Contemporary Mathematics
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UNIVERSAL WISER PUBLISHER

Research Article

Approximate Controllability for Fuzzy Fractional Evolution Equations of Order $\ell \in (1, 2)$

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Received: 20 June 2024; **Revised:** 29 July 2024; **Accepted:** 31 July 2024

Abstract: This study focuses on exploring the concept of approximate controllability in nonlocal Cauchy problems associated with fuzzy fractional evolution equations of order $\ell \in (1, 2)$. Our approach combines fuzzy theory, fractional calculus, and Krasnoselskii's fixed point theorem to establish key findings. Our primary objective is to establish the approximate controllability of these fuzzy fractional systems. Additionally, we provide a theoretical example to illustrate the implications of our findings.

Keywords: approximate controllability, fuzzy fractional evolution system, mild solution, fixed point theory

MSC: 34A07, 34A08, 47H10, 93B05

1. Introduction

A fractional differential equation extends ordinary differential equations to non-integer orders through integration. Recent studies have demonstrated the effectiveness of fractional-order derivatives in modeling a wide range of physical phenomena in science and engineering. The field of partial fractional differential equations and fractional-order ordinary differential equations has seen significant advancements in recent years. This progress has captured the interest of physicists, engineers, and mathematicians. Lakshmikantham and Vatsala [1] discuss key concepts such as existence, uniqueness, and stability of solutions, as well as methods for handling both linear and nonlinear fractional differential equations. Concepts of stability of linear infinite-dimensional continuous-time conformable systems with the conformable fractional derivatives using the theories of α-semigroup are studied in [2]. Their contributions have significantly advanced the mathematical theory underpinning fractional differential equations. Zhou [3] in the book "Basic Theory of Fractional Differential Equations," offers a comprehensive exploration of the fundamental principles underlying fractional differential equations. Ahmad et al. [4] explores the existence and uniqueness of solutions for complex boundary value problems. This research is pivotal for understanding systems with memory effects and spatially distributed influences, commonly found in disciplines such as thermodynamics and elasticity. Ahmad et al. [5] proposed initial and boundary value problems, enhancing our understanding of systems governed by complex fractional dynamics. Chalishajar et al. [6] investigates the conditions under which solutions to these complex equations exist and are unique. The study leverages Gronwall's inequality within the framework of Banach spaces to establish rigorous mathematical results.

DOI: https://doi.org/10.37256/cm.5320245162

Fractional calculus is now applied in various disciplines, including rheology, viscoelasticity, electrochemistry, and electromagnetics. Zhou [7] discuss the Analysis and Control explores the dynamics and control of systems described by fractional evolution equations and inclusions. The article emphasizes analytical techniques and control strategies, addressing the challenges posed by the fractional nature and inclusions in such equations within various application domains. A new type of fractional derivatives involving Riemann-Liouville and Caputo cotangent fractional derivatives is proposed in [8] and by utilizing this derivative, a nonlinear fractional differential problem with a nonlocal initial condition is investigated in [9]. These investigations show the important role that fractional differential equations play in explaining detailed real-world processes. They extend the theory of fractional calculus and its applications in fields such as environmental science, engineering, and physics.

Fuzzy set theory has seen a rise in popularity recently, with applications spanning mechanics, electrical engineering, and signal processing [10]. This theory was initially introduced by Zadeh in [11]. Chang and Zadeh later defined fuzzy derivatives [12], and Kandal and Byatt coined the term "fuzzy differential equation" in [13]. These equations are particularly useful for modeling the transmission of epistemic uncertainty in dynamic environments [14]. Extensive research on the existence and uniqueness of solutions to these equations has been carried out by Song and Wu [15], Seikkala [16], and Kaleva [17]. Karpagappriya et al. [18] introduces a novel approach using the cubic spline method to solve Fuzzy differential equations. Agarwal et al. [19] investigates the integration of fuzzy and stochastic methodologies to effectively handle the inherent uncertainties in dynamic systems, providing a robust theoretical foundation for further research and applications.

Allahviranloo et al. [20] introduces a novel method for solving fuzzy fractional differential equations using the generalized fuzzy Taylor expansion. Fuzzy initial value problem defined in [21] extend the classical initial value problems by incorporating fuzzy logic to handle uncertainty in initial conditions and system parameter and Fuzzy similarity measures proposed in [22] adapted for fuzzy sets, quantify the degree of overlap between the species compositions of different communities. By comparing these concept, the proposed fuzzy strucuture has the ability to capture a range of possible behaviors and outcomes, providing a comprehensive view of the system's dynamics under uncertainty. Stefanini and Bede [23] explores the concept of generalized Hukuhara differentiability for interval-valued functions and its application to interval differential equations. In recent days, fuzzy fractional theory has been extended with graph concepts to propose novel ideas like fuzzy fractional factors in fuzzy graph as described in [24].

In mathematical control theory, controllability is a key concept with significant relevance in both theoretical and applied mathematics. Its importance has grown within the context of fractional calculus [25]. Chalishajar et al. [26] explored the controllability of complex dynamical systems characterized by impulsive effects, fractional orders, and infinite delays. Gokul and Udhayakumar [27] focus on the concept of approximate controllability, which is crucial for understanding how small changes in control inputs can steer the system's state within a desired range over time. Researchers are increasingly investigating the application of control theory to fractional differential systems. Varun Bose and Udhayakumar [28] investigates the conditions under which systems governed by Caputo fractional differential equations can be controlled approximately. Recently, numerous scholars have aimed to deepen our understanding of the exact and approximate controllability of various dynamical systems, including those with delays. Exact controllability refers to a system's ability to reach any specified final state, whereas approximate controllability means the system can get arbitrarily close to any desired state. In recent years, Sadek has proposed numerous innovative ideas in controllability and observability contributing substantially to the advancement of fractional evolution equations [29–31].

Dinesh et al. [32] provides a detailed analysis of the conditions under which approximate controllability can be achieved in the systems, incorporating both stochastic elements and delay inclusions. Vijayakumar et al. [33] revisits the controllability of fractional Sobolev-type integro-differential systems characterized by Volterra-Fredholm operators with fractional orders between 1 and 2. Kumar et al. [34] explores the conditions under which boundary controllability can be achieved for nonlocal impulsive neutral integrodifferential evolution equations. Sakthivel et al. [35] deals with the concept of approximate controllability in the context of fractional differential inclusions, which are a generalization of fractional differential equations. This concept refers to the ability to steer the state of a system arbitrarily close to a desired final state, within a given tolerance. It is a relaxed version of exact controllability, which demands reaching the exact final state.

The study by Aziz El Ghazouani et al. [10] delves into the existence and uniqueness of fuzzy mild solutions for fractional evolution equations. To the best of our knowledge, the concept of approximate controllability with fuzzy fractional evolution equation has not been investigated. Inspired by this motivation, we introduce a approximate controllability in this paper to pursue this direction. This paper aims to close this gap, motivated by the aforementioned work. We prove the existence of a mild solution to the approximate controllability for fuzzy fractional evolution equation using the Caputo fractional derivative.

$$\begin{cases} {}^{C}_{gH}D^{\ell}_{0_+}h(\wp) = \Lambda h(\wp) + Bu(\wp) + f(\wp, h(\wp), {}^{C}_{gH}D^{\ell-1}_{0_+}h(\wp)), \quad \wp \in [0, a] = \mathbb{P}, \\ h(0) = h_0, \quad 1 < \ell < 2, \\ {}^{C}D^{\ell-i}_{0_+}h(0) = h_i, \quad i = 1, ..., |\ell|, \end{cases} \tag{1}$$

where ${}^{C}D^{\ell}$ is a Caputo fractional derivative of order $\ell \in (1, 2)$. The state variable $h(\cdot)$. Where Λ is an infinitesimal generator of a compact operator semigroup $\{T(\wp)\}_{\wp \geq 0}$ on J is the triangular fuzzy number in $\mathbb{R}_{\mathscr{F}}$ and f is a continuous function. $(J, d_\infty) \subseteq (\mathbb{R}_{\mathscr{F}}, d_\infty)$ is an metric space. Assume that H is a subset of $\mathbb{R}_{\mathscr{F}}$, $\mathbb{J} \subset \mathbb{R}$. Then, let the set of all continuous mappings $f : \mathbb{J} \to H$ represent $\mathscr{C}(\mathbb{J}, H)$.

