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Research Article

# On Equitable Colorings of Windmill Graphs

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**Abstract:** Let $G$ be an undirected simple graph. Graph coloring is a special case of labeling, and $G$ is said to admit a proper coloring if no two neighbouring vertices of it are given the identical color. The vertices of identical color constitute a color class. A graph is $p$-colorable if it has a $p$-coloring. The chromatic number of $G$, denoted by $\chi(G)$, is the minimum $p$ such that $G$ is $p$-colorable. A graph $G$ is equitably $p$-colorable if it has a $p$-coloring and the absolute difference in size between any two distinct color classes is at most 1. The equitable chromatic number of $G$, denoted by $\chi_{=}(G)$, is the minimum $p$ such that $G$ is equitably $p$-colorable. The equitable chromatic threshold of $G$, denoted by $\chi_{=}^{*}(G)$, is the minimum $p'$ such that $G$ is equitably $p$-colorable for all $p \geq p'$. A windmill graph $W_n^m$ consists of $m$ copies of the complete graph $K_n$, with every vertex connected to a common vertex. In this paper, we give exact values of $\chi_{=}(G)$ and $\chi_{=}^{*}(G)$ when $G$ is a windmill graph, bistar windmill graph, cycle windmill graph, and complete windmill graph.

***Keywords*:** equitable coloring, equitable chromatic number, equitable chromatic threshold, windmill graphs

**MSC:** 05C15, 05C40, 05C38, 05C76

## 1. Introduction

Consider $G = (V, E)$, or simply $G$ to be a finite, connected, undirected, and simple graph with $V(G)$ and $E(G)$ respectively, denoting the vertex set and edge set. For standard graph theory notations here, we refer to [1–3]. A (proper) $p$-coloring of $G$ is a labeling $\sigma : V(G) \to [p] = [1, 2, 3, \ldots, p]$ such that the neighbouring vertices admit distinct labels. The labels are conventionally referred to as colors, and the vertices of the same color constitute a color class. The chromatic number, $\chi(G) = \min\{p : G \text{ is a } p\text{-coloring}\}$. An equitable $p$-coloring of $G$ is a $p$-coloring such that any two disjoint color classes differ in cardinality, which does not exceed 1. A graph is called an equitably $p$-colorable if it has an equitable $p$-coloring. The equitable chromatic number, $\chi_{=}(G) = \min\{p : G \text{ is equitably } p\text{-colorable}\}$ and the equitable chromatic threshold, $\chi_{=}^{*}(G) = \min\{p' : G \text{ is equitably } p\text{-colorable for all } p \geq p'\}$ [4, 5]. Let deg $G(u)$, or deg$(u)$ in short, denote the degree of the vertex $u$ in $G$. The maximum degree of $G$, denoted by $\Delta(G)$, and the vertices that have degree one are known as pendent vertices. Let $\lceil u \rceil$ and $\lfloor u \rfloor$ stand for, respectively, the least integer not smaller than $u$ and the greatest integer not more than $u$. In 1973, Meyer [6] introduced first the notion of equitable colorability. The impetus for his desire stemmed from the application given by Tucker [7]. In certain independent industrial systems, one can run into the task of achieving an equitable splitting of a system characterized by conflicting relationships within sub-systems that are free from conflicts. Equitable graph coloring provides a viable approach to simulate such circumstances.


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For instance, a garbage collection problem where the vertices of the network represent the routes for trash collection. In cases where two routs must not be used on the same day, an edge is added to join the corresponding two vertices. Consequently, the task of selecting a certain daily for each route transforms into the intriguing problem of the six colorings of the graph. From a pragmatic standpoint, it may be more advantageous to distribute a comparable quantity of routes throughout the span of six consecutive days. Hence, it is essential to provide a uniform distribution of colors over the graph, using a total of six distinct colors. Timetabling scheduling is yet another scenario where the use of equitable edge coloring turns advantageous.

The equitable graph coloring progress is discussed in [8]. In [9], Erdős conjectured that every $G$ with maximum degree $\Delta(G) \leq p$, permits an equitable $(p+1)$-coloring. Hajnal and Szemerédi [10] proved it with a long and complicated proof. It is interesting to note that if a graph $G$ is equitably $p$-colorable, it does not imply that it is equitably $(p+1)$-colorable. For instance, $K_{3,\,3}$, in which two colors are permitted to be colored equitably but not three. One can observe that (i) $\chi_{=}(P_n) = 2$, (ii) $\chi_{=}(K_{n,\,n}) = 2$, (iii) $\chi_{=}(K_n) = n$ and hence (iv)

$$\chi_{=}(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

For some notable results on $\chi_{=}(G)$, see Table 1.

**Table 1.** Some existing results on $\chi_{=}(G)$

| Graph $G$ | $\chi_{=}(G)$ | Reference |
|---|---|---|
| $G$ is semi - planar graph and $m \geq 9$ with $\Delta \leq m$ | $m$ | [11] |
| $G$ is planar graph and $m \geq 8$ with $\Delta \leq m$ | $m$ | [11] |
| Connected bipartite graph $G \neq K_{n,\,n}$ | $\Delta$ | [12] |
| Bipartite graph with $\lvert E\rvert \leq \left\lfloor \frac{m}{n+1} \right\rfloor (m-n)+2m$ | upper bound : $\left\lceil \frac{m}{n+1} \right\rceil + 1$ | [12] |
| A tree with $\lvert E\rvert = m+n-1 \leq \left\lfloor \frac{m}{n+1} \right\rfloor (m-n)+2m$ | upper bound : $\left\lceil \frac{(m+n+1)}{(min\{m,\,n\}+1)} \right\rceil$ | [12] |
| A tree with maximum degree $\Delta$ | $\left\lceil \frac{\Delta}{2} \right\rceil + 1$ | [6] |
| A star graph $K_{1,\,n}$ is of order $n+1$ | $\left\lceil \frac{n}{2} \right\rceil + 1$ | [6] |

It is straightforward to check that $\chi(G) \leq \chi_{=}(G) \leq \chi^{*}_{=}(G)$ [4, 5, 13, 14]. In general, strict inequality is valid. For instance,

$$\chi(K_{1,\,4}) = 2 < \chi_{=}(K_{1,\,4}) = \chi^{*}_{=}(K_{1,\,4}) = 3,$$

$$\chi(K_{3,\,3}) = \chi_{=}(K_{3,\,3}) = 2 < \chi^{*}_{=}(K_{3,\,3}) = 4,$$

$$\chi(K_{5,\,8}) = 2 < \chi_{=}(K_{5,\,8}) = 3 < \chi^{*}_{=}(K_{5,\,8}) = 5.$$

**Definition 1** [1] A cycle $C_n$ is a graph on n vertices in which all the vertices are of degree 2.

**Definition 2** [1] A bipartite graph $G$ is one whose collection of vertices can be divided into two distinct subsets $U$ and $V$ in such a way that each edge in $G$ has one end in $U$ and the other end in $V$. The partition $(U, V)$ is said to be a bi partition of $G$.

