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Research Article

Estimation of Tensile Strength of Carbon Fibers Using Exponentiated Exponential Weibull-Dagum Distribution Model (EEWD) and Its Properties

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Abstract: The tensile strength of Carbon fibers was investigated. The ability of the tested EEWD distribution to fit the data was evaluated, and the skewness of the data was observed. The T-R{.} framework has been recently used to generalize various distributions, but the viability of using Dagum distribution has not been investigated. Three distributions are combined in the T-R{.}, with one serving as a baseline distribution. The combined potency of each distribution, which is a weighted hazard function of the baseline distribution, would have more parameters but would also be highly flexible in handling bimodality in datasets. Therefore, the quantile function of the Weibull distribution was used to generalize the Dagum distribution. In this present research work a novel generalized 6P (six parameter) model called Exponentiated Exponential Weibull Dagum Distribution (EEWD) has been introduced. The appropriate distributions including PDF, CDF, moments, Moment Generating Function (MGF) of EEWD distribution, stochastic ordering, Cumulant Generating Function (CGF) of EEWD distribution, mean, mean deviations and their sub-models have been Furthermore, the EEWD model has been applied to admissible real-time data.

***Keywords*:** tensile strength, carbon fiber, dagum distribution, series expansion, moment generating function (MGF), cumulant generating function (CGF), stochastic ordering, skewness, kurtosis

MSC: 62E10, 62N05, 62P30, 62E15, 60E05, 62F03,62E10,62H05

Abbreviation

PDF	Probability Density Function
CDF	Cumulative Distribution Function
DGM	Dagum
QntF	Quantile Function
SF	Survival Function
HF	Hazard Function
RV	Random Variable
RS	Random Sample

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1. Introduction

Carbon fibers are made by pyrolyzing a suitable precursor fiber and contain at least 90% carbon by weight. Graphite is one kind of carbon. Graphite has sp2 hybridized carbon atoms in two-dimensional hexagonal planes. Carbon atoms bind differently in-plane and out-of-plane, making graphitic planes extremely anisotropic [1]. The elastic modulus is larger in planes than perpendicular to them. Graphitic planes have Van der Waals bonding, allowing them to glide against one another. Graphitic planes aligned parallel to the fiber axis exhibit high tensile modulus, as well as electrical and thermal conductivity [2–4]. The present work focuses on a new deduction of the DGM (Dagum distribution), called T-DGM with family Y of members, and the six-parameter Dagum Weibull (EEWD) exponential distribution has been proposed. The quality of statistical model outputs is significantly influenced by the reliability of the probability distribution that is pretended to be present in the data. Many probabilistic distribution families and the associated statistical techniques have thus been created with great effort. But there are a lot of serious, real-world issues, especially in the fields of engineering, finance, and medicine, were current standards or newly developed distributions. In the model of income and wealth distribution ere data is not well accommodated by current standards or newly developed distributions. In the model of income and wealth distribution, the DGM [5] distribution holds great significance, particularly when examining the distribution of personal income. This proposed distribution will consider not only the high tolerance of the shapes and the scale parameters, tail variation, kurtosis, and skewness (right and left) and may be consistent for various parameter values. The motivation for this work is to claim the DGM which can be better and appropriate distributions for income and wealth modelling. The T-R. [6] structure allows for generalization. The T-R. structure associates the three different distributions T, R, and Y, with the QntF (quantile function) of Y serving as a framework for maintaining R's cdf generated by T, and the parameters of the entire distribution manipulating the newly framed distribution. One of the most significant benefits of contributing a new framed distribution through the quantification event, the new distribution is more flexible in dealing with datasets; it can be a WHF (Weighted Hazard Function) of the ordinary spread, which is the DGM. Each distribution's parameters influence the newly framed distribution. Although this is usually the most suitable, there are other three- parameter distributions that are also used to simulate the distribution of income. Beta-DGM distribution [7], Mc-DGM distribution [8], weighted DGM distribution [9], gamma DGM distribution [10], exponentiated Kumaraswamy DGM distribution [11] and extended DGM distribution [12] were the intended DGM distributions. They said that for the most part, using our parameter distributions was adequate and that, to model real data, at least three parameters were required. However, they doubted that adding the fifth or sixth parameter would result in meaningful improvement. But we were inspired by the work [13], so we dispersed the six parameters. Compared to its sub-models with fewer parameters, the distribution of its six parameters was superior. Additionally, other authors proved Johnson [14] et al. wrong demonstrating that a sub-model with fewer parameters has less flexibility in survival data and less model reliability than the one with more parameters. [15] Their claims were untrue. Despite the reality that the Dagum distribution is among the most significant and suitable distributions for modeling wealth and income, there is another reason for this work. The three distributions T, R and Y are combined to form the T-R. framework. R's CDF is generated by T and the parameters of each distribution influence a newly designed distribution. The quantile function of Y is used as a framework for maintaining R. A new distribution is provided through the quantification function of the existing distribution, and two of its main advantages are that it can handle bimodality in datasets and that it is a weighted hazard function of the baseline distribution in this case, the DGM distribution.

2. T-DGM {Y} class

Nicholas Eugene, Carl Lee and Felix Famoye in [16] generated the beta family and expressed [17] the T-X$\{.\}$ family and it was generalized in a novel technique for generating domestic continuous distributions by Ayman Alzaatreh [18]. Felix Famoye. Carl Lee. [19] to the T-X{ф} family further expanded in [5] to the T-X{ф} by defining $\phi\left\{F_X(x)\right\}$ as a QntF of an RV (Random Variable) Y and described as the T-X$\{.\}$ domestic [20].

$$GG(x) = \int_a^{Q_Y F(x)} r(t)dt = R\{Q_Y[F(x)]\}. \tag{1}$$

T-X{ф} [5] determined by Alzaatreh et al. They defined the T-R$\{.\}$ domestic. The CDF of the T-R$\{.\}$ group is expressed as:

$$F_X(x) = \int_a^{Q_Y F(x)} f(t)dt = R\{Q_y(F(x))\} = F_T\{Q_Y[F_R(x)]\}. \tag{2}$$

The PDF of the RV. T is denoted by $f_T(t)$ and $F_R(x)$ is the CDF of the RV. R, then the RV. Y has the quantile function $Q_Y[F_R(x)]$ can be distinguished and monotonically increased. It is required that $f_T(t)$ and $f_R(x)$ have the identical sustain. The pdf of EEWD distribution related to the cdf of EEWD distribution and the hazard function of EEWD distribution is determined by:

$$f_X(x) = f_R(x)\frac{f_T\{Q_Y[F_R(x)]\}}{f_Y\{Q_Y[F_R(x)]\}},$$

$$h_X(x) = h_R(x)\frac{h_T\{Q_Y[F_R(x)]\}}{h_Y\{Q_Y[F_R(x)]\}}. \tag{3}$$

Let R be a RV that complies DGM with three parameters then CDF of DGM is derived by:

$$F_R(x) = \left(1+\lambda x^{-\delta}\right)^{-\beta}. \tag{4}$$

Here the scale parameter $\lambda > 0$, the location parameter $\delta > 0$ shape parameter $\beta > 0$. T-DGM$\{$Y$\}$.
Family is established by applying eqn. (4) in (2) then The CDF defined of T-DGM$\{$Y$\}$:

$$F_X(x) = F_T\left\{Q_Y\left(1+\lambda x^{-\delta}\right)^{-\beta}\right\}. \tag{5}$$

Equation (5) is the CDF of proposed T-DGM$\{$Y$\}$ class distribution.
Let the dagum probability density function (DPDF):

$$f_R(x) = \beta\lambda\delta x^{-(\delta+1)}\left(1+\lambda x^{-\delta}\right)^{-\beta+1}. \tag{6}$$

From equation (3) the PDF of proposed class distribution defined by:

$$f_X(x) = \beta\lambda\delta x^{-(\delta+1)}\left(1+\lambda x^{-\delta}\right)^{-\beta+1}\frac{f_T\left\{Q_Y\left[\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]\right\}}{f_Y\left\{Q_Y\left[\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]\right\}}. \tag{7}$$

Remark 1 Let X be a RV follows T-DGM$\{.\}$ class of distributions provided by [21]. Eq. (2), then it can be observed that:

(i) $X^d = b\left\{[F_Y(T)]^{1/s} - 1\right\}^{-1/t}$;

(ii) $Qnt_X(s) = b\left\{F_Y\left(Qnt_X(s)\right\}^{-1/s}\right) - 1^{-1/t}$;

(iii) If $T^d = Y$ then $X^d = \text{Exponential}(b, s, t)$; and

(iv) If $Y^d = \text{Exponential}(\text{b, s, t})$ then $X^d = T$.