This article's primary benefit and contribution can be put up as follows:

1. We study the approximate controllability results of fuzzy fractional differential evolution systems in this work.
2. A new set of sufficient conditions has been determined for the system (1) to possess a mild solution. This research expands on multiple findings related to fuzzy fractional differential equations that utilize Caputo and Riemann-Liouville fractional derivatives.
3. Our approach is based on the fixed point theorem of Krasnoselskii, which has been successfully applied to prove our new results.
4. A theoretical example is used to demonstrate the recommended results.

Additionally, the remainder of the paper has been categorized as follows: We get the fundamental ideas of fuzzy fractional calculus relevant to our investigation from Sect 2. To put it briefly, Sect 3 explore the Fuzzy fractional derivative. Sect 4 presents evidence supporting the established existence of the mild solution. Sect 5 discusses the approximate controllability of the system under consideration. Sect 6 provides an example for helping in understanding. Finally, Sect 7 reflects our conclusion.

2. Preliminaries

In this section, we will cover some key ideas that will be important to the rest of the article.

Definition 1 [10] A fuzzy number is a fuzzy set $h : \mathbb{R} \to [0, 1]$ that satisfies the following conditions:

1. h is normal, i.e., there is a $\wp_0 \in \mathbb{R}$ s.t. $h(\wp_0) = 1$;
2. h is a fuzzy convex set;
3. h is upper semi-continuous;
4. h closure of $\wp \in \mathbb{R} : h(\wp) > 0$ is compact.

E^1 represents the space of all fuzzy numbers on $\mathbb{R}$.

$$M^1 = \{h : \mathbb{R} \to [0, 1]\}.$$

$\forall \varpi \in (0, 1]$ the ϖ-cut of an element of M^1 is defined by

$$x^{\varpi} = \{\wp \in \mathbb{R} : h(\wp) \geq \varpi\}.$$

We may write using the previous parcels

$$h^{\varpi} = [\underline{h}(\varpi), \bar{h}(\varpi)]. \tag{2}$$

Where,

• $\underline{h}(\varpi) \Rightarrow$ Lower bound of the function h at ϖ, representing the minimum possible value.

• $\bar{h}(\varpi) \Rightarrow$ Upper bound of the function h at ϖ, representing the maximum possible value.

The distance between two elements of M^1 is given by

$$d(h, g) = \sup_{\varpi \in (0, 1]} \max\{|\bar{h}(\varpi) - \bar{g}(\varpi)|, \ |\underline{h}(\varpi) - \underline{g}(\varpi)|\}. \tag{3}$$

Furthermore, the following characteristics are true:

1. $d(h+k, g+k) = d(h, g)$;
2. $d(\rho h, \rho g) = |\rho| d(h, g)$;
3. $d(h+g, j+k) \leq d(h, j) + d(g, k)$.

Definition 2 [10] Fuzzy number addition and scalar multiplication within $\mathbb{R}_{\mathscr{F}}$ are expressed as

$$[h \oplus g]^{\varpi} = [h]^{\varpi} + [g]^{\varpi} \quad \text{and} \quad [\rho \odot h]^{\varpi} = \rho [h]^{\varpi}, \quad \rho \in \mathbb{R}, \tag{4}$$

where

(i) $[h]^{\varpi} + [g]^{\varpi} = \{u + v : u \in [h]^{\varpi}, v \in [g]^{\varpi}\}$ is the Minkowski sum of $[h]^{\varpi}$ and $[g]^{\varpi}$,

(ii) $\rho [h]^{\varpi} = \{\rho u : u \in [h]^{\varpi}\}$.

Definition 3 [36] The gH difference of h and g, represented by $h \ominus_{gH} g$, for $h, g \in \mathbb{R}_F$, is defined as the element $k \in \mathbb{R}_{\mathscr{F}}$ such that

$$h \ominus_{gH} g = k \quad \Leftrightarrow \quad \{(i)\ h = g + k \quad \text{or} \quad (ii)\ g = h + (-1)k\}. \tag{5}$$

In terms of ϖ-levels we have

$$(h \ominus_{gH} y)^{\varpi} = [min\{\underline{h}(\varpi) - \underline{g}(\varpi), \bar{h}(\varpi) - \bar{g}(\varpi)\}, max\{\underline{h}(\varpi) - \underline{g}(\varpi), \bar{h}(\varpi) - \bar{g}(\varpi)\}],$$

The condition for $k = h \ominus_{gH} g$ to exist in M^1 is:

Case-(i)

$$\begin{cases} \underline{k}(\varpi) = \underline{h}(\varpi) - \underline{g}(\varpi) \text{ and } \bar{k}(\varpi) = \bar{h}(\varpi) - \bar{g}(\varpi), \\ \\ \text{with } \underline{k}(\varpi) \text{ increasing, } \bar{k} \text{ decreasing, } \underline{k}(\varpi) \le \bar{k}(\varpi); \quad \forall \varpi \in [0, 1]. \end{cases} \tag{6}$$

Case-(ii)

$$\begin{cases} \underline{k}(\varpi) = \bar{h}(\varpi) - \bar{g}(\varpi) \text{ and } \bar{k}(\varpi) = \underline{h}(\varpi) - \underline{g}(\varpi), \\ \\ \text{with } \underline{k}(\varpi) \text{ increasing, } \bar{k} \text{ decreasing, } \underline{k}(\varpi) \le \bar{k}(\varpi); \quad \forall \varpi \in [0, 1]. \end{cases} \tag{7}$$

Definition 4 [37] For a fuzzy value function $\mathcal{F}\colon (a, b) \to M^1$ at $\wp_0$, the generalized Hukuhara derivative is defined as follows: Let $\wp_0 \in (a, b)$ and h_1 be such that $\wp_0 + h_1 \in (a, b)$.

$$\lim_{h_1 \to 0} \left\| \frac{\mathcal{F}(\wp_0 + h_1) -_g \mathcal{F}(\wp_0)}{h_1} -_g \mathcal{F}'_{gH}(\wp_0) \right\|_1 = 0. \tag{8}$$

Definition 5 [10] The Gamma function $\Gamma(u)$, extending the factorial to complex and real numbers, is defined by

$$\Gamma(u) = \int_0^\infty \vartheta^{u-1} e^{-\vartheta} d\vartheta, \quad \kappa > 0, \tag{9}$$

For positive integers m, it holds that

$$\begin{aligned} \Gamma(m) &= (m-1)!, \\ \Gamma(m) &= m\Gamma(m-1). \end{aligned}$$

Definition 6 [37] Assume that $\mathcal{F}\colon [a, b] \to M^1$ and $\wp_0 \in (a, b)$, differentiable at $\wp_0$ by $\underline{\mathcal{F}}(\wp, \varpi)$ and $\bar{\mathcal{F}}(\wp, \varpi)$.
(a) $\mathcal{F}$ is$[(1) - gH]$-differentiable at $\wp_0$ if

$$\mathcal{F}'_{1,\, gH}(\wp_0) = [\underline{\mathcal{F}'}(\wp, \varpi), \bar{\mathcal{F}}'(\wp, \varpi)].$$

(b) $\mathcal{F}$ is $[(2) - gH]$ differentiable at $\wp_0$ if

$$\mathcal{F}'_{2,\, gH}(\wp_0) = [\bar{\mathcal{F}}'(\wp, \varpi), \underline{\mathcal{F}'}(\wp, \varpi)].$$

Theorem 1 [38] Let $\phi\colon \mathbb{J} \to \mathbb{R}$, $\mathcal{F}\colon \mathbb{J} \subset \mathbb{R} \to M^1$, and $\wp \in \mathbb{J}$. Assume that the fuzzy-valued function $\mathcal{F}(\wp)$ is gH-differentiable at $\wp$ and that $\phi(\wp)$ is a differentiable function at $\wp$. So,

$$(F\phi)_g'(\wp) = (F'\phi)_g(\wp) + (F\phi')_g(\wp).$$

Definition 7 [39] At $\wp_0 \in (a, b)$, let $F: [a, b] \to M^1$ and $F'_{gH}(\wp)$ be gH-differentiable. Furthermore, there isn't a switch on (a, b) and $\underline{F}(\wp, \varpi)$ and $\bar{F}(\wp, \varpi)$ both differentiable at $\wp_0$. We state that

(a) F' is $[(1)-gH]$-differentiable at $\wp_0$ if

$$F''_{1, gH}(\wp_0) = \left[\underline{F}''(\wp, \varpi), \bar{F}''(\wp, \varpi)\right].$$

(b) F' is $[(2)-gH]$-differentiable at $\wp_0$ if

$$F''_{2, gH}(\wp_0) = \left[\bar{F}''(\wp, \varpi), \underline{F}''(\wp, \varpi)\right].$$

3. Fuzzy fractional derivative

The features of the generalized fuzzy fractional derivative are introduced in this section.

