{z})\to C$ as $L(f)=<f(A)(1+T_n),\ (1+T_n)>_n$, with $f(A)$ the functional calculus of A. The functional L is correctly defined as $(1-AA^*)=0$, linear and satisfies (2-4).

Indeed, the functional calculus and the scalar product are linear, $L(\widehat{1})=<Id_{H_n}(1+T_n),\ (1+T_n)>_n=\|(1+T_n)\|_n^2=1$, $L(\widehat{\overline{f}})=<\overline{f}(A)(1+T_n),\ (1+T_n)>_n=<\sum_k\overline{f}(k)(A^*)^k(1+T_n),\ (1+T_n)>_n=<(1+T_n),\ (\sum_k f(k)A^k)(1+T_n)>_n=\overline{<(\sum_k f(k)A^k)(1+T_n),\ (1+T_n)>_n}=\overline{L(\widehat{f})}$. If $\widehat{f}\in C[z,\ \overline{z}]/(1-z\overline{z}),\ f\geq 0$, from Riesz Fejer lemma, $f=|q|^2$. That is $L(\widehat{f})=L(\widehat{|q|^2})=<q\overline{q}(A)(1+T_n),\ (1+T_n)>_n=<q(A)(1+T_n),\ q(A)(1+T_n)>_n=\|q(A)(1+T_n)\|_n^2\geq 0$.

Let us prove that L is an extension of $L_{2\cdot n}$, extension with properties (2-4) Let $\widehat{p}\in C_{2\cdot n}[z,\ \overline{z}]/I_{2\cdot n}$, $\widehat{p}(z,\ \overline{z})=\sum_{i,\ j,\ i+j\leq 2n}a_{ij}\widehat{z^i\overline{z}^j}$ be arbitrary and consider a representation of $\widehat{p}$ of the form $\widehat{p}(z,\ \overline{z})=\sum_{\alpha_i+\beta_i\leq n,\ \gamma_j+\delta_j\leq n}a_{ij}[\widehat{z^{\alpha_i}\overline{z}^{\beta_i}}][\widehat{z^{\gamma_j}\overline{z}^{\delta_j}}]$. From $L's$ definition and Proposition 2.10, we have:

$$L(\widehat{p})=<p(A)(1+T_n),\ (1+T_n)>_n=<\sum_{\alpha_i+\beta_i,\ \gamma_j+\delta_j}a_{ij}A^{\alpha_i-\beta_i}A^{\gamma_j-\delta_j}(1+T_n),\ (1+T_n)>_n$$

$$=<\sum_{\alpha_i+\beta_i,\ \gamma_j+\delta_j}a_{ij}\widehat{z^{\alpha_i-\beta_i}}(1+T_n),\ (\widehat{z^{\delta_j-\gamma_j}}(1+T_n)>_n=\sum_{\alpha_i+\beta_i,\ \gamma_j+\delta_j}a_{ij}L_{2\cdot n}(\widehat{z^{\alpha_i-\beta_i}\overline{z}^{\delta_j-\gamma_j}})=$$

$$=L_{2\cdot n}(\sum_{\alpha_i+\beta_i,\ \gamma_j+\delta_j}a_{ij}\widehat{z^{\alpha_i+\gamma_j}\overline{z}^{\beta_i+\delta_j}})=L_{2\cdot n}(\widehat{p}).$$

Consequently, $L_{|C_{2\cdot n}[z,\ \overline{z}]/I_{2\dot{n}}}=L_{2\cdot n|C_{2\cdot n}[z,\ \overline{z}]/I_{2\cdot n}}$. Let us prove that the extension L with properties (2-4) is unique. Let $L_1,\ L_2:\ C[z,\ \overline{z}]/(1-z\overline{z})$ be extensions of $L_{2\cdot n}$ with properties (2-4), $\{H_k^1\}_{k=0}^\infty=\{\frac{C_k[z,\ \overline{z}]/I_k}{T_k^1}\}_{k=0}^\infty$, $\{H_k^2\}_{k=0}^\infty=\{\frac{C_k[z,\ \overline{z}]/I_k}{T_k^2}\}_{k=0}^\infty$ the Hilbert spaces built via L_1, respectively L_2, $T_k^1,\ T_k^2$ the null spaces at step k of L_1 respectively L_2. We prove that for every $\widehat{z^\alpha\overline{z}^\beta}\in C[z,\ \overline{z}]/(1-z\overline{z}),\ \alpha+\beta=n+k\geq(n-1),\ k$ arbitrary, there exists $\widehat{p_{(n-1)}}\in C_{(n-1)}[z,\ \overline{z}]/I_{(n-1)}$ such that $(\widehat{z^\alpha\overline{z}^\beta}-\widehat{p_{(n-1)}})\in T_{n+k}^1\cap T_{n+k}^2$. When $k=0$, from $L_1|_{C_{2\cdot n}[z,\ \overline{z}]/I_{2\cdot n}}=L_2|_{C_{2\cdot n}[z,\ \overline{z}]/I_{2\cdot n}}=L_{2\cdot n}|_{C_{2\cdot n}[z,\ \overline{z}]/I_{2\cdot n}}$ and from $L_{2\cdot n}|_{C_{2\cdot n}[z,\ \overline{z}]/I_{2\cdot n}}$ stability at $(n-1)$, the assertion is true. Suppose the statement is true for $(n+k)$ and prove it for $(n+k+1)$. If $\alpha'+\beta'=n+k+1$, $\widehat{z^{\alpha'-1}\overline{z}^{\beta'}}$ or and $\widehat{z^{\alpha'}\overline{z}^{\beta'-1}}\in C_{n+k}[z,\ \overline{z}]/T_{n+k}^1\cap T_{n+k}^2$ from induction hypothesis, there exists $\widehat{q_{n-1}},\ \widehat{q^{'}_{n-1}} \in C_{n-1}[z,\ \bar{z}]/I_{n-1}$ such that $(\widehat{z^{\alpha'-1}\bar{z}^{\beta'}} - \widehat{q_{n-1}}) \in T^1_{n+k} \cap T^2_{n+k}$ and or $(\widehat{z^{\alpha'}\bar{z}^{\beta'-1}} - \widehat{q^{'}_{n-1}}) \in T^1_{n+k} \cap T^2_{n+k}$. If $z\bar{z} = 1$ and $L^i(|\widehat{f}|^2) = 0$, from lemma 2.6, we have also $L_i(|z\widehat{f}|^2) = L_i(|\bar{z}\widehat{f}|^2) = 0,\ i = 1,\ 2$; with result

$$z^{\alpha'}\widehat{\bar{z}^{\beta'} - zq_{n-1}} \in T^1_{n+k+1} \cap T^2_{n+k+1} \text{ or } z^{\alpha'}\widehat{\bar{z}^{\beta'} - \bar{z}q'_{n-1}} \in T^1_{n+k+1} \cap T^2_{n+k+1}. \tag{11}$$

If L is stable at n, there exists $\widehat{p_{n-1}}, \widehat{p^{'}_{n-1}} \in C_{n-1}[z,\bar{z}]/I_{n-1}$ such that

$$(\widehat{zq_{n-1}} - \widehat{p_{n-1}}) \text{ or } (\widehat{\bar{z}q^{'}_{n-1}} - \widehat{p^{'}_{n-1}}) \in T_n. \tag{12}$$

