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Research Article

Support Based Essential and Core Based Superfluous Fuzzy Modules

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Abstract: In this paper, we introduce and explore several novel concepts within the framework of fuzzy module theory. First, we define the notion of a support based essential fuzzy module, establishing its foundational properties. We then investigate the essentiality of alpha cuts and the quotient fuzzy modules arising from support based essential fuzzy modules. Additionally, we demonstrate that under specific conditions, the product of a fuzzy ideal and a fuzzy module results in a support based essential fuzzy module. Further, we define the concept of a support based essential fuzzy monomorphism and provide a detailed characterization. We also introduce the fuzzy injective hull and prove that the direct sum of the fuzzy injective hulls of a class of fuzzy modules coincides with the fuzzy injective hull of the direct sum of the same class. Finally, we define the core based superfluous fuzzy module as a dual concept to the support based essential fuzzy module and establish corresponding dual results.

***Keywords*:** support of a fuzzy module, core of a fuzzy module, injective hull, support based essential fuzzy module, fuzzy injective hull, core based superfluous fuzzy module

MSC: 08A72, 03E72

1. Introduction

In 1965, Lotfi A. Zadeh [1] defined fuzzy sets. Over the years, countless scholars have explored fuzzy set theory and its applications across various fields. Rosenfeld [2] pioneered the concept of a fuzzy group, laying the foundation for fuzzy algebra. The fuzzification of rings, ideals, and other algebraic structures has been rigorously studied by many researchers. Building on these developments, Negoita and Ralescu [3] defined fuzzy modules over classical modules, opening new avenues for investigating fuzzy aspects in module theory.

Fuzzy sets have found extensive applications in decision making problems, an area that has witnessed considerable advancements in recent years. Notably, the expansion of probabilistic hesitant fuzzy set structures has facilitated the development of Q-rung orthopair probabilistic hesitant fuzzy hybrid aggregating operators for use in multicriteria decision-making scenarios [4]. Recent studies have also introduced innovations such as fuzzy Aczél-Alsina weighted geometric operators [5], bipolar-valued complex hesitant fuzzy Dombi aggregating operators [6], Generalized Dice measures of

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single valued neutrosophic type-2 hesitant fuzzy sets [7] and bipolar-valued probabilistic hesitant fuzzy sets based on generalized hybrid operators [8]. Additionally, vector similarity measures of picture type-2 hesitant fuzzy sets have been explored, further enriching this field [9]. For more comprehensive insights into recent advancements in decision making, readers can consult other key works in the literature [10–14]. These broad applications of fuzzy set theories in decision making have motivated our exploration of their relevance in classical module theory. Specifically, this inspiration has led us to fuzzify the concepts of essential and superfluous submodules.

An essential submodule of a module is defined as one that has non-trivial intersections with all non-trivial submodules of the module. Traditionally, essential submodules have been extensively studied in the context of module theory. Their unique nature allows them to permeate the module structure, leaving a non-trivial impact on all significant submodules. This understanding has facilitated advancements in diverse fields, such as algebra, representation theory, and ring theory.

A submodule N of a module M is considered superfluous in M if, for any submodule $K \subseteq M$, the condition $N+K=M$ leads to $K=M$. We donte it by $K \ll M$. We explore the concept of a superfluous submodule as a dual notion to that of an essential submodule within the framework of module theory.

Recent research has further enriched our understanding and established the importance of the dual concepts of essential and superfluous submodules across diverse contexts. Notably, Quinch and Van [15] introduced the notion of a nilpotent-invariant module.and investigated essentiality and nilpotent-invariance properties. Ech-chaouy and Tribak [16] defined simple-separable modules and characterized this notion with the superfluous property, stating that a module M is simple-separable if every simple superfluous submodule of M is contained within a finitely generated direct summand of M. Bumpendee et al. [17] leveraged essentiality to define Pseudo NQ-principally projective modules. Meanwhile, Enochs et al. [18] provided a characterization of distributive modules over commutative Noetherian rings based on injective envelopes.

The study by Kir and Turkmen [19] explored semisimple modules exhibiting small cyclic behavior within their injective envelopes. Additionally, Tesdemir and Kosan [20] investigated kernel-endoregular modules, demonstrating that an essential submodule of a kernel-endoregular module need not be kernel-endoregular. Notably, Kasparian and Kirkor [21] discussed maximal essential extensions and extension theorems for codes in projective modules over finite Frobenius rings, while Ahmed and Moh'd [22] established that $\mathrm{Soc}(M)$ is an essential submodule of the simple intersection graph of module M. Fuchigami et al. [23] introduced a novel characterization of generalized N-injective modules using homomorphisms between their injective hulls, and examined conditions under which an N-almost invariant module qualifies as almost N-injective. Farzalipour et al. [24] extended the concept of S-semiannihilator small submodules and S-T-small submodules as generalizations of S-small submodules. Nilkandish and Amini [25] introduced and defined epi-superfluous submodules and epi-Artinian modules, investigating their fundamental properties within module theory. Rajni et al. [26] studied superfluous ideals of N-groups, contributing valuable insights into the structure and behavior of these algebraic entities. Rajaee [27] introduced and conducted a comprehensive study of essential submodules of an R-module M relative to an arbitrary submodule of M. Furthermore, Chaturvedi and Kumar [28] delved into modules characterized by finitely many small submodules. Nimbhorkar and Khubchandani [29] proved significant properties concerning fuzzy semi-essential submodules and fuzzy semi-closed submodules, Mahmood and Rehman [30] studied Bipolar complex fuzzy subalgebras and ideals of BCK/BCI-algebras shedding light on the nuanced connections between fuzzy set theory and module theory.

For further exploration of essential submodules, seminal papers such as [31–34], and for comprehensive studies on superfluous submodules, seminal works such as [35–37] offer detailed insights and analyses into the theoretical foundations and practical applications of these submodule concepts.

1.1 *Motivation*

The support of a fuzzy set is a critical concept in fuzzy set theory, encompassing all elements of the universal set with membership values greater than zero. This creates a crisp set that has been foundational in various applications. Drawing a parallel, the notion of an essential submodule within a module can be compared to the support of a fuzzy module. The essential submodule represents a large and significant subset within a module, while the support of a fuzzy module signifies a relatively larger submodule among its alpha cuts. This analogy underscores the potential impact of understanding support in the study of essentiality within fuzzy modules. Despite the importance of essentiality in classical module theory, this concept remains underexplored in the fuzzy module context. To address this gap, we propose a fresh perspective by introducing support as a key defining element in the study of essentiality within fuzzy modules and define the support based essential fuzzy module. This approach opens new possibilities for understanding and applying essentiality in fuzzy modules.

