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Research Article

Numerical Treatment for the Distributed Order Fractional Optimal Control Coronavirus (2019-nCov) Mathematical Model

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Abstract: In this paper, we presented the distributed order fractional optimal control of the Coronavirus (2019-nCov) mathematical model. The distributed order fractional operator is defined in the Caputo sense. Control variables are considered to reduce the transmission of infection to healthy people. The discretization of the composite Simpson's rule and Grünwald-Letnikov nonstandard finite difference method is constructed to solve the obtained optimality system numerically. The stability analysis of the proposed method is studied. Numerical examples and comparative studies for testing the applicability of the utilized method and to show the simplicity of this approximation approach are presented. Moreover, by using the proposed method we can conclude that the model given in this paper describes well the confirmed real data given in Spain and Wuhan.

***Keywords*:** coronavirus, composite simpson's rule, distributed order derivatives, Grünwald-Letnikov nonstandard finite difference method, optimal control problem

MSC: 37N25, 49J15, 65M06

1. Introduction

Distributed-order fractional derivatives indicate fractional integrated over the order of the differentiation within a given range. The concept of distributed order fractional derivative is expanded by [1], as well as Bagley and Torvik in [2]. There are a lot of researchers who took this concept and applied it to some fields [3–7].

The new coronavirus, which is classified as a very lethal virus that attacks the human respiratory system. The pandemic began in late December 2019 in Wuhan, China's capital, with individuals brought to hospitals with an initial diagnosis of pneumonia.

Several mathematical models have been proposed in the literature to examine and evaluate the complicated transmission for pandemic of COVID-19 see for example [8–12]. Where in these references, the authors presented various models of fractional order for the spread of the Coronavirus and used various modern definitions of fractional derivatives.

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In this work, we presented a new model offered for the first time from distributed fractional orders and the optimal control for this model since we can obtain the mathematical model of the fractional order as a special case of our model presented in this paper.

To improve the discretization of certain terms in investigated differential equations, a distinct method called the nonstandard finite difference method (NSFDM) has been established. It was first presented by mathematician Mickens [13, 14]. This strategy can provide better accuracy and stability than conventional approaches, depending on the denominator function chosen and the specific discretization technique [15]. Moreover, generating the NSFDM is not too difficult [14]. Positive applications of the NSFDM have been identified in physics, chemistry, and engineering, among other domains [16–18]. It has been particularly useful in mathematical ecology and biology, where its efficacy has been amply displayed [19, 20]. Furthermore, when the NSFDM is used to solve fractional-order systems, it has demonstrated exceptional dynamic-preserving features.

The fractional optimal control (FOC) of disease treatment has become popular in biology and refers to the minimization (maximization) of an objective function with dynamic constraints, on state and control variables, such that these conditions have a derivative of fractional order. Some numerical methods to find approximation solutions of some types of FOCPs were recorded [21–24] and the references cited therein.

It is complicated to acquire an exact solution for distributed-order differential models. To estimate the solutions of these models, it is necessary to create some numerical methods, such as [25, 26]. Also, there are some interesting references in efficient numerical methods to solve the nonlinear models such as [26–28].

The current study aims to extend the integer order model of COVID-19 as described in [29] to a distributed order fractional model. We will verify the present model's positivity, boundedness, stability, and reproduction number. We take into account the data that was gathered for Wuhan from 4 January to 9 March 2020 and the data that was accessible for daily confirmed cases in Spain from 25 February to 16 May 2020 [30]. We also employed the optimality conditions required in [21]. Then, to numerically solve the resulting optimality system, we will construct a numerical method using the discretization of the composite Simpson's rule and nonstandard finite difference (NSFD) with the discretization of Grünwald-Letnikov (GL) derivative to solve the obtained optimality system numerically. The stability analysis of the proposed method will be studied. Numerical simulations will be given to confirm the efficiency and wide applicability of the proposed method before and after the controlled case.

The remainder of this paper is organized as follows: mathematical preliminaries and important definitions of distributed order fractional calculus are discussed in section 2. In section 3, we develop a distributed order fractional model of Coronavirus (2019-nCov) discussed, also, some properties of the proposed model such as the basic reproduction, boundedness, positivity, and stability are studied. In section 4, the requirements that must be met in order for the optimal control problem is given. Derivation of the difference scheme is presented in section 5. Numerical simulation to validate our results is done in section 6. Section 7 provides a conclusion summarizing the key findings and contributions of the study.

2. Background information

In this section, some basic concepts and characteristics in the theory of distributed order fractional calculus are discussed.

The Riemann-Liouville's definition (RL) is given as [31]:

$$ {}^{RL}D_t^{\alpha} f(t) = \left[\frac{d^n}{dt^n} \int_0^t f(\tau)(t-\tau)^{(n-\alpha-1)} d\tau \right] \frac{1}{\Gamma(n-\alpha)}, $$

and, n denotes to the first integerand it is not less than α, that is, $n-1 < \alpha < n$ and $\Gamma(\cdot)$ denotes to Gamma function.

The derivative of Caputo fractional order for $f(t)$ is defined as [31]:

$$
{}_{0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left[\int_0^t f^{(n)}(\tau)(t-\tau)^{(n-\alpha-1)} d\tau \right].
$$

Proposition 1 [31] Let ${}_tD_b^{\alpha} f$, ${}_aD_t^{\alpha} f$ are the right-side and lift-side RL fractional derivatives of $f(t)$ and ${}_t^CD_b^{\alpha} f$, ${}_a^CD_t^{\alpha} f$ are the right-side and lift-side Caputo's fractional derivatives of $f(t)$, $\alpha \notin \mathbb{N}$, then for $n = [\Re(\alpha)] + 1$,

$$
{}_aD_t^{\alpha} f(t) = \sum_{k=0}^{n-1} f^k(a) \frac{(t-a)^{(k-\alpha)}}{\Gamma(k-\alpha+1)} + {}_a^C D_t^{\alpha} f(t),
$$

$$
{}_tD_b^{\alpha} f(t) = \sum_{k=0}^{n-1} f^k(b) \frac{(b-t)^{(k-\alpha)}}{\Gamma(k-\alpha+1)} + {}_t^C D_b^{\alpha} f(t).
$$

Proposition 2 If

$$
f(a) = f'(a) = f''(a) = ... = f^{n-1}(a) = 0,
$$

then

$$
{}_aD_t^{\alpha} f(t) = {}_a^CD_t^{\alpha} f(t),
$$

and

$$
f(b) = f'(b) = f''(b) = ... = f^{n-1}(b) = 0,
$$

then

$$
{}_tD_b^{\alpha} f(t) = {}_t^CD_b^{\alpha} f(t).
$$

If $q(\alpha)$ is a function of α, $\alpha \in (0,\ 1]$, $q \neq 0$, and $\int_0^1 q(\alpha) d\alpha = c_0 > 0$, the right and left sided distributed order fractional derivatives in Caputo sense are given by [32]:

$$
{}_t^CD_b^{q(\alpha)} f(t) = \int_0^1 q(\alpha)\, {}_t^CD_b^{\alpha} f(t) d\alpha,
$$

$$
{}_a^CD_t^{q(\alpha)} f(t) = \int_0^1 {}_a^CD_t^{\alpha} f(t)\, q(\alpha) d\alpha.
$$