From (11) and (12) it results $\widehat{z^{\alpha}}\overline{\widehat{z^{\beta}}} - \widehat{p^{'}_{n-1}} \in T^1_{n+k+1} \cap T^2_{n+k+1} + T_n$ or $\widehat{z^{\alpha}}\overline{\widehat{z^{\beta}}} - \widehat{p_{n-1}} \in T^1_{n+k+1} \cap T^2_{n+k+1} + T_n$. Thus, for all $\alpha,\ \beta \in N$, there exists $p_{n-1},\ p^{'}_{n-1} \in C_{n-1}[z,\ \bar{z}]/I_{n-1}$ with property $(\widehat{z^{\alpha}\bar{z}^{\beta}} - \widehat{p_{n-1}}) \in T^1_{n+k} \cap T^2_{n+k}$ and, $(\widehat{z^{\alpha}\bar{z}^{\beta}} - \widehat{p^{'}_{n-1}}) \in T^1_{n+k} \cap T^2_{n+k}$. Consequently, $L_1(\widehat{z^{\alpha}\bar{z}^{\beta}}) = L_2(\widehat{z^{\alpha}\bar{z}^{\beta}}) = L_1(\widehat{p_{n-1}}) = L_2(\widehat{p_{n-1}}) = L_{2\cdot n}(\widehat{p_{n-1}})$. Extending (11) and (12) by linearity to arbitrary functions $\widehat{f} \in C[z,\ \bar{z}]/(1 - z\bar{z})$, we have $L_1 = L_2 = L_{ext}$; the extension is unique. □

Let $L_{2\cdot n} : C_{2n}[z,\ \bar{z}]/I_{2\cdot n} \to C$ be a linear functional with properties (2-4), stable at $(n-1)$. The unique extension of $L_{2\cdot n}$ to $L : C[z,\ \bar{z}]/(1 - z\bar{z}) \to C$ like a linear functional with properties (2-4) is called the stable extension of $L_{2\cdot n}$.

Remark 2.12 Let $L : C[z,\ \bar{z}]/(1 - z\bar{z}) \to C$ be a linear functional with properties (2-4), the stable extension of the functional $L_{2\cdot n}$, stable at $(n-1)$. Then L is stable at any $(n+k) \geq (n-1),\ k \geq 0$.

Proof. Let $\{H_l\}^{\infty}_{l=0}$ the Hilbert spaces built via L, the stable extension of $L_{2\cdot n}$, $J_l : H_l \to H_{l+1}$ the associated isometries. We prove by induction that $J_l(H_l) = H_{l+1}$, for all $l \geq (n-1)$. The assertion is true for $l = n-1$. Assume the assertion is true for some l and let us get it for $(l+1)$. We fix an element $\widehat{z^{\alpha}\bar{z}^{\beta}} \in C_{l+2}[z,\ \bar{z}]/I_{l+2}, \alpha + \beta = (l+2)$. We have $\widehat{z^{\alpha}\bar{z}^{\beta}} = z \cdot \widehat{z^{\alpha-1}\bar{z}^{\beta}}$ or $\widehat{z^{\alpha}\bar{z}^{\beta}} = \bar{z}\widehat{z^{\alpha}\bar{z}^{\beta-1}},\ \alpha + \beta - 1 = (l+1)$. From induction hypothesis, for $\widehat{z^{\alpha-1}\bar{z}^{\beta}}$ or $\widehat{z^{\alpha}\bar{z}^{\beta-1}} \in C_{l+1}[z,\ \bar{z}1]/I_{l+1}$ there exists $\widehat{f},\ \widehat{g} \in C_l[z,\ \bar{z}]/I_l$ such that $(\widehat{z^{\alpha}\bar{z}^{\beta-1}} - \widehat{g}),\ (\widehat{z^{\alpha-1}\bar{z}^{\beta}} - \widehat{f}) \in T_{l+1}$. If $L(|(\widehat{z^{\alpha}\bar{z}^{\beta-1}} - \widehat{g})|^2) = 0 = L(|(\widehat{z^{\alpha-1}\bar{z}^{\beta}} - \widehat{f})|^2)$, we have also $L(|\bar{z}(\widehat{z^{\alpha}\bar{z}^{\beta-1}} - \widehat{g})|^2) = 0 = L(|z(\widehat{z^{\alpha-1}\bar{z}^{\beta}} - \widehat{f})|^2$. From above, we have $J_{l+1}(H_{l+1}) = H_{l+2}$. By the recurrence hypothesis, we get $J_l(H_l) = H_{l+1}$ for all $l \geq n-1$. □

Remark from remark 2.12 it results: the number $\mathbf{sd(L)}$ is unambiguously defined.

Remark 2.13 Let $L : C[z,\ \bar{z}]/I \to C$ be a linear functional with properties (2-4) extension of the functional $L_{2\cdot n}$, stable at $(n-1)$. Let also $\{H_l\}^{\infty}_{l=0}$ the Hilbert spaces built via L and let $J_{l,\ m} : H_l \to H_m,\ m \geq l$ the associated isometries, $J_{l,\ l+1} := J_l$. If L is stable at all $l \geq (n-1)$, we may construct for all $l \geq n$ the unitary operators $A_l : H_l \to H_l,\ A_l = M_{z_l}J^{-1}_l,\ B_l = M_{\bar{z}_l}J^{-1}_l$. As in propositions 2.8, 2.9, we have $A^{-1}_l = B_l,\ l \geq n$.

Whenever $\widehat{f} + T_{l-1} \in H_{l-1}$ arbitrary, we have $M_{z_l}J_{l-1}(\widehat{f} + T_{l-1}) = J_l M_{z_{(l-1)}}(\widehat{f} + T_{l-1}) = z\widehat{f} + T_{l+1}$. Therefore, $M_{z_l}J_{l-1} = J_l M_{z_{(l-1)}}$. Using these equalities, we obtain $M_{z_l}M_{z_{(l-1)}}(\widehat{f} + T_{(l-1)}) = z^2\widehat{f} + T_{(l+1)}$. Moreover, we infer $A_l \circ A_l(\widehat{f} + T_l) = M_{z_{(l-1)}}J^{-1}_{(l-1)}M_{z_{(l-1)}}J^{-1}_{(l-1)}(\widehat{f} + T_l) = J^{-1}_l M_{z_l}M_{z_{(l-1)}}J^{-1}_{(l-1)}(\widehat{f} + T_l) = J^{-1}_l M_{z_l}J^{-1}_l M_{z_l}(\widehat{f} + T_l) = J^{-1}_l J^{-1}_{(l+1)}M_{z_{(l+1)}}M_{z_l}(\widehat{f} + T_l) = J^{-1}_{l,(l+2)}(z^2\widehat{f} + T_{l+2})$. A recurrence argument leads to the formula:

$$A^k_l(\widehat{f} + T_l) = J^{-1}_{l,l+k}(z^k\widehat{f} + T_{l+k}) \text{ for all } l \geq n,\ k \geq 0,\ \widehat{f} \in C_n[z,\ \bar{z}]/I_n. \tag{13}$$

The same:

$$A^{-k}_l(\widehat{f} + T_l) = J^{-1}_{l,\ l+k}(\bar{z}^k\widehat{f} + T_{l+k}) \text{ when } l \geq n,\ k \geq 0,\ \widehat{f} \in C_n[z,\ \bar{z}]/I_n. \tag{14}$$

$\square$

Remark 2.14 Let $L_{2\cdot n} : C_{2\cdot n}[z, \bar{z}]/I_{2\cdot n} \to C$, be a linear functional with properties (2-4), stable at $(n-1)$, $l \geq n$, A the operator in remark 2.7. The assertions are true.

i) For all $\widehat{f}, \widehat{q} \in C_l[z, \bar{z}]/I_l$, $f(A)(\widehat{q}+T_l) = J^{-1}_{l,\, 2l}(\widehat{fq}+T_{2l})$ with $\widehat{fq}$ the usual product in $C[z, \bar{z}]/(1-z\bar{z})$ algebra.

ii) If $\widehat{f} \in T_n$, $\widehat{q} \in C_n[z, \bar{z}]/I_n$ arbitrary, we have $f(A)(\widehat{q}+T_n) = 0_{H_n}$, for all $\widehat{q}+T_n$. Consequently $f(A) = 0_{L(H_n)}$.