Similarly, in fuzzy set theory, the core of a fuzzy set-a crisp subset containing elements with the highest degree of membership-plays a crucial role. This core has been pivotal in various theoretical developments. Extending this idea to fuzzy modules, the core of a fuzzy module can be identified as a crisp submodule that often tends to be smaller than other submodules within the same module. This observation draws a parallel between the core of a fuzzy module and the concept of a superfluous submodule in classical module theory. Motivated by this analogy, we develop the concept of a core based superfluous fuzzy module, aiming to provide a dual perspective to the support based essential fuzzy module.

2. Preliminaries

Throughout the article R denotes a Commutative ring and all other capital alphabets stand for R-modules unless otherwise specified.

Definition 1 [38] Let M be an R-module. A submodule $K \subseteq M$ becomes essential in M if whenever $K \cap L = 0$ for any submodule L of M, we have $L = 0$. we denote it by $K \trianglelefteq M$.

Dually, a submodule $K \subseteq M$ becomes superfluous in M if whenever $K + L = M$, for any proper submodule L of M, we have $L = M$. We denote it by $K \ll M$.

Proposition 1 [38] Suppose N and H are contained in M and N contains K, then

(i) $K \trianglelefteq M$ if $K \trianglelefteq N$ and $N \trianglelefteq M$

(ii) $H \cap K \trianglelefteq M$ if $H \trianglelefteq M$ and $K \trianglelefteq M$.

Dually,

(i) $N \ll M$ if $K \ll M$ and $N/K \ll M/K$

(ii) $H + K \ll M$ if $H \ll M$ and $K \ll M$.

Definition 2 [3] Let R be a ring and M an R-module. A fuzzy set μ defined on M is termed a fuzzy module if it meets the criteria outlined below:

(i) $\mu(0) = 1$

(ii) $\mu(m_1 + m_2) \geq \min\{\mu(m_1), \mu(m_2)\}\ \forall\, m_1, m_2 \in M$

(iii) $\mu(rm) \geq \mu(m)\ \forall\, m \in M$ and $\forall\, r \in R$

If μ is a fuzzy module on M, it is denoted as $\mu \in F(M)$.

Definition 3 [39] A fuzzy subset μ on a ring R meets the following criteria, then, it is said to be a fuzzy ideal on R:

(i) $\mu(r_1 - r_2) \geq \min(\mu(r_1), \mu(r_2))$, for all $r_1, r_2 \in R$ and

(ii) $\mu(r_1 r_2) \geq \max(\mu(r_1), \mu(r_2))$, for all $r_1, r_2 \in R$

For the sake of convenience, we also take one more condition $\mu(0) = 1$.

Definition 4 [40] Let μ be a fuzzy set defined on M. For any $\alpha \in [0, 1]$, we define the set μ_α as:

$$\mu_\alpha = \{m \in M \mid \mu(m) \geq \alpha\}. \tag{1}$$

This set μ_α is referred to as the alpha-cut of the fuzzy set μ.

Definition 5 [40] If μ is a fuzzy set defined on a set M, then the set $\{m \in M \mid \mu(m) > 0\}$ is known as the support of μ, and is denoted by $\mu^\star$. Moreover, if $f : M \to N$ is a function, then $f(\mu^\star) = (f(\mu))^\star$.

The set $\{m \in M \mid \mu(m) = 1\}$ is referred to as the core of μ, and is denoted by $\mu_\star$.

Definition 6 [40] Let g represent a function from M to N. Assume μ and ν are fuzzy sets defined on M and N, respectively. Then for all $n \in N$,

$$g(\mu)(n) = \begin{cases} Sup\{\mu(m) \mid m \in M,\ g(m) = n\}, & \text{if } g^{-1}(n) \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

and for all $m \in M$,

$$g^{-1}(\nu)(m) = \nu(g(m)). \tag{3}$$

Definition 7 [38] Let M_i be R-module for $i \in \Lambda$ then the sequence of R-module homomorphisms:

$$\ldots \xrightarrow{h_{i-2}} M_{i-1} \xrightarrow{h_{i-1}} M_i \xrightarrow{h_i} M_{i+1} \xrightarrow{h_{i+1}} \ldots$$

is considered to be exact if $Imh_i = kerh_{i+1}$.

Proposition 2 [39] Assume μ is a fuzzy set on M then $\mu_\alpha = \{m \in M : \mu(m) \geq \alpha\}$, $\alpha \in Im\mu$, becomes a submodule of M if and only if μ is a fuzzy module on M.

Definition 8 [41] Suppose $\nu \in F(M)$ and N is a submodule of M. Now define a fuzzy module ξ on M/N as follows: $\xi(m+N) = \sup\{\nu(p) \mid p \in m+N\}\ \forall\, m \in M$. The fuzzy module ξ is known as the quotient (factor) fuzzy module of the fuzzy module ν on M.

Definition 9 [39] If μ, ν are fuzzy sets on M. Then their sum is defined by:

$$(\mu+\nu)(m) = \sup\{\min(\mu(m_1),\ \nu(m_2)) :\ m_1 + m_2 = m\} \tag{4}$$

If μ, ν are fuzzy modules on M, then $\mu+\nu$ is a fuzzy module on M as well.

Definition 10 [41] If $\mu_1 \in F(A)$, $\mu_2 \in F(B)$ then for each $x \in A \oplus B$, $\mu_1 \oplus \mu_2$ is defined on $A \oplus B$ by:

$$\mu_1 \oplus \mu_2(x) = \min\{\mu_1(x_1),\ \mu_2(x_2)\} \tag{5}$$

where $x_1 \in A,\ x_2 \in B$ such that $x_1 + x_2 = x$.

Definition 11 [39] Let us assume μ as the fuzzy ideal on R and θ as the fuzzy module on M. The product of μ and θ is defined as:

$$\mu\theta(m) = \sup_{m = \sum_{i<\infty} r_i m_i} \left(\min_i \left(\min\left(\mu\left(r_i\right),\ \theta\left(m_i\right)\right)\right)\right),\ m \in M. \tag{6}$$

Futher, $\mu\,\theta$ is fuzzy module on M.

Now, we present some definitions and a proposition, as detailed in [42].

Definition 12 The submodule A of M is known as the complement of a submodule B in M if A is maximal for all such submodules P of M for which $P \cap B = 0$.

Proposition 3 If the complement of a submodule B in M is A then $A \oplus B \trianglelefteq M$.

Proposition 4 If B is complement of A, D is complement of B in M with $A \subseteq D$, then $A \trianglelefteq D$.

Definition 13 M is called an essential extension of N if N is essential in M. The essential extension M of N is known as the maximal essential extension of N if no module properly containing M can be the essential extension of N.

Definition 14 A maxiamal essential extension of M is called the Injective hull of M. We denote it by $H(M)$.

Proposition 5 If $A_i \leq M$ for $i = 1, 2, 3, \ldots, n$ then

$$H(A_1 \oplus A_2 \oplus A_3 \oplus, \ldots, A_n) = H(A_1) \oplus H(A_2) \oplus H(A_3) \oplus, \ldots, H(A_n). \quad (7)$$

3. Support based essential fuzzy module

Definition 15 If $\mu \in F(M)$ then we say μ is a support based essential fuzzy module on M if, for any fuzzy module ν on M for which $\mu^\star \cap \nu^\star = 0$ we have $\nu^\star = 0$. It is denoted by $\mu \in SEF(M)$.