**Definition 3** [1] A complete bipartite $K_{m, n}$ is a simple bipartite graph with bipartition partition $(U, V)$ in which each vertex of $U$ is connected to each vertex of $V$.

**Definition 4** A bistar graph $B_{n, n}$, is formulated by connecting the central vertices of dual copies of $K_{1, n}$ by an edge, and $B_{n, n}$ has vertex set $V(B_{n, n}) = \{a, a_1, a_2, \ldots, a_n\} \cup \{b, b_1, b_2, \ldots, b_n\}$, where $a$ and $b$ are central vertices and $\{a_1, a_2, \ldots, a_n, b_1, b_1, \ldots, b_n\}$ are all pendent vertices.

**Definition 5** The bloom graph $B_{m, n}$, $m, n > 2$ is with $V(B_{m, n}) = \{(a, b) : 0 \le a \le m-1, 0 \le b \le n-1\}$ and two distinct vertices $(a_1, b_1)$ and $(a_2, b_2)$ are adjacent if and only if

(i) $a_1 = a_2 - 1$ and $b_1 = b_2$

(ii) $a_1 = a_2 = 0$ and $b_1 + 1 \equiv b_2 (mod\ n)$

(iii) $a_1 = a_2 = m-1$ and $b_1 + 1 \equiv b_2 (mod\ n)$

(iv) $a_1 = a_2 - 1$ and $b_1 + 1 \equiv b_2 (mod\ n)$.

**Definition 6** A $(k, n)$-barbell graph, $B(k, n)$, is of order $kn$ and constructed by joining $n$ copies of clique $K_k$, $k \ge 3$, each with a bridge.

**Proposition 1** [2] If $K_n \subset G$, then $\chi(G) \ge n$.

**Conjecture 1** (Equitable Coloring Conjecture (ECC) [6]) For any $G \ne K_n$ or $G \ne C_{2n+1}$, $\chi_=(G) \le \Delta(G)$.

It has been demonstrated that this hypothesis holds for any graphs that include six vertices or less. To demonstrate that ECC is valid for all bipartite graph. Lih and Wu [12] provided substantial evidence.

Additionally, There is one more conjecture.

**Conjecture 2** (Equitable $\Delta$-coloring Conjecture (E$\Delta$CC) [15]) For any connected graph $G$, that does not contain cycles of odd length ($C_{2n+1}$ for any positive integer $n$), complete graphs on $n$ vertices ($K_n$ for any positive integer $n$), or complete bipartite graphs with equal-sized partitions ($K_{2n+1, 2n+1}$ for any positive integer $n$), the equitable chromatic number $\chi_=(G) = \Delta(G)$.

It has been shown that E$\Delta$CC is valid for certain kinds of graphs, such as bipartite graphs [12], outerplanar graphs with $\Delta \ge 3$ [16], and planar graphs with $\Delta \ge 13$ [17].

This paper deals with the equitable coloring of three graph structures known as the bistar windmill graph $BW_n^m$, cycle windmill graph $CW_n^m$, and complete windmill graph $KW_n^m$. These graphs are the variations of windmill graph $W_n^m$, where the central vertex is swapped by a bistar graph, a cycle graph, and a complete graph, respectively.

## 2. Windmill graph

Certain generalizations of the windmill graph are explored in this section.

The windmill graph $W_n^m$ [18], where $n \ge 3$ and $m \ge 2$, is formed by taking $m$ copies of $K_n$ joined at a central vertex (see Figure 1). For our objective, we represent the complete subgraphs of $W_n^m$ as $W_n^j$, for $j = 1, 2, \ldots, m$, the central vertex is denoted as $a$, and the vertices of $W_n^j$ by $\{a, a_1^j, a_2^j, \ldots, a_{n-1}^j\}$. Thus $W_n^m$, has order $m(n-1)+1$ and size $\dfrac{mn(n-1)}{2}$.

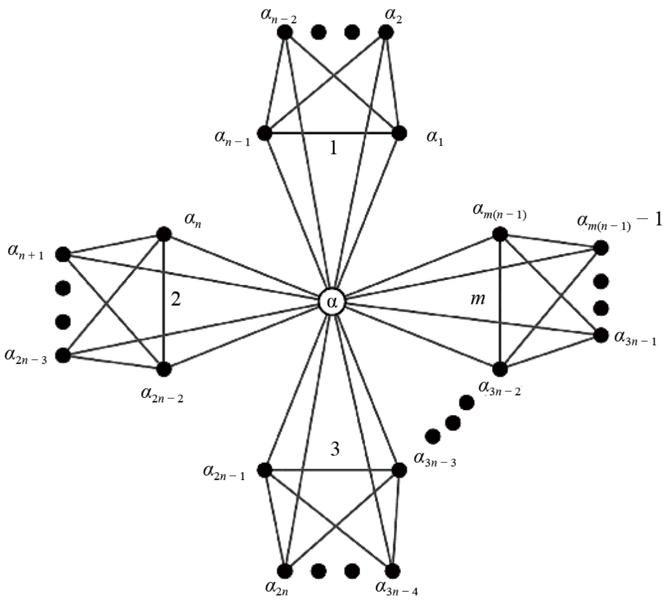


**Figure 1.** Windmill graph ($W_n^m$)

The subsequent theorem discusses $\chi_=(W_n^m)$ and $\chi_=^*(W_n^m)$.

