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Research Article

Structure of (σ, ρ)-n-Derivations on Rings

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Abstract: The goal of this research is to describe the structure of Jordan (σ, ρ)-n-derivations on a prime ring. By (σ, ρ)-n-derivations, we mean n-additive maps $\Im : R^n \longrightarrow R$ satisfying the following property in each n-slot:

$$\Im(pq, \varpi_1, \cdots, \varpi_{n-1}) = \Im(p, \varpi_1, \cdots, \varpi_{n-1})\sigma(q) + \rho(p)\Im(q, \varpi_1, \cdots, \varpi_{n-1}),$$

for every $p, q, \varpi_1, \cdots, \varpi_{n-1} \in R$. We find the conditions under which every Jordan (σ, ρ)-n-derivation becomes a (σ, ρ)-n-derivation. Moreover, the concept of $*$-n-centralizers on $*$-ring has given. The $*$-ring is also used for examining some outcomes, where left and right $*$-n-centralizers are significant.

Keywords: prime ring, (σ, ρ)-derivation, Jordan n-derivation, $*$-n-centralizers

MSC: 16N60, 16R50, 16W10, 17A36

1. Introduction

In the present study, R denotes an associative ring that has a nonzero center $Z(R)$. A ring R is called a p-torsion free ring, if $pr = 0$ entails $r = 0$ for every $r \in R$, and $p > 1$ is any fixed integer. A ring R is called a prime in case $r_1Rr_2 = \{0\}$ implies that either $r_1 = 0$ or $r_2 = 0$, and a semiprime ring if $rRr = \{0\}$ yields $r = 0$.

A map d is known as a Jordan derivation if $\mathrm{d}(r^2) = \mathrm{d}(r)r + r\mathrm{d}(r)$, for all $r \in R$. An additive map $\eth$ is said to be (σ, ρ)-derivation on R if $\eth$ satisfy the condition $\eth(qr) = \eth(q)\sigma(r) + \rho(q)\eth(r)$, for every $q, r \in R$. It is clear that (σ, ρ)-derivation is the natural generalization of derivation for σ and ρ being identity maps on R.

Numerous findings on biderivations and associated mappings of prime and semi-prime rings, as well as specific algebras, have been discovered (see [1–3] and its references). In addition to biderivations many studies were conducted on 3-derivations, n-derivations and Jordan n-derivations on prime and semiprime rings (Look in [4–6] and references therein). In particular, for symmetric n-derivations of rings, Park [7] derived findings comparable to the previous citations. The author in [7], proved that: let $n \geq 2$ be a fixed positive integer and let R be a noncommutative $n!$-torsion free semiprime ring. Suppose that there exists a symmetric n-derivation $\Lambda : R^n \longrightarrow R$ such that the trace $(\Lambda(\underbrace{k, k, \cdots, k}_{n-times}))$ of Λ is centralizing on R, then the trace is commuting on R. If R is a $n!$-torsion free prime ring and $\Lambda \neq 0$ under the same condition, then R is commutative. The purpose of this research is to investigate the characterization of Jordan (σ, ρ)-n-derivations on prime rings. First, we shall define the following terminology and discuss some fundamental concepts that we shall use throughout:

Definition 1 Let R be a ring and "σ, ρ" be automorphisms on R. An n-additive map $\Im$ (additive in all n slots) is said to be a (σ, ρ)-n-derivation if $\Im$ satisfies the following property in each n-slots:

$$\Im(pq, \varpi_1, \cdots, \varpi_{n-1}) = \Im(p, \varpi_1, \cdots, \varpi_{n-1})\sigma(q) + \rho(p)\Im(q, \varpi_1, \cdots, \varpi_{n-1}),$$

for every $p, q, \varpi_1, \cdots, \varpi_{n-1} \in R$.