Definition 8 [40] Let $f \in L^{M^1}([a, b])$, where $L^{M^1}([a, b])$ denote the space of all Lebesgue integrable fuzzy-valued functions. The following defines the fuzzy Riemann-Liouville integral of the fuzzy-valued function:

$$^{RL}I^{\ell} f(\wp) = \frac{1}{\Gamma(\ell)} \odot \int_a^{\wp} (\wp - \theta)^{\ell-1} \odot f(\theta) d\theta, \quad a < \theta < \wp, \quad 0 < \ell < 1.$$

Definition 9 [39] [R-L derivative] Let us consider the fuzzy number-valued function $f \in L^{M^1}([a, b])$.

$$^{RL}D^{\ell} f(\wp) = \begin{cases} \frac{1}{\Gamma(q-\ell)} \odot \left(\frac{d}{d\theta}\right)^q \int_a^{\theta} (\theta - \wp)^{q-\ell-1} \odot f(\wp) d\wp, & q-1 < \ell < q, \\ \left(\frac{d}{d\theta}\right)^{q-1} f(\theta), & \ell = q-1. \end{cases}$$

Definition 10 [39] [Caputo gH derivative] Suppose that the operator operating on the operand function inside the integral $f(\wp) \in M^1$ is the integer order of the derivative in the definition of the RL fractional derivative, and that $\wp \in [a, b]$ is a fuzzy number-valued function.

$$^{C}D^{\ell} f(\theta) = \begin{cases} \frac{1}{\Gamma(q-\ell)} \odot \int_a^{\theta} (\theta - \wp)^{q-\ell-1} \odot f_{gH}^{(q)}(\wp) d\wp, & q-1 < \ell < q, \\ \left(\frac{d}{d\theta}\right)^{q-1} f(\theta), & \ell = q-1. \end{cases}$$

Definition 11 [41] Consider the following: $\digamma : [0, +\infty) \to H \subset \mathbb{R}_{\mathscr{F}}$ While $e^{-\theta} \wp \odot \digamma(\wp)$ is integrable, is a continuous function. $\mathscr{L}[\digamma(\wp)]$ represents $\digamma$'s fuzzy Laplace transform, which is

$$\mathscr{L}[\digamma(\wp)] := F(\theta) = \int_0^{+\infty} e^{-\theta\wp} \odot \digamma(\wp) d\wp, \quad \theta > 0.$$

A fuzzy-valued function f is comparable to $M > 0$.

Lemma 1 [10] Mittag-Leffler function $E_{\ell,\,n}$ is a special function, a function which depends on two parameters ℓ and n. It is defined by,

$$E_{\ell,\,n}(\wp) = \sum_{\alpha=1}^{\infty} \frac{\wp^{\alpha}}{\Gamma(\ell\alpha + n)}.$$

Where $\Gamma(\cdot)$ is the gamma function.

(i) $\forall \ell > 0$, we get the following result

$$\int_0^{\wp} E_{\ell,\,1}(\Lambda\theta^{\ell}) d\theta = \wp E_{\ell,\,2}(\Lambda\wp^{\ell}).$$

(ii) $\forall \ell \in [1, 2]$ and $\theta > 0$, we have

1. $\theta^{\ell-1}(\theta^{\ell} - \Lambda)^{-1} = L(E_{\ell,\,1}(\Lambda\wp^{\ell}))(\theta)$,
2. $\theta^{\ell-2}(\theta^{\ell} - \Lambda)^{-1} = L(\wp E_{\ell,\,2}(\Lambda\wp^{\ell}))(\theta)$,
3. $(\theta^{\ell} - \Lambda)^{-1} = \frac{1}{\Gamma(\ell-1)} L\big(\int_0^{\wp} (\wp - \theta)^{\ell-2} E_{\ell,\,1}(\Lambda\theta^{\ell}) d\theta\big)$.

Lemma 2 [41]

1. Let $v, w : [0, +\infty) \to H$ be continuous functions, $p_1, p_2 \in \mathbb{R}^+$. Then

$$\mathscr{L}\left[p_1 \odot v(\wp) + p_2 \odot w(\wp)\right] = p_1 \odot \mathscr{L}[v(\wp)] + p_2 \odot \mathscr{L}[w(\wp)].$$

2. Assume that the function $v : [0, +\infty) \to H$ is continuous, then

$$\mathscr{L}\left[e^{a\wp} \odot f(\wp)\right] = F(\theta - a), \quad \theta - a > 0.$$

3. Let $v \in C^1([0, +\infty), H)$ be an exponent bounded of order β. Then,

(a) When h approaches $[(1) - gH]$, if it is differentiable, then

$$\mathscr{L}\left[D^1_{gH} h(\wp)\right] = \theta \odot \mathscr{L}[h(\wp)] \ominus f(0),$$

(b) When h approaches $[(2) - gH]$, if it is differentiable, then

$$\mathscr{L}\left[D^2_{gH} h(\wp)\right] = (-1) \odot f(0) \ominus (-\theta) \odot \mathscr{L}[h(\wp)].$$

Definition 12 The Cauchy problem (1) can be represented as the following integral equation, by utilizing the gH derivative's Newton-Leibnitz formula [14], Caputo's fuzzy fractional derivative definition, and the Riemann-Liouville mixed fractional integral operator ${}^{RL}I^{\ell}_{0+}$ member applied to (1), we derive

$$h(\wp)\ominus_{gH} h(0)\ominus_{gH} \wp\odot h^{'}(0) = {}^{RL}I^{\ell}_{0+}(\Lambda h(\wp)+Bu(\wp)+f(\wp,\, h(\wp),\, {}^{C}_{gH}D^{\ell}h(\wp)).$$

The following claims flow from the definition of gH difference and (1).

(a) Let h is $[(1)-gH]$-Caputo differentiable if

$$h(\wp) = h(0)+\wp\odot h^{'}(0)+{}^{RL}I^{\ell}_{0+}(\Lambda h(\wp)+Bu(\wp)+f(\wp,\, h(\wp),\, {}^{C}_{gH}D^{\ell}h(\wp)).$$

(b) Let h is $[(2)-gH]$-Caputo differentiable if

$$h(\wp) = h(0)+\wp\odot h^{'}(0)\ominus(-1)\odot {}^{RL}I^{\ell}_{0+}(\Lambda h(\wp)+Bu(\wp)+f(\wp,\, h(\wp),\, {}^{C}_{gH}D^{\ell}h(\wp)).$$

Here is how we get the exact integral formula of the Cauchy issue (1) by using the fuzzy Laplace transform from [41].