**Theorem 1** If $m \geq 2$ and $n \geq 3$, then $\chi_=(W_n^m) = \chi_=^*(W_n^m) = \left\lceil \dfrac{m(n-1)+2}{2} \right\rceil$.

**Proof.** Let $V(W_n^m) = \{a\} \cup \{a_i : 1 \leq i \leq m(n-1)\}$ where $a$ represents the central vertex of $V(W_n^m)$. By the definition of windmill graph, the central vertex $a$ is adjacent to each $\{a_i : 1 \leq i \leq m(n-1)\}$. There is a color (say $c_1$), which is assigned to $a$, so that $c_1$ cannot be assigned to any vertex of $\{a_i : 1 \leq i \leq m(n-1)\}$, the color $c_1$ can be assigned at most one time in $W_n^m$. According to the principle of equitable coloring, the number of vertices of $W_n^m$, that are assigned an identical color is either 1 or 2. Therefore, $\chi_=(W_n^m) \geq \left\lceil \dfrac{m(n-1)+2}{2} \right\rceil$.

To prove : $\chi_=(W_n^m) \leq \left\lceil \dfrac{m(n-1)+2}{2} \right\rceil$.

Now, look at the following partitions to color the vertices of $W_n^m$ equitably in the next two cases:

**Case 1** when $m$ is even and $n$ is odd or even

Now use the partitions to color the vertices of $W_n^m$.

(a) $W_3^2$

$$V_1 = \{a\},\ V_2 = \{a_1,\ a_3\},\ V_3 = \{a_2,\ a_4\}$$

(b) $W_6^4$

$$V_1 = \{a\},\ V_2 = \{a_1,\ a_{11}\},\ V_3 = \{a_2,\ a_{12}\},$$

$$V_4 = \{a_3,\ a_{13}\},\ V_5 = \{a_4,\ a_{14}\},\ \ldots$$

$$V_{10} = \{a_9,\ a_{19}\},\ V_{11} = \{a_{10},\ a_{20}\}$$

Split $V(W_n^m)$ as follows:
$V(W_n^m) = V' \cup V''$ where $V' = V_1 = \{a\}$ and $V'' = \{a_i : 1 \le i \le n-1\} \cup \{a_j : n \le j \le m(n-1)\}$.
Obviously, $V' \cup V'' = V(W_n^m)$ and $V' \cap V'' = \phi$ with $|V'| = 1$ and $|V''| = m(n-1)$.
Now, partition $V''$ into $\left\lceil \frac{m(n-1)+2}{2} \right\rceil - 1$ color classes each having two vertices of $V''$ as follows.

$$V_i = \left\{\{a_{i-1},\, a_j\} :\ 2 \le i \le \left\lceil \frac{m(n-1)+2}{2} \right\rceil - 1,\ \left\lceil \frac{m(n-1)+2}{2} \right\rceil \le j \le m(n-1)\right\}$$

It is evident that $\bigcup_{i=2}^{\left\lceil \frac{m(n-1)+2}{2} \right\rceil} (V_i) = V'' = V_2 \cup V_3 \cup \ldots, \cup V_{\left\lceil \frac{m(n-1)+2}{2} \right\rceil}$ and $V_i \cap V_j = \phi$ and $||V_i| - |V_j|| \le 1\ \forall$ $i \ne j$.

**Case 2** when $m$ is odd and $n$ is even.
Now, use the partitions to color the vertices of $W_n^m$.
(a) $W_4^3$

$$V_1 = \{a\},\ V_2 = \{a_1,\, a_5\},\ V_3 = \{a_2,\, a_6\},\ V_4 = \{a_3,\, a_7\},\ V_5 = \{a_4,\, a_8\},\ V_6 = \{a_9\}$$

(b) $W_6^3$

$$V_1 = \{a\},\ V_2 = \{a_1,\, a_8\},\ V_3 = \{a_2,\, a_9\},$$

$$V_4 = \{a_3,\, a_{10}\},\ V_5 = \{a_4,\, a_{11}\},\ V_6 = \{a_5,\, a_{12}\},$$

$$V_7 = \{a_6,\, a_{13}\},\ V_8 = \{a_7,\, a_{14}\},\ V_9 = \{a_{15}\}$$

Divide $V(W_n^m)$ as follows:
$V(W_n^m) = V' \cup V'' \cup V'''$ where $V' = V_1 = \{a\}$ and $V'' = \{a_i : 1 \le i \le n-1\} \cup \{a_j : n \le j \le mn-m-1\}$ and $V''' = \{a_{mn-m}\}$.
It is clear that $V' \cup V'' \cup V''' = V(W_n^m)$ and $V' \cap V'' \cap V''' = \phi$ with $|V'| = 1$ and $|V''| = m(n-1)$ and $|V'''| = 1$.
Now, partition $V''$ into $\left\lceil \frac{m(n-1)+2}{2} \right\rceil - 2$ color classes each having two vertices of $V''$ as follows:

$$V_i = \left\{\{a_i,\, a_j\} :\ 2 \le i \le \left\lceil \frac{m(n-1)+2}{2} \right\rceil - 1,\ \left\lceil \frac{m(n-1)+2}{2} \right\rceil \le j \le mn-m-1\right\}$$

It is evident that, $\bigcup_{i=2}^{\left\lceil \frac{m(n-1)+2}{2} \right\rceil - 1} V_i = V''$ and $V_i \cap V_j = \phi$ and absolute difference in size between any pairwise color classes not exceeding one for all $i \ne j$. Hence, $W_n^m$ is equitably-$\left\lceil \frac{m(n-1)+2}{2} \right\rceil$ colorable.