Definition 2 Let R be a ring and "σ, ρ" be automorphisms on R. An n-additive map $\Im$ (additive in all n slots) is said to be a Jordan-(σ, ρ)-n-derivation if $\Im$ satisfy the following property in each n-slots:

$$\Im(p^2, \varpi_1, \cdots, \varpi_{n-1}) = \Im(p, \varpi_1, \cdots, \varpi_{n-1})\sigma(p) + \rho(p)\Im(p, \varpi_1, \cdots, \varpi_{n-1}),$$

for every $p, \varpi_1, \cdots, \varpi_{n-1} \in R$.

The above definitions become n-derivation and Jordan n-derivation in case σ and ρ are identity mappings on R.

The role of the automorphisms and some other mappings is crucial to developing the algebraic structure of the domain of the underlying mapping. Many algebraists have studied the correlation between certain unique kinds of mappings on a ring R and the commutativity of R. Divinsky's proof that an automorphism of an Artinian ring R must also be commutative if it is nontrivial, and commutative facilitated the first accomplishment in this field. Divinsky's case was expanded to prime rings by Luh. Mayne discovered that it needs to be a commutative ring if a prime ring has a non-identity and centralizing automorphism. We have now extended these findings to other algebraic structures. Posner affirmed that the commutative structure of the prime ring must exist once a derivation occurs on a centralizing and nonzero prime ring. Numerous researchers, such as Bresar, Luh, Johnson, Mayne, etc., have altered and enhanced their findings in different ways throughout the past few decades.

In [2], the author established that every Jordan derivation on a 2-torsion free prime ring is a derivation. Bresar [4] extended the last result on semiprime ring and obtained: Let R be a 2-torsion free semiprime ring, and let τ be a Jordan derivation on R. In this case, τ will be a derivation on R. Recently, an extension of this idea is established in [8] on generalized Jordan n-derivations.

Motivated by these interpretations, we shall show that every Jordan-(σ, ρ)-n-derivation on a ring R will be a (σ, ρ)-n-derivation on R in Section 2. The third section of the paper opens up with an involution ring. Left and right $*$-n-centralizers play a crucial part in the investigation of certain outcomes using a $*$-ring. We also required the following lemmas:

Lemma 1 [6] Let R be a non-commutative prime ring and $\pi : R^2 \longrightarrow R$ be a bi-additive map. If $\pi(p, q)[p, q] = 0$ for all $p, q \in R$, then $\pi = 0$.

Lemma 2 [9] Let σ be a nontrivial automorphism on a prime ring R. If $[\sigma(t), t] = 0,$ for every $t \in R$, then R will be a commutative ring.

2. Main theorems

These lemmas are required in order to demonstrate the main theorems.

Lemma 3 Let the mappings σ, ρ be automorphisms on a prime ring R with $char \neq 2$ and $\Im$ be a symmetric Jordan-(σ, ρ)-n-derivation on R. Then $\Im(pq+qp, \varpi_1, \cdots, \varpi_{n-1}) = \Im(p, \varpi_1, \cdots, \varpi_{n-1})\sigma(q) + \rho(p)\Im(q, \varpi_1, \cdots, \varpi_{n-1}) + \Im(q, \varpi_1, \cdots, \varpi_{n-1})\sigma(p) + \rho(q)\Im(p, \varpi_1, \cdots, \varpi_{n-1})$, for every $p, q, \varpi_1, \cdots, \varpi_{n-1} \in R$.