Definition 16 μ is referred to as a support based essential fuzzy module on M if $\mu^\star \trianglelefteq M$.

Proposition 6 Above two definitions are equivalent.

Proof. Definition 2 to Definition 1 is obvious. Conversely, suppose $\{M_i | i \in \Lambda\}$ is a collection of all non-trivial submodules of M. Define fuzzy modules $\nu_i' s$ on M which assume positive membership values only on M_i then $\mu^\star \cap \nu_i^\star (= M_i) \neq 0$ or $\mu^\star \trianglelefteq M$. □

Example 1 Suppose $P, Q \leq M$ such that $P \cap Q = 0$ and Q is maximal among all those submodules N of M for which $N \cap P = 0$. Now define fuzzy set, μ on M as $\mu(m) = p \ \forall \ m \in P \setminus \{0\}$, $\mu(m) = q \ \forall \ m \in Q \setminus \{0\}$, $\mu(m) = r$ $\forall \ m \in L(P \cup Q) \setminus P \cup Q$ and $\mu(0) = 1$ such that p, q, r lies between 0 and 1 then $\mu \in SEF(M)$.

There is no requirement for all alpha-cuts of μ to be essesntial in the way the definition of support based essential fuzzy module is coined. The next theorem ensures essentiality of an alpha-cut with some condition.

Theorem 1 Let $\mu \in SEF(M)$ and μ_α be an alpha cut of μ such that whenever $\mu_\alpha \cap L = 0$ for L being non-zero submodule of M implies $\mu(l) = 0 \ \forall \ 0 \neq l \in L$ then $\mu_\alpha \trianglelefteq M$.

Proof. To prove essentiality of μ_α we claim that for any $P \leq M$ with $\mu_\alpha \cap P = 0$ we get $P = 0$. Assume $P \neq 0$ i.e. $\exists$ $0 \neq p \in P$ such that $p \notin \mu_\alpha$ and $\mu(p) = 0$.

Now define fuzzy set ν on M by $\nu(0) = 1$, $\nu(m) = 0$ if $m \in M \setminus P$ and $\nu(m) = r$ if $m \in P \setminus \{0\}$ then clearly $\nu \in F(M)$ and also $\mu^\star \cap \nu^\star = 0$ without $\nu^\star$ being zero which is contrary to the fact that $\mu \in SEF(M)$. □

Proposition 7 If $\mu, \nu \in SEF(M)$, then the following hold:

(i) $\mu \cap \nu \in SEF(M)$.

(ii) $\mu + \nu \in SEF(M)$.

Proof.

(i) This follows from the fact that $(\mu \cap \nu)^\star = \mu^\star \cap \nu^\star$.

(ii) This follows from the fact that $\mu^\star \subseteq (\mu + \nu)^\star$ and $\nu^\star \subseteq (\mu + \nu)^\star$. □

Proposition 8 Let $\mu, \nu \in F(M)$ such that two collections of submodules of M, $\{K \leq M : \mu^\star \cap K \neq 0\}$ and $\{L \leq M : \nu^\star \cap L \neq 0\}$ are equal to each other. Then $\mu \in SEF(M)$ if $\nu \in SEF(M)$.

Proof. Suppose $\mu \in SEF(M)$. Then, for any non-zero submodule K of M, we have $\mu^\star \cap K \neq 0$. By the given condition, this implies that $\nu^\star \cap K \neq 0$, meaning $\nu \in SEF(M)$. The converse can be proven similarly. □

Theorem 2 Consider N as a submodule of an R-module M. Suppose $\nu \in SEF(M)$ such that for any submodule K of M that srictly contains N, $\nu^\star \cap (K - N)$ is non-empty then $\xi \in SEF(M/N)$, where ξ is a quotient fuzzy module on M/N with respect to ν.

Proof. Since $\nu \in SEF(M)$ therefore $\nu^\star \trianglelefteq M$ i.e. for any $H \leq M$ such that $\nu^\star \cap H = 0$ we have $H = 0$. Now consider the module M/N. We know that submodules of M/N are of the form P/N where $N \subseteq P \subseteq M$. Let $P/N \leq M/N$ such that $\xi^\star \cap P/N = \{N\}$ but $P/N \neq \{N\}$. Then $\xi(p_i + N) = 0 \ \forall \ p_i \in P - N$ that implies $\sup\{\nu(z) : z \in p_i + N\} = 0 \ \forall \ p_i \in P - N$ or $\nu(p_i) = 0 \ \forall \ p_i \in P - N$ contradiction. □

Example 2 Consider the $\mathbb{Z}$-module $\mathbb{Z}$, and define a fuzzy set ν on it as follows:

$$\nu(z) = \begin{cases} 1 & \text{if } z = 0, \\ p & \text{if } z \in 2\mathbb{Z} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\nu^\star = 2\mathbb{Z}$, which is an essential submodule of $\mathbb{Z}$. Thus, $\nu \in SEF(\mathbb{Z})$.

Next, consider the $\mathbb{Z}$-module $\mathbb{Z}/4\mathbb{Z}$ and define a quotient fuzzy module ξ on it with respect to ν. By the above result, since $\nu^\star \cap (K - 4\mathbb{Z}) \neq \emptyset$ for all submodules K of M that strictly contain $4\mathbb{Z}$, it follows that $\xi \in SEF(\mathbb{Z}/4\mathbb{Z})$.

Theorem 3 Let μ denote a fuzzy ideal on R, and let θ be a fuzzy module on the R-module M such that $\theta^\star = M$. If μ is not two valued then $\mu\theta$ is support based essential fuzzy module on M.

Proof. Suppose $(\mu\theta)^\star \cap H = 0$ for a submodule H of M such that $H \neq 0$. Then

$$\mu\theta(h) = 0 \; \forall \; 0 \neq h \in H$$

$$\Rightarrow \sup_{h=\sum_{i<\infty} r_i m_i} \left(\min_i (\min(\mu(r_i), \; \theta(m_i))) \right) = 0 \; \forall \; 0 \neq h \in H$$

$$\Rightarrow \min(\mu(r_i), \theta(m_i)) = 0 \text{ at least for one } i$$

$$\Rightarrow \mu(r_i) = 0 \text{ or } \theta(m_i) = 0$$

Now, because H is a submodule of M, so for any $h \in H$ and $\forall \; r_i \in R$, $r_i h \in H$. Let $r_i h = k_i$ then $r_i h$ is one representation of k_i. Thus according to above argument

$$\min(\mu(r_i), \; \theta(h)) = 0$$

$$\Rightarrow \mu(r_i) = 0 \text{ or } \; \theta(h) = 0$$

Because $\theta^\star = M$ so $\mu(r_i) = 0 \; \forall \; r_i \in R \setminus \{0\}$. This allows us to define μ as;

$$\mu(r) = \begin{cases} 1, & \text{if } r = 0 \\ 0, & \text{otherwise} \end{cases}$$

This implies μ is two valued. Hence there exists $r_i \in R$ such that $\mu(r_i) \neq 0$ or $\min(\mu(r_i), \; \theta(h)) \neq 0$ for a non-zero $h \in H$ i.e. $\mu\theta(r_i h) \neq 0$ or $r_i h = k_i \in (\mu\theta)^\star$.