Proof. If $l \geq n$, from (13), $A_l^k(\widehat{f}+T_l) = J^{-1}_{l,\, l+k}(z^k\widehat{f}+T_{l+k})$, $k \geq 0$, $\widehat{f} \in C_l[z, \bar{z}]/I_l$. As result, $f(A)(\widehat{q}+T_l) = J^{-1}_{l,\, 2l}(\widehat{fq}+T_{2l})$, $\widehat{q}$, $\widehat{f} \in C_l[z, \bar{z}]/I_l$ arbitrary.

ii) From (i), when $\widehat{f} \in T_n$, $\widehat{q} \in C_n[z, \bar{z}]/I_n$ arbitrary, the following occur: $f(A)(\widehat{q}+T_n) = J^{-1}_{n,\, 2n}(\widehat{fq}+T_{2n})$. Let us prove that $\widehat{fq} \in T_{2n}$. Indeed, we have $\widehat{fq} \in T_{2n}$, equivalently with $L(\widehat{|fq|}^2) = 0$. If $0 \leq L(\widehat{|fq|^2}) =< (\widehat{fq\overline{fq}})(A)(1+T_n),\ (1+T_n) >_n =< f(A)(1+T_n),\ q^*(A)f(A)q(A)(1+T_n) >_n \leq \|f(A)(1+T_n)\|_n \|q^*(A)f(A)q(A)(1+T_n)\|_n = 0$, if when $\widehat{f} \in T_n$, we have $f(A) = 0_{L(H_n)}$ (from (i)). This imply $\widehat{fq} \in T_{2n}$. $\square$

The next result is a substitute for Theorem 2.10 from [5].

Theorem 2.15 Let $L_{2\cdot n} : C_{2n}[z, \bar{z}]/I_{2n} \to C$ a linear functional with (2-4), stable at $(n-1)$. Endowed with an equivalent norm, H_n has a structure of a unital commuting C^* algebra.

Proof. Let $L : C[z, \bar{z}]/(1-z\bar{z}) \to C$ be the stable extension of $L_{2\cdot n}$, $\{H_k\}_{k=0}^{\infty}$, the Hilbert spaces built via it, $J_{k,\, l}$, $n \leq k < l$ the associated isometries, A the operator in remark 2.7. Let $X = \{f(A),\ f \in C[z, \bar{z}]/I \text{ arbitrary}\}$ be a commutative C^* sub-algebra in the C^* algebra $L(H_n)$ of the linear operators on H_n. We consider the map $\pi : H_n \to X$, defined by the equation $\pi(\widehat{f}+T_n) = f(A)$, and check the correctness of the definition. Indeed, if $\widehat{f} \in T_n$, for all $\widehat{q} \in C_n[z, \bar{z}]/I_n$, from remark 2.14, $f(A)(\widehat{q}+T_n) = J^{-1}_{n,\, 2n}(\widehat{fq}+T_{2n}) = 0_{H_n}$, implying $\widehat{fq} \in T_{2n}$, that is $f(A) = 0_{L(H_n)}$. The map π is linear and injective since $\pi(\widehat{f}+T_n) = f(A) = 0_{L(H_n)}$ implies $f(A)(1+T_n) = J^{-1}_{n,\, 2n}(\widehat{f}+T_{2n}) = \widehat{f}+T_n = 0_{H_n}$, with consequence $\widehat{f} \in T_n$. Let us prove that the map π is surjective too. In case $l \geq n$, from stability of $L|_{C_{2n}[z,\, \bar{z}]/I_{2n}} = L_{2\cdot n}$ at $(n-1)$, we can find $r \in C_{n-1}[z, \bar{z}]/I_{n-1}$ such that $\widehat{f-r} \in T_l$. Therefore, $f(A)(\widehat{q}+T_n) = J^{-1}_{n,\, n+l}(\widehat{fq}+T_{n+l}) = J^{-1}_{n,\, n+l}(\widehat{rq}+T_{n+l}) = r(A)(\widehat{q}+T_n)$, $\widehat{q} \in C_n[z, \bar{z}]/I_n$ arbitrary; that is π is surjective. The map π is an isomorphism of vector spaces, implying $dim(H_n) = dim(X) = s$. Moreover, we prove that the map π is an isometry too. Indeed,

$$\left\|\pi\left(\hat{f}+T_n\right)\right\|_{L(H_n)} = f(A)_{L(H_n)} = \sup_{\|\hat{q}+T_n\|\leq 1}\left\|J^{-1}_{n,2n}\left(\widehat{fq}+T_{2n}\right)\right\|_{H_n}$$

$$= \sup_{\|\hat{q}+T_n\|\leq 1}\left\|\left(\widehat{fq}+T_{2n}\right)\right\|_{H_{2n}} \geq \left\|\hat{f}+T_n\right\|_{H_n}. \tag{15}$$

Moreover,

$$\left\|\pi\left(\hat{f}+T_n\right)\right\|_{L(H_n)} = \sup_{\|\hat{q}+T_n\|\leq 1}\left\|J^{-1}_{n,2n}\left(\widehat{fq}+T_{2n}\right)\right\|_{H_n}$$

$$= \sup_{\|\hat{q}+T_n\|\leq 1}\left(L\left(\left|\widehat{fq}\right|^2\right)\right)^{\frac{1}{2}} \leq \sup_{\|\hat{q}+T_n\|\leq 1}\|f(A)(1+T_n)\|_{H_n}^{\frac{1}{2}}\|q^*(A)f(A)q(A)(1+T_n)\|_{H_n}^{\frac{1}{2}}$$

$$\leq \left\|\left(\hat{f}+T_n\right)\right\|_{H_n}^{\frac{1}{2}} \sup_{\|\hat{q}+T_n\|\leq 1}\|q^*(A)q(A)\|_X^{\frac{1}{2}}\left\|\hat{f}+T_n\right\|_{H_n}^{\frac{1}{2}} \leq \left\|\hat{f}+T_n\right\|_{H_n} \tag{16}$$

If $\|q(A)\|_X \leq 1$ when $\|\widehat{q}+T_n\|_{H_n} \leq 1$, from (15) and (16), it results $\left\|\pi(\widehat{f}+T_n)\right\|_{L(H_n)} = \left\|\widehat{f}+T_n\right\|_{H_n}$ when $\widehat{f} \in C_n[z, \bar{z}]/I_n$. If $\pi : H_n \to X$ is a linear isomorphism and an isometry too, it identifies X with H_n obtaining a structure of commuting C^* algebra on H_n. $\square$

The next result is a substitute for Theorem 2.11 in [5].