Proof. [6] Consider the below term for each $p, q, \varpi_1, \cdots, \varpi_{n-1} \in R$

$$\begin{aligned}\Im((p+q)^2,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) =&\Im((p+q)(p+q),\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\ =&\Im((p+q),\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p+q)\\ &+\rho(p+q)\Im(p+q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\end{aligned} \tag{1}$$

Simplify the right hand side of (1) by using the additivity of ρ, and σ

$$\begin{aligned}\Im((p+q)^2,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) =&\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p)+\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)\\ &+\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)+\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p)\\ &+\rho(p)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})+\rho(p)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\ &+\rho(q)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})+\rho(q)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\end{aligned} \tag{2}$$

for all $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$. Next, evaluate the left hand side of (1) for every $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$

$$\begin{aligned}\Im((p+q)^2,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) =&\Im((p^2+pq+qp+q^2,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\ =&\Im(p^2,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\ &+\Im(pq+qp,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\ &+\Im(q^2,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\ =&\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p)\\ &+\rho(p)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\ &+\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)\\ &+\rho(q)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\ &+\Im(pq+qp,\ \varpi_1,\ \cdots,\ \varpi_{n-1}).\end{aligned} \tag{3}$$

Comparing (1) and (2), we find

$$
\begin{aligned}
\Im(pq+qp,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) =& \Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q) \\
& + \rho(p)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) \\
& + \Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p) \\
& + \rho(q)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1}),
\end{aligned} \tag{4}
$$

for all $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$. This completes the proof of the lemma.

Lemma 4 Let the mappings $\sigma,\ \rho$ be automorphisms on a prime ring R with $char \neq 2$ and $\Im$ be a symmetric Jordan-$(\sigma,\ \rho)$-n-derivation on R. Then $\Im(pqp,\ \varpi_1,\ \cdots,\ \varpi_{n-1})$=$\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)\sigma(p)+\rho(p)\rho(q)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})+\rho(p)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p)$, for every $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$.

Proof. We will take a quick start by replacing q by $pq+qp$ in (4) and observe that

$$
\begin{aligned}
\Im(p(pq+qp)+(pq+qp)p,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) =& \Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(pq+qp) \\
& + \Im(pq+qp,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p) \\
& + \rho(pq+qp)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) \\
& + \rho(p)\Im(pq+qp,\ \varpi_1,\ \cdots,\ \varpi_{n-1}),
\end{aligned} \tag{5}
$$

for all $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$. Explore the right side of (5) making use of Lemma 3 to obtain

$$
\begin{aligned}
=& \Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p)\sigma(q) + \Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)\sigma(p) \\
& + \Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)\sigma(p) + \Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p)\sigma(p) \\
& + \rho(q)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p) + \rho(p)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p) \\
& + \rho(p)\rho(q)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) + \rho(q)\rho(p)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) \\
& + \rho(p)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q) + \rho(p)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p) \\
& + \rho(q)\rho(p)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) + \rho(p)\rho(q)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1}),
\end{aligned} \tag{6}
$$

for all $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$. Next, Simplify (5) in light of (4) from left to find

$$
\begin{aligned}
\Im(p(pq+qp)+(pq+qp)p,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) =&\Im(p^2q+2pqp \\
&+qp^2,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) \\
=&\Im(p^2q,\ s_1,\ ..,\ s_{n-1}) \\
&+2\Im(pqp,\ s_1,\ ..,\ s_{n-1}) \\
&+\Im(qp^2,\ s_1,\ ..,\ s_{n-1}) \\
=&\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p)\sigma(q) \\
&+\rho(p)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q) \\
&+\rho(p)\rho(p)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) \\
&+\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p)\sigma(p) \\
&+\rho(q)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p) \\
&+\rho(q)\rho(p)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) \\
&+2\Im(pqp,\ \varpi_1,\ \cdots,\ \varpi_{n-1}),
\end{aligned} \tag{7}
$$

for all $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$. Confront the two equation (6) and (7) and apply characteristic condition to get

$$
\begin{aligned}
\Im(pqp,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) =&\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)\sigma(p) \\
&+\rho(p)\rho(q)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) \\
&+\rho(p)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p),
\end{aligned} \tag{8}
$$

for all $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$.