Lemma 3 [10]

1. Assuming that h is Caputo $[(1)-gH]$ differentiable satisfying the conditions of the Cauchy problem (1), then

$$\begin{aligned} h(\wp) =& E_{\ell,\,1}(\Lambda\wp^{\ell})\odot h_0+\wp\odot E_{\ell,\,2}(\Lambda\wp^{\ell})\odot h_1 \\ &+\int_0^{\wp}\int_{\theta}^{\wp}\frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,\,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\wp)\oplus f(\theta,\, h(\theta),\, {}^{C}_{gH}D^{\ell-1}h(\theta))]d\delta d\theta. \end{aligned} \tag{10}$$

2. Assuming that h is Caputo $[(2)-gH]$ differentiable satisfying the conditions of the Cauchy problem (1), then

$$\begin{aligned} h(\wp) =& E_{\ell,\,1}(\Lambda\wp^{\ell})\odot h_0+\wp\odot E_{\ell,\,2}(\Lambda\wp^{\ell})\odot h_1\ominus \\ &(-1)\odot\int_0^{\wp}\int_{\theta}^{\wp}\frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,\,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\wp)\oplus f(\theta,\, h(\theta),\, {}^{C}_{gH}D^{\ell-1}h(\theta))]d\delta d\theta. \end{aligned} \tag{11}$$

Theorem 2 [27] Consider the Banach space $\mathtt{H}$. If $\mho_1,\, \mho_2$ be mappings from $\mathscr{B}\to\mathtt{H}$, then $\mathscr{B}$ is a closed, bounded, and convex subset of a Banach space $\mathtt{H}$ such that

(a) $\mho_1 h+\mho_2 g\in\mathscr{B}$ for all pair of $h,\, g\in\mathscr{B}$,

(b) $\mho_1$ is contraction mapping,

(c) $\mho_2$ is compact and continuous,

then $\mho_1(h)+\mho_2(h)=h$ has a solution in $\mathscr{B}$.

Now, let us introduce the pertinent operator for the approximate controllability,

$$\Gamma_0^b = \int_0^{\wp}\int_{\theta}^{\wp} E_{\ell,n}(\Lambda\wp^{\ell})BB^*E^*_{\ell,n}(\Lambda\wp^{\ell})d\delta d\theta,\, \mathscr{R}(\varpi,\Gamma_0^b) = (\varpi I + \Gamma_0^b)^{-1}, \quad \varpi > 0.$$

Here, B^* and $E^*_{\ell,n}(\Lambda\wp^{\ell})$ are the adjoint of B and $E_{\ell,n}(\Lambda\wp^{\ell})$, and Γ_0^b be the linear bounded operator. Next, for every $\varpi > 0$, and $h_1 \in H$, take

$$u(\wp) = B^*E^*_{\ell,n}(\Lambda\wp^{\ell})\mathscr{R}(\varpi,\Gamma_0^b)p(u(\cdot)),$$

where

$$\begin{aligned} p(h^{\beta}) = & h_b - E_{\ell,1}(\Lambda\wp^{\ell})\odot h_0 + \wp\odot E_{\ell,2}(\Lambda\wp^{\ell})\odot h_1 \\ & + \int_0^{\wp}\int_{\theta_1}^{\wp}\frac{(\wp-\delta_1)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta_1-\theta_1)^{\ell})\odot f(\theta_1, h^{\beta}(\theta_1), {}_{gH}^{C}D^{\ell-1}h^{\beta}(\theta_1))d\delta_1 d\theta_1. \end{aligned}$$

4. Existence of mild solutions

We define a function $h \in \mathscr{C}([0, a], J)$ that fulfills (10) and (11) as a mild solution in types 1 and 2 of the Cauchy problem (1).

The next findings will make use of the following hypotheses:

(H1) Let $\sup_{\rho\in[0,\infty)} ||E_{\ell,n}(\Lambda\wp^{\ell})|| = M_E$, where $M_E \geq 1$, and $||R(\varpi,\Gamma_0^b)|| \leq 1$ for every $\varpi > 0$.

(H2) For any $\wp \in \mathbb{P}$, the function $f \in \mathscr{C}(\mathbb{P}\times J\times J, J)$ is almost continuous, and for every $z \in \mathscr{C}(\mathbb{P}, J)$, the function $f(\cdot,\ z,\ {}_{gH}^{C}D_{0_+}^{\ell-1}z)\colon \mathbb{P}\to J$ is strongly measurable.

(H3) There exist $\ell_2 \in [0,\ \ell)$, $B_r\colon = \{h \in J\colon d_{\infty}(h,\ \hat{0}) \leq r\} \subset J,\ r > 0,$ and $\rho(\cdot) \in L^{\frac{1}{\ell_2}}(\mathbb{P},\ \mathbb{R}^+)$ such that for any $h,\ g \in \mathscr{C}(\mathbb{P},\ B_r)$, we have

$$d_{\infty}\big(f\big(\wp, h(\wp), {}_{gH}^{C}D^{\ell-1}h(\wp)\big), f\big(\wp, g(\wp), {}_{gH}^{C}D^{\ell-1}g(\wp)\big)\big) \leq \rho(\wp)d_{\infty}(h(\wp), g(\wp)), \quad \wp\in\mathbb{P}.$$

(H4) There exists a constant $\ell_1 \in [0,\ \ell)$ and $m \in \rho(\cdot) \in L^{\frac{1}{\ell_1}}(\mathbb{P},\ \mathbb{R}^+)$ such that

$$d_{\infty}\big(f\big(\wp, h(\wp), {}_{gH}^{C}D^{\ell-1}h(\wp)\big), \hat{0}\big) \leq m(\wp),$$

$\forall z \in \mathscr{C}(\mathbb{P},\ J)$ and for almost all $\wp \in \mathbb{P}$.

(H5) For each $\wp > 0$ and $n \in \mathbb{N}$, $E_{\ell,n}(\Lambda\wp^{\ell})$ is a compact operator.

Theorem 3 The Cauchy problem (1) admits a mild fuzzy solution of type 1 in the space $C(\mathbb{P},\ J)$, satisfying assumptions (H1)-(H5).

Proof. Assume, $h \in \mathrm{C}(\mathbb{P},\ J)$. A measurable function on $\mathbb{P}$ is $f(\theta,\ h(\theta),\ {}_{gH}^{C}D^{\ell-1}h(\theta))$, since h is continuous with regard to $\wp$ and hypotheses (H1).

Let,

$$\mu = \frac{\ell - 1}{1 - \ell_1} \quad \text{and we consider} \quad 1 + \mu = \nu$$

$$M_1 = ||m||_{L^{\frac{1}{\ell_1}}\mathbb{P}}.$$

With Holder's inequality and (H4) applied for $\wp \in \mathbb{P}$, we obtain

$$d_\infty(E_{\ell,1}(\Lambda\wp^\ell) \odot h_0 + t \odot E_{\ell,2}(\Lambda\wp^\ell) \odot h_1) \leq M_E||h_0|| + aM_E||h_1||.$$

$$d_\infty\left(\int_0^\wp (\wp - \theta)^{\ell-1} \odot [Bu(\theta) \oplus f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta))]d\theta, \hat{0}\right)$$

$$\leq \int_0^\wp (\wp - \theta)^{\ell-1} \odot d_\infty(Bu(\theta) \oplus f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta)), \hat{0})d\theta$$

$$\leq \int_0^\wp (\wp - \theta)^{\ell-1} d_\infty(Bu(\theta), \hat{0})d\theta + \int_0^\wp (\wp - \theta)^{\ell-1} d_\infty(f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta)), \hat{0})d\theta$$

$$\leq \int_0^\wp (\wp - \theta)^{\ell-1} ||m||_{L^{\frac{1}{\ell_1}}[0,a]} d\theta + \int_0^\wp (\wp - \theta)^{\ell-1} B[B^* E_{\ell,1}^*(\Lambda(\delta - \theta)^\ell) R(\varpi, \Gamma_0^\delta) p(x(.))]d\theta.$$