The T-E {Y} class in Eq. (5) can generate many different extended exponential families. Some generalized exponential families using some $Qnt_Y(.)$ are defined. Many authors including Alzatraah et al. [5, 22]; Nasir et al. [21, 23]; Jamal et al. [24, 25]; Aljarrah et al. [5]; Jamal et al. [26]; Famoye et al. [16, 17, 25–32]; and Jamal and Nasir [27] have developed probability distributions using this framework [33–37]. None of these authors have applied this framework to a generalization of the Dagum distribution.

Let Y be an RV with WPDF (Weibull Probability Density Function) derived by:

$$f_Y(x) = \frac{\kappa}{\lambda^\kappa} x^{\kappa-1} e^{-(x/\lambda)^\kappa}, \tag{8}$$

where $\lambda > 0,\ \kappa > 0$ are the scale parameter, shape parameter, respectively.

Weibull Cumulative Distribution Function (WCDF) given by:

$$F_Y(x) = 1 - e^{-(x/\lambda)^\kappa}. \tag{9}$$

The Weibull Distribution Quantile Function (WQF) is given by:

$$Q_Y(p,\ \kappa,\ \lambda) = \lambda(-\ln(1-p))^{1/\kappa}, \tag{10}$$

where $\lambda > 0,\ \kappa > 0$ are the scale and the shape parameter, respectively.

From equation (5) and (7),

$$F_X(x) = F_T\left\{Q_Y\left(1 + \lambda x^{-\delta}\right)^{-\beta}\right\}, \tag{11}$$

$$F_X(x) = F_T\left\{\lambda\left(-\ln\left[1 - \left(1 + \lambda x^{-\delta}\right)^{-\beta}\right]\right)^{1/k}\right\}, \tag{12}$$

$$f_X(x) = \beta\lambda\delta x^{-(\delta+1)}\left(1 + \lambda x^{-\delta}\right)^{-\beta+1} \frac{f_T\left\{\lambda\left(-\ln\left[1 - \left(1 + \lambda x^{-\delta}\right)^{-\beta}\right]\right)^{1/k}\right\}}{f_Y\left\{\lambda\left(-\ln\left[1 - \left(1 + \lambda x^{-\delta}\right)^{-\beta}\right]\right)^{1/\kappa}\right\}}. \tag{13}$$

The Weibull quantile function is used to define the CDF for the T-DGM (Weibull) class of distributions in (12). A novel method of generalizing the Dagum distribution is by using equation (13). Therefore, T supported by any [0, ∞) univariate probability distribution.

3. Properties

Some properties of T-D {Weibull} class of distribution. Some general properties of the T-D {Weibull} class are discussed in this section.

Lemma 1 Given any random variable T with pdf $f_T(x)$, then the random variable.

$Qnt_X(.) = \left\{ b\left[1-(1+\text{ Qnt }_X(p)/\theta)^{-\gamma}\right]^{-1/p} \right\}^{-1/q}$ follows T-Dagum {Weibull} distribution in equation.

Proof. The result of Remark 1(i) can be easily seen. Lemma 1 shows the relationship between the random variables X and T. The random variable X can be generated by using these relationships from the random variable T. For example, if a random variable T is a standard random variable with a known quantile function, it can simulate the random variable X by first simulating the T value.

Lemma 2 The Quantile function for the T-D {Weibull} distribution is given by:

$$X = \left[b\left\{ 1-\left(1+\frac{Q_T(P)}{\theta}\right)^{-\gamma} \right\}^{-\frac{1}{p}} - 1 \right]^{-1/q}.$$

The result of Remark 1 (ii) can easily be seen.

Claude E. Shannon invented the concept of entropy, a measure of Uncertainity, in the context of information theory. Entropy, which deals with the collection of events and self-entropy, which is connected to a single event, are two related quantities that can be distinguished in this context. There is no Uncertainty surrounding the event, so the entropy of the system is zero. The Shannon entropy is given by $'\mathrm{H}_X(x) = -\sum_{i=1}^{n} p_i \cdot \ln p_i$.

Theorem 1 T-DGM {Weibull} distribution Shannon's entropy (ShEn) can be stated as:

$$\eta_x = \eta_T + E\{\ln[f_Y(T)]\} + \ln\beta + \ln\lambda + \ln\delta - (\delta+1)E\{\ln x\} - (\beta+1)E\left\{\ln\left(1+\lambda x^{-\delta}\right)\right\}.$$

Proof. Since Lemma 1 (i) $X^d = b\left\{[F_Y(T)]^{1/p} - 1\right\}^{-1/q}$.

It follows that $t = Q_Y(F_R(x))$.

Hence, based on the PDF in the equation $f_X(x) = f_R(x)\dfrac{f_T\{Q_Y(F_R(x)\}}{f_Y\{Q_Y(F_R(x)\}}$.

This implies that $\eta_x = \eta_T + E\{\ln[f_Y(T)]\} + E\{\ln[(f_R(x)]\}$,

$$\eta_x = \eta_T + E\{\ln[f_Y(T)]\} + \ln\beta + \ln\lambda + \ln\delta - (\delta+1)E\{\ln x\} - (\beta+1)E\left\{\ln\left(1+\lambda x^{-\delta}\right)\right\},$$

Thus, the ShEn of the newly framed T-DGM {Weibull} class of distribution.

4. EEWD exponentiated exponential weibull dagum distribution with six parameter

4.1 *CDF of EEWD distribution (EEWDCDF)*

The PDF of an exponential distribution was described by Gupta and Kundu [28] as:

$$f_T(x) = \theta\lambda e^{-\eta x}\left(1-e^{-\eta x}\right)^{\theta-1}. \tag{14}$$

$$F_T(x) = \left(1 - e^{-\eta x}\right)^{\theta}. \tag{15}$$

Substituting equations (14) and (15) in (12) to have

$$F_X(x) = \left\{1 - \exp\left\{\eta\lambda\left\{\ln\left(1 - \left[1 + \lambda x^{-\delta}\right]^{-(\beta)}\right)^{\frac{1}{\kappa}}\right\}\right\}\right\}^{\theta}, \tag{16}$$

$$F_X(x) = \left\{1 - \left(1 - \left[1 + \lambda x^{-\delta}\right]^{-(\beta)}\right)^{\frac{\alpha}{\kappa}}\right\}^{\theta}. \tag{17}$$

Thus, eqn. (17) is the CDF of the novel framed probability distribution function which is entitled the EEWD Distribution, here $\alpha,\ \beta,\ \kappa,\ \lambda,\ \delta$ and θ are non negative parameters; with the shape parameters $\beta,\ \kappa > 0,\ \alpha$ defines such as (skewness, kurtosis, and mode), and the scale parameter λ which states the extend of the distribution, $\delta > 0$ is a location parameter and the tail variation parameter θ.