Lemma 5 Let the mappings $\sigma,\ \rho$ be automorphisms on a prime ring R with $char \neq 2$ and $\Im$ be a symmetric Jordan-$(\sigma,\ \rho)$-n-derivation on R. Then $\Im(pqt+tqp,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) = \Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)\sigma(t)+\rho(p)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})$ $\sigma(t)+\Im(t,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)\sigma(p)+\rho(t)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p)+\rho(t)\rho(q)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})+\rho(p)\rho(q)\Im(t,\ \varpi_1,$ $\cdots,\ \varpi_{n-1})$, for every $p,\ q,\ t,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$.

Proof. By considering the given hypothesis, let us substitute $p+t$ for p in (8) to find

$$
\begin{aligned}
\Im((p+t)q(p+t),\, \varpi_1,\, \cdots,\, \varpi_{n-1}) =& \Im(p+t,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\sigma(q)\sigma(p+t) \\
&+\rho(p+t)\rho(q)\Im(p+t,\, \varpi_1,\, \cdots,\, \varpi_{n-1}) \\
&+\rho(p+t)\Im(q,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\sigma(p+t),
\end{aligned} \tag{9}
$$

for all $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$. Simplify the above equation from left to obtain

$$
\begin{aligned}
\Im((p+t)q(p+t),\, \varpi_1,\, \cdots,\, \varpi_{n-1}) =& \Im(pqp+pqt+tqp+tqt, \varpi_1,\, \cdots,\, \varpi_{n-1}) \\
=& \Im(pqp,\, s_1,\, ..,\, s_{n-1}) \\
&+\Im(pqt+tqp,\, s_1,\, ..,\, s_{n-1}) \\
&+\Im(tqt,\, s_1,\, ..,\, s_{n-1}) \\
=& \Im(p,\, s_1,\, ..,\, s_{n-1}))\sigma(q)\sigma(p) \\
&+\rho(p)\rho(q)\Im(p,\, s_1,\, ..,\, s_{n-1}) \\
&+\rho(p)\Im(q,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\sigma(p) \\
&+\Im(pqt+tqp,\, \varpi_1,\, \cdots,\, \varpi_{n-1}) \\
&+\Im(t,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\sigma(q)(t) \\
&+\rho(t)\rho(q)\Im(t,\, \varpi_1,\, \cdots,\, \varpi_{n-1}) \\
&+\rho(t)\Im(q,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\sigma(t),
\end{aligned} \tag{10}
$$

for all $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$. Next, solving (9) from right in such a way for every $p,\ t,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$,

$$\begin{aligned}\Im((p+t)q(p+t),\, \varpi_1,\, \cdots,\, \varpi_{n-1}) =& \Im(p,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\sigma(q)\sigma(p)\\ &+\Im(t,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\sigma(q)\sigma(p)\\ &+\Im(p,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\sigma(q)\sigma(t)\\ &+\Im(t,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\sigma(q)\sigma(t)\\ &+\rho(p)\rho(q)\Im(p,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\\ &+\rho(t)\rho(q)\Im(p,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\\ &+\rho(p)\rho(q)\Im(t,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\\ &+\rho(t)\rho(q)\Im(t,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\\ &+\rho(p)\Im(q,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\sigma(p)\\ &+\rho(t)\Im(q,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\sigma(p)\\ &+\rho(p)\Im(q,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\sigma(t)\\ &+\rho(t)\Im(q,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\sigma(t).\end{aligned} \tag{11}$$

Evaluate the last two equations (10) and (11) to get

$$\begin{aligned}\Im(pqt+tqp,\, \varpi_1,\, \cdots,\, \varpi_{n-1}) =& \Im(p,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\sigma(q)\sigma(t)\\ &+\rho(p)\Im(q,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\sigma(t)\\ &+\Im(t,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\sigma(q)\sigma(p)\\ &+\rho(t)\Im(q,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\sigma(p)\\ &+\rho(t)\rho(q)\Im(p,\, \varpi_1,\, \cdots,\, \varpi_{n-1})\\ &+\rho(p)\rho(q)\Im(t,\, \varpi_1,\, \cdots,\, \varpi_{n-1}),\end{aligned} \tag{12}$$

for every $p,\ q,\ t,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$. This completes the proof.