For $p \geq [\left\lceil \frac{m(n-1)+2}{2} \right\rceil]$, let

$$\sigma_k = \left\lfloor \frac{(m(n-1)+k}{p} \right\rfloor$$

and for $k \in [p]$. Since

$$\sigma_p = \left\lfloor \frac{(m(n-1))+p}{p} \right\rfloor \leq 2 \; or \; 3,$$

one can divide $V(W_n^m)$ into $p$-color class of sizes $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_p$. Hence, $W_n^m$ is equitably $p$-colorable for all $p \geq \left\lceil \frac{m(n-1)+2}{2} \right\rceil$. The theorem then follows. □

## 2.1 *Bistar windmill graph*

Let $BW_n^m$ be a bistar windmill graph formulated by replacing the central vertices of $(W_n^m)$, with a bistar graph $B_{m,n}$ (see, Figure 2). For our intention, we denote $W_n^j$, as the complete subgraphs of $BW_n^m$ and the vertices of $W_n^j$, by $\{a_1^j, a_2^j, \ldots, a_{n-1}^j, a_n^j\}$ or $\{b_1^j, b_2^j, \ldots, b_{n-1}^j, b_n^j\}$ for $j = 1, 2, \ldots, m$. Thus, $BW_n^m$ has order $2mn+2$ and size $2mn(n-1)+2m+1$. $\chi_=(BW_n^m)$ and $\chi_=^*(BW_n^m)$ are determined in the subsequent results.

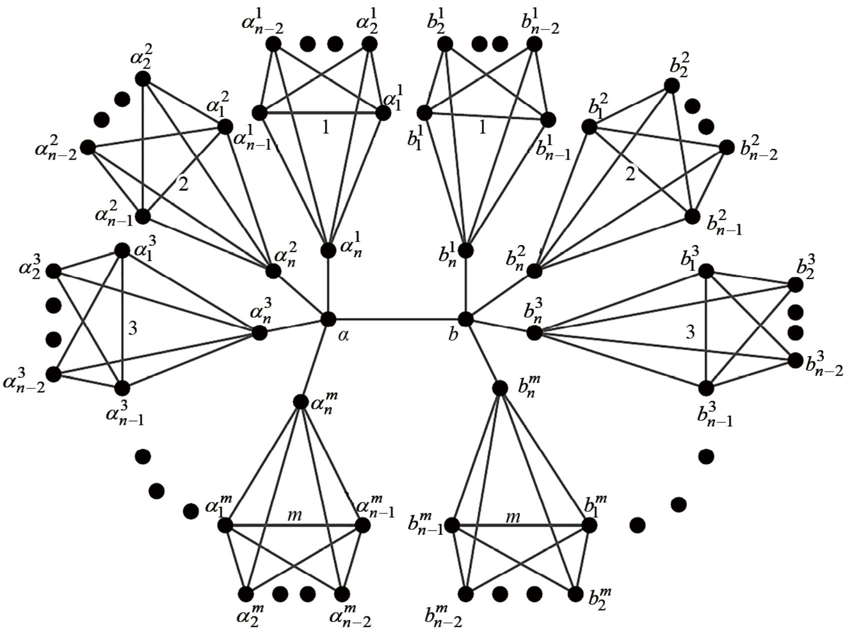


**Figure 2.** Bistar windmill graph $(BW_n^m)$

**Theorem 2** Let $BW_n^m$, be a bistar windmill graph. Then, $\chi_=(BW_n^m) = \chi_=^*(BW_n^m) = n$.

**Proof.** Let $V(BW_n^m) = \{a, b\} \cup \{a_i^j, b_i^j : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$, where $a$ and $b$ are the central vertices of $V(BW_n^m)$ and $E(BW_n^m) = \{aa_n^j : j = 1, 2, \ldots, m\} \cup \{a_i^j a_k^j : i \neq k, i, k = 1, 2, \ldots, n, j = 1, 2, \ldots, m\} \cup \{bb_n^j : j = 1, 2, \ldots, m\} \cup \{b_i^j b_k^j : i \neq k, i, k = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$. Also the vertex subset $\{a_1^j, a_2^j, \ldots, a_{n-1}^j, a_n^j\}$ or

$\{b_1^j,\ b_2^j,\ \ldots,\ b_{n-1}^j,\ b_n^j\}$ induces a complete subgraph of order $n$. It's fine to conclude the lower bound of $n$ immediately, since it follows from the existence of the $n$-clique.

Now, consider the following $n$ color classes for $V(BW_n^m)$ as follows:

$$V_1 = \{a\} \cup \{a_1^1,\ a_1^2,\ \ldots,\ a_1^m\} \cup \{b_1^1,\ b_1^2,\ \ldots, b_1^m\}$$

$$V_2 = \{a_2^1,\ a_2^2,\ \ldots,\ a_2^m\} \cup \{b\} \cup \{b_2^1,\ b_2^2,\ \ldots,\ b_2^m\}$$

$$V_3 = \{a_3^1,\ a_3^2,\ \ldots,\ a_3^m\} \cup \{b_3^1,\ b_3^2,\ \ldots,\ b_3^m\}$$

$$\vdots$$

$$V_{n-1} = \{a_{n-1}^1,\ a_{n-1}^2,\ \ldots,\ a_{n-1}^m\} \cup \{b_{n-1}^1,\ b_{n-1}^2,\ \ldots,\ b_{n-1}^m\}$$

$$V_n = \{a_n^1,\ a_n^2,\ \ldots,\ a_n^m\} \cup \{b_n^1,\ b_n^2,\ \ldots,\ b_n^m\}.$$

It is evident that $V(BW_n^m) = \bigcup\limits_{i=1}^{n} V_i$ and each $V_i's$ are disjoint color classes with $|V_1| = 2m+1$, $|V_2| = 2m+1$, $|V_3|$ $=,\ \ldots,\ |V_n| = 2m$, satisfying the required coloring condition $||V_i| - |V_j|| \leq 1, \forall\ i \neq j$. Therefore, $\chi_{=}(BW_n^m) \leq n$.

For $p \geq [n]$, let

$$\sigma_k = \left\lfloor \frac{2mn+2+k}{p} \right\rfloor$$

and for $k \in [p]$. Since

$$\sigma_p = \left\lfloor \frac{2mn+2+p}{p} \right\rfloor \leq \left\lceil \frac{2mn+2}{p} \right\rceil,$$

one can divide $V(W_n^m)$ into $p$-color classes of sizes $\sigma_1,\ \sigma_2,\ \sigma_3,\ \ldots,\ \sigma_p$. Hence $BW_n^m$ is equitably $p$-colorable for all $p \geq n$.