Considering,

$$d_\infty\left(\int_0^\wp (\wp - \theta)^{\ell-1} \odot [Bu(\theta) \oplus f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta)d\theta, \hat{0})\right) \leq p_1 + p_2.$$

$$p_1 = \int_0^\wp (\wp - \theta)^{-\ell 1} ||m||_{L^{\frac{1}{\ell_1}}\mathbb{P}} d\theta = \left(\int_0^\wp (\wp - \theta)^{\frac{-\ell 1}{1-\ell_1}} d\theta\right)^{1-\ell_1} ||m||_{L^{\frac{1}{\ell_1}}[0,a]}$$

$$\leq \frac{M_1 a^{(\nu)(1-\ell_1)}}{(\nu)^{1-\ell_1}}.$$

We obtain

$$d_\infty\left(\int_0^\wp \int_\theta^\wp \frac{(\wp - \delta)^{\ell-2}}{\Gamma(\ell - 1)} \odot E_{\ell,1}(\Lambda(\delta - \theta)^\ell) \odot f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta))d\delta d\theta, \hat{0}\right)$$

$$\leq \int_0^\wp \int_\theta^\wp \frac{(\wp - \delta)^{\ell-2}}{\Gamma(\ell - 1)} \odot d_\infty(E_{\ell,1}(\Lambda(\delta - \theta)^\ell) \odot f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta)), \hat{0})d\delta d\theta$$

$$\leq \frac{M_E}{\Gamma(\ell)} \odot \int_0^{\wp} (\wp - \theta)^{\ell-1} d_\infty(f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta)), \hat{0}) d\theta$$

$$\leq \frac{M_E M_1 a^{(\nu)(1-\ell_1)}}{\Gamma(\ell)(\nu)^{1-\ell_1}}, \quad \text{for all } \wp \in \mathbb{P}.$$

$$p_2 = \int_0^{\wp} (\wp - \theta)^{\ell-1} B[B^* E_{\ell,1}^*(\Lambda(\delta - \theta)^{\ell}) R(\varpi, \Gamma_0^{\delta}) p(h(\cdot)) d\theta.$$

$$\leq \frac{{M_B}^2 M_E}{\varpi} \int_0^{\wp} (\wp - \theta)^{\ell-1} \left[h_b - M_E||h_0|| + aM_E||h_1|| + \frac{M_E M_1 a^{(\nu)(1-\ell_1)}}{\Gamma(\ell)(\nu)^{1-\ell_1}} \right] d\theta.$$

Here,

$$\Delta = h_b - M_E||h_0|| + aM_E||h_1|| + \frac{M_E M_1 a^{(\nu)(1-\ell_1)}}{\Gamma(\ell)(\nu)^{1-\ell_1}}.$$

Since,

$$p_2 \leq \frac{{M_B}^2 M_E \Delta}{\varpi} \left(\int_0^{\wp} (\wp - \theta)^{\frac{\ell-1}{1-\ell_1}} d\theta \right)^{1-\ell_1}$$

$$\leq \frac{{M_B}^2 M_E \Delta a^{(\nu)(1-\ell_1)}}{\varpi(\nu)^{1-\ell_1}}.$$

Therefore, we obtain

$$d_\infty \left(\int_0^{\wp} \int_{\theta}^{\wp} \frac{(t-\delta)^{\ell-2}}{\Gamma(\ell-1)} \odot E_{\ell,1}(\Lambda(\delta-\theta)^{\ell}) \odot Bu(\theta) d\delta d\theta, \hat{0} \right)$$

$$\leq \int_0^{\wp} \int_{\theta}^{\wp} \frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)} \odot d_\infty(E_{\ell,1}(\Lambda(\delta-\theta)^{\ell}) \odot Bu(\theta), \hat{0}) d\delta d\theta$$

$$\leq \frac{M_E}{\Gamma(\ell)} \odot \int_0^{\wp} (\wp - \theta)^{\ell-1} B[B^* E_{\ell,1}^*(\Lambda(\delta - \theta)^{\ell}) R(\varpi, \Gamma_0^{\delta}) p(x(.)) d\theta$$

$$\leq \frac{M_E}{\Gamma(\ell)} \frac{{M_B}^2 M_E \Delta a^{(\nu)(1-\ell_1)}}{\varpi(\nu)^{1-\ell_1}}$$

$$\leq \frac{{M_B}^2 {M_E}^2 \Delta a^{(\nu)(1-\ell_1)}}{\Gamma(\ell)\varpi(\nu)^{1-\ell_1}}.$$

Then

$$\int_0^{\wp}\int_\theta^{\wp}\frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta,$$

is bounded for any $\wp\in\mathbb{P}$. Regarding $h\in C(\mathbb{P}, J)$, we specify

$$(\mho_1 h)(\wp)=E_{\ell,1}(\Lambda\wp^{\ell})\odot h_0+\wp\odot E_{\ell,2}(\Lambda\wp^{\ell})\odot h_1.$$

$$(\mho_2 h)(\wp)=\int_0^{\wp}\int_\theta^{\wp}\frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta.$$

Set

$$K_0=M_E||h_0||+aM_E||h_1||+\frac{M_E M_1 a^{(\nu)(1-\ell_1)}}{\Gamma(\ell)(\nu)^{1-\ell_1}}+\frac{{M_B}^2{M_E}^2\Delta a^{(\nu)(1-\ell_1)}}{\Gamma(\ell)\varpi(\nu)^{1-\ell_1}}$$

$$K_0=||h_b||+\left(1+\frac{{M_B}^2{M_E}^2 a^{(\nu)(1-\ell_1)}}{\Gamma(\ell)\varpi(\nu)^{1-\ell_1}}\right)\Delta$$

and $B_{k_0}:=\{h(\cdot)\in C(\mathbb{P}, J)\colon d_\infty(h(\wp),\hat{0})\leq k_0 \text{ for all } \wp\in\mathbb{P}\}$. We will prove that F_1h+F_2h has a fixed point on B_{k_0}.

Step-1 We demonstrate that for every $h\in B_{k_0}$, $F_1h+F_2h\in B_{k_0}$. Indeed, with $0\leq\wp_1\leq\wp_2\leq a$, we have

$$d_\infty((F_2h)(\wp_2), (F_2h)(\wp_1))$$

$$=d_\infty\Bigg(\int_0^{\wp_2}\int_\theta^{\wp_2}\frac{(\wp_2-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta,$$

$$\int_0^{\wp_1}\int_\theta^{\wp_1}\frac{(\wp_1-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta\Bigg)$$

$$=d_\infty\Bigg(\int_{\wp_1}^{\wp_2}\int_\theta^{\wp_2}\frac{(\wp_2-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta$$

$$+\int_0^{\wp_1}\int_\theta^{\wp_1}\frac{(\wp_1-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta$$

$$+\int_0^{\wp_1}\int_\theta^{\wp_2}\frac{(\wp_2-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta,\hat{0}\Bigg)$$

$$
\begin{aligned}
=&d_\infty\Bigg(\int_{\wp_1}^{\wp_2}\int_{\theta}^{\wp_2}\frac{(\wp_2-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta\\
&+\int_{0}^{\wp_1}\int_{\theta}^{\wp_1}\frac{(\wp_2-\delta)^{\ell-2}-(\wp_1-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta\\
&+\int_{0}^{\wp_1}\int_{\wp_1}^{\wp_2}\frac{(\wp_2-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta,\hat{0}\Bigg)\\
=&d_\infty\Bigg(\int_{\wp_1}^{\wp_2}\int_{\theta}^{\wp_2}\frac{(\wp_2-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta,\hat{0}\Bigg)\\
&+d_\infty\Bigg(\int_{0}^{\wp_1}\int_{\theta}^{\wp_1}\frac{(\wp_2-\delta)^{\ell-2}-(\wp_1-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta,\hat{0}\Bigg)\\
&+d_\infty\Bigg(\int_{0}^{\wp_1}\int_{\wp_1}^{\wp_2}\frac{(\wp_2-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta,\hat{0}\Bigg)\\
=&\Xi_1+\Xi_2+\Xi_3,
\end{aligned}
$$

where,

$$
\Xi_1=d_\infty\Bigg(\int_{\wp_1}^{\wp_2}\int_{\theta}^{\wp_2}\frac{(\wp_2-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta ds,\hat{0}\Bigg).
$$

$$
\begin{aligned}
\Xi_2=&d_\infty\Bigg(\int_{0}^{\wp_1}\int_{\theta}^{\wp_1}\frac{(\wp_2-\delta)^{\ell-2}-(\wp_1-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\\
&\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta,\hat{0}\Bigg).
\end{aligned}
$$

$$
\Xi_3=d_\infty\Bigg(\int_{0}^{\wp_1}\int_{\wp_1}^{\wp_2}\frac{(\wp_2-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta,\hat{0}\Bigg).
$$