Definition 17 Assume K and M as R-modules and $\mu \in F(K)$. Let $f : K \to M$ be a monomorphism. We say f is a support based essential fuzzy monomorphism with respect to μ if $f(\mu) \in SEF(M)$.

Example 3 Consider a fuzzy monomorphism $f : \mathbb{Z} \to \mathbb{Q}$ defined by $f(z) = z$ for all $z \in \mathbb{Z}$. Define a fuzzy set μ on $\mathbb{Z}$ as $\mu(z) = p$ for $z \neq 0$ and $\mu(0) = 1$. It is clear that μ is a fuzzy module on $\mathbb{Z}$.

Consider the fuzzy set μ under the map f. The membership value of an element $q \in \mathbb{Q}$ in $f(\mu)$ is defined as follows:

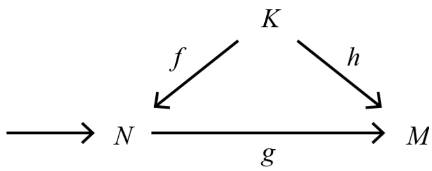
$$f(\mu)(q)=\begin{cases}\sup\{\mu(z)\mid f(z)=q\} & \text{if } q\in f(\mathbb{Z}),\\ 0 & \text{if } q\notin f(\mathbb{Z}).\end{cases}$$

Thus, we obtain $(f(\mu))^\star=\mathbb{Z}$, and since $\mathbb{Z}$ is an essential submodule in $\mathbb{Q}$, it follows that f is a support based essential fuzzy monomorphism with respect to μ.

Theorem 4 We say that monomorphism $f: K\to M$ is the support based essential fuzzy monomorphism wrt $\mu\in F(K)$ if for every module N and for each $h\in Hom_R(M,\ N)$, $kerh\cap(f(\mu))^\star=0$ implies $kerh=0$.

Proof. Ist part is trivial. Conversely, Suppose for any $\nu\in F(M)$ we have $\nu^\star\cap(f(\mu))^\star=0$. We claim $\nu^\star=0$. Let us consider a natural homomorphism $h: M\to M/\nu^\star$ defined by $h(m)=m+\nu^\star$. Clearly $kerh=\nu^\star$. Now we have kerh $\cap(f(\mu))^\star=0$ this implies $kerh=0$ or $\nu^\star=0$. □

Proposition 9 Consider the following commutative diagram



If g is monic and h is support based essential fuzzy monomorphism wrt $\mu\in F(K)$ then so is f.

Proof. According to given specifics we have $(h(\mu))^\star\trianglelefteq M$. Now we claim that f is support based essential fuzzy monomorphism wrt μ, i.e. $(f(\mu))^\star\trianglelefteq N$. Let us assume that $(f(\mu))^\star\cap N_1=0$ where $N_1\leq N$ such that $N_1\neq 0$. Then we have,

$$g((f(\mu)))^\star\cap g(N_1)=0$$

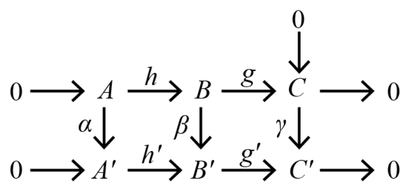
$$\Rightarrow h(\mu)^\star\cap g(N_1)=0$$

$$\Rightarrow (h(\mu))^\star\cap g(N_1)=0$$

$$\Rightarrow g(N_1)=0$$

Because g is 1-1 so we obtain $N_1=0$. This concludes the argument. □

Proposition 10 Consider the commutative diagram given below:



Assume that two rows and C-column are exact. Let $\mu\in F(A')$ such that $\mu^\star=A'$. If h' is support based essential fuzzy monomorphism wrt μ then so is h wrt $\alpha^{-1}(\mu)$.

Proof. We claim that $(h(\alpha^{-1}(\mu)))^{\star} \trianglelefteq B$. We start with the assumption that $0 \neq B_1 \subseteq B$ be submodule of B for which;

$$\left(h\left(\alpha^{-1}(\mu)\right)\right)^{\star} \cap B_1 = 0$$

$$\Rightarrow \left(h\left(\alpha^{-1}(\mu)\right)^{\star}\right) \cap B_1 = 0$$

$$\Rightarrow h(A) \cap B_1 = 0 \left(\text{Since} \alpha^{-1}(\mu)(x) = \mu(\alpha(x)) \forall x \in A\right)$$

$$\Rightarrow \text{kerg} \cap B_1 = 0$$

By above equation we get $g(b) \neq 0 \; \forall \; 0 \neq b \in B_1$. Since γ is one-one therefore $\gamma \circ g(b) \neq 0$ or $g' \circ \beta(b) \neq 0$. This implies $\beta(b) \neq 0$ and $\beta(b) \notin kerg' \; \forall \; 0 \neq b \in B_1$. Thus we obtain $\beta(B_1) \cap kerg' = 0$ or $\beta(B_1) \cap h'(A') = 0$ or $\beta(B_1) \cap h'(\mu^{\star}) = 0$ or $\beta(B_1) \cap (h'(\mu))^{\star} = 0$. This is contrary to the fact that $h'(\mu) \in SEF(B')$. Hence $(h(\alpha^{-1}(\mu)))^{\star} \trianglelefteq B$ or $h(\alpha^{-1}(\mu)) \in SEF(B)$. □

Proposition 11 Let $\mu \in SEF(M)$. Suppose ν is a fuzzy set defined on ${}_RR$ as:

$$\nu(r) = \begin{cases} 1, & \text{if } r = 0 \\ 0, & \text{if } r \notin \rho_m \\ p_1, & \text{if } r \in H_1 \setminus \{0\} \\ p_2, & \text{if } r \in H_2 \setminus \{H_1\} \\ . & \\ . & \\ . & \\ p_n, & \text{if } r \in H_n \setminus \{H_{n-1}\} \end{cases}$$

Where $\rho_m = \{r \in R : rm \in \mu^{\star}$, for m be fixed in $M\}$ and $0 \subseteq H_1 \subseteq H_2 \subseteq H_3 \subseteq ... \subseteq H_n = \rho_m$ be a series of submodules of ${}_RR$ and $1 > p_1 > p_2 > ... > p_n > 0$. Then ν is support based essential fuzzy module on ${}_RR$.