If $0 < \alpha < 1$, we obtain [21]:

$$\int_a^b g(t){}_a^C D_t^{q(\alpha)} f(t)dt = \int_a^b f(t){}_t D_b^{q(\alpha)} g(t)dt + I_t^{1-q(\alpha)} f(t)g(t)|_a^b,$$

$$\int_a^b g(t){}_t^C D_b^{q(\alpha)} f(t)dt = \int_a^b f(t){}_a D_t^{q(\alpha)} g(t)dt - I_t^{1-q(\alpha)} f(t)g(t)|_a^b.$$

Consider a constant point for the distributed order fractional Caputo system say S^*, which is known to be its equilibrium point then,

$${}_a^C D_t^{q(\alpha)} S(t) = g(t,\ S),\ 0 < \alpha < 1,$$

if and only if $g(t,\ S^*) = 0$.

3. Distributed order fractional coronavirus model

In this section, we will extend the COVID-19 model described in [29] to a distributed order fractional-order model. This model consists of eight nonlinear differential equations. The distributed order fractional operator is updated with an auxiliary parameter μ to ensure dimensional matching between both sides, where the resulting distributed order fractional equations on the left side have the dimension of day^{-1} [33]. Three controls, u_I, u_P, and u_h, are introduced in order to health care in order to provide soothing therapies on a regular basis, isolate patients in private health rooms, and provide respirators. Table 1 summarizes the variables for the proposed model. Table 2 provides a summary of all simulation settings and values.

In the following, the updated nonlinear distributed order fractional differential mathematical model:

$$\frac{1}{\mu^{1-q(\alpha)}}{}_0^C D_t^{q(\alpha)} S = -\frac{IS}{N}\beta - L\beta\frac{HS}{N} - \beta_1\frac{PS}{N},$$

$$\frac{1}{\mu^{1-q(\alpha)}}{}_0^C D_t^{q(\alpha)} E = \beta\frac{IS}{N} + L\beta\frac{HS}{N} + \beta_1\frac{PS}{N} - KE,$$

$$\frac{1}{\mu^{1-q(\alpha)}}{}_0^C D_t^{q(\alpha)} I = K\rho_1 E - (\gamma_a + \gamma_i)I - \delta_i I - \nu u_I I,$$

$$\frac{1}{\mu^{1-q(\alpha)}}{}_0^C D_t^{q(\alpha)} P = K\rho_2 E - (\gamma_a + \gamma_i)P - \delta_p P - \nu u_P P,$$

$$\frac{1}{\mu^{1-q(\alpha)}}{}_0^C D_t^{q(\alpha)} A = K(1 - \rho_1 - \rho_2)E,$$

$$\frac{1}{\mu^{1-q(\alpha)}}{}_0^C D_t^{q(\alpha)} H = \gamma_a(I + P) - \gamma_r H - \delta_h H - \nu u_h H + 0.5\nu u_I I + 0.5\nu u_P P,$$

$$\frac{1}{\mu^{1-q(\alpha)}}{}_0^C D_t^{q(\alpha)} R = \gamma_i (I+P) + \gamma_r H + 0.5\nu u_I I + 0.5\nu u_P P + \nu u_h H,$$

$$\frac{1}{\mu^{1-q(\alpha)}}{}_0^C D_t^{q(\alpha)} F = \delta_i I + \delta_p P + \delta_h H, \tag{1}$$

where, $0 < \nu \leq 1$,

$$N(t) = I(t) + P(t) + S(t) + E(t) + A(t) + F(t) + H(t) + R(t),$$

with,

$$0 \leq I(0) = i_0,\ 0 \leq S(0) = s_0,\ 0 \leq E(0) = e_0,\ 0 \leq P = p_0\ 0 \leq A = a_0,\ 0 \leq R(0) = r_0,$$

$$0 \leq H = h_0,\ 0 \leq F = f_0. \tag{2}$$

Table 1. All variable description given in (1) [29]

Variable	Definition
A	Denotes the class of infectious but asymptomatic.
F	Denotes the class of fatality.
H	Denotes the class of hospitalized.
S	Indicates the group of susceptible.
E	S: Indicates the group of exposed.
I	S: Indicates the group of symptomatic and infectious.
R	Denotes the group of recovery.
P	Denotes the group of super-spreaders.

Table 2. Parameters used in the model and simulations [29]

Parameters	Description	Value (per day^{-1})
L	Hospitalized patients relative transmissibility	1.56 dimensionless
β	Coefficient of infected individual	2.55
β_1	Denotes the super-spreaders coefficient	7.65
K	Denotes the rate of exposure become infectious	0.25
ρ_1	Denotes rate at which expose people become infected I	0.580 dimensionless
ρ_2	The rate at which expose people become super-spreaders	0.001 dimensionless
γ_i	Rate of recovery without hospitalization	0.27
γ_r	hospitalized patients' pace of recovery	0.5
γ_a	Hospitalization rate	0.94
δ_i	Death from disease as a result of a high rate of infection	3.5
δ_h	Hospitalized class rate of disease-induced mortality	0.3
δ_p	Super-spreaders of death caused by illness	1

3.1 *Boundedness and positivity of solutions*

Boundedness of the proposed model solution can be verified by adding all equations of system (1) as follows:

$$ {}_{0}^{C}D_{t}^{q(\alpha)}N(t)=0, \quad N(0)=A, \tag{3}$$

and $A \geq 0$ is constant, N is total summation of population in (1). The solution of (3) is as follows: $N(t) \geq 0$. Additionally, $N(t) \geq 0$, as $t \longrightarrow \infty$. The solutions of the system (1) are bounded.

Lemma 1 Under the initial conditions (2), all the solutions of system (1) remain nonnegative for $t \geq 0$.