Theorem 1 Let the maps $\sigma,\ \rho$ be automorphisms on a prime ring R with $char \neq 2$ and $\Im$ be a symmetric Jordan-$(\sigma,\ \rho)$-n-derivation on R that satisfy the properties in Lemma 3, 4 and 5. Then either any symmetric Jordan-$(\sigma,\ \rho)$-n-derivation is symmetric $(\sigma,\ \rho)$-n-derivation on R, or R is commutative.

Proof. Assumes the hypothesis of Lemma 5 and begin with (12) by putting pq in place of t, we find

$$\begin{aligned}\Im(pqpq+pqqp,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) =& \Im(pq,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)\sigma(p)\\ &+\rho(pq)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p)\\ &+\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)\sigma(pq)\\ &+\rho(pq)\rho(q)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\ &+\rho(p)\rho(q)\Im(pq,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\ &+\rho(p)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(pq),\end{aligned} \tag{13}$$

for all $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$. Simplifying the above to observe the right side as follows

$$\begin{aligned}\Im(pqpq+pqqp,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) =& \Im(pq,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)\sigma(p)\\ &+\rho(p)\rho(q)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p)\\ &+\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)\sigma(p)\sigma(q)\\ &+\rho(p)\rho(q)\rho(q)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\ &+\rho(p)\rho(q)\Im(pq,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\ &+\rho(p)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p)\sigma(q),\end{aligned} \tag{14}$$

for all $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$. Next, reword the equation (13) as below

$$
\begin{aligned}
\Im(pqpq+pqqp,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) =&\Im((pq)^2+pq^2p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\
=&\Im((pq)^2,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\
&+\Im(pq^2p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\
=&\Im(pq,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p)\sigma(q)\\
&+\rho(p)\rho(q)\Im(pq,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\
&+\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)\sigma(q)\sigma(p)\\
&+\rho(p)\rho(q)\rho(q)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\
&+\rho(p)\Im(q^2,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p),
\end{aligned}
\tag{15}
$$

for all $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$. This implies that

$$
\begin{aligned}
\Im((pq)^2+pq^2p,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) =&\Im((pq)^2,\ \varpi_1,\ \cdots,\ \varpi_{n-1})+\Im(pq^2p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\
=&\Im(pq,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p)\sigma(q)\\
&+\rho(p)\rho(q)\Im(pq,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\
&+\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)\sigma(q)\sigma(p)\\
&+\rho(p)\rho(q)\rho(q)\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\
&+\rho(p)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)\sigma(p)\\
&+\rho(p)\rho(q)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p),
\end{aligned}
\tag{16}
$$

for all $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$. Confronting the equation (16) and (14), we have

$$\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)\sigma(p)\sigma(q)+\rho(p)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p)\sigma(q)$$
$$+\Im(pq,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)\sigma(p)$$
$$=\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)\sigma(q)\sigma(p) \tag{17}$$
$$+\rho(p)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)\sigma(p)$$
$$+\Im(pq,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(p)\sigma(q),$$

for all $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1}\in R$. Equate the two expression to conclude

$$\begin{aligned}&\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)[\sigma(p),\ \sigma(q)]+\rho(p)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})[\sigma(p),\ \sigma(q)]\\&-\Im(pq,\ \varpi_1,\ \cdots,\ \varpi_{n-1})[\sigma(p),\ \sigma(q)]=0,\end{aligned} \tag{18}$$

for all $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1}\in R$. A usual manipulation in (18) yields that

$$\begin{aligned}&\{\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)+\rho(p)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\\&-\Im(pq,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\}R[\sigma(p),\ \sigma(q)]=0,\end{aligned} \tag{19}$$