Conversely, it is obvious, $\chi_{=}(BW_n^m) \geq n$. Suppose that $\chi_{=}(BW_n^m) < n$. Since $BW_n^m$ consists of a complete subgraph of order $n$, and so $\chi(BW_n^m) \geq n$. It is clear from this contradiction that $\chi_{=}(BW_n^m) \geq n$. The result of the theorem follows.□

### 2.2 *Cycle windmill graph*

Let $CW_n^m$ be a cycle windmill graph obtained by changing the central vertex of $W_n^m$ with $C_n$ (see, Figure 3) [18]. For our motive, we name after the complete subgraphs of $CW_n^m$ by $W_n^j$, and the vertices as $\{a_1^j,\ a_2^j,\ \ldots,\ a_{n-1}^j,\ a_n^j:\ 1 \leq j \leq m\}$. Thus, $CW_n^m$ has order $mn$ and size $\dfrac{mn(n-1)}{2} + m$.

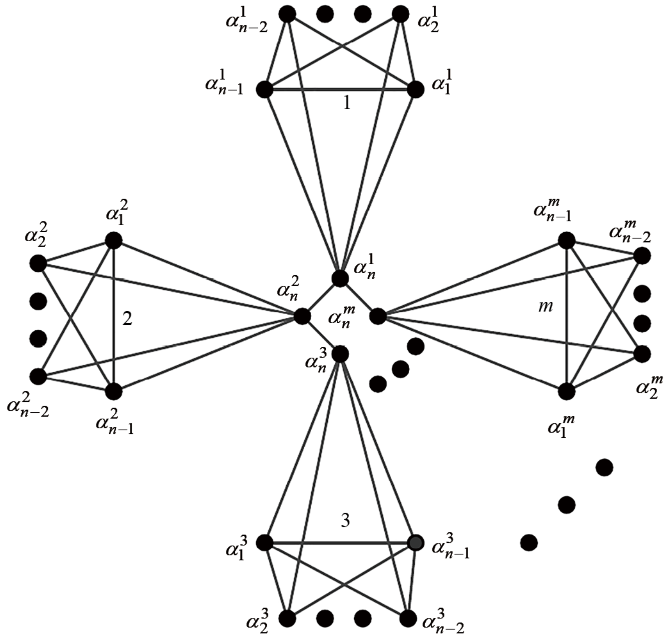


**Figure 3.** Cycle windmill graph ($CW_n^m$)

In the subsequent theorem, we have $\chi_{=}(CW_n^m)$ and $\chi^*_{=}(CW_n^m)$.

**Theorem 3** For $m \geq 3$, $n \geq 3$, $\chi_{=}(CW_n^m) = \chi^*_{=}(CW_n^m) = n$.

**Proof.** Let $V(CW_n^m) = \{a_i^j : 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and $E(CW_n^m) = \{a_n^j a_n^{j+1} : 1 \leq j \leq m-1\} \cup \{a_n^m a_n^1\} \cup \{a_i^j a_k^j :\ i \neq k,\ i,\ k = 1,\ 2,\ \ldots,\ n,\ 1 \leq j \leq m\}$. Also the vertex subset $\{a_1^j,\ a_2^j,\ \ldots,\ a_{n-1}^j a_n^j\}$ induces a clique of order $n$.

Now we investigate the color class for $V(CW_n^m)$ in the subsequent two cases:

**Case 1** when $m$ is odd integer and $n \geq 3$.

**Subcase 1** when $m = 6k - 3$, $k \in \mathbb{N}$.

Now, explore the subsequent $n$ color classes for $V(CW_n^m)$

$$V_1 = \{a_n^1,\ a_{n-1}^2,\ a_{n-2}^3,\ a_n^4,\ a_{n-1}^5,\ a_{n-2}^6,\ \ldots,\ a_{n-1}^{m-1},\ a_{n-2}^m\}$$

$$V_2 = \{a_1^1,\ a_n^2,\ a_{n-1}^3,\ a_1^4,\ a_n^5,\ a_{n-1}^6,\ \ldots,\ a_1^{m-2},\ a_n^{m-1},\ a_{n-1}^m\}$$

$$V_3 = \{a_2^1,\ a_1^2,\ a_n^3,\ a_2^4,\ a_1^5,\ a_n^6,\ \ldots,\ a_2^{m-2},\ a_1^{m-1},\ a_n^m\}$$

$$\vdots$$

$$V_{n-1} = \{a_{n-2}^1,\ a_{n-3}^2,\ a_{n-4}^3,\ a_{n-2}^4,\ a_{n-3}^5, a_{n-4}^6,\ \ldots,\ a_{n-3}^{m-1},\ a_{n-4}^m\}$$

$$V_n = \{a_{n-1}^1,\ a_{n-2}^2,\ a_{n-3}^3,\ a_{n-1}^4,\ a_{n-2}^5,\ a_{n-3}^6,\ \ldots,\ a_{n-2}^{m-1},\ a_{n-3}^m\}$$

It is clearly observed that all $V_i's$ are mutually disjoint color classes with $|V_1| = |V_3| = |V_2| =, \ldots, = |V_n| = m$. These partitions of $V(CW_n^m)$ satisfy the required constrain that the absolute value of the difference between the sizes of the disjoint sets $V_i$ and $V_j$ is less than or equal to one, for all $i$ and $j$.