Proof. Suppose $\nu \notin SEF({}_RR)$, so there is a submodule R_1 of ${}_RR$ which is not equal to zero and $\nu^{\star} \cap R_1 = 0$. Here $\nu^{\star} = \rho_m$ so we have $\rho_m \cap R_1 = 0$. Consider $R_1m = \{km : k \in R_1\}$. Clearly $R_1m \neq 0$ and $R_1m \leq M$. So we have $\mu^{\star} \cap R_1m \neq 0$ i.e. $\exists \; 0 \neq km \in R_1m$ for which $\mu(km) > 0$ i.e. $k \in \rho_m$ but $k \in R_1$ thus $k \in \rho_m \cap R_1$. Hence our assumption is wrong.

4. Fuzzy injective hull

Definition 18 For $\mu, \; \nu \in F(M)$, μ is the fuzzy complement of ν in M if $Im\mu = Im\nu$ and for each $\alpha \in Im\mu = Im\nu$, μ_{α} is a complement of ν_{α}.

Theorem 5 If μ becomes fuzzy complement of ν in M then $\mu + \nu \in SEF(M)$.

Proof. We know $(\mu+\nu)^{\star} = \{m \in M | (\mu+\nu)(m) > 0\} = \{m \in M | \sup\{\min(\mu(m_1), \; \nu(m_2)) | m_1 + m_2 = m\} > 0\}$. First we claim $(\mu+\nu)^{\star} = \mu^{\star} + \nu^{\star}$. If $m \in (\mu+\nu)^{\star}$ then there exist $m_1, \; m_2 \in M$ such that $m_1 + m_2 = m$ and $\min\{\mu(m_1), \nu(m_2))\} > 0$. Which implies $m_1 \in \mu^{\star}$ and $m_2 \in \nu^{\star}$. Because $\mu^{\star} + \nu^{\star} \leq M$ so we get $m_1 + m_2 = m \in \mu^{\star} + \nu^{\star}$. Converse is clear. Thus $(\mu+\nu)^{\star} = \mu^{\star} + \nu^{\star}$. Now μ is fuzzy complement of ν in M implies $\mu^{\star}$ is a complement of $\nu^{\star}$ so

$(\mu+\nu)^\star = \mu^\star + \nu^\star = \mu^\star \oplus \nu^\star$. Let us assume there exists a submodule N of M which is non-zero and intersects $\mu^\star + \nu^\star$ trivially. Then we get $\nu^\star \cap N = 0$ or $N \subseteq \mu^\star$ or $N \cap \mu^\star \neq 0$ a contradiction. □

Proposition 12 Assume μ is the fuzzy complement of ν, and ξ is the fuzzy complement of μ in M with $\nu_\alpha \subseteq \xi_\alpha$ for all $\alpha \in Im\mu = Im\nu = Im\xi$ then $\nu_\alpha \trianglelefteq \xi_\alpha$.

Proof. Since μ is fuzzy complement of ν and ξ is fuzzy complement of μ therefore each considerable μ_α is complement of ν_α and ξ_α is complement of μ_α. Rest of the proof flows with proposition 4. □

Proposition 13 If μ is the fuzzy complement of ν in M, then all of its alpha-cuts are essentially closed.

Proof. We start with the supposition that there exist $\alpha \in Im\mu = Im\nu$ such that μ_α is not essentially closed i.e. a submodule $U \neq \mu_\alpha$ exists and $\mu_\alpha \trianglelefteq U$. Because μ_α is a complement of ν_α so $U \cap \nu_\alpha \neq 0$. Suppose $U \cap \nu_\alpha = K$ then $K \leq U$ so $K \cap \mu_\alpha \neq 0$ therefore $U \cap \nu_\alpha \cap \mu_\alpha \neq 0$. Thus we obtain all alpha-cuts of μ are essentially closed. □

Definition 19 If $\mu,\ \nu \in F(M)$ then μ is called Fuzzy injective hull of ν if $Im\mu = Im\nu$ and μ_α is a maximal essential extension of ν_α $\forall\ \alpha \in Im\mu = Im\nu$. We denote it by $\mu = H(\nu)$.

Theorem 6 If μ is fuzzy complement of fuzzy complement of ν in M with $\nu_\alpha \subseteq \mu_\alpha\ \forall\ \alpha \in Im\mu = Im\nu$ then $\mu = H(\nu)$.

Proof. For any arbitrary α from $Im\mu = Im\nu$ we claim μ_α is a maximal essential extension of ν_α. By proposition 12 it is clear that $\nu_\alpha \trianglelefteq \mu_\alpha$. For maximality suppose there exists an essential extension K of ν_α with $\mu_\alpha \leq K$. By proposition 13, μ_α is essentially closed in M therefore we surely have a non-zero submodule S of K with $S \cap \mu_\alpha = 0$. Now $\nu_\alpha \subseteq \mu_\alpha$ thus we get $S \cap \nu_\alpha = 0$ or ν_α is not essential in K. Hence $\mu = H(\nu)$.

Theorem 7 Let $A_i \leq M$ and $\mu_i \in F(A_i)$ for $i = 1,\ 2,\ 3,\ \dots,\ n$. Then;

$$H(\oplus_{i=1}^{n}\mu_i) = \oplus_{i=1}^{n} H(\mu_i).$$

Proof. Consider,

$$(\mu_1 \oplus \mu_2 \oplus \mu_3 \oplus \dots \oplus \mu_n)_\alpha = \{m \in A_1 \oplus A_2 \oplus A_3 \oplus \dots A_n : (\mu_1 \oplus \mu_2 \oplus \mu_3 \oplus \dots \oplus \mu_n)(m) \geq \alpha\}$$

$$= \{m \in A_1 \oplus A_2 \oplus A_3 \oplus \dots A_n : min\{\mu_1(m_1),\ \mu_2(m_2),\ \dots,\ \mu_n(m_n)\} \geq \alpha\}$$

where $m_1 \in A_1,\ m_2 \in A_2,\ m_3 \in A_3 \dots,\ m_n \in A_n$ and $m_1 + m_2 + \dots + m_n = m$.

$$= \{m \in A_1 \oplus A_2 \oplus A_3 \oplus \dots A_n : \mu_1(m_1) \geq \alpha,\ \mu_2(m_2) \geq \alpha,\ \dots,\ \mu_n(m_n) \geq \alpha\}$$

$$= (\mu_1)_\alpha \oplus (\mu_2)_\alpha \oplus (\mu_3)_\alpha \oplus \dots \oplus (\mu_n)_\alpha$$

Since each $(H(\mu_i))_\alpha$ is maximal essential extension of $(\mu_i)_\alpha$ therefore by proposition 5 $(H(\mu_1))_\alpha \oplus (H(\mu_2))_\alpha \oplus \dots \oplus (H(\mu_n))_\alpha$ is maximal essential extension of $(\mu_1)_\alpha \oplus (\mu_2)_\alpha \oplus (\mu_3)_\alpha \oplus \dots \oplus (\mu_n)_\alpha$ or we can say that $(H(\mu_1) \oplus H(\mu_2) \oplus \dots \oplus H(\mu_n))_\alpha$ is maximal essential extension of $(\mu_1 \oplus \mu_2 \oplus \mu_3 \oplus \dots \oplus \mu_n)_\alpha$ for each α. Thus we obtain;

$$H(\oplus_{i=1}^{n}\mu_i) = \oplus_{i=1}^{n} H(\mu_i).$$

5. Core based superfluous fuzzy module

Definition 20 If $\mu \in F(M)$, we say that μ is a core based superfluous fuzzy module on M if, for any fuzzy module ν on M for which $\mu_\star + \nu_\star = M$ we have $\nu_\star = M$. It is denoted as $\mu \in CSF(M)$.