4.2 *PDF of EEWD distribution (EEWDPDF)*

Equation (17) can be differentiated with esteem to x. To obtain EEWDPDF (the PDF of the new EEWD distribution), or it can be solved by directly substituting (8) and (14) into equation (10),

$$f_X(x) = \frac{\alpha\beta\theta\lambda\delta}{\kappa} x^{-(\delta+1)}\left[1 + \lambda x^{-\delta}\right]^{-(\beta+1)}\left(1 - \left[1 + \lambda x^{-\delta}\right]^{-(\beta)}\right)^{\frac{\alpha}{\kappa}1}\left\{1 - \left(1 - \left[1 + \lambda x^{-\delta}\right]^{-(\beta)}\right)^{\frac{\alpha}{\kappa}}\right\}^{\theta-1}. \tag{18}$$

Thus eqn. (18) is the PDF of the innovative framed probability distribution EEWD, called EEWDPDF. Any survival rate, environmental hazard, and failure time data parameters will be well-modeled by the sixparameter distribution. Here $\eta\lambda$ is constant which can be replaced by α without loss of generality, apply the series expansion,

$$(1-\phi)^{\tau-1} = \sum_{j=0}^{\infty} \frac{(-1)^j r(\tau)}{r(\tau-j)j!}\phi^j, \quad \text{for } \tau > 0 \text{ and } /\phi/ < 1, \tag{19}$$

$$f_X(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty} w(i,\ j,\ r) x^{-(\delta+1)}\left[1 + \lambda x^{-\delta}\right]^{-\beta(j+r+1)-1}, \tag{20}$$

where $w(i,\ j,\ r) = \frac{\alpha\beta\theta\lambda\delta}{\kappa}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^{i+j}\Gamma\left(\frac{\alpha}{\kappa}+r-1\right)r\left(\frac{\alpha}{\kappa}\right)r(\theta)}{r\left(\frac{\alpha}{\kappa}-i\right)r(\theta-j)j!r!}$ here the shape parameters $\beta,\ \kappa$, the scale parameter λ and θ the tail variation parameter.

If

$$1-\left[1+\lambda x^{-\delta}\right]^{-\beta}=\omega, \tag{21}$$

in eqn. (18) then,

$$f_X(x)=\frac{\alpha\beta\theta\lambda\delta}{\kappa}x^{-(\delta+1)}\omega^{\frac{\alpha}{k}-1}\left(1-\omega^{\frac{\alpha}{\kappa}}\right)^{\theta-1}(1-\omega)^{\left(1+\frac{1}{\beta}\right)}. \tag{22}$$

Then the CDF of EEWD function is

$$F_X(x)=\left\{1-(\omega)^{\frac{\alpha}{\kappa}}\right\}^{\theta}, \tag{23}$$

where ω is a function of x.

Figure 1 indicates the Probability density function of EEWD distribution for diverse parameters and indicates how the distribution may be stable, skewed positively or negatively. The Cumulative distribution function of EEWD distribution is depicted in Figure 2 as being normal, unimodal, or bimodal, positively skewed or negatively skewed. Any data containing environmental hazards or having a high degree of variability benefits from the behavior of the EEWD model.

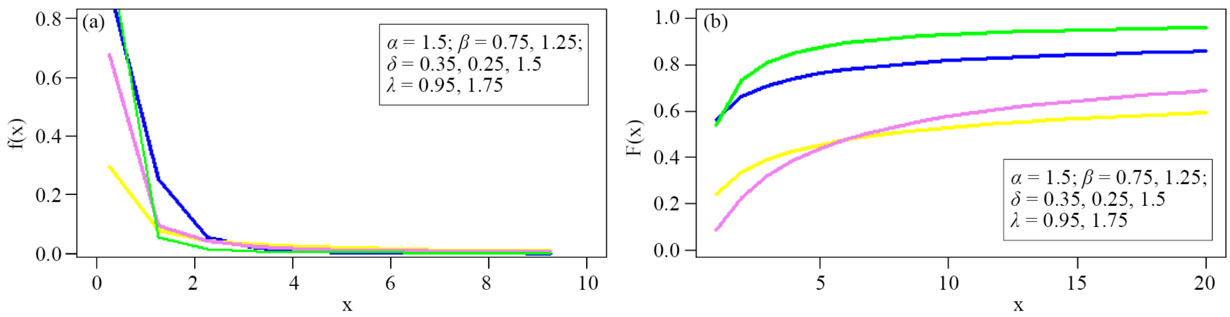

Figure 1. (a) PDF of EEWD distribution (WDPDF) with different parameter (b) CDF of EEWD distribution (WDCDF) with different parameter

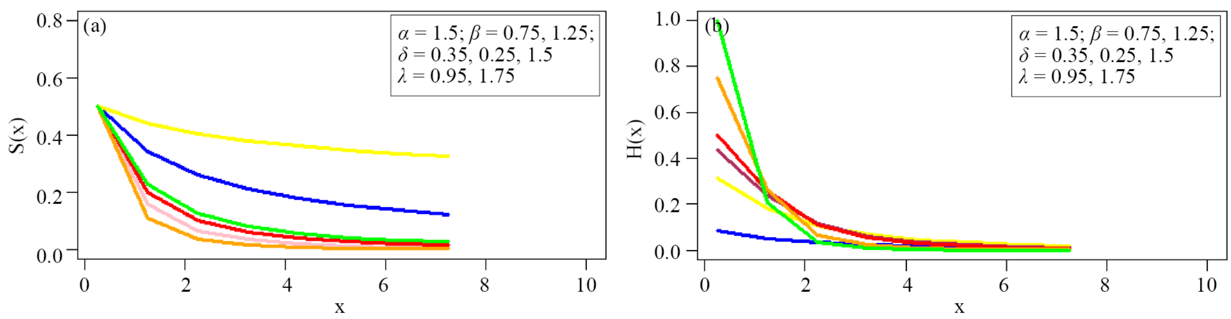

Figure 2. (a) Survival Function of EEWD distribution with various parameters (b) Hazard Function of EEWD distribution with various parameter

4.3 *Stochastic ordering theorem of EEWD distribution*

Definition Let X and Y be univariate RV over the distributives F_1 and F_2, survival functions $\overline{F_1}$ and $\overline{F_2}$, density functions f_1 and f_2; and hazard rates $r_{F_1}\left(=\frac{f_1}{\overline{F1}}\right)$ and $r_{F_2}\left(=\frac{f_2}{\overline{F_2}}\right)$ respectively [29]. X is said to be stochastically smaller than Y denoted by $(X \leq Y)$ is $\left(\overline{F_1}(x) \leq \overline{F_2}(x)\right)$ ∀x. This is equivalent to saying that $E(X) \leq E(Y)$ for any increasing function for which expectations exist [29–31].

4.4 *EEWDGoS (general order statistics of EEWD) distribution*

Theorem 1 The EEWDGoS distribution with PDF $f_{X_r}(x)$ is obtained by [31],

$$f_{X_r}(x) = \frac{n!\alpha\beta\theta\lambda\delta}{(r-1)!(n-r)!\kappa}\left[\frac{\left(\omega^{\frac{\eta\lambda}{\kappa}-1}\right)\left(1-\left(\omega^{\frac{\eta\lambda}{\kappa}-1}\right)^{\theta r-1}\right)\Big]\left[1-\left(1-\left(\omega^{\frac{\eta\lambda}{\kappa}-1}\right)^{\theta}\right)\right]^{n-r}}{x^{(\delta+1)}}\right].$$

Proof. Let $X_1,\ X_2,\ X_3,\ \ldots\ldots X_r$ be a continuous RS which follows the EEWD distribution, with $F_X(x)$ and $f_X(x)$ [31]. Then the PDF X_r is

$$f_{X_r}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(\mathrm{x})\left[F_X(\mathrm{x})\right]^{r-1}\left[1-F_X(\mathrm{x})\right]^{n-r}, \tag{24}$$

$$f_{X_r}(x) = \frac{n!\alpha\beta\theta\lambda\delta}{(r-1)!(n-r)!k}\left[\frac{\left[\left(\omega^{\frac{\alpha}{k}-1}\right)\left(1-\omega^{\frac{\alpha}{k}-1}\right)^{-(\theta r+1)}(1-\omega)^{\left(1+\frac{1}{\beta}\right)}\right]\left[1-\left(1-\omega^{\frac{\alpha}{k}}\right)^{\theta}\right]^{n-r}}{x^{(\delta+1)}}\right], \tag{25}$$

$$f_{X_r}(x) = \frac{n!\alpha\beta\theta\lambda\delta}{(r-1)!(n-r)!\kappa}\left[\frac{\left[\left(\omega^{\frac{\alpha}{k}-1}\right)(1-\omega)^{\left(1+\frac{1}{\beta}\right)}\right]\left[1-\left(1-\omega^{\frac{\alpha}{k}}\right)^{\theta}\right]^{n-r}}{x^{(\delta+1)}\left(1-\omega^{\frac{\alpha}{k}-1}\right)^{(\theta r+1)}}\right]. \tag{26}$$

Eqn. (26) completes the proof of EEWDGoS with PDF. Let X and Y be two RV which follows EEWD distribution. $X < Y$ if $P(X > x) \leq P(Y > y)$, $\forall x \in (-\infty,\ \infty)$, where P(.) represents the probability function.