**Subcase 2** when $m = 6k - 1$, $k \in \mathbb{N}$.

Now, study the subsequent $n$ color classes of $V(CW_n^m)$

$$V_1 = \{a_n^1,\ a_{n-1}^2,\ a_{n-2}^3,\ a_n^4,\ a_{n-1}^5,\ \ldots,\ a_n^{m-1},\ a_{n-1}^m\}$$

$$V_2 = \{a_1^1,\ a_n^2,\ a_{n-1}^3,\ a_1^4,\ a_n^5,\ \ldots,\ a_1^{m-1},\ a_n^m\}$$

$$V_3 = \{a_2^1,\ a_1^2,\ a_n^3,\ a_2^4,\ a_1^5,\ \ldots,\ a_2^{m-1},\ a_1^m\}$$

$$\vdots$$

$$V_{n-1} = \{a_{n-2}^1,\ a_{n-3}^2,\ a_{n-4}^3,\ a_{n-2}^4,\ a_{n-3}^5,\ \ldots,\ a_{n-2}^{m-1},\ a_{n-3}^m\}$$

$$V_n = \{a_{n-1}^1,\ a_{n-2}^2,\ a_{n-3}^3,\ a_{n-1}^4,\ a_{n-2}^5,\ \ldots,\ a_{n-1}^{m-1},\ a_{n-2}^m\}.$$

Obviously, all $V_i's$ are mutually disjoint color classes with size $m$. These color classes of $V(CW_n^m)$ satisfy the constrain that the absolute value of the difference between the sizes of the disjoint sets $V_i$ and $V_j$ is less than or equal to one, for all $i$ and $j$.

**Subcase 3** when $m = 6k + 1$, $k \in \mathbb{N}$.

Now, suggest the following $n$ color classes for $V(CW_n^m)$.

$$V_1 = \{a_n^1,\ a_{n-1}^2,\ a_{n-2}^3,\ a_n^4,\ a_{n-1}^5,\ a_{n-2}^6,\ a_n^7,\ \ldots,\ a_{n-2}^{m-1},\ a_n^m\}$$

$$V_2 = \{a_1^1,\ a_n^2,\ a_n^3,\ a_1^4,\ a_n^5,\ a_{n-1}^6,\ a_n^7,\ \ldots,\ a_{n-1}^{m-1},\ a_1^m\}$$

$$V_3 = \{a_2^1,\ a_1^2,\ a_n^3,\ a_2^4,\ a_1^5,\ a_n^6,\ a_n^7,\ \ldots,\ a_n^{m-1},\ a_2^m\}$$

$$\vdots$$

$$V_{n-1} = \{a_{n-2}^1,\ a_{n-3}^2,\ a_{n-4}^3,\ a_{n-2}^4,\ a_{n-3}^5, a_{n-4}^6,\ \ldots,\ a_{n-4}^{m-1},\ a_{n-2}^m\}$$

$$V_n = \{a_{n-1}^1,\ a_{n-2}^2,\ a_{n-3}^3,\ a_{n-1}^4,\ a_{n-2}^5,\ a_{n-3}^6,\ \ldots,\ a_{n-3}^{m-1},\ a_{n-1}^m\}$$

Clearly, all the above subcases satisfy the required constrain, $||V_i| - |V_j|| \le 1, \forall\ i \ne j$.

**Case 2** when $m$ is an even integer and $n \ge 3$.

Now, look at the following $n$ color classes for $V(CW_n^m)$.

$$V_1 = \{a_n^1,\, a_{n-1}^2,\, a_n^3,\, a_{n-1}^4,\, \ldots,\, a_n^{m-1},\, a_{n-1}^m\}$$

$$V_2 = \{a_1^1,\, a_n^2,\, a_1^3,\, a_n^4,\, \ldots,\, a_1^{m-1},\, a_n^m\}$$

$$V_3 = \{a_2^1,\, a_1^2,\, a_2^3,\, a_1^4,\, \ldots,\, a_2^{m-1},\, a_1^m\}$$

$$\vdots$$

$$V_{n-1} = \{a_{n-2}^1,\, a_{n-3}^2,\, a_{n-2}^3,\, a_{n-3}^4,\, \ldots,\, a_{n-2}^{m-1},\, a_{n-3}^m\}$$

$$V_n = \{a_{n-1}^1,\, a_{n-2}^2,\, a_{n-1}^3,\, a_{n-2}^4,\, \ldots,\, a_{n-1}^{m-1},\, a_{n-2}^m\}.$$

Clearly, all $V_i's$ are mutually disjoint color classes with $|V_1| = |V_3| = |V_2| = \ldots, = |V_n| = m$. Also the required equitable coloring condition $||V_i| - |V_j|| \leq 1$ for all $i \neq j$ holds in above two cases. Therefore $CW_n^m$ is equitably $n$-colorable.

For $p \geq [n]$, let

$$\sigma_k = \left\lfloor \frac{mn+k-1}{p} \right\rfloor$$

and for $k \in [p]$. Since

$$\sigma_p = \left\lfloor \frac{mn+p-1}{p} \right\rfloor \leq \left\lceil \frac{mn}{p} \right\rceil,$$

one can divide $V(W_n^m)$ into $p$ - color classes of sizes $\sigma_1,\, \sigma_2,\, \sigma_3,\, \ldots,\, \sigma_p$. Hence $W_n^m$ is equitably $p$-colorable for all $p \geq n$.

Conversely, it is obvious, $\chi_{=}(CW_n^m) \geq n$. Suppose that $\chi_{=}(CW_n^m) < n$. Since $CW_n^m$ consists of a complete subgraph of order $n$, and so $\chi(CW_n^m) \geq n$. It is clear from this contradiction that $\chi_{=}(CW_n^m) \geq n$. $\square$

## 2.3 *Complete windmill graph*

In this section, we analyze the equitable chromatic number and equitable chromatic threshold of the complete windmill graph and its generalizations. The complete windmill graph, denoted as $KW_n^m$, is obtained by swapping the central vertex of $W_n^m$ with $K_n$, (see Figure 4) [18]. The graph $KW_n^m$ has a total of $mn$ vertices and $\frac{mn(n-1)}{2} + \frac{m(m-1)}{2}$ lines. To analyze its properties, consider the complete subgraphs of $KW_n^m$ as $W_n^j$, where the vertices are represented as $\{a_1^j,\, a_2^j,\, \ldots,\, a_{n-1}^j,\, a_n^j :\ 1 \leq j \leq m\}$.