Definition 21 A fuzzy module μ on M is known as core based superfluous fuzzy module on M, if $\mu_\star \ll M$.

Example 4 Let M be a finitely generated module. Define a fuzzy set ν on M as follows:

$$\nu(m) = \begin{cases} 1 & \text{if } m = 0, \\ p & \text{if } m \in \mathrm{Rad}(M) \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathrm{Rad}(M)$ is a superfluous submodule of M for M being finitely generated, and $\nu_\star = \mathrm{Rad}(M)$, it follows that $\nu \in CSF(M)$.

Proposition 14 Above two definitions are equivalent.

Proof. Definition 20 to 21 is obvious. Conversely, suppose $\{M_i | i \in \lambda\}$ is a collection of all non-trivial submodules of M. Define fuzzy modules ν_i's on M which assume membership values 1 only for members of M_i's then $\mu_\star + \nu_\star \neq M$ or $\mu_\star << M$. □

Proposition 15 Assume $N \leq M$ and $\nu \in CSF(M)$, such that $M - N \subseteq \nu_\star$. Then, the quotient fuzzy module ξ defined on M/N with respect to ν is a core based superfluous fuzzy module on M/N, i.e. $\xi \in CSF(M/N)$.

Proof. Assuming $\xi \notin CSF(M/N)$, there exists a submodule P/N of M/N, such that $\xi_\star + P/N = M/N$ but $P/N \neq M/N$. This implies $P \neq M$.

Next, we claim that $\nu_\star + P = M$. For each $m \in M$, we consider two cases:

(i) If $m \in M$ but $m \notin N$, then $m \in M - N \subseteq \nu_\star$.

(ii) If $m \in N$, then since $N \subseteq P$, we have $m \in P$.

In both cases, every $m \in M$ can be expressed as a combination of elements from $\nu_\star$ and P. Therefore, we conclude that $\nu_\star + P = M$, but $P \neq M$, contradicting the assumption that $\nu_\star \ll M$. This completes the proof. □

Next, we will investigate the dual statement of the theorem 3. For our convenience, we define a R-module M as a finite representable module if each element $m \in M$ can be represented by elements of M in only finitely many ways, i.e. $m = \sum_i r_i m_i$ for finitely many indices i, where $r_i \in R$ and $m_i \in M$.

Proposition 16 Consider $\mu \in FI(R)$ and $\theta \in F(M)$. Let M be a finite representable module such that $\theta_\star = \{0\}$. Then, $\mu\theta \in CSF(M)$.

Proof. Assuming there exists $H \leq M$ such that $(\mu\theta)_\star + H = M$ but $H \neq M$, we can find an element $m \in M$ that is not in H. Since $(\mu\theta)_\star + H = M$, there must exist $m_1 \in (\mu\theta)_\star$ and $m_2 \in H$ such that $m = m_1 + m_2$. Notably, $m_1 \neq 0$ because if $m_1 = 0$, then $m = m_2 \in H$, contradicting that $m \notin H$. Now,

$$\mu\theta(m_1) = 1$$

$$\Rightarrow \sup_{m_1 = \sum_{i<\infty} r_i m_i} \left(\min_i (\min(\mu(r_i),\ \theta(m_i))) \right) = 1$$

In this scenario, there always exists a combination $m_1 = \sum_i r_i n_i$ such that :

$$\mu(r_i) = 1 \text{ and } \theta(n_i) = 1$$

Since $\theta_\star = \{0\}$ this implies $n_i = 0$. Thus we get $m_1 = 0$ a contradiction. Hence $\mu\theta \in CSF(M)$. □

Definition 22 We say that an epimorphism, $f : M \to N$ is core based superfluous epimaorphism with respect to $\nu \in F(N)$ if $(f^{-1}(\nu))_\star$ is superfluous in M.

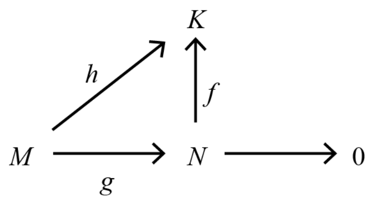
Example 5 Consider the identity module homomorphism f on the $\mathbb{Z}$-module $\mathbb{Z}$ and a fuzzy module ν on $\mathbb{Z}$, where ν assumes the value 1 only for the additive identity "0". It follows that $\nu_\star = 0$. Since $(f^{-1}(\nu))_\star = f^{-1}(\nu_\star) = f^{-1}(0) = 0$, the submodule $(f^{-1}(\nu))_\star$ is superfluous in $\mathbb{Z}$. Therefore, the mapping f is a core based superfluous epimorphism with respect to ν.

In theorem 4, a characterization of support based essential fuzzy monomorphisms was provided. Now, in a dual approach to this result, we present the characterization of core based superfluous epimorphisms.

Proposition 17 We say that an epimorphism, $f : M \to N$ is core based superfluous epimaorphism with respect to $\nu \in F(N)$ iff for every module K and for each $h \in Hom_R(K, M)$, $Imh + (f^{-1}(\nu))_\star = M$ implies $Imh = M$.

Proof. Ist part is obvious. Conversely, we claim that for any $H \leq M$, such that $H + (f^{-1}(\nu))_\star = M$, we have $H = M$. Let us consider a natural inclusion map $i : H \to M$ then $i \in Hom_R(H, M)$ with $Imi = H$, so $H + (f^{-1}(\nu))_\star = M$ implies $H = M$. Thus $(f^{-1}(\nu))_\star \ll M$ or f is a core based superfluous epimorphism with respect to $\nu \in F(N)$. □

Proposition 18 Consider the following commutative diagram:



If g is epic and h is core based superfluous fuzzy epimorphism with respect to $\nu \in F(K)$ then so is f.