Theorem 2 Let Y_1 and Y_2 be a RV's which follows the EEWD distribution. If $Y_1 \leq Y_2$ and $E(Y_1) = E(Y_2)$, then $Y^d \leq Y$ [31].

Proof. The PDF of the EEWDGoS is derived by eqn. (26) set $r = 1$ to get the first order statistics is given by:

$$f_{Y_1}(x) = \frac{n\alpha\beta\theta\lambda\delta}{\kappa}\left[\frac{\left[\left(\omega^{\frac{\alpha}{k}-1}\right)(1-\omega)^{\left(1+\frac{1}{\beta}\right)}\right]\left[1-\left(1-\omega^{\frac{\alpha}{k}}\right)^{\theta}\right]^{n-1}}{x^{(\delta+1)}\left(1-\omega^{\frac{\alpha}{k}-1}\right)^{(\theta+1)}}\right], \tag{27}$$

also set $r = 2$ to get second order statistics given by:

$$f_{Y_1}(x) = \frac{n(n-1)\alpha\beta\theta\lambda\delta}{\kappa}\left[\frac{\left[\left(\omega^{\frac{\alpha}{k}-1}\right)(1-\omega)^{\left(1+\frac{1}{\beta}\right)}\right]\left[1-\left(1-\omega^{\frac{\alpha}{k}}\right)^{\theta}\right]^{n-2}}{x^{(\delta+1)}\left(1-\omega^{\frac{\alpha}{k}-1}\right)^{(2\theta+1)}}\right]. \tag{28}$$

$Y_1 \leq Y_2$ show that $E(Y_1) = E(Y_2)$: using series expansion $(1-Z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^k r(b)}{r(b-k)k!} Z^k$, for $b > 0$ and $|Z| < 1$ the above inequality expanded as:

$$\sum_{i,\ k=0}^{\infty} (-1)^{i+k} \binom{n-1}{i} \binom{\theta i+\theta-1}{k} \left(\omega^{\frac{\alpha}{k}}\right)^{k+1} \leq \sum_{i,\ k=0}^{\infty} (-1)^{i+k} \binom{n-2}{i} \binom{\theta i+2\theta-1}{k} \left(\omega^{\frac{\alpha}{k}}\right)^{k+1}, \tag{29}$$

take expectation on both sides to have:

$$E\left[\sum_{i,\ k=0}^{\infty} (-1)^{i+k} \binom{n-1}{i} \binom{\theta i+\theta-1}{k}\right] \leq E\left[\sum_{i,\ k=0}^{\infty} (-1)^{i+k} \binom{n-2}{i} \binom{\theta i+2\theta-1}{k}\right]. \tag{30}$$

$E(c) = c$, (constant) test the equality $i = k = 0$ to get

$$E\left[\sum_{i,\ k=0}^{\infty} (-1)^{i+k} \binom{n-1}{i} \binom{\theta i+\theta-1}{k}\right] = 1 \text{ and } E\left[\sum_{i,\ k=0}^{\infty} (-1)^{i+k} \binom{n-2}{i} \binom{\theta i+2\theta-1}{k}\right] = 1. \tag{31}$$

Eqn. (31) completes the proof as a result Y_1 and Y_2 are random samples drawn from the EEWD distribution.

5. Moments of EEWD distribution

The moments of EEWD distributions are one of the valuable properties in describing a distribution. It can be used to derive the mean of EEWD distribution, the variance of EEWD distribution, the standard deviation of EEWD distribution, a measure of skewness, kurtosis of EEWD distribution and other parametric measures that specified the distribution.

Let

$$h = \left[1+\lambda x^{-\delta}\right]^{-1}, \tag{32}$$

$$E\left(x^h\right) = \int_0^{\infty} x^h f_x(x)dx,$$

$$E\left(x^h\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty} w(i,\ j,\ r) \int_0^{\infty} x^{h-\delta-1)} \left[1+\lambda x^{-\delta}\right]^{-\beta(j+r+1)-1} dx. \tag{33}$$

By using Beta function transformation formula

$$B(p,\ q)=\int_0^{\infty}\frac{x^{(q-1)}}{(1+x)^{p+q}}dx,$$

$$E\left(x^h\right)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}w(i,\ j,\ r)B\left(\beta(r+j+1)-1+\frac{h-1}{\delta},\ 2+\frac{1-h}{\delta}\right),\ h<\delta,$$

$$\text{(i.e.) } w(i,\ j,\ r)=\frac{\alpha\beta\theta\delta}{\kappa}(\lambda)^{\frac{h-1}{\delta}}\frac{(-1)^{i+j}\Gamma\left(\frac{\alpha}{k}+r-1\right)r\left(\frac{\alpha}{k}\right)r(\theta)}{r\left(\frac{\alpha}{\kappa}-i\right)r(\theta-j)j!r!}. \tag{34}$$

6. MGF of EEWD distribution

The EEWDMGF (Moment Generating Function of EEWD) distribution is provided by

$$M_x(g)=E\left(e^{gx}\right)=\int_0^{\infty}e^{gx}f_x(x)dx, \tag{35}$$

$$M_x(g)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}w(i,\ j,\ r,\ k)\frac{t^k}{k!}B\left(\beta(j+r+1)-1+\frac{g-1}{\delta},\ 2+\frac{1-g}{\delta}\right),\ k<\delta,$$

where

$$w(i,\ j,\ r)=\frac{\alpha\beta\theta\delta}{\kappa}(\lambda)^{\frac{g-1}{\delta}}\frac{(-1)^{i+j}\Gamma\left(\frac{\alpha}{\kappa}+r-1\right)\Gamma\left(\frac{\alpha}{\kappa}\right)\Gamma(\theta)}{\Gamma\left(\frac{\alpha}{\kappa}-i\right)\Gamma(\theta-j)j!r!}. \tag{36}$$

7. Mean deviation and median deviations of EEWD distribution

Let X be a RV, which follows the EEWD distribution, then the MD (Mean Deviation) with the mean $\mu_w=E(x)$ and the MDD (Median Deviation) with the median μ_w from $\delta_1=\int_0^{\infty}/x-\mu_w/f_x(x)dx$ and $\delta_2=\int_0^{\infty}/x-M_w/f_x(x)dx$ can be derived respectively. The mean μ_w can be obtained from $E\left(x^t\right)$ with $t=1$ and the median M_w is given by the equation of Quantile function with $p=\frac{1}{2}$, where $Qnt_x(p)=\lambda^{1/\delta}\left[\left\{\left(1-p^{1/\theta}\right)^{\kappa/\alpha}-1\right\}^{-1/\beta}-1\right]^{-1/\delta}$.