for all $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1}\in R$. In view of primeness of R and (19), we have either $\{\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)+\rho(p)$ $\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})-\Im(pq,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\}=0$ or $[\sigma(p),\ \sigma(q)]=0$, for all $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1}\in R$. Considering the last case that $[\sigma(p),\ \sigma(q)]=0$, for all $p,\ q\in R$. Replace q by $\sigma^{-1}(q)$ to find $[\sigma(p),\ q]=0$, for all $p,\ q\in R$. Hence R is commutative by Lemma 2. Next if $\{\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)+\rho(p)\Im(q,\ \varpi_1,\ \cdots,\ \varpi_{n-1})-\Im(pq,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\}=0$, for all $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1}\in R$, then we arrive at $\Im(pq,\ \varpi_1,\ \cdots,\ \varpi_{n-1})=\Im(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})\sigma(q)+\rho(p)\Im(q,\ \varpi_1,\ \cdots,$ $\varpi_{n-1})$. for all $p,\ q,\ \varpi_1,\ \cdots,\ \varpi_{n-1}\in R$. That is nothing but the definition of $(\sigma,\ \rho)$-n-derivation. This is the desired conclusion.

Example 1 Consider a field $\digamma$, automorphisms $\sigma,\ \rho$. The set

$$\Re=\left[\left(\begin{array}{cc}v&\omega\\0&0\end{array}\right)\middle|,\ \omega\in\digamma\right]$$

is a ring with matrix operations. Define a map $\zeta_1:\Re\longrightarrow\Re$ such that

$$\zeta_1\left[\left(\begin{array}{cc}v&\omega_1\omega_2\cdots\omega_{n-1}\\0&0\end{array}\right)\right]=\left(\begin{array}{cc}\sigma(v)&\rho\left(\omega_1\omega_2\cdots\omega_{n-1}\right)\\0&0\end{array}\right),$$

and $\zeta_2 : \mathfrak{R} \longrightarrow \mathfrak{R}$ such that

$$\zeta_2 \left[\begin{pmatrix} v & \omega_1 \omega_2 \cdots \omega_{n-1} \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} \sigma(v) & 0 \\ 0 & 0 \end{pmatrix}.$$

In this setting ζ_1, ζ_2 are automorphisms on R. Define a map $\mathfrak{I} : R^n \longrightarrow R$ such that

$$\mathfrak{I} \left[\begin{pmatrix} v & \omega_1 \omega_2 \cdots \omega_{n-1} \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & \sigma(v) \rho (\omega_1, \omega_2, \cdots, \omega_{n-1}) \\ 0 & 0 \end{pmatrix}.$$

This is (ζ_1, ζ_2)-n-derivation on $\mathfrak{R}$.

3. *-n-multipliers on *-rings

A $*$-algebra ($*$-ring) or algebra with involution is an algebra (ring) that has an involution attached to it (ring with involution). A $\mathscr{C}^*$-algebra $\mathscr{B}$ with the additional norm constraint $||a^*a|| = ||a||^2$, is a Banach $*$-algebra for each $a \in \mathscr{B}$. With thorough study of $*$-mapping and its propertiesone refers the reader to [10–12].

Inspired by the concepts presented in [13], the notion of $*$-n-multipliers are now introduced in the following manner: A symmetric left $*$-n-multiplier (or symmetric right $*$-n-multiplier) is defined as a symmetric n-additive mapping $\mathfrak{T} : R^n \longrightarrow R$ if

$$\mathfrak{T}(pq, \varpi_1, \cdots, \varpi_{n-1}) = \mathfrak{T}(p, \varpi_1, \cdots, \varpi_{n-1}) q^*$$

$$(\text{ or } \mathfrak{T}(pq, \varpi_1, \cdots, \varpi_{n-1}) = p^* \mathfrak{T}(q, \varpi_1, \cdots, \varpi_{n-1}))$$

holds for all $p, q, \varpi_1, \cdots, \varpi_{n-1} \in R$. $\mathfrak{T}$ will be called $*$-n-multiplier if it's both left as well as right $*$-n-multiplier.