Proof. According to the given specifics, we have $(h^{-1}(\nu))_\star \ll M$. Now we claim that f is a core based superfluous fuzzy epimorphism with respect to ν, i.e., $(f^{-1}(\nu))_\star \ll N$. Let us assume $(f^{-1}(\nu))_\star + N_1 = N$ where $N_1 \leq N$ and $N_1 \neq N$. Then we have:

$$g^{-1}((f^{-1}(\nu))_\star + N_1) = g^{-1}(N)$$

$$\Rightarrow \; (g^{-1} \circ f^{-1}(\nu))_\star + g^{-1}(N_1) = M$$

$$\Rightarrow \; (h^{-1}(\nu))_\star + g^{-1}(N_1) = M$$

$$\Rightarrow \; (h^{-1}(\nu))_\star + g^{-1}(N_1) = M$$

Thus we get $g^{-1}(N_1) = M$. Since g is an epimorphism, therefore $N_1 = N$, which yields a contradiction. Hence $(f^{-1}(\nu))_\star \ll N$. □

Now, we will discuss the dual of Proposition 10. To get the dual result, we see that the required condition makes the statement meaningless.

Proposition 19 Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \alpha\big\downarrow & & \beta\big\downarrow & & \gamma\big\downarrow & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

Assume that both rows are exact and α is epic. Then if g is core based superfluous fuzzy epimorphism with respect to $\nu \in F(C)$ such that $\nu_\star = \{0\}$ then so is g' with respect to $\gamma(\nu)$, provided g' is one-one also.

Proof. We claim that $(g'^{-1}(\gamma(\nu)))_\star \ll B'$. Let us start with the assumption that $\exists\, B'_1 \leq B'$ such that,

$$(g'^{-1}(\gamma(\nu)))_\star + B'_1 = B'$$

$$\implies g'^{-1}(\gamma(\nu_\star)) + B'_1 = B'$$

$$\implies g'^{-1}(\gamma(0)) + B'_1 = B'$$

$$\implies g'^{-1}(0) + B'_1 = B'$$

$$\implies \operatorname{Ker} g' + B'_1 = B'$$

$$\implies f'(A') + B'_1 = B'$$

$$\implies f' \circ \alpha(A) + B'_1 = B'$$

$$\implies \beta \circ f(A) + B'_1 = B'$$

$$\implies \beta(\operatorname{Ker} g) + B'_1 = B'$$

$$\implies g'(\beta(\operatorname{Ker} g)) + g'(B'_1) = g'(B') = C'$$

$$\implies \gamma \circ g(\operatorname{Ker} g) + g'(B'_1) = g'(B') = C'$$

$$\implies g'(B'_1) = C'$$

Since g' is an isomorphism, therefore we conclude that $B'_1 = B'$. This completes the proof. □

As we see in the above proposition, for the dual statement to hold, we need g' to be one-to-one, which makes $A' = 0$. Therefore, the diagram does not make any sense.

Proposition 20 Consider $f : M \to E$ as an R-module homomorphism. Suppose $\mu \in CSF(M)$ with the supremum property. Then it follows that $f(\mu) \in CSF(E)$.

Proof. First we claim that $(f(\mu))_\star = f(\mu_\star)$. Take $e \in f(\mu_\star)$ then, there exists $x \in \mu_\star$, such that $e = f(x)$. Now,

$$f(\mu)(e) = \sup\{\mu(x) \mid f(x) = e\} = 1$$

Thus we have $e \in (f(\mu))_\star$, or equivalently, $f(\mu_\star) \subseteq (f(\mu))_\star$. Furthermore, since μ has the supremum property, whenever we have, $\sup\{\mu(x) \mid f(x) = e\}$, there exists an $x \in M$ with $f(x) = e$ such that $\mu(x) = 1$ or $x \in \mu_\star$. This implies the reverse containment, i.e., $(f(\mu))_\star \subseteq f(\mu_\star)$. Thus $(f(\mu))_\star = f(\mu_\star)$. Now, $\mu \in CSF(M)$, so $\mu_\star \ll M$, and f is an R-module homomorphism, therefore, $f(\mu_\star) \ll E$. This completes the proof. □

Definition 23 $\mu \in F(M)$ is called a core based small fuzzy module if $\mu_\star$ is isomorphic to a small submodule of any module.

Definition 24 Let $f : M \to N$ be an injective homomorphism and $\mu \in F(M)$. Define a fuzzy set μ_f on N as follows:

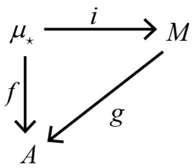
$$\mu_f(n) = \begin{cases} \mu(m) & \text{if } f(m) = n, \\ 0 & \text{otherwise.} \end{cases}$$

Then, it is easy to check that $\mu_f \in F(N)$ and $(\mu_f)_\star \cong \mu_\star$.

Proposition 21 The following are equivalent:

(i) $\mu \in CSF(M)$

(ii) If there is a commutative diagram:

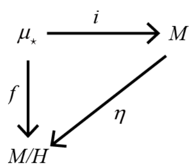


with f an epimorphism, where $i : \mu_\star \to M$ is the inclusion map then $A = \{0\}$.

Proof.

(i) $\implies (ii)$: We have $g(\mu_\star) = g \circ i(\mu_\star) = f(\mu_\star) = A$, since f is an epimorphism. Thus, $g(\mu_\star) + \{0\} = A$. Given that $g(\mu_\star)$ is small in A, it follows $A = \{0\}$.

(ii) $\implies (i)$: Consider H as a submodule of M suchthat $\mu_\star + H = M$. Then restriction of the natural homomorphism $\pi : M \to M/H$ to $\mu_\star$ is an epimorphism. If $i : \mu_\star \to M$ is the inclusion map and $f = \eta|_{\mu_\star}$, then we have a commutative diagram as shown below:



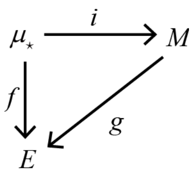
Because, f is an epimorphism, we conclude that $M/H = \{0\}$, implying $M = H$. □

As an immediate consequence of the above proposition, we state the following corollary without proof.

Corollary 1 $\mu \in F(M)$ is core based small iff $\mu_\star$ is small in its injective hull.

Proposition 22 A fuzzy module $\mu \in F(U)$ is core based small if μ_f is core based small in an injective extension of $\mu_\star$. (Here f is a monomorphism from $\mu_\star$ to its injective extension.)

Proof. Suppose $\mu \in F(U)$ is core based small. Then $\mu_\star$ is a small submodule of some module M. Let $f : \mu_\star \to E$ be a monomorphism into an injective module E. Then, there exists a homomorphism $g : M \to E$ such that the following diagram commutes:



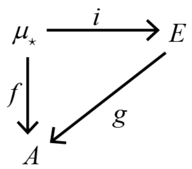
Here, $i : \mu_\star \to M$ is the inclusion map. We then have $g(\mu_\star) = g \circ i(\mu_\star) = f(\mu_\star)$, which is small in E since f is a monomorphism. Therefore, $\mu_\star \cong f(\mu_\star) \cong (\mu_f)_\star$, implying $(\mu_f)_\star \ll E$ or $\mu_f \in CSF(E)$.

Conversely, suppose $\mu_\star$ is small in an injective extension E. By definition, $\mu_\star$ being small implies that μ is core based small as a fuzzy module. □

Corollary 2 Following statements are equivalent:

(i) $\mu \in F(M)$ is core based small fuzzy module.