8. Cumulant generating function of EEWD distribution

The CGF of EEWD distribution is defined by

$$K_x(t)=\ln M_x(t), \tag{37}$$

$$K_x(t)=\ln\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}w(i,\ j,\ r,\ k)\frac{t^k}{k!}B\left(\beta(r+j+1)-1+\frac{g-1}{\delta},\ 2+\frac{1-g}{\delta}\right)\right\},\ k<\delta,$$

where

$$w(i,\ j,\ r,\ k)=\frac{\alpha\beta\theta\delta}{\kappa}(\lambda)^{\frac{s-1}{\delta}-1}\frac{(-1)^{i+j}\Gamma\left(\frac{\alpha}{k}+r-1\right)r\left(\frac{\alpha}{\kappa}\right)r(\theta)}{r\left(\frac{\alpha}{K}i\right)r(\theta-j)j!r!}. \tag{38}$$

9. Central moment of EEWD distribution

Let $\delta>0$ then, the δ^{th} moment α_δ of the EEWD distribution is derived by,

$$\alpha_\delta=E\left(X^\delta\right)=\int_\gamma^\infty x^\delta f_X(x)dx, \tag{39}$$

$$\alpha_\delta=\int_\gamma^\infty x^\delta\frac{\alpha\beta\theta\lambda\delta}{\kappa}x^{-(\delta+1)}\left[1+\lambda x^{-\delta}\right]^{-(\beta+1)}\left(1-\left[1+\lambda x^{-\delta}\right]^{-(\beta)}\right)^{\frac{\alpha}{\kappa}-1},$$

$$\left\{1-\left(1-\left[1+\lambda x^{-\delta}\right]^{-(\beta)}\right)^{\frac{\alpha}{\kappa}}\right\}^{\theta-1}dx.$$

Let

$$x=\left\{\frac{1}{\lambda}\left[(1-\omega)^{-1/\beta}-1\right]\right\}^{-1/\delta}, \tag{40}$$

$$1-\left[1+\lambda x^{-\delta}\right]^{-(\beta)}=\omega, \tag{41}$$

$$dx=\frac{\left(x^{(\delta+1)}\right)d\omega}{(-\beta\lambda\delta)\left[1+\lambda x^{-\delta}\right]^{-(\beta+1)}}, \tag{42}$$

$$\alpha_\delta=E\left(X^\delta\right)=\int_\gamma^\infty x^\delta\frac{\alpha\beta\theta\lambda\delta}{\kappa}x^{-(\delta+1)}\omega^{\frac{\alpha}{\kappa}-1}\left(1-\omega^{\frac{\alpha}{\kappa}}\right)^{\theta-1}(1-\omega)^{\left(\frac{1}{\beta}+1\right)}\frac{\left(x^{(\delta+1)}\right)d\omega}{(-\beta\lambda\delta)[1-\omega]^{\left(\frac{1}{\beta}+1\right)}}. \tag{43}$$

Using binomial expansion,

$$\alpha_\delta=E\left(x^\delta\right)=\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{r=0}^\infty w(i,\ j,\ r)B\left(\beta(j+r+1)-1+\frac{\delta-1}{\delta},\ 2+\frac{1-\delta}{\delta}\right),\ k<\delta. \tag{44}$$

where $w(i,\ j,\ r) = \dfrac{\alpha\beta\theta\lambda\delta}{\kappa}(\lambda)^{\left(\frac{(\delta-1)}{\delta}-1\right)} \dfrac{(-1)^{i+j}\Gamma\left(\frac{\alpha}{\kappa}+r-1\right)\Gamma\left(\frac{\alpha}{K}\right)\Gamma(\theta)}{\Gamma\left(\frac{\alpha}{\kappa}-i\right)\Gamma(\theta-j)j!r!}$, $\alpha,\ \beta,\ \theta,\ \lambda,\ \delta > 0$ and $0 \le r \le \delta,\ \delta > 0$,

$$\alpha_1 = E\left(X^\delta\right) \text{ with } \delta = 1.$$

9.1 δ^{th} central moment

The δ^{th} central moment of EEWD distribution can be easily derived as follows.

$$\beta_\delta = E[X-E(x)]^\delta = \int_r^\infty \left[E[X-E(x)]^\delta\right] f_X(x)dx,$$

$$\beta_\delta = \sum_{t=0}^{\delta}(-1)^t\binom{\delta}{t}E(X)^t E(X)^{\delta-t}, \tag{45}$$

where $E(X)^t$ and $E(X)^{\delta-t}$ can be attained from the equation (44).

9.2 Mean, variance, coefficient of skewness and kurtosis of EEWD distribution

Mean: set $\delta = 1$ in (44) then mean $= \alpha_1 = E(X)$ can be easily attained.
Variance: set $\delta = 2$ in (45) then,

$$\beta_2 = E[X-E(X)]^2 = \int_\gamma^\infty [X-E(X)]^2 f_X(x)dx = E(X)^2 - [E(X)]^2. \tag{46}$$

9.3 Coefficient of skewness and kurtosis of EEWD distribution

By set $\delta = 3$ and $\delta = 4$ in (45) can be obtained central moments of 3^{rd} and 4^{th} respectively [16, 17, 32].
That is

$$\beta_3 = E[X-E(X)]^3 = \sum_{t=0}^{3}(-1)^t\binom{3}{t}E(X)^t E(X)^{3-t}, \tag{47}$$

$$\beta_4 = E[X-E(X)]^4 = \sum_{t=0}^{4}(-1)^t\binom{4}{t}E(X)^4 E(X)^{4-t}. \tag{48}$$

Using (47) and (48) the measures of skewness γ_1 and kurtosis γ_2 are respectively given by

$$\gamma_1 = \frac{\sum_{t=0}^{3}(-1)^t \binom{3}{t} E(X)^t E(X)^{3-t}}{\left(E(X)^2 - [E(X)]^2\right)^{3/2}};\ \gamma_1 = \frac{\beta_3}{(\beta_2)^{3/2}}, \tag{49}$$

$$\gamma_2 = \frac{\sum_{t=0}^{4}(-1)^t \binom{4}{t} E(X)^t E(X)^{4-t}}{\left(E(X)^2 - [E(X)]^2\right)^{2}};\ \gamma_2 = \frac{\beta_4}{(\beta_2)^{2}}. \tag{50}$$

The infinite series equation (44) is convergent for all $\alpha > 0,\ \beta > 0,\ \theta > 0,\ \lambda > 0,\ \delta > 0$ and $0 \leq r \leq \delta,\ \delta > 0,\ x > 0$.

10. Survival function of EEWD distribution (SFEEWD)

If X follows an EEWD distribution and $F_X(x)$ be the probability that any given device of interest will survive to a given point in time x, such that $x \in X$ that is, $P(X \leq x)$, then the survival function $S_X(x)$ is a function that gives the probability which surviving beyond x. Suppose that $F_X(x)$ is the cdf of EEWD distribution defined on the interval $[0, \infty)$, then the survival function of EEWD is given by:

$$S_X(x) = P(X > x) = 1 - \left[1 - \left(1 - \left[1 + \lambda x^{-\delta}\right]^{-\beta}\right)^{\alpha/k}\right]^{\theta}. \tag{51}$$

In terms ω, equation (51) becomes,

$$S_X(x) = P(X > x) = 1 - \left[1 - (\omega)^{\alpha/k}\right]^{\theta}. \tag{52}$$

11. Hazard function of EEWD (HFEEWD) distribution

Let X be a RV that follows an EEWD distribution with SF and PDF stated in (2) and (12) respectively, then HF of EEWD distribution is obtained by $h_X(x) = \dfrac{f_X(x)}{S_X(x)}$.

$$\begin{aligned} h_X(x) = {} & \frac{\alpha\beta\theta\lambda\delta}{\kappa} x^{-(\delta+1)} \left[1 + \lambda x^{-\delta}\right]^{-(\beta+1)} \left(1 - \left[1 + \lambda x^{-\delta}\right]^{-(\beta)}\right)^{\frac{\alpha}{\alpha}-1} \\ & \left\{1 - \left(1 - \left[1 + \lambda x^{-\delta}\right]^{-(\beta)}\right)^{\frac{\alpha}{\kappa}}\right\}^{\theta-1} \left\{1 - \left[1 - \left(1 - \left[1 + \lambda x^{-\delta}\right]^{-\beta}\right)^{\alpha/k}\right]^{\theta}\right\}^{-1}. \end{aligned} \tag{53}$$

If

$$1 - \left[1 + \lambda x^{-\delta}\right]^{-\beta} = \omega, \tag{54}$$

$$h_X(x) = \frac{\alpha\beta\theta\lambda\delta}{\kappa} x^{-(\delta+1)} \omega^{\frac{\alpha}{\kappa}-1} (1-\omega)^{1+\frac{1}{\beta}} \left(1-\left(1-\omega^{\frac{\alpha}{\kappa}}\right)^{-1}\right.. \tag{55}$$

12. The sub-models of EEWD distribution

The EEWD distribution model subsist of some significant sub- models which are extensively used in lifetime modelling.