Theorem 2.16 Let $L_{2\cdot n} : C_{2n}[z,\ \bar{z}]/I_{2n} \to C,\ n \geq 1$ be a linear functional with (2-4), stable at $(n-1)$ $\{H_k\}_{k=0}^{n}$ the Hilbert spaces associated with it. There exists an s-atomic measure μ on T_1, where s=dimH_n, such that

$$L_{2\cdot n}(\widehat{z^k}) = \int_{\sigma_A} z^k \mathrm{d}\mu(z),\ \text{for all}\ k \in Z,\ |k| \leq 2n.$$

Proof. Whenever $L_{2\cdot n}$ is stable at $(n-1)$, from remark 2.7, respectively theorem 2.11, there exists the unitary operator A on H_n, and the unique extension of it, $L : C[z,\ \bar{z}]/(1-z\bar{z}) \to C$ such that $L_{2\cdot n}(\widehat{z^i\bar{z}^j}) = L(\widehat{z^i\bar{z}^j}) =< A^i(A^*)^j(1+T_n),\ (1+T_n) >_{H_n},\ 0 \leq i+j \leq 2n,\ z\bar{z} = 1$. If E is the spectral measure of A, concentrated on the spectrum $\sigma_A \subset T_1$, with $\mu(*) =< E(*)(1+T_n),\ (1+T_n) >_{H_n}$ we have

$$L_{2\cdot n}(\widehat{z^p\bar{z}^q}) = L(\widehat{z^p\bar{z}^q}) =< A^p(A^*)^q(1+T_n),\ (1+T_n) >_{H_n} = \int_{\sigma_A} z^{p-q}\mathrm{d}\mu(z),\ 0 \leq p,\ q \leq (p+q) \leq 2n.$$

Let us prove that σ_A has $s = \dim H_n$ points. Indeed, if $\mathbf{X} = \{f(A), f \in C[z,\ \bar{z}]/(1-z\bar{z})\}$ is a commutative unital C^* sub-algebra in $L(H_n)$ and, from theorem 2.15, $H_n \cong X$ (are isomorphic), we have $\dim H_n\ = \dim\mathbf{X}$, consequently $\mathbf{X}$ has s generators, consequently s characters, say $\{\Phi_1,\ \ldots,\ \Phi_s\}$. We have $\sigma_A = \{(\Phi_1(A),\ \ldots,\ \Phi_s(A))\} \subset T_1$. Thus, $\mu(*)$ is an s-atomic positive measure with $support\mu \subset \sigma_A$. Finally we prove that $supp\mu$ is exactly σ_A. Indeed, if $\xi \in \sigma_A$ is an arbitrary point of σ_A, and χ_ξ is the characteristic function of ξ related to σ_A, we may construct the spectral projection $\chi_\xi(A)$, built via the functional calculus of A. We have then, $< \chi_\xi(A)(1+T_n),\ (1+T_n) >_{H_n} = \int_{\sigma_A} \chi_\xi(z)\mathrm{d}\mu(z) = \mu(\{\xi\})$. Assuming that $\mu(\{\xi\}) = 0$, we have $\chi_\xi(A)(\widehat{f}+T_n) = \chi_\xi^2(A)(\widehat{f}+T_n) = \chi_\xi(A)f(A)(1+T_n) = 0$, for all $f \in C_n[z,\ \bar{z}]/I_n$ Consequently, $\chi_\xi(A) = 0_{L(H_n)}$ which is a contradiction; it follows $supp\mu = \sigma_A$. □

The next result is an assertion in the spirit of Theorem 2.12 of [5].

Theorem 2.17 Let $L : C[z,\ \bar{z}]/(1-z\bar{z}) \to C$ be a linear functional with properties (2-4), $\{H_k\}_{k=0}^{\infty}$ be the Hilbert spaces constructed via L. There exists an s-atomic, positive representing measure μ on T_1 for L, if and only if L is dimensionally stable at $(n-1) \geq 0$ with $s = sd(L) = \dim H_n$.

Proof. Whenever L is stable, with $s\ =\ sd(L)$, there exists $n \geq 1$ such that $L|_{C_{2n}[z,\ \bar{z}]/I_{2n}}$ is stable at l, $0 \leq l < n$. Via theorem 2.16, there exists an s-atomic representing measure with $s\ = \dim H_l = \dim H_n\ =\ sd(L)$ such that

$$L(\widehat{z^k}) = \int_{T_1} z^k \mathrm{d}\mu(z),\ k \in \{-2n,\ \ldots,\ 2n\}.$$

Let be $L'(\widehat{z^k}) = \int_{T_1} z^k\mathrm{d}\mu(z),\ \forall k \in Z$. We infer L' is a linear functional on $C[z,\ \bar{z}]/(1-z\bar{z})$, with properties (2-4) and $L'|_{C_{2\cdot n}[z,\ \bar{z}]/I_{2\cdot n}} = L|_{C_{2\cdot n}[z,\ \bar{z}]/I_{2\cdot n}}$. Indeed, $L'(\overline{\widehat{f}}) = \int_{T_1} \overline{f(z)}\mathrm{d}\mu(z) = \overline{\int_{T_1} f(z)\mathrm{d}\mu(z)} = \overline{L'(\widehat{f})}$, $L'(\widehat{1}) = 1$ and $L'(|\widehat{f}|^2) \geq 0$. If the integral is linear, L' is the same. Consequently L' is a linear functional with properties (2-4), extension of $L|_{C_{2\cdot n}[z,\ \bar{z}]/I_{2\cdot n}}$ to $C[z,\ \bar{z}]/(1-z\bar{z})$. If such an extension is unique, it results $L = L'$, implying L admits on $C[z,\ \bar{z}]/(1-z\bar{z})$ a representing measure.

Conversely, we assume that an s-atomic represent1ing measure μ exists such that $L(\widehat{z^k}) = \int_{T_1} z^k\mathrm{d}\mu(z), k \in Z$. We prove that L is stable at $(n-1)$ with $s = \dim H_n$.

If $s = 1$, $\int_{T_1} z^k\mathrm{d}\mu(z) = \xi_1^k\mu(\{\xi_1\})$, with result $T_n = \{\widehat{f} = \sum_{k=0}^{n} b_k\widehat{z^k},\ z\bar{z} = 1,\ L(\widehat{|f|^2}) = 0\} = \{\widehat{f} = \sum_{k=0}^{n} b_k\widehat{z^k} \in C_n[z,\ \bar{z}]/I_n,\ |\sum_{k=0}^{n} b_k\xi_1^k|^2 = 0\}$. Because if $n \geq 1$ arbitrary, $\dim T_n = \dim((C_n[z,\ \bar{z}]/I_n) - 1) \Leftrightarrow \dim H_n = 1,\ n \geq 1$. Therefore, L is stable at $n = 1$ and sd$L$=dim$H_n = s = 1$.

Let be the case $s \geq 2$ and $\tau = \{\xi_1,\ \ldots,\ \xi_s\} \subset T_1$, such that the representing s-atomic measure is concentrated on τ. Let $\{H_m\}_{m=0}^{\infty} = \{\frac{C_m[z,\ \bar{z}]/I_m}{T_m}\}_{m=0}^{\infty}$ be the Hilbert spaces built via L. We have $T_m = \{\widehat{f} \in C_m[z,\ \bar{z}]/I_m,\ L(\widehat{|f|^2}) = 0\} =$

$\{\widehat{f} \in C_m[z, \bar{z}]/I_m, \mid \int_{T_1} |f|^2 \mathrm{d}\mu(z) = 0\}$. Let us prove that in case $s \geq 2$, we have for all $l \geq (s-1)$, $\dim H_l = s$. Whenever $\tau = \{\xi_1, \ldots, \xi_s\} = supp(\mu)$, $|\xi_k| = 1$, $1 \leq k \leq s$, we consider for all k the Lagrange interpolation polynomial

$$\widehat{\Xi_k(z)} = (\prod_{j=1,\ j\neq k}^{j=s} \|\widehat{z} - \xi_j\|^2)(\prod_{j=1,\ j\neq k}^{j=s} \|\xi_k - \xi_j\|^2)^{-1}.$$