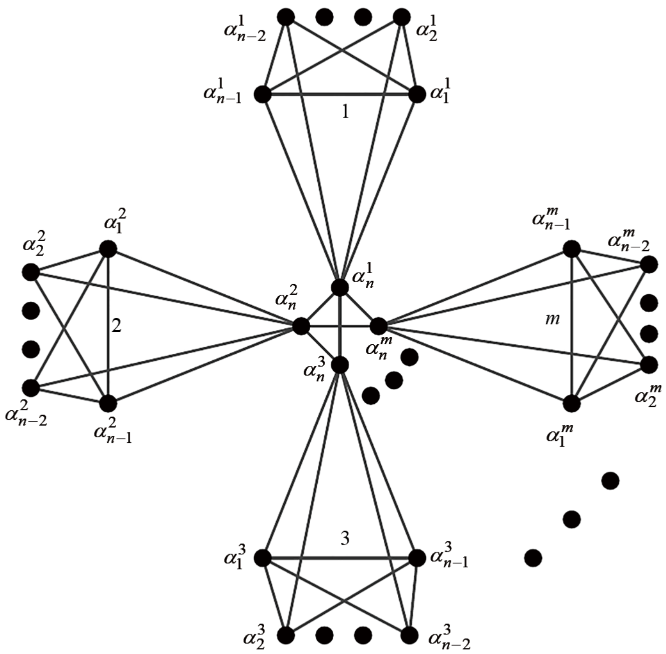


**Figure 4.** Complete windmill graph $(KW_n^m)$

The following theorems determine $\chi_=(KW_n^m)$ and $\chi^*_=(KW_n^m)$.

**Theorem 4** If $m,\ n \in \mathbb{N}$ with $m \geq 2$, $n \geq 3$, and $m < n$, then $\chi_=(KW_n^m) = \chi^*_=(KW_n^m) = n$.

**Proof.** Let $V(KW_n^m) = \{a_i^j:\ 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and $E(KW_n^m) = \{a_i^j a_k^j:\ i \neq k,\ 1 \leq i,\ k \leq n,\ 1 \leq j \leq m\} \cup \{a_n^j a_n^k:\ j \neq k,\ j,\ k = 1,\ 2,\ \ldots,\ m\}$. Also the vertex subset $\{a_1^j,\ a_2^j,\ \ldots,\ a_{n-1}^j,\ a_n^j\}$ of $V(KW_n^m)$ induces a clique of order $n$, $\chi(KW_n^m) \geq n$ by equitable coloring, concludes that $\chi_=(KW_n^m) \geq n$. To prove : $\chi_=(KW_n^m) \leq n$.

Now, investigate the following $n$ color classes of $V(KW_n^m)$ as follows.

$$V_1 = \{a_n^1,\ a_{n-1}^2,\ a_{n-2}^3,\ \ldots,\ a_{n-(m-1)}^m\}$$

$$V_2 = \{a_1^1,\ a_n^2,\ a_{n-1}^3,\ \ldots,\ a_{n-(m-2)}^m\}$$

$$V_3 = \{a_2^1,\ a_1^2,\ a_n^3,\ \ldots,\ a_{n-(m-3)}^m\}$$

$$\vdots$$

$$V_{n-1} = \{a_{n-2}^1,\ a_{n-3}^2,\ a_{n-4}^3,\ \ldots,\ a_{n-(m+1)}^m\}$$

$$V_n = \{a_{n-1}^1,\ a_{n-2}^2,\ a_{n-3}^3,\ \ldots,\ a_{n-m}^m\}.$$

Evidently, $\bigcup_{i=1}^{n} V_i = V(KW_n^m)$ are mutually disjoint color classes such that $|V_1| = |V_2| =, \ldots, = |V_n| = m$ and also satisfy the criteria that the absolute value of the difference between the sizes of sets $V_i$ and $V_j$ is less than or equal to 1, for all $i \neq j$. Hence $KW_n^m$ is equitably $n$-colorable.

For $p \geq [n]$, let

$$\sigma_k = \left\lfloor \frac{mn+k-1}{p} \right\rfloor$$

and for $k \in [p]$, Since

$$\sigma_p = \left\lfloor \frac{mn+p-1}{p} \right\rfloor \leq \left\lceil \frac{mn}{p} \right\rceil,$$

one can divide $V(W_n^m)$ into $p$ color classes of sizes $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_p$. Hence $KW_n^m$ is equitably $p$-colorable for all $p \geq n$.

Conversely, it is obvious, $\chi_=(KW_n^m) \geq n$. Suppose that $\chi_=(KW_n^m) < n$. Since $KW_n^m$ consists of a complete subgraph of order $n$, and so $\chi(KW_n^m) \geq n$. It is clear from this contradiction that $\chi_=(KW_n^m) \geq n$. □

**Theorem 5** For $n \geq 3$, and $m = 2n$, $\chi_=(KW_n^m) = \chi_=^*(KW_n^m) = m$.

**Proof.** Let $V(KW_n^m) = \{a_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(KW_n^m) = \{a_i^j a_k^j : i \neq k, i, k = 1, 2, \ldots, n, j = 1, 2, \ldots, m\} \cup \{a_n^j a_n^k : j \neq k, j, k = 1, 2, \ldots, m\}$. Also the vertex subset $\{a_n^1, a_n^2, \ldots, a_n^{m-1}, a_n^m\}$ of $V(KW_n^m)$ induces a clique of order $m$, $\chi(KW_n^m) \geq m$ by equitable coloring $\chi_=(KW_n^m) \geq m$.

To prove : $\chi_=(KW_n^m) \leq m$. Now, investigate the following $m$ color classes of $V(KW_n^m)$.

$$V_1 = \{a_n^1, a_n^3, a_n^5, \ldots, a_n^{m-1}\}$$

$$V_2 = \{a_1^1, a_1^3, a_1^5, \ldots, a_1^{m-1}\}$$

$$V_3 = \{a_2^1, a_2^3, a_2^5, \ldots, a_2^{m-1}\}$$

$$\vdots$$

$$V_{n-1} = \{a_{n-2}^1, a_{n-2}^3, a_{n-2}^5, \ldots, a_{n-2}^{m-1}\}$$

$$V_n = \{a_{n-1}^1, a_{n-1}^3, a_{n-1}^5, \ldots, a_{n-1}^{m-1}\}$$

$$V_{n+1} = \{a_n^2, a_n^4, a_n^6, \ldots, a_n^m\}$$

$$V_{m-1} = \{a_{n-2}^2, a_{n-2}^4, a_{n-2}^6, \ldots, a_{n-2}^m\}$$

$$V_m = \{a_{n-1}^2, a_{n-1}^4, a_{n-1}^6, \ldots, a_{n-1}^m\}.$$

All $V_i's$, $(i = 1,\ 2,\ 3,\ \ldots,\ m)$ are mutually disjoint color classes with size $n$ and also satisfy the equitable coloring criterion $||V_i| - |V_j|| \leq 1$ for $i \neq j$. Hence $KW_n^n$ is equitably $m$-colorable.