Proof. By the initial conditions (2), it was discovered that

$$\frac{1}{\mu^{1-q(\alpha)}}{}_{0}^{C}D_{t}^{q(\alpha)}S\mid_{S=0}=0,$$

$$\frac{1}{\mu^{1-q(\alpha)}}{}_{0}^{C}D_{t}^{q(\alpha)}E\mid_{E=0}=\beta\frac{IS}{N}+L\beta\frac{HS}{N}+\beta_1\frac{PS}{N}\geq 0,$$

$$\frac{1}{\mu^{1-q(\alpha)}}{}_{0}^{C}D_{t}^{q(\alpha)}I\mid_{I=0}=K\rho_1E\geq 0,$$

$$\frac{1}{\mu^{1-q(\alpha)}}{}_{0}^{C}D_{t}^{q(\alpha)}P\mid_{P=0}=K\rho_2E\geq 0,$$

$$\frac{1}{\mu^{1-q(\alpha)}}{}_{0}^{C}D_{t}^{q(\alpha)}A\mid_{A=0}=K(1-\rho_1-\rho_2)E\geq 0,$$

$$\frac{1}{\mu^{1-q(\alpha)}}{}_{0}^{C}D_{t}^{q(\alpha)}H\mid_{H=0}=\gamma_a(I+P)+0.5\nu u_I I+0.5\nu u_P P\geq 0,$$

$$\frac{1}{\mu^{1-q(\alpha)}}{}_{0}^{C}D_{t}^{q(\alpha)}R\mid_{R=0}=\gamma_i(I+P)+\gamma_r H+0.5\nu u_I I+0.5\nu u_P P+\nu u_h H\geq 0,$$

$$\frac{1}{\mu^{1-q(\alpha)}}{}_{0}^{C}D_{t}^{q(\alpha)}F\mid_{F=0}=\delta_i I+\delta_p P+\delta_h H\geq 0,$$

3.2 *Reproduction number*

We will utilize the next generation method [34] to check the system's fundamental reproduction number (R_0). Consider the matrices F and V, where F denotes the new infection terms and V signifies the leftover transfer terms [34]. The matrices are provided as follows:

$$F=\mu^{1-q(\alpha)}\begin{pmatrix} 0 & \beta & \beta & \beta_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V=\mu^{1-q(\alpha)}\begin{pmatrix} -K & 0 & 0 & 0 \\ -K & -(\gamma_a+\gamma_i) & 0 & 0 \\ -K\rho_1 & 0 & -(\gamma_a+\gamma_i+\delta_p) & 0 \\ 0 & \gamma_a & \gamma_a & -(\gamma_r+\delta_h) \end{pmatrix}.$$

Then,

$$R_0=\rho(FV^{-1})=\mu^{1-q(\alpha)}\left[\frac{\beta\rho_1\left(\gamma_a L+(\gamma_r+\delta_h)\right)}{(\gamma_r+\delta_h)(\gamma_a+\gamma_i+\delta_i)}+\frac{\left(\beta\gamma_a L+\beta_1(\gamma_r+\delta_h)\right)\rho_2}{(\gamma_r+\delta_h)(\gamma_a+\gamma_i+\delta_p)}\right],$$

where, ρ denotes to the spectral radius of FV^{-1}.

3.3 *Stability analysis of the proposed model*

Consider the linear system of fractional distributed order can be wrote as:

$$
\begin{aligned}
&{}_0^C D_t^{q(\alpha)} Q = BQ(t),\\
&Q(0)=Q_0, \qquad (4)
\end{aligned}
$$

where, $B\in R^{n\times n}$, $Q(t)=(A(t),\ S(t),\ E(t),\ I(t),\ P(t),\ H(t),\ R(t),\ F(t))\in R^8$, $q(\alpha)$, denoted to the density function, $0<\alpha\leq 1$. The general solution of ((4)), is given as follows [35]:

$$Q(t)=Q(0)+\frac{1}{\pi}\int_0^t\int_0^\infty\int_0^\infty \sin(\rho\sin(\pi\gamma))\sin(BQ(0))e^{-rt+B\tau-\rho\cos(\pi\gamma)}drd\tau dt,$$

and $\rho=|X(s)|$, $r=e^{i\pi}$, $X(s)=\rho\cos(\pi\gamma)+i\rho\sin(\pi\gamma)$, $\gamma=\arg[X(s)]\dfrac{1}{\pi}$.

Theorem 1 [35] The system (4) be asymptotically stable $\Leftrightarrow$ the zeros of $det(-B+X(s)I)=0$, have negative real parts.

Remark 1 The characteristic function of B with respect to the distributed function is $det(X(s)I-B)=0$, where $\int_0^1 s^\alpha q(\alpha)d\alpha$ is the distributed function with respect to the density function and $X(s)=s^\alpha$. Moreover, we can write $det(X(s)I-B)=0$, as $s^\alpha I-B=0$. Let $\varpi=s^\alpha$, then $s=\varpi^{\frac{1}{\alpha}}$, and we have $|\arg\varpi^{\frac{1}{\alpha}}|>\dfrac{\pi}{2}$. Thus, the zeros ϖ of $det(\varpi I-B)=0$, satisfy $\alpha\dfrac{\pi}{2}<|\arg\varpi^{\frac{1}{\alpha}}|$.

The inertia of (4) is:

$$(\Theta_{nX(s)}B,\ \vartheta_{nX(s)}B,\ \Phi_{nX(s)}B) = \Upsilon_{nX(s)}B,$$

where, $\Theta_{nX(s)}B$, $\vartheta_{nX(s)}B$ and $\Phi_{nX(s)}B$ be the number of zeros of $det(X(s)I - X) = 0$, with negative, zero and positive, real parts, also, $X(s) = (X_1(s),\ X_2(s),\ ...,\ X_n(s))^T$ be the function of distributed with respect to $q(\alpha)$.

Theorem 2 [35] The linear system (4) be asymptotically stable ⇔ the following conditions holds:

1. The roots of the characteristic function of B with respect to $X(s) = (X_1(s),\ X_2(s),\ ...,\ X_n(s))^T$ satisfy $|\arg(s)| > \dfrac{\pi}{2}$,
2. $\Theta_{nX(s)}(B) = \Phi_{nX(s)}B = 0$.

A nonlinear distributed order fractional system's stability analysis is covered here. Consider the nonlinear distributed order fractional system is given by:

$$ {}_0^C D_t^{q(\alpha)} Q = G(Q(t)),$$

$$Q(0) = Q_0, \tag{5}$$

$$G(Q(t)) = \begin{pmatrix} g_1\left(y_1(t),\ y_2(t),\ \dots,\ y_N(t)\right)\mu^{1-q(\alpha)} \\ g_2\left(y_1(t),\ y_2(t),\ \dots,\ y_N(t)\right)\mu^{1-q(\alpha)} \\ . \\ . \\ . \\ g_n\left(y_1(t),\ y_2(t),\ \dots,\ y_N(t)\right)\mu^{1-q(\alpha)} \end{pmatrix}$$

Theorem 3 Let $Q^* = (y_1^*,\ y_2^*,\ ...,\ y_n^*)^T$ is the equilibrium of (5) satisfied ${}_0^C D_t^{q(\alpha)} Q^* = G(Q^*) = 0$ and $J = (\dfrac{\partial G}{\partial Q})|_{Q=Q^*}$ is the matrix of Jacobian at Q^* then Q^* be asymptotically stable ⇔ the zeros of the characteristic function of J with respect to $X(s) = (X_1(s),\ X_2(s),\ ...,\ X_n(s))^T$ satisfy $\dfrac{\pi}{2} < |\arg(s)|$.