We have

$$\Xi_1 = d_\infty\left(\int_{\wp_1}^{\wp_2}\int_{\theta}^{\wp_2}\frac{(\wp_2-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta, \hat{0}\right)$$

$$\leq\left(\frac{M_E M_1}{\Gamma(\ell)}+\frac{{M_B}^2{M_E}^2\Delta}{\varpi\Gamma(\ell)}\right)\left(\int_{\wp_1}^{\wp_2}(\wp_2-\theta)^b d\theta\right)^{1-\ell_1}$$

$$\leq\left(\frac{M_E M_1}{\Gamma(\ell)}+\frac{{M_B}^2{M_E}^2\Delta}{\varpi\Gamma(\ell)}\right)\frac{(\wp_2-\wp_1)^{(\nu)(1-\ell_1)}}{(\nu)^{1-\ell_1}}.$$

Also

$$\Xi_2 = d_\infty\left(\int_{0}^{\wp_1}\int_{\theta}^{\wp_1}\frac{(\wp_2-\delta)^{\ell-2}-(\wp_1-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta, \hat{0}\right)$$

$$\leq\frac{M_E}{\Gamma(\ell)}\left(\int_{0}^{\wp_1}\left[(\wp_2-\theta)^{\ell-1}-(\wp_1-\theta)^{\ell-1}-(\wp_2-\wp_1)^{\ell-1}\right]\odot d_\infty(Bu(\theta)\oplus f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta))d\theta, \hat{0})\right)$$

$$\leq\frac{M_E}{\Gamma(\ell)}\left(\int_{0}^{\wp_1}\left[(\wp_2-\theta)^{\ell-1}-(\wp_1-\theta)^{\ell-1}\right]\odot d_\infty(Bu(\theta)\oplus f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta))d\theta, \hat{0})\right.$$

$$\left.-\int_{0}^{\wp_1}(\wp_2-\wp_1)^{\ell-1}\odot d_\infty(Bu(\theta)\oplus f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta))d\theta, \hat{0})\right)$$

$$\leq\left(\frac{M_E M_1}{\Gamma(\ell)}+\frac{{M_B}^2{M_E}^2\Delta}{\varpi\Gamma(\ell)}\right)\frac{\left(-(\wp_2-\wp_1)^{\nu}+\wp_2^{\nu}-\wp_1^{\nu}-(\wp_2-\wp_1)^{\ell-1}(\nu)^{1-\ell_1}\right)}{(\nu)^{1-\ell_1}}.$$

Likewise

$$\Xi_3 = d_\infty\left(\int_{0}^{\wp_1}\int_{\wp_1}^{\wp_2}\frac{(\wp_2-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta, \hat{0}\right)$$

$$\leq\frac{M_E}{\Gamma(\ell)}\left(\int_{0}^{\wp_1}(\wp_2-\wp_1)^{\ell-1}\odot d_\infty(Bu(\theta)\oplus f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta))d\theta, \hat{0})\right)$$

$$\leq\left(\frac{M_E M_1}{\Gamma(\ell)}+\frac{{M_B}^2{M_E}^2\Delta}{\varpi\Gamma(\ell)}\right)(\wp_2-\wp_1)^{\ell-1}\wp_1^{1-\ell_1}.$$

Then, it is straightforward that Ξ_1, Ξ_2 and Ξ_3 tend to 0 as $(\wp_2-\wp_1)\to 0$. So, $(\mho_2 h)(\wp)$ is continuous in $\wp\in\mathbb{P}$. $(\mho_1 h)(\wp)$ is likewise continuous in $\wp\in\mathbb{P}$, as can be easily observed. Let us now take any $h\in B_{k_0}$ and $\wp\in\mathbb{P}$ such that

$$d_\infty\big((\mho_1 h)(\wp)+(\mho_2 h)(\wp),\,\hat{0}\big)\leq ||h_b||+\left(1+\frac{{M_B}^2{M_E}^2a^{(\nu)(1-\ell_1)}}{\Gamma(\ell)\varpi(\nu)^{1-\ell_1}}\right)\Delta\leq k_0.$$

Then $\mho_1+\mho_2$ is an operator from B_{k_0} into B_{k_0}.
Step-2 We establish that $\mho_1(h(\wp))$ acts as a contraction on B_{k_0}. Consider $h,\, g\in B_{k_0}$. Then

$$\begin{aligned}
d_\infty\big(\mho_1(h(\wp))-\mho_1(g(\wp)),\,\hat{0}\big)&\leq d_\infty\big(E_{\ell,1}(\Lambda\wp^\ell)\odot h_0+\wp\odot E_{\ell,2}(\Lambda\wp^\ell)\odot h_1-E_{\ell,1}(\Lambda\wp^\ell)\odot g_0\\
&\quad-\wp\odot E_{\ell,2}(\Lambda\wp^\ell)\odot g_1,\,\hat{0}\big)\\
&\leq d_\infty\big(E_{\ell,1}(\Lambda\wp^\ell)\odot(h_0-g_0)+\wp\odot E_{\ell,2}(\Lambda\wp^\ell)\odot(h_1-g_1),\,\hat{0}\big)\\
&\leq d_\infty\big(M_E\odot(h_0-g_0)+a\odot M_E\odot(h_1-g_1),\,\hat{0}\big)\\
&\leq M_E d_\infty\big((h_0-g_0)+a\odot(h_1-g_1),\,\hat{0}\big)\\
&\leq M_E d_\infty\big(h(\wp)-g(\wp),\,\hat{0}\big).
\end{aligned}$$

Thus, $\mho_1(h(\wp))$ is a contraction.
Step-3 First, we show that $\mho_2$ is Continuous on B_{k_0}.
Let $h_n,u_n\subseteq B_{k_0}$ with $h_n\to h$ and $u_n\to u$ on B_{k_0}. Applying hypothesis (H3), we get

$$f\big(\theta,\,h_n(\theta),\,{}^C_{gH}D^{\ell-1}h_n(\theta)\big)\to f\big(\theta,\,h(\theta),\,{}^C_{gH}D^{\ell-1}h(\theta)\big),\quad \text{as}\quad n\to+\infty,$$

$$Bu_n(\theta)\to Bu(\theta),\quad \text{as}\quad n\to+\infty.$$

Based on the hypothesis (H4), nearly everywhere $\wp\in\mathbb{P}$,

$$d_\infty\big(f\big(\theta,\,h_n(\theta),\,{}^C_{gH}D^{\ell-1}h_n(\theta)\big),\,f\big(\theta,\,h(\theta),\,{}^C_{gH}D^{\ell-1}h(\theta)\big)\big)\leq 2m(\theta).$$

Therefore, by the domination convergence theorem, we get

$$\begin{aligned}
&d_\infty((\mho_2 h_n)(\wp),\,(\mho_2 h)(\wp))\\
=&d_\infty\Bigg(\int_0^\wp\int_\theta^\wp\frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^\ell)\odot[Bu_n(\theta)\oplus f(\theta,\,h_n(\theta),\,{}^C_{gH}D^{\ell-1}h_n(\theta))]d\delta d\theta,
\end{aligned}$$

$$\int_0^{\wp}\int_{\theta}^{\wp}\frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta\Big)$$

$$\leq\int_0^{\wp}(\wp-\theta)^{\ell-1}\frac{\ell M_E}{\Gamma(1+\ell)}\odot$$

$$d_\infty\bigg([Bu_n(\theta)\oplus f(\theta,h_n(\theta),{}_{gH}^{C}D^{\ell-1}h_n(\theta))],[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]\bigg)d\theta$$

$$\longrightarrow 0.$$

When $n\to+\infty$, This means $\mho_2$ is continuous.