For $p \geq [m]$, let

$$\sigma_k = \left\lfloor \frac{mn + k - 1}{p} \right\rfloor$$

and for $k \in [p]$. Since

$$\sigma_p = \left\lfloor \frac{mn + p - 1}{p} \right\rfloor \leq \left\lceil \frac{mn}{p} \right\rceil,$$

one can divide $V(W_n^m)$ into $p$ color classes of sizes $\sigma_1,\ \sigma_2,\ \sigma_3,\ \ldots,\ \sigma_p$. Hence $KW_n^m$ is equitably $p$-colorable for all $p \geq m$.

Conversely, it is obvious, $\chi_=(KW_n^m) \geq m$. Suppose that $\chi_=(KW_n^m) < m$. Since $KW_n^m$ consists of a complete subgraph of order $m$, and so $\chi(KW_n^m) \geq m$. It is clear from this contradiction that $\chi_=(KW_n^m) \geq m$.This completes the proof. □

**Theorem 6** For $n \geq 2$, $\chi_=(KW_n^n) = \chi_=^*(KW_n^n) = n$.

**Proof.** Let $V(KW_n^n) = \{a_i^j : 1 \leq i,\ j \leq n\}$ and $E(KW_n^n) = \{a_i^j a_k^j : i \neq k,\ i,\ j,\ k = 1,\ 2,\ \ldots,\ n\} \cup \{a_n^j a_n^k :\ j \neq k,\ j,\ k = 1,\ 2,\ \ldots,\ n\}$. Also the vertex subset $\{a_1^j,\ a_2^j,\ \ldots,\ a_{n-1}^j,\ a_n^j\}$ of $V(KW_n^n)$ induces a complete subgraph of order $n$, and so $\chi(KW_n^n) \geq n$, which concludes that $\chi_=(KW_n^n) \geq n$. To prove : $\chi_=(KW_n^n) \leq n$.

Now, the following color classes of $V(KW_n^n)$ are proposed.

$$V_1 = \{a_n^1,\ a_{n-1}^2,\ a_{n-2}^3,\ \ldots,\ a_2^{n-1},\ a_1^n\}$$

$$V_2 = \{a_1^1,\ a_n^2,\ a_{n-1}^3,\ \ldots,\ a_3^{n-1},\ a_2^n\}$$

$$V_3 = \{a_1^1,\ a_n^2,\ a_{n-1}^3,\ \ldots,\ a_4^{n-1},\ a_3^n\}$$

$$\vdots$$

$$V_{n-1} = \{a_{n-2}^1,\ a_{n-3}^2,\ a_{n-4}^3,\ \ldots,\ a_n^{n-1},\ a_{n-1}^n\}$$

$$V_n = \{a_{n-1}^1,\ a_{n-2}^2,\ a_{n-3}^3,\ \ldots,\ a_1^{n-1},\ a_n^n\}.$$

Note that $V_1,\ V_2,\ \ldots,\ V_n$ are distinct color classes with size $n$. In addition, any distinct pairwise color sets differ in size by at most 1. Hence $KW_n^m$ is equitably $n$-colorble.

For $p \geq [n]$, let

$$\sigma_k = \left\lfloor \frac{mn + k - 1}{p} \right\rfloor$$

and for $k \in [p]$. Since

$$\sigma_p = \left\lfloor \frac{mn+p-1}{p} \right\rfloor \leq \left\lceil \frac{mn}{p} \right\rceil,$$

one can divide $V(W_n^m)$ into $p$ color classes of sizes $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_p$. Hence $KW_n^m$ is equitably $p$-colorable for all $p \geq n$.

Conversely, it is obvious, $\chi_=(KW_n^m) \geq n$. Suppose that $\chi_=(KW_n^m) < n$. Since $KW_n^m$ consists of a complete subgraph of order $n$, and so $\chi(KW_n^m) \geq n$. It is clear from this contradiction that $\chi_=(KW_n^m) \geq n$.This completes the proof. □

## 3. Conclusion

In this paper, the following results of $W_n^m$, $BW_n^m$, $CW_n^m$ and $KW_n^m$ in which the central vertex of $W_n^m$ is swapped by bistar graph, cyclic graph and complete graph, respectively are obtained.The equitable chromatic number and equitable chromatic threshold of generalizations in windmill graphs, bistar windmill graphs, cycle windmill graphs and complete windmill graphs are also confirmed to be the same.

1. $\chi_=(W_n^m) = \chi_=^*(W_n^m) = \left\lceil \dfrac{m(n-1)+2}{2} \right\rceil$, for $m \geq 2$ and $n \geq 3$.
2. $\chi_=(BW_n^m) = \chi_=^*(BW_n^m) = n$, for $m \geq 2$ and $n \geq 3$.
3. $\chi_=(CW_n^m) = \chi_=^*(CW_n^m) = n$, for $m \geq 3$, $n \geq 3$.
4. $\chi_=(KW_n^m) = \chi_=^*(KW_n^m) = n$, for $m \geq 2$, $n \geq 3$, and $m < n$.
5. $\chi_=(KW_n^m) = \chi_=^*(KW_n^m) = m$, for $n \geq 3$, and $m = 2n$.
6. $\chi_=(KW_n^n) = \chi_=^*(KW_n^n) = n$, for $n \geq 2$.

We put forth the following open problems:

**Problem 1** Calculate the equitable chromatic number and equitable chromatic threshold for bloom graphs $B(m, n)$, and barbell graphs $B(k, n)$.

**Problem 2** Derive the equitable chromatic number and equitable chromatic threshold of different graph products, namely Cartesian product, lexicographic product, rooted product, etc.


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## Conflict of interest

The authors declare there is no conflict of interest at any point with reference to research findings.

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