1. When $\theta = 1,\ k = 1$, This EEWD distribution attains Kumaraswamy-Dagum distribution with cdf:

$$F_X(x) = 1 - \left(1 - \left[1 + \lambda x^{-\delta}\right]^{-\beta}\right)^{\alpha} \text{ for } \alpha > 0,\ \beta > 0,\ \lambda > 0,\ \delta > 0 \text{ and } x > 0.$$

2. When $\alpha = 1,\ \theta = 1,\ k = 1$, This EEWD distribution attains Dagum distribution with cdf:

$$F_X(x) = \left[1 + \lambda x^{-\delta}\right]^{-\beta} \text{ for } \beta > 0,\ \lambda > 0,\ \delta > 0 \text{ and } x > 0.$$

3. When $\lambda = 1,\ k = 1$, it attains Exponentiated Kumaraswamy-Burr III distribution with cdf:

$$F_X(x) = \left\{1 - \left(1 - \left[1 + x^{-\delta}\right]^{-\beta}\right)^{\alpha}\right\}^{\theta} \text{ for } \alpha > 0,\ \beta > 0,\ \delta > 0,\ \theta > 0 \text{ and } x > 0.$$

4. When $\lambda = 1,\ k = 1,\ \theta = 1$, it attains Kumaraswamy-Burr III distribution with cdf :

$$F_X(x) = 1 - \left(1 - \left[1 + x^{-\delta}\right]^{-\beta}\right)^{\alpha} \text{ for } \beta > 0,\ \delta > 0,\ \alpha > 0 \text{ and } x > 0.$$

5. When $\alpha = 1,\ \lambda = 1,\ k = 1,\ \theta = 1$, it attains Burr III distribution with cdf:

$$F_X(x) = \left(\left[1 + x^{-\delta}\right]^{-\beta}\right) \text{ for } \beta > 0,\ \delta > 0 \text{ and } x > 0.$$

6. When $\beta = 1,\ k = 1$, it attains EKF (Exponentiated Kumaraswamy Fisk) or KLoL (Kumaraswamy-Log-Logistic) distribution with cdf:

$$F_X(x) = \left\{1 - \left(1 - \left[1 + \lambda x^{-\delta}\right]^{-1}\right)^{\alpha}\right\}^{\theta} \text{ for } \alpha > 0,\ \delta > 0,\ \theta > 0,\ \lambda > 0 \text{ and } x > 0.$$

7. When $\beta = 1,\ k = 1,\ \theta = 1$, it attains Kumaraswamy Fisk distribution with cdf:

$$F_X(x) = 1 - \left(1 - \left[1 + \lambda x^{-\delta}\right]^{-1}\right)^{\alpha} \text{ for } \alpha > 0,\ \beta > 0,\ \delta > 0 \text{ and } x > 0.$$

8. When $\beta = 1,\ \alpha = 1,\ \theta = 1,\ k = 1$, it attains Dagum distribution with cdf:

$$F_X(x) = \left[1 + \lambda x^{-\delta}\right]^{-1} \text{ for } \lambda > 0,\ \delta > 0 \text{ and } x > 0.$$

13. Parameter estimation

Mathematical Formulation for MLE of EEWD distribution

The moment of EEWD distribution and resultant the log-likelihood function (LLHF). The MLEs of the unknown parameter for the EEWD distribution are determined based on whole samples. Let $X_1,\ X_2,\ X_3 \ldots\ldots\ldots\ldots\ldots X_t$ be a RS of size t as of this distribution through vector parameters $\mathrm{E} = (\alpha,\ \beta,\ \lambda,\ \delta,\ \theta,\ \kappa)^T$. The corresponding LHF is

$$l = \ln\{L(E)\} = t \ln \left\{ \prod_{i=1}^{t} \frac{\alpha\beta\theta\lambda\delta}{\kappa} x_i^{-(\delta+1)} \left(1 + \lambda x_i^{-(\delta)}\right)^{-(\beta+1)} \left(1 - \left[1 + \lambda x^{-\delta}\right]^{-(\beta)}\right)^{\frac{\alpha}{k} - 1} \right. \tag{56}$$

$$\left. \left\{ \left(1 - \left(1 - \left(1 + \lambda x_i^{-(\delta)}\right)^{-(\beta)\frac{\alpha}{\kappa}} \right\}^{\theta - 1} \right\} \right. .$$

Now differentiating in terms of corresponding parameters and equal to zero, we attain the corresponding ML estimates, respectively.

$$\frac{dl}{d\alpha} = \frac{t}{\alpha} + \frac{\beta}{\kappa} \sum_{i=1}^{t} \ln\left(1 + \lambda x_i^{-(\delta)}\right) - \frac{\theta - 1}{\kappa} \sum_{i=1}^{t} \beta \ln\left(1 + \lambda x_i^{-(\delta)}\right).$$

(i)

$$\frac{dl}{d\alpha} = 0 \text{ then } \frac{t}{\alpha} = \frac{\beta(\theta - 2)}{\kappa} \sum_{i=1}^{t} \ln(Z_i); \tag{57}$$

(ii)

$$\frac{dl}{d\beta} = 0 \text{ then } \frac{t}{\beta} = \frac{2[1 - \alpha(\theta - 1)]}{\kappa} \sum_{i=1}^{t} \ln(Z_i); \tag{58}$$

(iii)

$$\frac{dl}{d\theta} = 0 \text{ then } \frac{t}{\theta} = \frac{\alpha\beta}{\kappa}\sum_{i=1}^{n}\ln(Z_i); \tag{59}$$

(iv)

$$\frac{dl}{d\lambda} = 0 \text{ then } \frac{t}{\lambda} = 1 + \frac{\alpha\beta\theta}{\kappa}\sum_{i=1}^{t}\frac{x_i - (\delta)}{z_i}; \tag{60}$$

(v)

$$\frac{dl}{d\delta} = 0 \text{ then } \frac{t}{\delta} = \sum_{i=1}^{t}\ln(x_i) + \left[1 + \frac{\alpha\beta\theta}{\kappa}\right]\sum_{i=1}^{t}\frac{y_i \log x_i}{z_i}; \tag{61}$$

(vi)

$$\frac{dl}{d\kappa} = 0 \text{ then } \kappa = (\alpha\beta\theta)\frac{1}{t}\sum_{i=1}^{t}\ln Z_i. \tag{62}$$

13.1 *Simulation study*

Simulation studies are conducted to study the performance of estimators. The standard estimate error (SE), average absolute bias (AAB), and average square root error (RMSE) of the maximum probability estimator of EEWD parameter parameters were investigated [33–37]. The result shows that, when the sample size approaches infinity, the AAB and RMSE are progressively reduced to zero and demonstrate their consistency. It is consistent in the sense that, as the number of observations increases and the error reduces to zero, it converges to the real parameter value. Further study about simulations will discuss the future.