Proof. Consider $\rho(t) = Q(t) - Q^*(t)$. Then

$$ {}_0^C D_t^{q(\alpha)} \rho(t) = {}_0^C D_t^{q(\alpha)} (Q(t) - Q^*), \tag{6}$$

since, ${}_0^C D_t^{q(\alpha)} (Q(t) - Q^*) = {}_0^C D_t^{q(\alpha)} Q(t) - {}_0^C D_t^{q(\alpha)} Q^* = 0$; thus,we have

$$ {}_0^C D_t^{q(\alpha)} \rho(t) = {}_0^C D_t^{q(\alpha)} Q(t) = G(Q(t)) = G(\rho(t) + Q^*),$$

$$= higher\ order\ terms + G(Q^*) + J\rho(t) \approx J\rho(t).$$

We can write (6) as

$$ {}_0^C D_t^{q(\alpha)} \rho(t) = J\rho(t), \tag{7}$$

and $\rho(0) = Q_0 - Q^*$. The trajectories of the nonlinear system in the neighborhood of the equilibrium point have the same form as the trajectories of (7), if J has no purely imaginary eigenvalues, Hence, by using Theorem 2, (7) is asymptotically stable ⇔ the zeros of the characteristic function of J with respect to $X(s) = (X_1(s), X_2(s), \ldots, X_n(s))^T$ satisfy $\frac{\pi}{2} < |\arg(s)|$, which implies that Q^* of (5) is asymptotically stable. □

Remark 2 The system (5) in Q^* be asymptotically stable ⇔ $0 = \Theta_{nX(s)}B = \Phi_{nX(s)}B$.

4. Prerequisite optimum conditions for the examined problem

Consider the system (1) in $\mathbb{R}^8$, let

$$\Omega = \left\{ (u_1(.), u_2(.))|u_1, u_2(.) \text{ are Lebsegue measurable on } [0, 1], 0 \le u_1(.), u_2(.) \le 1, \text{ for all } t \in [0, T_f] \right\},$$

is the set of admissible control. We will define the objective functional as follows :

$$J(u_1, u_2) = \int_0^{T_f} (P(t) + H(t) + I(t) + B_1 u_I^2(t) + B_2 u_P^2(t) + B_3 u_h^2(t))dt, \tag{8}$$

where, B_1 and B_2 are weight constants. The goal is to identify u_I, u_P, u_h that minimize the following cost function:

$$J(u_1, u_2) = \int_0^{T_f} \eta(t, I, P, S, E, R, F, A, H, u_I, u_P, u_h)dt,$$

Depending on the objective functional (8) and the constraints equations (1), the Hamiltonian function of the studied distributed order optimal control problem is introduced, using a Lagrange multiplier technique. For simplicity we can re-write the constraints (1) as follows:

$${}_a^C D_t^{q(\alpha)} \Psi_j = \xi_i.$$

Where

$$\xi_i = \xi_i(t, I, P, S, E, R, F, A, H, u_P, u_I, u_h), \; i, j = 1, \ldots, 8,$$

$$\Psi_j = \{P, I, S, E, R, F, A, H\},$$

$\Psi_1(0) = I_0$, $\Psi_2(0) = P_0$, $\Psi_3(0) = S_0$, $\Psi_4(0) = E_0$, $\Psi_5(0) = A_0$, $\Psi_6(0) = H_0$, $\Psi_7(0) = F_0$, $\Psi_8(0) = R_0$.

To formulate the optimal control problem, we use a kind of Pontryagin's maximum principle in distributed order fractional case which is given by Nda ïrou and Torres in [21]:

We define the Hamiltonian in the following form:

$$H(t,\ I,\ P,\ S,\ E,\ R,\ F,\ A,\ H,\ u_I,\ u_P,\ u_h,\ \lambda_i) = \eta(t,\ I,\ P,\ S,\ E,\ R,\ F,\ A,\ H,\ u_I,\ u_P,\ u_h,\ \lambda_i)$$

$$+\sum_{i=1}^{8}\lambda_i\xi_i(t,\ I,\ P,\ S,\ E,\ R,\ F,\ A,\ H,\ u_I,\ u_P,\ u_h). \tag{9}$$

From (8) and (9), we have the necessary optimality conditions:

$$D_{t_f}^{q(\alpha)}\lambda_\iota = \frac{\partial H}{\partial \vartheta_\iota},\ \iota = 1,\ \ldots,\ 8, \tag{10}$$

where,

$$\vartheta_\iota = \{t,\ I,\ P,\ S,\ E,\ R,\ F,\ A,\ H,\ u_I,\ u_P,\ u_h,\ \iota = 1,\ \ldots,\ 8\},$$

$${}_0^C D_t^{q(\alpha)}\vartheta_\iota = \frac{\partial H}{\partial \lambda_\kappa},\ \iota = 1,\ \ldots,\ 8, \tag{11}$$

It is required that the transversality conditions satisfies:

$$I_{t_f^-}^{1-q(\alpha)}\lambda_\iota(T_f) = 0,\ \iota = 1,\ 2,\ \ldots,\ 8. \tag{12}$$

$$0 = \frac{\partial H}{\partial u_k},\ k = I,\ P,\ h,$$

$$u_I^* = \min\{1,\ \max\{0,\ \frac{\nu I^*(\lambda_3 - 0.5\lambda_6 - 0.5\lambda_7)}{B_1}\}\},$$

$$u_P^* = \min\{1,\ \max\{0,\ \frac{\nu P^*(\lambda_4 - 0.5\lambda_6 - 0.5\lambda_7)}{B_2}\}\},$$

$$u_h^* = \min\{1,\ \max\{0,\ \frac{\nu H^*(\lambda_6 - \lambda_7)}{B_3}\}\}.$$

5. Derivation of the difference scheme

Consider the following distributed order fractional derivative equation:

$$ {}_{0}^{C}D_t^{q(\alpha)}y(t) = f(t,\ y(t)),\ T_f \geq t > 0,\ 1 \geq q(\alpha) > 0, \tag{13}$$

$$y(0) = y_o.$$

and,

$$ {}_{0}^{C}D_t^{q(\alpha)}y(t) = \int_0^1 q(\alpha){}_{0}^{C}D_t^{\alpha}y(t)d\alpha, \tag{14}$$