Step-4

Next, we demonstrate that $\mho_2$ is equicontinuous. Let $h\in B_{k_0}$, where $\wp\in\mathbb{P}$, we proved this for all $h\in B_{k_0}$ and $0\leq\wp_1\leq\wp_2\leq a$,

$$d_\infty\big((\mho_2h)(\wp_2),(\mho_2h)(\wp_1)\big)\leq\Xi_1+\Xi_2+\Xi_3.$$

We now have

$$\Xi_1\leq\left(\frac{M_EM_1}{\Gamma(\ell)}+\frac{M_B{}^2M_E{}^2\Delta}{\varpi\Gamma(\ell)}\right)\frac{(\wp_2-\wp_1)^{(u)(1-\ell_1)}}{(u)^{1-\ell_1}}.$$

$$\Xi_2\leq\left(\frac{M_EM_1}{\Gamma(\ell)}+\frac{M_B{}^2M_E{}^2\Delta}{\varpi\Gamma(\ell)}\right)\frac{\big(-(\wp_2-\wp_1)^u+\wp_2^{p+1}-\wp_1^{p+1}-(\wp_2-\wp_1)^{\ell-1}(u)^{1-\ell_1}\big)}{(u)^{1-\ell_1}}.$$

$$\Xi_3\leq\left(\frac{M_EM_1}{\Gamma(\ell)}+\frac{M_B{}^2M_E{}^2\Delta}{\varpi\Gamma(\ell)}\right)(\wp_2-\wp_1)^{\ell-1}\wp_1^{1-\ell_1}.$$

From Step-1, it is easy to see that $\mho_2(B_{k_0})$ is equicontinuous and independent of $h\in B_{k_0}$ as $\wp_2\to\wp_1$. Thus, $\mho_2$ is equicontinuous on B_{k_0}.

Step-5 Lastly, we need to prove $V(h)=(\mho_2h)(\wp):h\in B_{k_0}$ is relatively compact. For any given $0<\wp\leq a$, for all $\varepsilon\in(0,t)$ and $\delta>0$, let the operater $\mho_{\varepsilon,\delta}$ be defined as

$$(\mho_{\varepsilon,\delta}h)(\wp)$$

$$=\int_0^{\wp-\varepsilon}\int_{\theta+\varpi-}^{\wp}\frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta$$

$$=\int_{0}^{\wp-\varepsilon}\int_{\theta+\varpi}^{\wp}\frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell}-\Lambda(\varpi-\varepsilon)+\Lambda(\varpi-\varepsilon))\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta$$

$$=E_{\ell,1}\Lambda(\varpi-\varepsilon)\int_{0}^{\wp-\varepsilon}\int_{\theta+\varpi-}^{\wp}\frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell}-\Lambda(\varpi-\varepsilon))\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta.$$

Where $h\in B_{k_0}$. From hypotheses (H5), $E_{\ell,1}(\Lambda(\varpi-\varepsilon))$ is the compact operater, then $V_{\varepsilon,\varpi}(\wp)=\mho_{\varepsilon,\varpi}h:h\in B_{k_0}$ is relatively compact. Moreover, for all $h\in B_{k_0}$, and we have

$$d_{\infty}((\mho_2 h)(\wp),(F_{\varepsilon,\varpi}h)(\wp))$$

$$=d_{\infty}\Bigg(\int_{0}^{\wp}\int_{\theta}^{\wp}\frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta,$$

$$\int_{0}^{\wp-\varepsilon}\int_{\theta+\varpi}^{\wp}\frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta\Bigg)$$

$$=d_{\infty}\Bigg(\int_{0}^{\wp}\int_{\theta}^{\theta+\varpi}\frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta$$

$$+\int_{0}^{\wp}\int_{\theta+\varpi}^{\wp}\frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta$$

$$-\int_{0}^{\wp-\varepsilon}\int_{\theta+\varpi}^{\wp}\frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta,\hat{0}\Bigg)$$

$$\leq d_{\infty}\Bigg(\int_{0}^{\wp}\int_{\theta}^{\theta+\varpi}\frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta,\hat{0}\Bigg)$$

$$+d_{\infty}\Bigg(\int_{\wp-\varepsilon}^{\wp}\int_{\theta+\varpi}^{\wp}\frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^{\ell})\odot[Bu(\theta)\oplus f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))]d\delta d\theta,\hat{0}\Bigg)$$

$$\leq\left(\frac{M_E M_1}{\Gamma(\ell)}+\frac{M_B{}^2 M_E{}^2\Delta}{\varpi\Gamma(\ell)}\right)\frac{\left[\left((-\varpi)^u-(a-\varpi)^u+a^u\right)^{1-\ell_1}-\left((-\varpi)^u+(-\varepsilon-\varpi)^u\right)^{1-\ell_1}\right]}{(u)^{1-\ell_1}}$$

$$\longrightarrow 0,$$

when ϖ, $\varepsilon\rightarrow 0$. Then we obtain a relatively compact set arbitrarily that is close to $V(\wp)$, $\wp>0$, which means that $V(\wp)$, $\wp>0$, is also relatively compact. By applying the Ascoli-Arzela theorem, we observe that $\mho_2(B_{k_0})$ is relatively compact. The function $\mho_2$ is fully continuous. Since, $\mho_2$ is continuous and $\mho_2(B_{k_0})$ is significantly compact.

The function $\mho_2$ is continuous. Based on Theorem 2, we thus conclude that $\mho$ has a fixed point in B_{k_0}, the mild solution of system (1). Thus, the nonlocal Cauchy problem 10 has a mild fuzzy solution of type 1. □

Set

$$\hat{\mho}[x](t) = E_{\ell,1}(\Lambda\wp^{\ell}) \odot h_0 + \wp \odot E_{\ell,2}(\Lambda\wp^{\ell}) \odot h_1$$

$$\ominus (-1) \odot \int_0^{\wp} \int_{\theta}^{\wp} \frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)} E_{\ell,1}(\Lambda(\delta-\theta)^{\ell}) \odot [Bu(\wp) \oplus f(\theta, h(\theta), {}_{gH}^{C}D^{\ell-1}h(\theta))] d\delta d\theta.$$

$\hat{C}([0, a], J) = \{ h \in C(\mathbb{P}, J)$: $\hat{\mho}[h](\wp)$ exists for all $\wp \in \mathbb{P}\}$.

The following results show that there exists a mild fuzzy solution for type 2 in the space C($\mathbb{P}$, J).

Theorem 4 A mild fuzzy solution of the Cauchy problem (1) exists in type 2 in space $C(\mathbb{P}, J)$ under the assumptions (H1)-(H5).

Proof. For $h \in \hat{C}(\mathbb{P}, J)$,

$$\hat{\mho}[h](\wp) = (\mho_1 h)(\wp) \ominus (-1) \odot (\mho_2 h)(\wp).$$

$$K_0 = ||h_b|| + \left(1 + \frac{{M_B}^2 {M_E}^2 a^{(\Theta)(1-\ell_1)}}{\Gamma(\ell)\varpi(\Theta)^{1-\ell_1}}\right)\Delta.$$

Using the same procedure as before with the Caputo $[(1)-gH]$ derivative, we get: $(\mho_1 h) \ominus (-1) \odot (\mho_2 h) \in B_{k_0}$, for any pair h, $g \in B_{k_0} \subset \hat{C}(\mathbb{P}, J)$, where $\mho_1$ and $\mho_2$ are continuous in $\wp \in \mathbb{P}$. We now obtain for each h, $g \in B_{k_0}$

$$d_{\infty}\big((\mho_1 h)(\wp) \ominus (-1) \odot (\mho_2 h)(\wp), \hat{0}\big)$$

$$\leq d_{\infty}((\mho_1 h)(\wp), \hat{0}) + d_{\infty}((\mho_2 h)(\wp), \hat{0})$$

$$\leq M_E ||h_0|| + aM_E ||h_1|| + \frac{M_E M_1 a^{(\Theta)(1-\ell_1)}}{\Gamma(\ell)(\Theta)^{1-\ell_1}} + \frac{{M_B}^2 {M_E}^2 \Delta a^{(\Theta)(1-\ell_1)}}{\Gamma(\ell)\varpi(\Theta)^{1-\ell_1}}$$

$$\leq ||h_b|| + \left(1 + \frac{{M_B}^2 {M_E}^2 a^{(\Theta)(1-\ell_1)}}{\Gamma(\ell)\varpi(\Theta)^{1-\ell_1}}\right)\Delta.$$

$$= k_0.$$

Which means that $\mho_1 \ominus (-1) \odot \mho_2$ is an operator from B_{k_0} into B_{k_0}.