Clearly, for $l \geq s-1$ we have $\{\widehat{\Xi_k}\}_{k=1}^{k=s} \subset \frac{C_{2s}[z, \bar{z}]}{I_{2s}} \subset L^2(\mu)$, with main property $\widehat{\Xi_k}(\xi_j) = \delta_{kj}$, k, $j \in \{1, \ldots, s\}$. Since $f \in L^2(\mu)$ can be written on τ as $f = \sum_{j=1}^{j=s} f(\xi_j)\widehat{\Xi}_j$, $\{\widehat{\Xi}_j\}_{j=1}^{j=s}$ is an orthogonal basis in $L^2(\mu)$. Consequently, for all $l \geq (s-1)$, for $\widehat{f} \in C_l[z, \bar{z}]/I_l$, $|\widehat{f(z)} - \sum_{j=1}^{j=s} f(\xi_j)\widehat{\chi_j(z)}|^2 = 0$ on τ. Moreover,

$$0 = \int_\tau |f(z) - \sum_{j=1}^{j=s} f(\xi_j)\widehat{\chi_j(z)}|^2 \mathrm{d}\mu(z)$$

$$= \int_\tau |f(z)|^2 \mathrm{d}\mu(z) - \int_\tau f(z)\overline{(\sum_{j=1}^{j=s} f(\xi_j)\widehat{\chi_j(z)})}\mathrm{d}\mu(z) - \int_\tau \overline{f(z)}(\sum_{j=1}^{j=s} f(\xi_j)\widehat{\chi}_j(z))\mathrm{d}\mu(z) + \sum_{j=1}^{j=s} |f(\xi_j|^2.$$

It follows, for every $l \geq s-1$, we have: $T_l = \{\widehat{f} \in C_l[z, \bar{z}]/I_l,\ \int_\tau |f(z)|^2 \mathrm{d}\mu(z) = 0\} = \{\widehat{f} \in C_l[z, \bar{z}]/I_l$, with $\widehat{f}|_\tau = 0\}$. Thus, for all $l \geq s-1$, $\{\widehat{\xi_k} + T_l\}_{k=1}^{k=s}$ is an orthogonal basis in H_l and, for all $l \geq s-1$, $\dim H_l = \dim L^2(\mu) = s$. That is L is stable and $sd(L) = s$. □

3. Application: The quadratic moment problem on $\{z, z\bar{z} = 1\}$

The data of the problem are: $\gamma := \{\gamma_0 := \gamma_{0,\,0} = \gamma_{1,\,1} := \gamma_{1-1},\ \gamma_{-1} := \gamma_{1,\,0} := \gamma_{0-1},\ \overline{\gamma_{1,\,0}} = \gamma_{0,\,1} := \gamma_1,\ \gamma_{-2} := \gamma_{2,0} := \gamma_{0-2},\ \overline{\gamma_{2,\,0}} = \gamma_{0,\,2} = \gamma_2\}$.

Let $L_{2\cdot 1} := L_2$, $C_{2\cdot 1}[z, \bar{z}]/I_2 := C_2[z, \bar{z}]/I_2$, $L_2 :\ C_2[z, \bar{z}]/I_2 \to C$, be a linear functional, with properties (2-4) such that $L_2(1) = L_2(\widehat{z\bar{z}}) = L_2(\widehat{z^2\bar{z}^2}) = \gamma_{0,\,0} = \gamma_{1,\,1} = \gamma_{2,\,2}$, $L_2(\widehat{z}) = \overline{L_2(\widehat{\bar{z}})} = \gamma_{0,\,1} = \overline{\gamma_{1,\,0}}$; $L_2(\widehat{z^2}) = \overline{L_2(\widehat{\bar{z}^2})} = \gamma_{0,\,2} = \overline{\gamma_{2,\,0}} \geq 0$. The positivity of $L_2 \geq 0$ is a necessary condition for the existence of a representing measure for γ. Thus, we assume $L_2 \geq 0$. If $I = (1 - z\bar{z})$, $I_k = I \cap C_k[z, \bar{z}]$, $k = 0,\ 1,\ 2$, we have $I_0 = \{0\}$, $I_1 = \{0\}$, $I_2 = \{\alpha(1 - z\bar{z}),\ \alpha \in C\}$. The Hilbert spaces built via L_2 are: $\{H_k\}_{k=0}^{k=1}$, $T_0 = \{\widehat{f} \in C_0[z, \bar{z}],\ L_2(|\widehat{f}|^2 = 0\} = \{0\}$, $T_1 = \{f \in C_1[z, \bar{z}],\ L_2(|\widehat{f}|^2) = 0\} = \{0\}$, $H_0 = \dfrac{C_0[z, \bar{z}]/I_0}{T_0} = C_0[z, \bar{z}]$, $\dim H_0$=1. $H_1 = \dfrac{C_1[z, \bar{z}]/I_1}{T_1} = C_1[z, \bar{z}]$.

Case 1 L_2 is given. If $\dim H_1 = 1$, case in which $\widehat{z}$, $\widehat{\bar{z}}$ are linear dependent of 1, we have $\dim H_0 = 1 = \dim H_1$, consequently L_2 is stable at $(n-1) = 0$, we find a representing measure for γ with $s = 1 = sd(L_2)$ atoms (theorem 2.16).

Case 2 L_2 is given. Let be $\dim H_1 = 2$. In this case, L_2 is unstable at $(n-1) = 0$. We look for the existence of a functional $L_{2\cdot 2} := L_4$, $C_{2\cdot 2}[z, \bar{z}]/I_4 := C_4[z, \bar{z}]/I_4$; $L_4 :\ C_4[z, \bar{z}]/I_4 \to C$, such that $L_4(1) = \gamma_{0,\,0} = \gamma_{1,\,1} = 1$, $L_4(\overline{\widehat{f}}) = \overline{L_4(\widehat{f})}$, $\forall \widehat{f} \in C_4[z, \bar{z}]/I_4$, $L_4(|\widehat{f}|^2) \geq 0$, $\forall \widehat{f} \in C_2[z, \bar{z}]/I_2$, (respectively properties (2-4) of L_4). We have than: $L_4(1) = L_4(\widehat{z\bar{z}}) = 1 = \gamma_{0,\,0} = \gamma_{1,\,1}$. If $L_4(\overline{\widehat{f}}) = \overline{L_4(\widehat{f})}$, $\forall \widehat{f} \in C_4[z, \bar{z}]/I_4$, we have: $L_4(\widehat{\bar{z}}) = \overline{L_4(\widehat{z})}$, equivalently with $\gamma_{1,\,0} = \overline{\gamma_{0,\,1}}$; moreover $L_4(\widehat{\bar{z}^2}) = \overline{L_4(\widehat{z^2})} \geq 0$, equivalently with $\gamma_{2,\,0} = \overline{\gamma_{0,\,2}} \geq 0$; moreover $L_4(\widehat{\bar{z}^3}) = \overline{L_4(\widehat{z^3})}$, equivalently with $\gamma_{3,\,0} = \overline{\gamma_{0,\,3}}$; moreover $L_4(\widehat{\bar{z}z^2}) = \overline{L_4(\widehat{z\bar{z}^2})} = L_4(\widehat{z})$, equivalently with $\gamma_{1,\,2} = \overline{\gamma_{2,\,1}} = \gamma_{0,\,1} = \overline{\gamma_{1,\,0}}$. We have also $L_4(\widehat{\bar{z}^4}) = \overline{L_4(\widehat{z^4})} \geq 0$, equivalently $\gamma_{4,\,0} = \overline{\gamma_{0,\,4}} \geq 0$, $L_4(\widehat{\bar{z}^3 z}) = \overline{L_4(\widehat{z^3\bar{z}})} = L_4(\widehat{\bar{z}^2}) = \overline{L_4(\widehat{z^2})} \geq 0$, equivalently with $\gamma_{3,\,1} = \overline{\gamma_{1,\,3}} = \gamma_{2,\,0} = \overline{\gamma_{0,\,2}} \geq 0$. A linear functional, with (2-4) always exists.