To approximate the following integration:

$$\int_0^1 q(\alpha)d\alpha,$$

We will use the composite Simpson's rule as follows [24]:
Let $\triangle\alpha = \dfrac{1}{2j}$, and $\alpha_i = i\triangle\alpha$.

$$\int_0^1 q(\alpha)d\alpha = \triangle\alpha\sum_{i=0}^{2j}\gamma_i q(\alpha_i) - \frac{(\triangle\alpha)^4}{180}q^4(\xi),\ \xi \in [0,\ 1],$$

and,

$$\gamma_i = \begin{cases} \dfrac{1}{3},\ i = 0,\ 2j, \\ \dfrac{2}{3},\ i = 2,\ 4,\ ...,\ 2j-4,\ 2j-2, \\ \dfrac{4}{3},\ i = 1,\ 3,\ ...,\ 2j-3,\ 2j-1. \end{cases}$$

The discretization of fractional derivative is given by GL approach [36, 37] :

$$ {}_{0}^{C}D_t^{\alpha}y(t)|_{t=t^k} = \frac{1}{\triangle t^{\alpha}}\left(y_{k+1} - \sum_{m=1}^{k+1}\mu_i y_{k+1-m} - q_{k+1}y_0\right),$$

where $t^k = k\triangle t,\ \triangle t = \dfrac{T_f}{S_k}$, where S_k is the mesh points. $\mu_m = (-1)^{m-1}\binom{\alpha}{i}$, $\mu_1 = \alpha$, $q_m = \dfrac{m^{\alpha}}{\Gamma(1-\alpha)}$ Assume that [38]:

$$0 < \mu_{m+1} < \mu_m < ... < \mu_1 = \alpha < 1,$$

$$0 < q_{m+1} < q_m < ... < q_1 = \frac{1}{\Gamma(1-\alpha)}.$$

Using the GL approximation and the NSFD framework, defined by Mickens [39]:

$${}_0^C D_t^{\alpha} y(t)|_{t=t^k} = \frac{1}{\phi(\triangle t)^{\alpha}} \left(y_{k+1} - \sum_{m=1}^{k+1} \mu_i y_{k+1-m} - q_{k+1} y_0 \right),\ 1 > \phi(\triangle t) > 0,$$

when

$$\triangle(t) \longrightarrow 0,\ \phi(\triangle t) = \triangle(t) + O(\triangle(t)^2).$$

We discretize (13) as follows:

$$\triangle\,\alpha \sum_{i=0}^{2j} \gamma_i q(\alpha_i) {}_0^C D_t^{\alpha_i} y(t)|_{t=t^k} - \frac{(\triangle\alpha)^4}{180} \omega^4(\alpha;\ \xi)|_{\alpha=\xi_k}$$

$$= \triangle\,\alpha \sum_{i=0}^{2j} \gamma_i q(\alpha_i) {}_0^C D_t^{\alpha_i} y(t)|_{t=t^k} + O(\triangle\alpha)^4 = f(t_k,\ y_k),$$

where, $\xi_k \in [0,\ 1]$,

$$\triangle\,\alpha \sum_{i=0}^{2j} \gamma_i q(\alpha_i) \frac{1}{\phi(\triangle t)^{\alpha}} \left(y_{k+1} - \sum_{m=1}^{k+1} \mu_i y_{k+1-m} - q_{k+1} y_0 \right) + O(\phi(\triangle t)^2 + \triangle\alpha^4)$$

$$= f(t_k,\ y_k),$$

Put $K = \triangle\alpha \sum_{i=0}^{2j} \gamma_i q(\alpha_i) \dfrac{1}{\phi(\triangle t)^{\alpha}}$,

$$K \left(y_{k+1} - \sum_{m=1}^{k+1} \mu_m y_{k+1-m} - q_{k+1} y_0 \right) + O(\phi(\triangle t)^2 + \triangle\alpha^4) = f(t_k,\ y_k), \tag{15}$$

$$y_{k+1} = \frac{\left(K \sum_{m=1}^{k+1} \mu_m y_{k+1-m} + K q_{k+1} y_0 + f(t_k,\ y_k) \right)}{K}, \tag{16}$$

5.1 *Stability of method*

To evaluate the stability of the suggested method, consider the following test problem for a linear differential equation with distributed order:

$$({}_{0}^{C}D_t^{q(\alpha)})y(t) = Ay(t),\ A < 0,\ t > 0,\ 0 < q(\alpha) < 1,$$

$$y(0) = y_0,$$

using (14) and (15), we have:

$$y_{k+1} = \frac{\left(K\sum_{m=1}^{k+1}\mu_m y_{k+1-m} + Kq_{k+1}y_0 + Ay_k)\right)}{K},$$

Since $K > 1$, and $\left(K\sum_{m=1}^{k+1}\mu_m y_{k+1-m} + Kq_{k+1}y_0 + f(t_k,\ y_k)\right) > 0$, we have,

$$y_{k+1} < y_k < y_{k-1} < \ldots < y_0,$$

hence the proposed method is stable.

Table 3. The value of objective functional J and $t \in [0,\ 100]$ using GL-NFDM at different value $q(\alpha)$

$q(\alpha)$	J values without controls	J values with controls
$q(\alpha) = \delta(\alpha - 0.99)$	5.3241×10^4	2.9058×10^4
$q(\alpha) = \delta(\alpha - 0.90)$	7.3956×10^4	3.9262×10^4
$q(\alpha) = \delta(\alpha - 0.80)$	1.0973×10^5	5.1799×10^4
$q(\alpha) = \delta(\alpha - 0.70)$	1.4902×10^5	5.9303×10^4
$q(\alpha) = \delta(\alpha - 0.50)$	1.59904×10^5	2.1366×10^4
$q(\alpha) = \Gamma(3 - \alpha)$	7.3939×10^4	4.0097×10^4
$q(\alpha) = \Gamma(2 - \alpha)$	4.9095×10^4	2.6756×10^4
$q(\alpha) = 0.75\alpha$	1.9796×10^4	1.3360×10^4

6. Numerical simulations

In this section to approximate the optimality system which given by (10) and (11) with (12) numerically, we use proposition proposition (1) and Lemma (2), subsequently, we can approximate numerically the optimality system (11) and (10) with (12) as follows:

$$ {}_{t_f}^{C}D_t^{q(\alpha)}\lambda_\iota = \frac{\partial H}{\partial \vartheta_\iota},\ \iota = 1,\ \ldots,\ 8, \tag{17}$$

where,

$$\vartheta_\iota = \{t,\ S,\ E,\ I,\ P,\ A,\ H,\ R,\ F,\ u_I,\ u_P,\ u_h,\ l = 1,\ \ldots,\ 8\},$$

$${}_0^C D_t^{q(\alpha)} \vartheta_l = \frac{\partial H}{\partial \lambda_\kappa},\ = 1,\ \ldots,\ 8, \tag{18}$$

and it is also required:

$$\lambda_\iota(T_f) = 0,\ \iota = 1,\ 2,\ \ldots,\ 8. \tag{19}$$

To solve the optimality system (17) and (18) with (19) numerically, we constructed a new method from nonstandard finite difference method and the approximation of the composite Simpson's rule, this method given in (16).