14. Application

The data from the following example of 100 carbon fibres' tensile strength, as testified by Nichols and Padgett [30], are shown in Table 1. The data provided details the tensile strength of 100 individual carbon fibers, expressed in units likely to be gigapascals (GPa) or megapascals (MPa), as measured by Nichols and Padgett. Tensile strength indicates how much pulling or tension a fibre can withstand before breaking

Table 1. Tensile strength of 100 carbon fibers data [30, 32]

Tensile Strength of 100 Carbon fibers (Gba)									
(1) 3.7	(11) 2.74	(21) 2.73	(31) 2.5	(41) 3.6	(51) 3.11	(61) 3.27	(71) 2.87	(81) 1.47	(91) 3.11
(2) 4.42	(12) 2.41	(22) 3.19	(32) 3.22	(42) 1.69	(52) 3.28	(62) 3.09	(72) 1.89	(82) 3.15	(92) 4.9
(3) 3.75	(13) 2.43	(23) 2.95	(33) 2.97	(43) 3.39	(53) 2.96	(63) 2.53	(73) 2.67	(83) 2.93	(93) 3.22
(4) 3.39	(14) 2.81	(24) 4.2	(34) 3.33	(44) 2.25	(54) 3.31	(64) 3.31	(74) 2.85	(84) 2.56	(94) 3.56
(5) 3.15	(15) 2.35	(25) 2.55	(35) 2.59	(45) 2.38	(55) 2.81	(65) 2.77	(75) 2.17	(85) 2.83	(95) 1.92
(6) 1.41	(16) 3.68	(26) 2.97	(36) 1.36	(46) 0.98	(56) 2.76	(66) 4.91	(76) 3.68	(86) 1.84	(96) 1.59
(7) 3.19	(17) 1.57	(27) 0.81	(37) 5.56	(47) 1.73	(57) 1.59	(67) 2	(77) 1.22	(87) 1.12	(97) 1.71
(8) 2.17	(18) 1.17	(28) 5.08	(38) 2.48	(48) 1.18	(58) 3.51	(68) 2.17	(78) 1.69	(88) 1.25	(98) 4.38
(9) 1.84	(19) 0.39	(29) 3.68	(39) 2.48	(49) 0.85	(59) 1.61	(69) 2.79	(79) 4.7	(89) 2.03	(99) 1.8
(10) 1.57	(20) 1.08	(30) 2.03	(40) 1.61	(50) 2.12	(60) 1.89	(70) 2.88	(80) 2.82	(90) 2.05	(100) 3.65

14.1 *Histogram, PDF, Q-Q plot of EEWD distribution fitted with carbon fibers data*

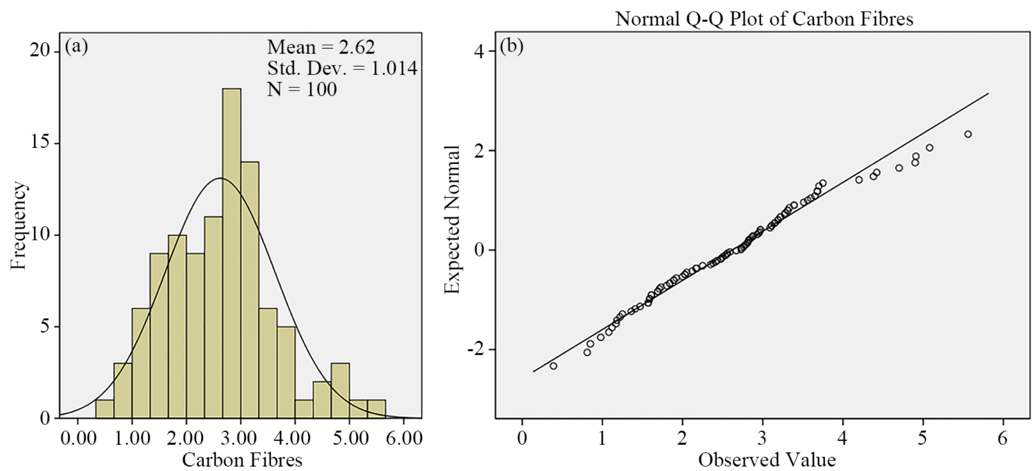

Figure 3. (a) Histogram for tensile strength of 100 carbon fibers data (b) Normal Q-Q plot of carbon fibers

Table 2. The tensile strength of 100 carbon fibers of descriptive statistics values

Descriptives			Statistic	Std. Error (SE)
Carbon fibers	Mean (α)		2.6186	0.10144
	95% Confidence interval for mean	Lower bound	2.4173	
		Upper bound	2.8199	
	5% Trimmed mean		2.5844	
	Median		2.7000	
	Variance		1.029	
	Std. Deviation (σ)		1.01440	
	Minimum		0.39	
	Maximum		5.56	
	Range		5.17	
	Interquartile range		1.38	
	Skewness		0.381	0.241
	Kurtosis		0.171	0.478

The PDF of EEWD distribution has been overlaid on the histogram of Carbon fibers data, given by Figure 3. We draw Quantile-Quantile plot which is linear with EEWD distribution and its detailed theoretically EEWD is a better suitable model.

These statistics provide a summary of the distribution and variability of tensile strength within a sample of 100 carbon fibers. The mean tensile strength of 2.6186 represents the average value, while the standard deviation of 1.01440 reflects the variation of values around this mean. The confidence interval offers an estimate of the range in which the true mean for the population of carbon fibers is likely to fall. Additionally, the skewness and kurtosis values suggest a minor asymmetry and indicate that the distribution is relatively normal.

14.2 *Estimation of confidence interval and test hypothesis*

We observed that from Figures 3 and 4 (Q-Q plot) the shape of tensile strength of carbon fibers data is positively skewed such as the observation of skewness (0.381), kurtosis (0.171). We estimate 95% of confidence interval (CI) mean, SD and Variance of the above data as follows.

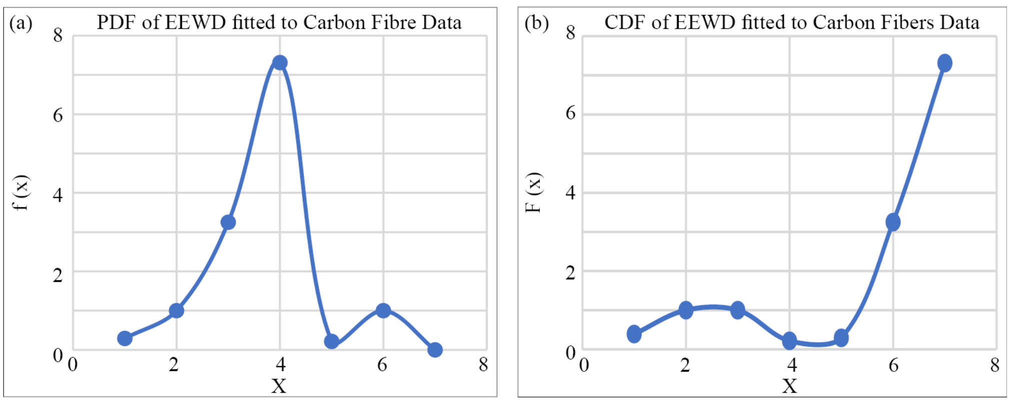

Figure 4. PDF and CDF fitted to Carbon Fibers Data

Table 3. Test of hypotheses

95% CI for mean	95% CI for SD	95% CI for Variance
2.42 < mean < 2.823	0.89 < SD < 1.18	0.80 < Variance < 1.39

The table 3 provide presents 95% confident intervals for the mean, standard deviation and variance of the carbon fibers data set. The above intervals of mean, standard deviation and variance shows consideration of sample variability, individual measurement, idea of dispersion of data respectively.

14.3 *CI analysis for skewness*

Observed from the estimation of parameters skewness is positively skewed and kurtosis > 3, indicates that EEWD is heavily tailed. Therefore, we compute skewness and kurtosis for the sample by using Cramer [31] and Joanes and Gill [32, 38–40] formula,

Sample skewness

$$\text{SSK} = \left[\frac{\sqrt{n(n-1)}}{n-2}\right] * \gamma_1, \text{here} n = 100,\ \gamma_1 = 0.56$$

So that SSK $= 0.5685$.

Standard sample Error of Skewness

$$\text{SESK} = \sqrt{\frac{6n(n-1)}{(n-2)(n+1)(n+3)}} \approx 0.2414$$

Test statistic for skewness

$$Z_{SK} = \frac{SSK}{SESK} \approx 2.3551.$$

Since Z > 2, it is positively skewed.

CI for skewness at 95 % is defined by

$SSK \pm 2\ SESK$, we estimate 95 % CI of Skewness (0.0857, 1.0513).