Let $L_4 : C_4/I_4 \to C$ be a linear functional with properties (2-4). The Hilbert spaces built via L_4 are: $\{H_k\}_{k=0}^{k=2}$. In this case, $I_0 = C_0[z,\ \bar{z}] \cap (1 - z\bar{z}) = \{0\},\ C_0[z,\ \bar{z}]/I_0 = C_0[z,\ \bar{z}];\ T_0 = \{\widehat{f} \in C_0[z,\ \bar{z}] | L_4(|\widehat{f^2}| = 0\} = \{0\};\ H_0 = \dfrac{C_0[z,\ \bar{z}]/I_0}{T_0} = C_0[z,\ \bar{z}]$. $I_1 = C_1[z,\ \bar{z}] \cap (1 - z\bar{z}) = \{0\},\ T_1 = \{\widehat{f} \in C_1[z,\ \bar{z}]/I_1,\ L_4(|\widehat{f}|^2) = 0\} = \{0\},\ H_1 = \dfrac{C_1[z,\ \bar{z}]/I_1}{T_1} = C_1[z,\ \bar{z}]; I_2 = C_2[z,\ \bar{z}] \cap (1 - z\bar{z}) = \alpha(1 - z\bar{z}),\ \alpha \in C, T_2 = \{\widehat{f} \in C_2[z,\ \bar{z}]/\alpha(1 - z\bar{z}),\ |L_4(|\widehat{f}|^2) = 0\},\ H_2 = \dfrac{C_2[z,\ \bar{z}]/\alpha(1 - z\bar{z})}{T_2}$.

As previous, the positivity $L_4 \geq 0$ is a necessary condition for the existence of a representing measure for γ. Thus, we assume $L_4 \geq 0$. We suppose $L_4 \geq 0$ on $C_4[z,\ \bar{z}]/I_4$ equivalently with $L_4(\widehat{f}) \geq 0$ if $\widehat{f} \in C_4[z,\ \bar{z}]/I_4, \widehat{f} \geq 0$ (that is $f \geq 0,\ z\bar{z} = 1$). From Riesz-Fejer lemma, $\widehat{f} \geq 0,\ \widehat{f} = \widehat{\overline{|q|}^2}$. Consequently $L_4(\widehat{\overline{|q|}^2}) \geq 0$ if $\widehat{q} = \sum_{i=0}^{i=2} \underline{a_i \widehat{z^i}}$, equivalently with $\sum_{i,\ j} a_i \overline{a_j} L_4(\widehat{z^i \overline{z^j}}) \geq 0$. From this, $L_4(\sum_{i,\ j} a_i \overline{a_j} \widehat{z^i \overline{z^j}}) = L_4(\sum_{i,\ j} a_i \overline{a_j} \widehat{z^i} \widehat{\overline{z^j}}) = L_4(\sum_{i,\ j} a_i \overline{a_j} \widehat{z^i} \widehat{\overline{z^j}})$. That is $L_4(\widehat{z^i \overline{z^j}}) = L_4(\widehat{z^i \overline{z^j}})$ for all $i,\ j \in \{0,\ 1,\ 2,\ i + j \leq 4\}$. The functional L_4 is completely determined by the moments $L_4(\widehat{z^k}),\ k \in \{-2,\ ...,\ 2\}$. If on T_1, $\bar{z} = z^{-1}$, $L_4(\widehat{z^i \overline{z^j}}) = L_4(\widehat{z^i z^{-j}}) = \gamma_{j,\ i} : = \gamma_{i-j}$ and $L_4(\widehat{\overline{z^i \overline{z^j}}}) = \overline{\gamma_{i-j}} = L_{22}(\widehat{z^i z^j}) = \gamma_{j-i}$. The matrix associated with L_4 on H_1 is: $A_1 = \begin{bmatrix} \gamma_{0,\ 0} & \gamma_{0,\ 1} & \gamma_{1,\ 0} \\ \gamma_{1,\ 0} & \gamma_{1,\ 1} & \gamma_{2,\ 0} \\ \gamma_{0,\ 1} & \gamma_{0,\ 2} & \gamma_{1,\ 1} \end{bmatrix} \geq 0.$

We divide the proof according to the values of $s = \dim H_1 \leq 3$.

Case I $\dim H_1$=1 It implies there exists $\alpha \in C, \alpha \neq 0$ such that $\widehat{z} = \alpha \cdot 1$, that is: $L_4(|\widehat{z} - \alpha 1|^2) = 0 = (\|(\widehat{z} - \alpha)\|_1)^2$. We have also $\widehat{\bar{z}} = \overline{\alpha} \cdot 1$, and $z\bar{z} = 1$. From the construction of the Hilbert spaces associated with of L_4, we have $\dim H_0$ = 1 = $\dim H_1$ = 1 = s, implying L_4 is stable at $(n - 1) = 0$. Moreover, if $H_1 \cong X$, the C^* algebra $X = \{f(A),\ f \in C_1[z,\ \bar{z}]/I_1\}$ has $s = \dim H_0\ = \dim H_1$ = 1 characters. From theorem 2.17, there exists an 1-atomic representing measure $\mu(*) =< E_A(*)(1 + T_2),\ (1 + T_2) >\geq 0,\ \mu(*) = \rho\delta_w$ such that $L_4(\widehat{z^k}) = \int_{T_1} z^k \mathrm{d}\mu(z),\ k \in \{-2,\ -1,\ 0,\ 1,\ 2\}$. If $\mu(w) = \rho > 0$, μ is a representing measure for γ. From theorem 2.11 there exists an unique non-negative extension $L_{ext} : C[z,\ \bar{z}]/(1 - z\bar{z})$ of L_4. Thus, we have for L_{ext} an unique representing measure on the compact T_1. Consequently, the representing measure for L_4 is unique. In the same time, the rank of A_1 is in this case $r = 1$ (the second and the third column in A_1 are linear combinations of the first one). Moreover, from theorem 2.11 and remark 2.12, we have also $\dim H_0$ = $\dim H_1$ = $\dim H_2 =\ s\ =\ 1$.