Next, we shall show the numerical simulations in two cities Spain and Wuhan. To fit the model (1) to real data, we used parameters from [29] and fitted the remaining values to data collected in Spain from 25 February to 16 May 2020 and Wuhan from 4 January to 9 March 2020. For all these cases we have considered the official data published by the WHO. The total population of Wuhan is about $11 million$. During the COVID-19 outbreak, there was a restriction of movements of individuals due to quarantine in the city. As a consequence, there was a limitation on the spread of the disease. In agreement, in our model, we consider the total population, $N = \dfrac{11,000,000}{250}$. Also, since in some parts of Spain there is more concentrated population and intensive use of public transportation, we consider $N = \dfrac{47,000,000}{425}$. Figure 1 showed the comparison between real data from WHO for Spain versus the simulation of the proposed model at $q(\alpha) = \Gamma(2-\alpha)$ and $q(\alpha) = \delta(\alpha - 0.93)$. Figure 2 showed the comparison between real data from WHO for Wuhan versus the simulation of the proposed model at $q(\alpha) = \Gamma(2-\alpha)$ and $q(\alpha) = \delta(\alpha - 0.983)$. Figure 3 compares the solutions of the proposed model with and without control cases using the proposed method (16) and $q(\alpha) = \Gamma(3-\alpha)$. We noted that the number of infected people is reducing in control cases. We noticed from Figure 4 that the solutions in the case of the fractional order $\alpha = 0.90$. are completely identical to the solutions in case $q(\alpha) = \delta(\alpha - 0.90)$. This means that we can obtain the fractional order derivatives as a special case from the distributed order fractional in case $q(\alpha) = \delta(\alpha - a)$, $0 < a < 1$. Figure 5 shows how the approximation solutions in the controlled case are changed using the introduced NSFDM change when $q(\alpha)$ takes different values of the Dirac function. Also, Figure 6 shows how the behavior of solutions is changed using different values of $q(\alpha)$ takes different values. Table 3 provides the value of the cost functional for various values of $q(\alpha)$ with and without controls.