(ii) If $i : \mu_\star \to E$ is a monomorphism of $\mu_\star$ into an injective module E and there is a commutative diagram:

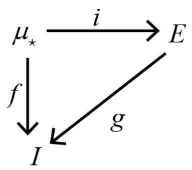


with f an epimorphism, then $A = \{0\}$.

Proof. Proof follows from Proposition 21. □

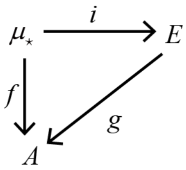
Proposition 23 Let $\mu \in F(M)$ be such that, $\mu_\star$ is a semi-simple module. Then the μ is a core based small fuzzy module if $\mu_\star$ does not contain a non zero injective submodule.

Proof. Suppose if possible $\mu_\star$ contains a non-zero injective module I. Let $p : \mu_\star \to I$ be the projection on I and E be an injective module containing I. If $i : \mu_\star \to E$ is the inclusion map, then as I is injective there exists a homomorphism $g : E \to I$ such that the following diagram is commutative:



Now, since $I \neq 0$, by Proposition 22, $\mu_\star$ is not small or μ is not core based small fuzzy module, which is a contradiction. Hence $\mu_\star$ does not contain a non-zero injective module.

Conversely, Suppose $\mu_\star$ is not small. Then again by Proposition 22, there exists a monomorphism $i : \mu_\star \to E$ of $\mu_\star$ into an injective module E and a commutative diagram:



with f an epimorphism and $A \neq 0$. Since $\mu_\star$ is semi-simple, $Kerf$ is direct summand of $\mu_\star$ and so there exists a submodule A' of $\mu_\star$ such that $\mu_\star = Kerf \oplus A'$ and $f \restriction_{A'}$ is an isomorphism. Now,

$$(g \circ i)(A') = f(A') \cong A'$$

Thus $i(A')$ is a direct summand of E. Since E is injective therefore A' is also injective. Again $A \neq 0$, $Kerf \neq \mu_\star$ and so $A' \neq 0$. Thus A' is a non-zero injective submodule of $\mu_\star$, a contradiction. Hence $\mu_\star$ is small. □

Proposition 24 Let $\nu \in F(M)$ be such that $\nu_\star \subseteq Rad(M)$ with finite cardinality. Then $\nu \in CSF(M)$.

Proof. We first claim that each $a \in \nu_\star$ generates a small submodule of M.

Let $a \in \nu_\star$ and V be the submodule of M generated by a. We claim that V is small in M. Suppose it is possible V is not small in M. Then there exists a proper submodule N of M such that $V + N = M$. Let $\{N_i | i \in \lambda\}$ be a family of submodules of M such that $V + N_i = M$. Then it is clear that this family is non-empty. Let $\{L_j | j \in \lambda\}$ be a totally ordered subset of $\{N_i | i \in \lambda\}$ and let $L = \cup_{j\in\lambda} L_j$. Then $L \in \{N_i | i \in \lambda\}$ is an upper bound for $\{L_j | j \in \lambda\}$. Since every totally ordered subset of $\{N_i \mid i \in \lambda\}$ has an upper bound, Zorn's lemma implies the existence of a maximal submodule S of M such that $\nu_\star + S = M$. Again, S is a maximal submodule of M, $\nu_\star \subseteq Rad(M) \subseteq S$, therefore $S = \nu_\star + S = M$.

Now, $\nu_\star$ is the sum of small submodules V generated by $a \in \nu_\star$ and $|\nu_\star| < \infty$, therefore we get $\nu_\star \ll M$ or $\nu \in CSF(M)$. □

Proposition 25 Consider $\mu, \nu \in F(M)$ such that $\mu \subseteq \nu$ and $\mu \in CSF(M)$. Then $\nu \in CSF(M)$ iff $\nu/\mu \in CSF(M/\mu_\star)$, provided, $(\nu/\mu)_\star = \nu_\star/\mu_\star$. (Here, ν/μ is quotient fuzzy module on $M/\mu_\star$ with respect to μ.)

Proof. Assume that $(\nu/\mu)_\star + N/\mu_\star = M/\mu_\star$. Since we have $(\nu/\mu)_\star = \nu_\star/\mu_\star$, which implies $(\nu_\star + N)/\mu_\star = M/\mu_\star$. Thus, $\nu_\star + N = M$. Since $\nu \in CSF(M)$, therefore $N = M$. Hence, $\nu/\mu \in CSF(M/\mu_\star)$.

Conversely, suppose $\nu_\star + N = M$. Then, $(\nu_\star/\mu_\star) + (N + \mu_\star)/\mu_\star = M/\mu_\star$. Since $(\nu/\mu)_\star = \nu_\star/\mu_\star \ll M/\mu_\star$, we conclude that $(N + \mu_\star)/\mu_\star = M/\mu_\star$, which implies $N + \mu_\star = M$ or $N = M$. □

6. Conclusion

In this article, we have focused on support based essential fuzzy modules and core based superfluous fuzzy modules, exploring phenomena such as intersection, sum, and alpha cuts within this framework. We have demonstrated the essential and superfluous properties of quotient fuzzy modules and the product of a fuzzy ideal by a fuzzy module under certain conditions. Additionally, we have introduced support based essential fuzzy monomorphisms and core based superfluous fuzzy epimorphisms. Furthermore, we have established the concept of a fuzzy injective hull.

In algebra, quotient structures play a pivotal role, with their applications, clearly demonstrated in the Fundamental Theorem of Homomorphisms. They allow us to explore and understand various properties of different module structures. When we extend this approach by applying fuzzy sets to classical modules, a deeper study and analysis of fuzzy quotient modules become essential. In this research, we have conducted a thorough investigation and analysis of quotient structures in the context of support based essential and core based superfluous fuzzy modules.

However, it is important to acknowledge the limitations of this research. Firstly, the study of support based essential fuzzy modules and core based superfluous fuzzy modules is still in its early stages, and further investigations are needed to uncover additional properties and applications. Secondly, while we have provided conditions for the essentiality and superfluous properties of certain operations, there may be other factors affecting them that require further exploration.

Moreover, the concept of support based essential fuzzy monomorphisms and core based superfluous fuzzy epimorphisms is new, and its full implications and practical effectiveness require further validation. Lastly, the broader applicability and connections of the fuzzy injective hull concept in fuzzy module theory need additional study.

Although there are limitations, this research advances the understanding of these dual concepts and suggests promising directions for future investigation. The framework presented here can be extended to various generalizations of fuzzy sets, such as intuitionistic fuzzy sets, interval-valued fuzzy sets, type-2 fuzzy sets, hesitant fuzzy sets, picture fuzzy sets, pythagorean fuzzy sets, neutrosophic sets, and rough fuzzy sets. By applying these extensions to support based essential and core based superfluous fuzzy modules, we can further explore their properties and analyze their impact within algebraic structures.

Conflict of interest

The authors declare no competing financial interest.

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