Case II $\dim H_1\ =\ 2\ =$ s. The data that will be interpolated are: $\gamma : \gamma_0\ : =\ \gamma_{0,\ 0}\ = \gamma_{1,\ 1}\ : =\ \gamma_{1-1},\ \gamma_1\ : = \gamma_{0,\ 1},\ \gamma_2\ : =\ \gamma_{0,\ 2},\ \overline{\gamma_1}\ =\ \gamma_{-1},\ \overline{\gamma_2}\ =\ \gamma_{-2}$. If $\dim H_1\ =\ 2$, we have $\alpha,\ \beta \in C,\ (\alpha \cdot \beta) \neq 0$ such that $\widehat{\bar{z}} = \alpha \cdot 1 + \beta \cdot \widehat{z}$, $\{1,\ \widehat{z}\}$ are linear independent in H_1. That is $L_4((\alpha \cdot 1 + \beta \cdot \widehat{z})\overline{(\alpha \cdot 1 + \beta \cdot \widehat{z})}) = 0$ if and only if $\alpha = \beta = 0$. Implying $|\alpha|^2 + |\beta|^2 + 2Re(\alpha\overline{\beta \cdot \gamma_{0,\ 1}}) = 0$ if and only if $\alpha = \beta = 0$. Suppose $|\gamma_{0,\ 1}| = |\gamma_{1,\ 0}| = 1$. We have $0 = L_4((\alpha \cdot 1 + \beta \cdot \widehat{z})\overline{(\alpha \cdot 1 + \beta \cdot \widehat{z})}) = |\alpha|^2 + |\beta|^2 + 2Re(\alpha\overline{\beta \cdot \gamma_{0,\ 1}}) \geq |\alpha|^2|\gamma_{0,\ 1}|^2 + |\beta|^2 - 2|\alpha||\gamma_{0,\ 1}||\beta| = (|\alpha||\gamma_{0,\ 1}| - |\beta|)^2 \geq 0$. Consequently, $|\alpha||\gamma_{0,\ 1}| = |\beta|$. If $\alpha,\ \beta \neq 0$ we have a contradiction. If $L_4 \geq 0$, $\{1,\ \widehat{z}\}$ are linear independent in H_1 if and only if $|\gamma_{0,\ 1}| < 1$. The matrix $A_0 = \begin{bmatrix} \gamma_{0,\ 0} & \gamma_{0,\ 1} \\ \gamma_{1,\ 0} & \gamma_{1,\ 1} \end{bmatrix}$ is invertible and $\det A_0 = \delta > 0$. To find $\alpha, \beta \in C$ we take in H_1 two equation with two unknown, respectively $\widehat{\bar{z}} = \alpha \cdot 1 + \beta\widehat{z}$ and $\widehat{\bar{z}^2} = (\alpha \cdot 1 + \beta\widehat{z})\widehat{\bar{z}} = (\alpha^2 + \beta) \cdot 1 + \alpha\beta\widehat{z}$; consequently $\begin{cases} \gamma_{1,\ 0} = \alpha\gamma_{0,\ 0} + \beta\gamma_{0,\ 1} \text{ and} \\ \gamma_{2,\ 0} = \alpha\beta \cdot \gamma_{1,\ 0} + (\alpha^2 + \beta)\gamma_{0,\ 0} = \alpha\gamma_{1,\ 0} + \beta\gamma_{0,\ 0} \end{cases}$. If A_0 is invertible, we have $\alpha = \dfrac{\begin{vmatrix} \gamma_{1,\ 0} & \gamma_{0,\ 1} \\ \gamma_{2,\ 0} & \gamma_{0,\ 0} \end{vmatrix}}{(\gamma_{0,\ 0}^2 - |\gamma_{0,\ 1}^2|)}, \beta = \dfrac{\begin{vmatrix} \gamma_{0,\ 0} & \gamma_{1,\ 0} \\ \gamma_{1,\ 0} & \gamma_{2,\ 0} \end{vmatrix}}{(\gamma_{0,\ 0}^2 - |\gamma_{0,\ 1}^2|)}$. If $\{1,\ \widehat{z}\}$ are linear independent in H_1, they are linear independent in H_2 too. We have $\widehat{\bar{z}} = \alpha \cdot 1 + \beta \cdot \widehat{z}$ implying $\widehat{\bar{z}^2} = (\alpha \cdot 1 + \beta \cdot \widehat{z})\widehat{\bar{z}} = (\alpha^2 + \beta) \cdot 1 + \beta \cdot \widehat{z}$ and $\widehat{z^2} = \widehat{z}(\overline{\alpha} \cdot 1 + \overline{\beta \cdot \widehat{z}}) = \overline{\alpha}\widehat{z} + \overline{\beta} \cdot 1,\ z\bar{z} = 1$. The above equalities are equivalent with $L_4(|\widehat{\bar{z}^2} - [(\alpha^2 + \beta) + \alpha\beta\widehat{z}]|^2) = 0,\ L_4(|\widehat{z^2} - [(\overline{\alpha}\widehat{z} + \overline{\beta}]|^2) = 0$; equalities that are true. Indeed, $0 \leq< \widehat{\bar{z}^2} - [(\alpha^2 + \beta) + \alpha\beta\widehat{z}],\ \widehat{\bar{z}^2} - [(\alpha^2 + \beta) + \alpha\beta\widehat{z})] >\leq< \widehat{\bar{z}^2} - \widehat{\bar{z}}(\alpha + \beta\widehat{z}),\ \widehat{\bar{z}^2} - [(\alpha^2 + \beta) + \alpha\beta\widehat{z} > + < \widehat{\bar{z}}(\alpha + \beta\widehat{z}) - [(\alpha^2 + \beta) + \alpha\beta\widehat{z}],\ \widehat{\bar{z}^2} - [(\alpha^2 + \beta) + \alpha\beta\widehat{z}] >=< \widehat{\bar{z}}(\widehat{\bar{z}} - (\alpha + \beta\widehat{z})),\ \widehat{\bar{z}^2} - [(\alpha^2 + \beta) + \alpha\beta\widehat{z}] > + < \alpha(\widehat{\bar{z}} - \beta\widehat{z} - \alpha,\ \widehat{\bar{z}^2} - [(\alpha^2 +$

$\beta) + \alpha\beta\bar{z}] >\leq 0$. The same $< \widehat{z^2} - (\overline{\alpha}\widehat{z} + \overline{\beta}),\ \widehat{z^2} - (\overline{\alpha}\widehat{z} + \overline{\beta}) >=< \widehat{z^2} - \widehat{z}(\overline{\alpha} + \overline{\beta}\widehat{\bar{z}}),\ \widehat{z^2} - (\overline{\alpha}\widehat{z} + \overline{\beta}) > + < \widehat{z}(\overline{\alpha}\widehat{z} + \overline{\beta}\widehat{\bar{z}}) - (\overline{\alpha}\widehat{z} + \overline{\beta})),\ \widehat{z^2} - (\overline{\alpha}\widehat{z} + \overline{\beta}) >=< \widehat{z}(\widehat{z} - (\overline{\alpha} + \overline{\beta}\widehat{\bar{z}})),\ \widehat{z^2} - (\overline{\alpha}\widehat{z} + \overline{\beta}) >\leq \left\|\widehat{z}(\widehat{z} - (\overline{\alpha} + \overline{\beta}\widehat{\bar{z}})\right\| \left\|\widehat{z^2} - (\overline{\alpha}\widehat{z} + \overline{\beta})\right\| + \|0\| \left\|\widehat{z^2} - (\overline{\alpha}\widehat{z} + \overline{\beta}\right\| \leq 0$. Consequently, $\dim H_2 = 2 = \dim H_1$ and L_4 is stable at $(n-1) = 1$. From theorem 2.15 the commutative C^* algebra $X = \{f(A),\ f \in C[z,\ \bar{z}]/I\}$ is isomorphic with H_2 with result X has two characters and the scalar measure $\mu(*) =< E_A(*)(1+T_2),\ (1+T_2) >\geq 0$ has two atoms. From theorem 2.16, $L_4(\widehat{z^k}) = \int_{T_1} z^k d\mu(z),\ k \in \{-2,\ -1,\ 0,\ 1,\ 2\}$. That is $\mu(*)$ is a representing measure for γ respectively $\mu(*) = \rho_0\delta_{w_0} + \rho_1\delta_{w_1}$. If $\mu(w_i) \geq 0$, the weights $\rho_i \geq 0,\ i \in \{0,\ 1\}$. In the same time, if A_0 is invertible and the third column in A_1 is linear dependent of the first two, the rank of A_1 is $r = 2$.