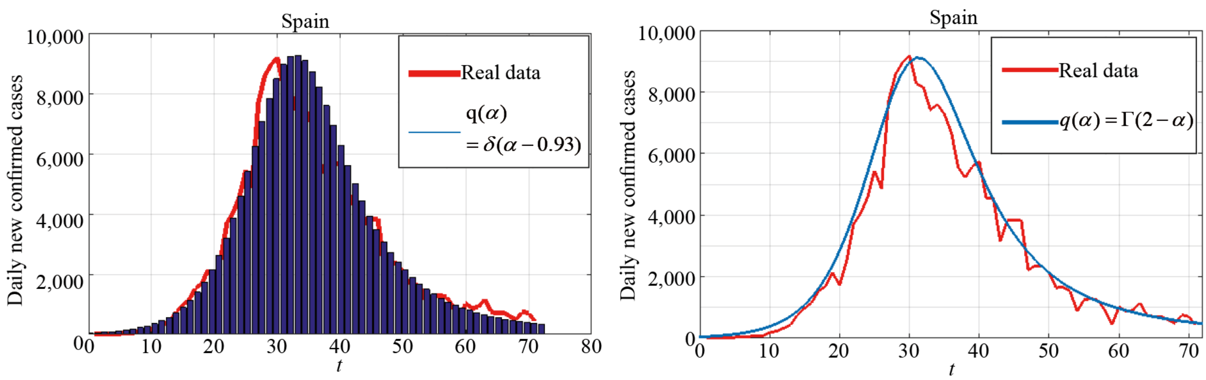

Figure 1. Number of confirmed cases per day in Spain

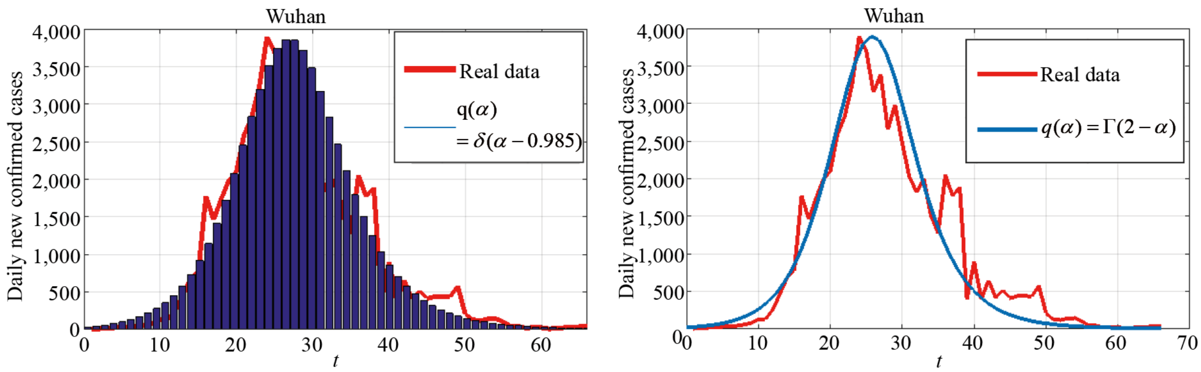

Figure 2. Number of confirmed cases per day in Wuhan

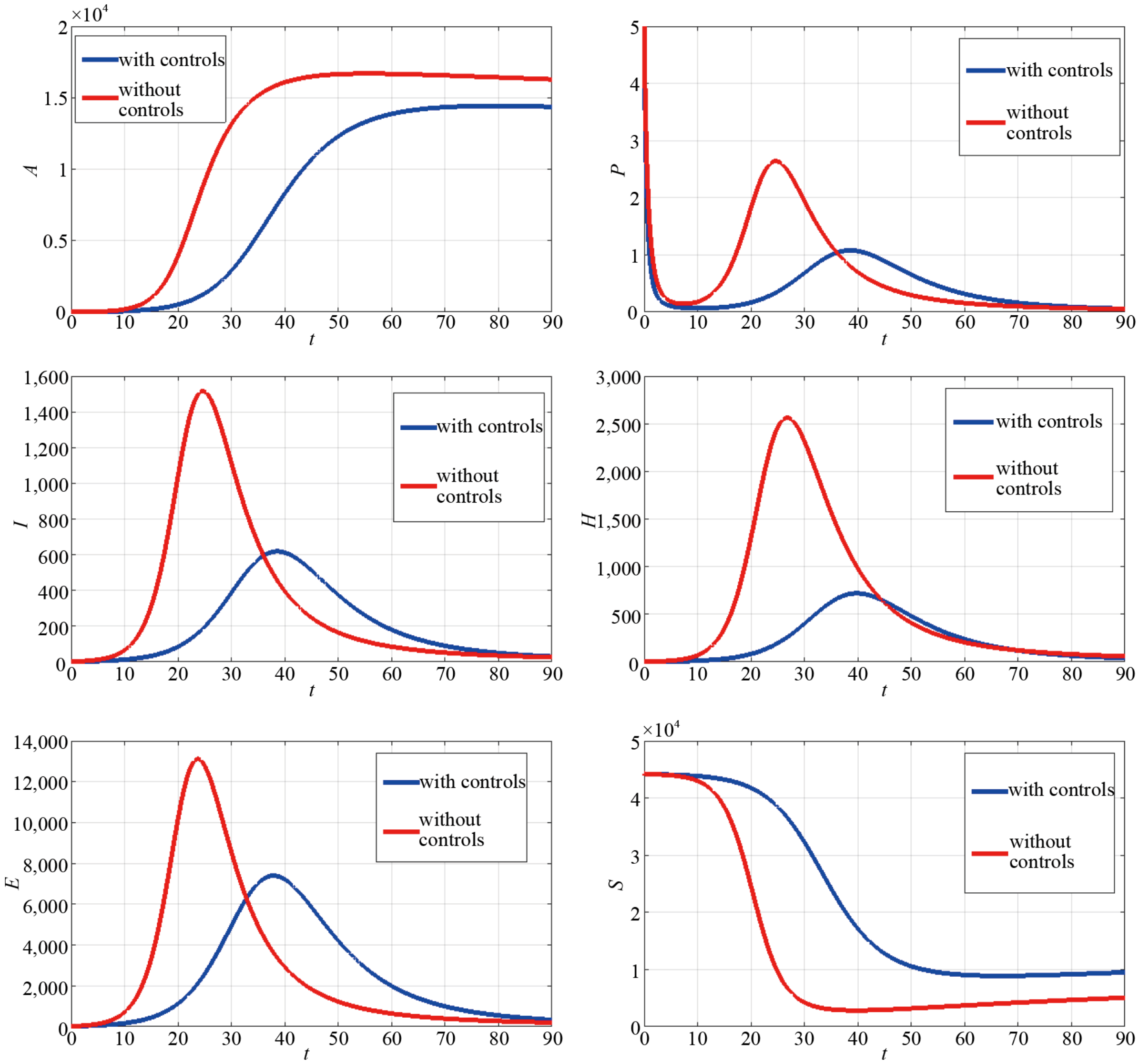

Figure 3. Comparison of the suggested model's solutions at $q(\alpha) = \Gamma(3 - \alpha)$, both with and without control

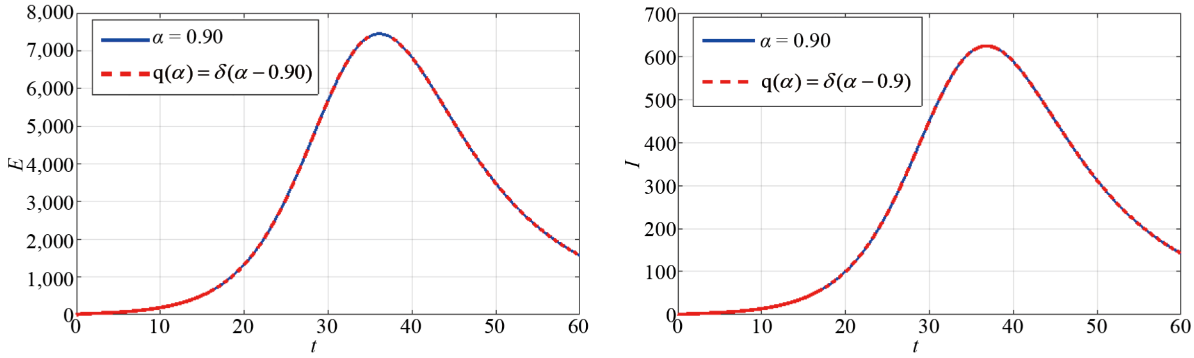

Figure 4. The numeical soluation of E and I in fractional order system and distributed order fractional system