14.4 *CI analysis for kurtosis*

We compute excess kurtosis for sample by using Cramer [31] and Joanes and Gill [32] formula,

$$SK = \gamma_2 - 3 = 1.782.$$

Then excess Kurtosis for tensile strength of Carbon data

$$\text{ESK} = \left[\frac{n-1}{(n-2)(n-3)}\right][(n+1)Sk+6] \approx 1.938.$$

The standard error of Kurtosis

$$\text{SEK} = 2 \times (SESK) \times \sqrt{\frac{(n^2-1)}{(n-3)(n+5)}} \approx 0.4784$$

$$Z_{SK} = \frac{ESK}{SEK} \approx 4.051 \tag{63}$$

$Z_{SK} > 2$ therefore, the sample has positively excess Kurtosis. CI for Kurtosis at 95 % obtained by $ESK \pm 2SEK$. we obtained 95 % CI of Kurtosis as (0.9814, 2.8948) .

Table 4. test statistics

One sample T-Test							
						95% CI for Location difference	
	Test	Statistic	Degrees of freedom	p	Location difference	Lower	Upper
Carbon fiber	Student	-0.014	99	0.989	-0.001	-0.203	0.2
	Z	-0.014		0.989	-0.001	-0.2	0.197

Note. For the student t-test and Z-test, location difference estimate is given by the sample mean difference d

There is no significant difference between population mean (2.6192) and Sample mean (2.6186), (Table 2, 4 and 5) the alternative hypothesis specifies (Table 3) that the mean is different from 2.62. So accepted null hypothesis.

The p-value of 0.989 is significantly higher than the common significance threshold (e.g., 0.05), suggesting that there is no statistically significant difference between the sample mean and the hypothesized population mean. Additionally, the confidence intervals for both the Student's T-test and the Z-test encompass zero, reinforcing the finding that the difference is not statistically significant. The test statistics, which are very near zero indicate minimal deviation from the hypothesized mean.

Table 5. The tensile strength of 100 carbon fibers of descriptive statistics values

	Test of normality (shapiro-wilk)		Descriptives				
	W	p	N	Mean	SD	SE	Coefficient of variation
Carbon fiber	0.982	0.193	100	2.6192	1.014	10.101	0.387

Note. Significant results suggest a deviation from normality

The proposed distribution was compared to the Fisk model after the EEWD distribution was fitted to the data set using MLE. In the model selection procedure, the AIC (Akaike information criterion), the BIC (Bayesian information criterion), the CAIC (Consistent Akaike information criterion), and the HQIC (Hannan- Quinn information criterion) are all employed.

Where,

$$AIC = -2l(\vartheta) + 2m$$

$$BIC = -2l(\vartheta) + m\log(N)$$

$$HQIC = -2l(\vartheta) + 2m\log(\log(N))$$

$CAIC = -2l(\vartheta) + \dfrac{2mN}{N-m-1}$, where m is the numeral of parameters, N is the size of sample.

Where $l(\vartheta)$ denotes the LLHF estimated at the MLE, the above-mentioned information criterion with lowest value is elected as the finest fit model to the data.

An ordered RS $X_1,\ X_2,\ X_3,\ X_4 \ldots\ldots X_t$ from EEWD with 6 parameters, the Kolmogorov-Smirnov k_s, Cramer-von Mises Cr_M test statistics are computed and which is given below.

$$k_s = \max_i \left[\frac{1}{t} - F(x_i,\ \vartheta) - \frac{i-1}{t}\right]$$

$$Cr_M = \frac{1}{12t} + \sum_{i=1}^{t}\left[F(x_i,\ \vartheta) - \frac{2i-1}{2t}\right]^2$$

Table 6. AIC, BIC, HQIC, CAIC, Kolmogorov-Smirnov k_s, Cramer-von Mises Cr_M test statistics

Statistics			Measures			
Models	k_s	Cr_M	AIC	BIC	HQIC	CAIC
EEWD	0.0738	0.0634	289.18	297.43	290.23	287.231
FISK	0.0916	0.1579	296.42	301.63	298.53	296.55

This outcome proves the test statistics yield the least value for the data set further down the EEWD distribution. Thus, the EEWD distribution model is an alternative model to the Fisk model.

15. Results and discussion

Tested the EEWD distribution's ability to fit the data on 100 carbon fibers' tensile strength. A distribution ability to fit the data observed can be evaluated, the PDF of the EEWD distribution has been overlaid along the histogram of the tensile strength of 100 carbon fibers data as shown in Figure 3 and 4. It is easy to plot the CDF of EEWD values against the data for the tensile strength of 100 carbon fibers. It has also been confirmed that the data's skewness and kurtosis indicate that the tensile strength of 100 is not normally distributed. Thus, observed that it is skewed.

Result 1: Obtained PDF and CDF of EEWD.

Result 2: Obtained Moments, survival, failure rate functions and some properties.

Result 3: Estimated descriptive Statistics of the real time Data.

Result 4: Estimated AIC, BIC, CAIC, HQIC for EEWD (Table 6) using MLE and compared with Fisk Model.

Result 5: Fitted the real time data to the PDF and CDF curve of EEWD model.

16. Conclusion

The EEWD distribution which is a new class of distribution introduced and examined with this sub-model. Some properties of this class distributions together with the series expansions with PDF of EEWD distribution, CDF of EEWD distribution, Moments and EEWDMGF distribution, stochastic ordering, Cumulant Generating Function (CGF) of EEWD distribution, Mean, deviations of EEWD distribution that are provided. The EEWD model is motivated by the widespread application of the Dagum distribution in practice. In this section, the MLEs of the EEWD distribution were presented and their presentation in relation to sample size t (10, 20, 50 and 100) was examined. Future work includes Quantile Function of EEWD distribution, Survival Function of EEWD distribution, HFEEWD distribution (Hazard Function), InFEEWD distribution (Inverse Hazard Function), CHFEEWD (Cumulative Hazard Function) distribution, Asymptotes of EEWD distribution, Residual life Functions of EEWD distribution, Shannon Entropy, Estimation of Parameters with Applications.

Conflict of interest

The authors declare no competing financial interest.

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Appendix

R Codes

EEWD CDF

```
EEWD_CDF < -function ((x, alpha, lambda, beta, theta, delta, kappa)) {
A < -(1 + (lambda*x^(-delta)))^(-beta)
B < -(1-A) ^(alpha/kappa)
C < -(1-B) ^(theta)
fxn < -(C)
return (fxn)
}
```

EEWD PDF

```
EEWD_PDF < -function (x, alpha, lambda, beta, theta, delta, kappa) {
A < -(1 + (lambda*x^(-delta)))^(-beta)
B < -(1 + (lambda*x^(-delta)))^(-(beta + 1))
C < -(alpha*beta*theta*lambda*delta)/(kappa)
D < -x^(-(delta + 1))
E < -(1-A)^((alpha/kappa)-1)
F < -(1-(1-(A^(alpha/kappa)))) ^ (theta-1)
fxn < -(C*D*B*E*F)
return (fxn)
}
```

EEWD_QF

```
EEWD_QF < -function (p, alpha, lambda, beta, delta, kappa) {
A < -(1- p ^(1/theta))^(kappa/alpha)
B < -(A-1) ^(-1/beta)
C < -(lambda^1/delta)*(B-1)^(-1/delta)
fxn < -(C)
return (fxn)
}
```

Hazard function HF EEWD

```
EEWD_HF < -function (x, alpha, lambda, beta, theta, delta, kappa) {
A < -(1 + (lambda*x^(-delta)))^(-beta)
B < -(1 + (lambda*x^(-delta)))^(-(beta + 1))
C < -(alpha*beta*theta*lambda*delta)/(kappa)
D < -x^(-(delta + 1))
E < -(1-A)^((alpha/kappa)-1)
F < -(1-(1-(A^(alpha/kappa)))) ^(theta-1)
G < -(1 + (lambda*x^(-delta))) ^ (-beta)
H < -(1-G) ^ (alpha/kappa)
I < -(1-H) ^ (theta)
fxn < -(C*D*B*E*F*I)
return (fxn)
}
```

EEWD SF

```
EEWD_SF < -function (x, alpha, lambda, beta, theta, delta, kappa) {
A < -(1 + (lambda*x ^ (-delta))) ^ (-beta)
B < -(1-A) ^ (alpha/kappa)
C < -(1-B) ^ (theta)
D < -(1-C)
fxn < -(D)
return (fxn)
}
```