Case III $\dim H_1$=s=3. We have $\{1,\ \widehat{z},\ \widehat{\bar{z}}\}$ are linear independent in H_1 equivalently $L_4((a \cdot 1 + b \cdot \widehat{z} + c \cdot \widehat{\bar{z}})\overline{(a \cdot 1 + b \cdot \widehat{z} + c \cdot \widehat{\bar{z}})}) = 0$ if and only if $a = b = c = 0$. The associated matrix of $L_4 \geq 0$ on H_1 is: $A_1 = \begin{bmatrix} \gamma_{0,\,0} & \gamma_{0,\,1} & \gamma_{1,\,0} \\ \gamma_{1,\,0} & \gamma_{1,\,1} & \gamma_{2,\,0} \\ \gamma_{0,\,1} & \gamma_{0,\,2} & \gamma_{1,\,1} \end{bmatrix}$ $\gamma_{ij} := \gamma_{i-j} = L_4(z^i\bar{z}^j),\ 0 \leq i, j \leq 1$. If $L_4 \geq$, we have $A_1 \geq 0$. If we suppose $\operatorname{rank} A_1 = 2$, it follows that there exists $a,\ b \in C$, such that we have the system: $\begin{cases} \gamma_{1,\,0} = a\gamma_{0,\,0} + b\gamma_{0,\,1} \\ \gamma_{2,\,0} = a\gamma_{1,\,0} + b\gamma_{1,\,1} \\ \gamma_{1,\,1} = a\gamma_{0,\,1} + b\gamma_{0,\,2} \end{cases}$. The first equation in the above system is equivalent with: $L_4(\widehat{\bar{z}} - a \cdot 1 - b \cdot \widehat{z}) = 0 =< (\widehat{\bar{z}} - a \cdot 1 - b\widehat{z}), 1 >$. Consequently, from second and third equation of the system, we have also $L_4(\widehat{z}(\widehat{\bar{z}} - a \cdot 1 - b \cdot \widehat{z})) = 0$, $L_4(\widehat{\bar{z}}(\widehat{\bar{z}} - a \cdot 1 - b \cdot \widehat{z})) = 0$. If L_4 is linear, we have $L_4(\widehat{\bar{z}} - a \cdot 1 - b \cdot \widehat{z})\overline{(\widehat{\bar{z}} - a \cdot 1 - b \cdot \widehat{z})}) = 0$ for $c = 1,\ a \cdot b \cdot c \neq 0$. That is a contradiction with $\{1,\ \widehat{z},\ \widehat{\bar{z}}\}$ are linear independent in H_1.

We suppose $\operatorname{rank} A_1 = 1$, for example we have 1) $\gamma_{0,\,1} = a\gamma_{0,\,0}$, 2) $\gamma_{1,\,1} = a\gamma_{1,\,0}$ 3) $\gamma_{0,\,2} = a\gamma_{0,\,1},\ a \in C^*$. Also for $b \in C^*$, $1')$ $\gamma_{1,\,0} = b\gamma_{0,\,0}$, $2')$ $\gamma_{2,\,0} = b\gamma_{1,\,0}$ $3')$ $\gamma_{1,\,1} = b\gamma_{0,\,1}$. From 1) and $1')$ we have $L_4(1 - a' \cdot \widehat{z} - b'\widehat{\bar{z}}) = 0$; from 2) and $2')$ we have $L_4(\widehat{\bar{z}}(1 - a' \cdot \widehat{z} - b'\widehat{\bar{z}})) = 0$; from 3) and $3')$ we have $L_4(\widehat{z}(1 - a' \cdot \widehat{z} - b'\widehat{\bar{z}})) = 0$. Consequently, $L_4((1 - a' \cdot \widehat{z} - b' \cdot \widehat{\bar{z}})\overline{(1 - a' \cdot \widehat{z} - b' \cdot \widehat{\bar{z}})}) = 0,\ a' \cdot b' \neq 0$; that is a contradiction with $\{1,\ \widehat{z},\ \widehat{\bar{z}}\}$ are linear independent in H_1.

We have $\operatorname{rank} A_1 = 3$; A_1 is invertible. Let $\widehat{z^2} = a \cdot 1 + b \cdot \widehat{z} + c \cdot \widehat{\bar{z}}$ that is 4) $L_4(\widehat{z^2} - (a \cdot 1 + b \cdot \widehat{z} + c \cdot \widehat{\bar{z}})) = 0$, 5) $L_4(\widehat{z}(\widehat{z^2} - (a \cdot 1 + b \cdot \widehat{z} + c \cdot \widehat{\bar{z}})) = 0$ and 6) $L_4(\widehat{\bar{z}}(\widehat{z^2} - (a \cdot 1 + b \cdot \widehat{z} + c \cdot \widehat{\bar{z}})) = 0$. If A_1 is invertible, we find the unique solution $a,\ b,\ c \in C$, of the system 4), 5), 6) with $a,\ b,\ c$ not simultaneous 0. In the same way we find $\alpha,\ \beta,\ \delta \in C,\ \alpha,\ \beta,\ \delta$ not simultaneous 0, such that $\widehat{\bar{z}^2} = \alpha \cdot 1 + \beta \cdot \widehat{z} + \delta \cdot \widehat{\bar{z}}$. If $z\bar{z} = 1$, we have $\dim H_2 = \dim H_1 = 3$, consequently L_4 is stable at $(n-1) = 1$. When the C^* commuting algebra X is isomorphic with H_2, X has three characters, the scalar, positive measure $\mu(*) =< E_A(*)(1+T_2),\ (1+T_2) >\geq 0$ has three atoms, respectively $\mu(*) = \rho_0\delta_{w_0} + \rho_1\delta_{w_1} + \rho_2\delta_{w_2}$. If $\mu(w_i) \geq 0$, the weights $\rho_i \geq 0,\ i \in \{0,\ 1,\ 2\}$, $\mu(*)$ is a representing atomic measure for γ with three atoms. □

We have proved the following:

Theorem 3.1 Let $L_4 : C_4[z,\ \bar{z}]/I_4 \to C$ be a linear functional with properties (2-4), $\{H_k\}_{k=0}^{k=2}$ the Hilbert spaces associated with it, A_1 the matrix associated with it on H_1. For the quadratic moment problem on T_1 are equivalent:

i) γ has an $s = r$ atomic representing measure with $s = sd(L_4) = \dim H_1$, the stable dimension of L_4 and $r = rankA_1$.

ii) γ has a representing measure.

iii) The non-negative functional L_4 is stable at $(n-1) = 1$. If $\dim H_1 = s_i = r_i = rankA_1$, the representing measure has $s_i = r_i,\ i \in \{1,\ 2,\ 3\}$ atoms. □

4. Conclusions

The present paper presents a truncated and a full trigonometric moment problem. It gives in the main theorems conditions such that a finite complex sequence and a full complex sequence to admit integral representations as moments with respect to an atomic, positive defined measures on the unit circle.

The conditions that allow the sequences to be moments of an atomic positive measure on the unit circle are: 1) the positivity of the given sequences and 2) the stability of the assignment “Riesz functional”. The technical construction of the spaces on which the Riesz functional acts assures the representing measure to have the support on the unit circle.

References

[1] Aryal S, Jafari F, Putinar M. Sparse moment sequences. *arXiv: 1203: 5859v2*. 2012.

[2] Curto R, Fialkow LA. *Solution of Truncated Complex Moment Problem for Flat Data*. Memoirs of the American Mathematical Society; 1996.

[3] Helton JW, Putinar M. *Positive Polynomials in Scalar and Matrix Variables, the Spectral Theorem, and Optimization*. Operator Theory, Structured Matrices, and Dilations, Theta; 2007.

[4] Lemnete-Ninulescu L. Stability in truncated trigonometric scalar moment problems. *Complex Analysis and Operator Theory.* 2023; 17(5): 75.

[5] Vasilescu FH. Dimensional stability in truncated moment problems. *Journal of Mathematical Analysis and Applications*. 2012; 388(1): 219-230.