A prime $*$-ring R is commutative if and only if it allows a $*$-derivation (or, alternatively, a reverse $*$-derivation). This was shown by Bresar and Vukman in [11]. AdditionallyAshraf and Ali [10], expanded the semiprime $*$-ring finding previously discussed.

This section's primary objective is to demonstrate the following theorem for left $*$-n-multiplier. In further detail, the outcomes will be demonstrated as follows:

Theorem 2 Let a semiprime $*$-ring R and $\mathfrak{T} : R^n \longrightarrow R$ be a symmetric left $*$-n-multiplier on R. Then $\mathfrak{T}$ maps R^n into $Z(R)$.

Proof. Based on the defined criteria of $*$-n-multiplier, for every $p, q, \varpi_1, \cdots, \varpi_{n-1} \in R$, we have

$$\mathfrak{T}(pq, \varpi_1, \cdots, \varpi_{n-1}) = \mathfrak{T}(p, \varpi_1, \cdots, \varpi_{n-1}) q^*. \tag{20}$$

Substitute qv for q and use the property $(pq)^* = q^* p^*$ in (20) to obtain

$$\mathfrak{T}(p(qv), \varpi_1, \cdots, \varpi_{n-1}) = \mathfrak{T}(p, \varpi_1, \cdots, \varpi_{n-1}) v^* q^*, \tag{21}$$

for every $p,\ q,\ \nu,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$. Another way to expand the equation (21) is as

$$\mathfrak{T}((pq)\nu,\ \varpi_1,\ \cdots,\ \varpi_{n-1}) = \mathfrak{T}(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})q^*\nu^*, \tag{22}$$

for every $p,\ q,\ \nu,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$. On subtraction of (21) and (22), we get

$$\mathfrak{T}(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})[q^*,\ \nu^*] = 0, \text{ for every } p,\ q,\ \nu\varpi_1,\ \cdots,\ \varpi_{n-1} \in R. \tag{23}$$

This implies that

$$\mathfrak{T}(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})[q,\ \nu] = 0, \text{ for every } p,\ q,\ \nu,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R. \tag{24}$$

Reword the above equation by putting $q\mathfrak{T}(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})$ for q in (24), we find

$$\mathfrak{T}(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1})q[\mathfrak{T}(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1}),\ \nu] = 0, \text{ for every } p,\ q,\ \nu,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R. \tag{25}$$

With an additional alteration to the equation above, we may attain the form

$$[\mathfrak{T}(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1}),\ \nu]q[\mathfrak{T}(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1}),\ \nu] = 0, \tag{26}$$

for every $p,\ q,\ \nu,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$. Semiprimeness of R reveals that $[\mathfrak{T}(p,\ \varpi_1,\ \cdots,\ \varpi_{n-1}),\ \nu] = 0$, for every $p,\ q,\ \nu,\ \varpi_1,\ \cdots,\ \varpi_{n-1} \in R$. Hence $\mathfrak{T}$ maps R^n into $Z(R)$. This is the intended result.

Theorem 3 Let a semiprime $*$-ring R and $\mathfrak{T} : R^n \longrightarrow R$ be a symmetric right $*$-n-multiplier on R. Then $\mathfrak{T}$ maps R^n into $Z(R)$.

Proof. The proof is parallel as the above theorem.

4. Conclusion

The present research explores some of results on the structure of Jordan $(\sigma,\ \rho)$-n-derivations satisfying particular conditions on a prime ring. In third section, we extend the results on $*$-rings utilizing the structure of left and right $*$-n-multiplier. The future research scope in the context of presented research is to obtain the continuity theorems on additional algebraic structures, such as Banach algebra, semi-simple Banach algebra, Lie algebra, C^* algebra etc. It would be fascinating to explore the present idea in light of [12] using the tools of algebra of linear operators (transformations).

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Conflict of interest

The authors declared that there are no conflicts of interest.

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