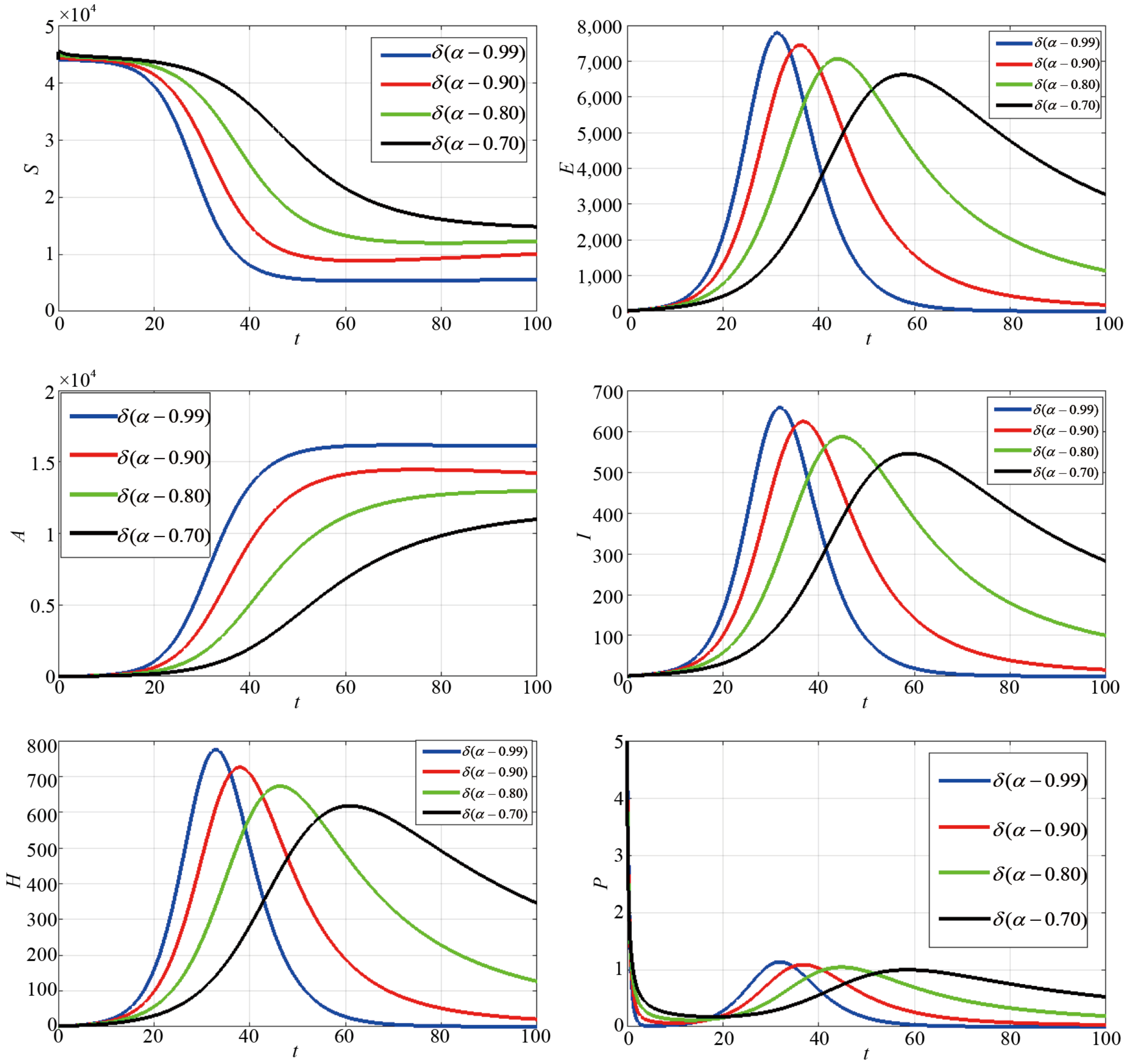

Figure 5. Behavior of the solution at different Dirac function

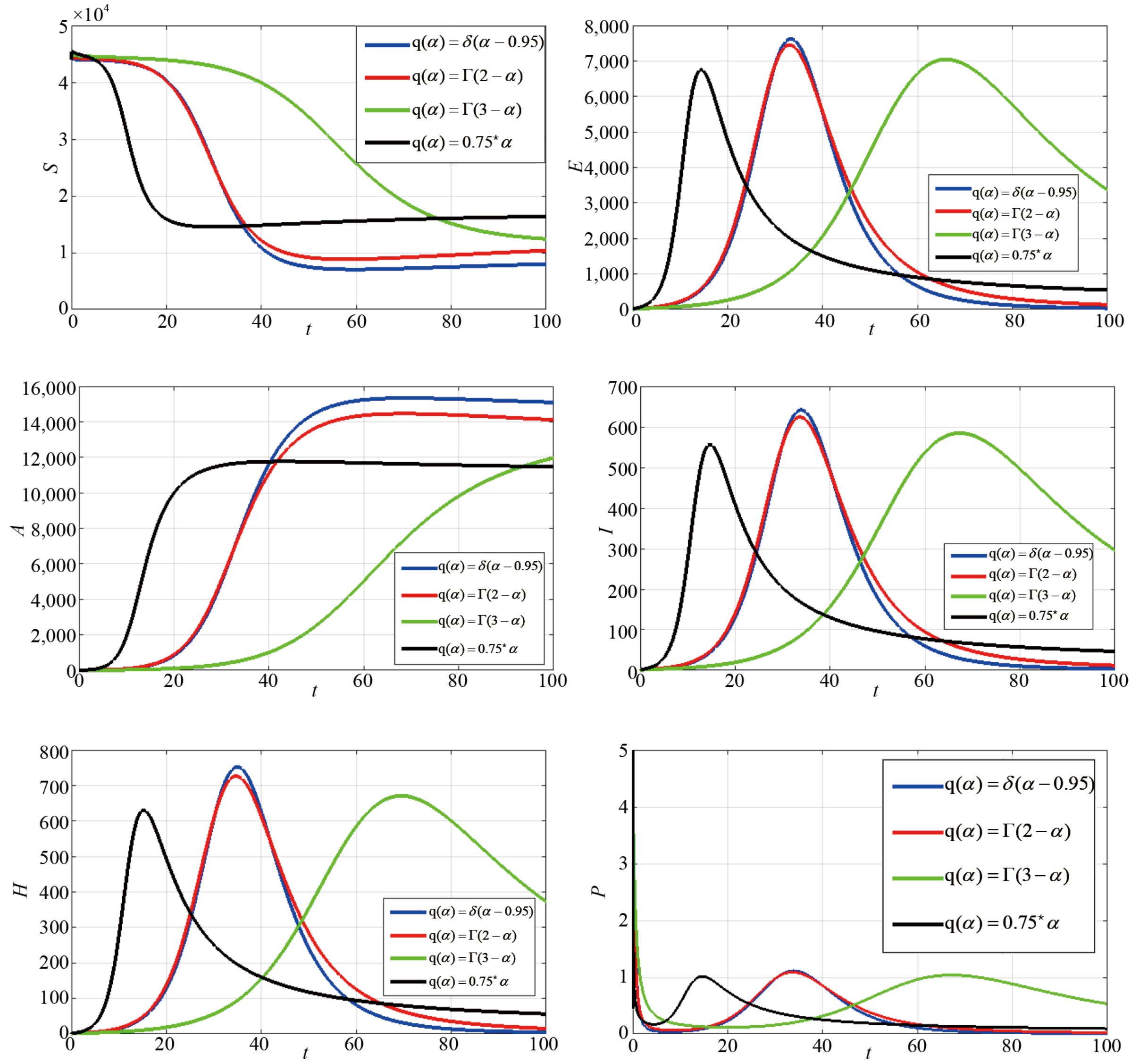

Figure 6. Behavior of the solution at differnt $q(\alpha)$

7. Concluding remarks

This study contributes to the use of distributed fractional optimum control approaches for epidemic models. The combination of distributed order fractional derivative and optimum control in the model improves the dynamics while increasing its complexity. This article presents a novel model of dispersed order fractional Coronavirus.

Many properties of the proposed model were analyticity studied. A new nonstandard finite difference scheme is constructed to solve the optimality system. Moreover, the fractional order system is obtained as a special case from the distributed fractional system when we put $q(\alpha) = \delta(\alpha - a)$, $0 < a < 1$. We've used a kind of Pontryagin's maximum principle to reduce the spread of illness among healthy people successfully. The suggested COVID-19 model accurately describes WHO data from Spain and Wuhan, with varying $q(\alpha)$. In order to maintain consistency with the physical model problem, a new parameter $\mu^{1-q(\alpha)}$ is given. Numerical simulations are provided to show how the proposed model is an extension of the fractional-order model, and how this dynamical system is more suited to describing the biological problems with memory than the fractional-order model. Furthermore, numerical results demonstrate the validity and application of the suggested approach. In future work, we will extend this work to crossover model with optimal control. All codes were written and debugged by Matlab program On Computer core i7.

Confilict of interest

The authors declare no competing interest

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