Given that $\mho_2$ is a completely continuous operator, the Cauchy problem (1) has a mild fuzzy solution of type 2, since $\mho_1 \ominus (-1) \odot \mho_2$ has a fixed point on B_{k_0}, as proved by Theorem 2. □

5. Approximate controllability

Theorem 5 The fuzzy fractional evolution system (1) is approximately controllable on $\mathbb{P}$ in type 1, if hypotheses (H1)-(H5) are true.

Proof. Suppose there exists a fixed point $\mho_1 + \mho_2$ in B_{k_0} is h^β. The system (1) has a mild solution at any fixed point h^γ, according to Theorem 3, such that

$$h^\beta(\wp) = E_{\ell,1}(\Lambda\wp^\ell)\odot h_0 + \wp\odot E_{\ell,2}(\Lambda\wp^\ell)\odot h_1 + \int_0^\wp\int_\theta^\wp \frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^\ell)$$

$$\odot[B^*E^*_{\ell,1}(\Lambda(\delta-\theta)^\ell)\mathscr{R}(\varpi,\Gamma_0^b)p(h^\beta)\oplus f(\theta,h^\beta(\theta),{}^C_{gH}D^{\ell-1}h^\beta(\theta))]d\delta d\theta.$$

Where

$$p(h^\beta) = h_b - E_{\ell,1}(\Lambda\wp^\ell)\odot h_0 + \wp\odot E_{\ell,2}(\Lambda\wp^\ell)\odot h_1 + \int_0^\wp\int_{\theta_1}^\wp \frac{(\wp-\delta_1)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta_1-\theta_1)^\ell)$$

$$\odot f(\theta_1,h^\beta(\theta_1),{}^C_{gH}D^{\ell-1}h^\beta(\theta_1))d\delta_1 ds_1.$$

We have $[I-\Gamma_0^b\mathscr{R}(\varpi,\Gamma_0^b)] = \varpi\mathscr{R}(\varpi,\Gamma_0^b)$, then

$$\begin{aligned}
h^\beta(b) &= E_{\ell,1}(\Lambda b^\ell)\odot h_0 + b\odot E_{\ell,2}(\Lambda b^\ell)\odot h_1\\
&\quad+\int_0^b\int_\theta^b \frac{(b-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^\ell)\odot f(\theta,h^\beta(\theta),{}^C_{gH}D^{\ell-1}h^\beta(\theta))d\delta d\theta\\
&\quad+\int_0^b\int_\theta^b \frac{(b-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^\ell)BB^*E^*_{\ell,1}(\Lambda(\delta-\theta)^\ell)\mathscr{R}(\varpi,\Gamma_0^b)\\
&\quad\times\Bigg[h_b - E_{\ell,1}(\Lambda b^\ell)\odot h_0 + b\odot E_{\ell,2}(\Lambda b^\ell)\odot h_1\\
&\quad+\int_0^b\int_{\theta_1}^b \frac{(b-\delta_1)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta_1)^\ell)\odot f(\theta_1,h^\beta(\theta_1),{}^C_{gH}D^{\ell-1}h^\beta(\theta_1))d\delta_1 d\theta_1\Bigg]d\delta d\theta.\\
&= E_{\ell,1}(\Lambda b^\ell)\odot h_0 + b\odot E_{\ell,2}(\Lambda b^\ell)\odot h_1\\
&\quad+\int_0^b\int_\theta^b \frac{(b-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^\ell)\odot f(\theta,h^\beta(\theta),{}^C_{gH}D^{\ell-1}h^\beta(\theta))d\delta ds + \Gamma_0^b\mathscr{R}(\varpi,\Gamma_0^b)p(h^\beta)
\end{aligned}$$

$$= E_{\ell,1}(\Lambda b^{\ell}) \odot h_0 + b \odot E_{\ell,2}(\Lambda b^{\ell}) \odot h_1$$

$$+ \int_0^b \int_\theta^b \frac{(b-\delta)^{\ell-2}}{\Gamma(\ell-1)} E_{\ell,1}(\Lambda(\delta-\theta)^{\ell}) \odot f(\theta, h^{\beta}(\theta), {}_{gH}^{C}D^{\ell-1}h^{\beta}(s))d\delta ds$$

$$+ p(h^{\beta}) - \varpi \mathscr{R}(\varpi, \Gamma_0^b)p(h^{\beta})$$

$$= h_b - \varpi \mathscr{R}(\varpi, \Gamma_0^b)p(h^{\beta}).$$

According to the Dunford-Pettis theorem, there exists a subsequence $f(\theta, h^{\beta}(\theta), gH^C D^{\ell-1}h^{\beta}(\theta))$ that converges weakly to $f(\theta, h(\theta), gH^C D^{\ell-1}h(\theta))$ in $L^1[\mathbb{P}, J]$. Consider

$$w = h_b - E_{\ell,1}(\Lambda \wp^{\ell}) \odot h_0 + \wp \odot E_{\ell,2}(\Lambda \wp^{\ell}) \odot h_1$$

$$+ \int_0^{\wp} \int_{\theta_1}^{\wp} \frac{(\wp-\delta_1)^{\ell-2}}{\Gamma(\ell-1)} E_{\ell,1}(\Lambda(\delta_1-\theta_1)^{\ell}) \odot f(\theta_1, h(\theta_1), {}_{gH}^{C}D^{\ell-1}h(\theta_1))d\delta_1 d\theta_1.$$

We obtain

$$d_\infty(p(h^{\beta}) - w, \hat{0}) = d_\infty\Bigg(\int_0^{\wp} \int_{\theta_1}^{\wp} \frac{(\wp-\delta_1)^{\ell-2}}{\Gamma(\ell-1)} E_{\ell,1}(\Lambda(\delta_1-\theta_1)^{\ell}) \odot f(\theta_1, h^{\beta}(\theta_1), {}_{gH}^{C}D^{\ell-1}h^{\beta}(\theta_1))d\delta_1 d\theta_1$$

$$- \int_0^{\wp} \int_{\theta_1}^{\wp} \frac{(\wp-\delta_1)^{\ell-2}}{\Gamma(\ell-1)} E_{\ell,1}(\Lambda(\delta_1-\theta_1)^{\ell}) \odot f(\theta_1, h(\theta_1), {}_{gH}^{C}D^{\ell-1}h(\theta_1))d\delta_1 d\theta_1, \hat{0}\Bigg)$$

$$\leq \int_0^{\wp} \int_{\theta_1}^{\wp} \frac{(\wp-\delta_1)^{\ell-2}}{\Gamma(\ell-1)} E_{\ell,1}(\Lambda(\delta_1-\theta_1)^{\ell}) \odot d_\infty\Bigg(\big[f(\theta_1, h^{\beta}(\theta_1), {}_{gH}^{C}D^{\ell-1}h^{\beta}(\theta_1))d\delta_1 d\theta_1$$

$$- f(\theta_1, h(\theta_1), {}_{gH}^{C}D^{\ell-1}h(\theta_1))d\delta_1 d\theta_1\big], \hat{0}\Bigg).$$

From the boundedness of $f^{\beta}(\wp)$, there exists some $f(\wp) \in L^1[\mathbb{P}, J]$, such that

$$f(\theta_1, h^{\beta}(\theta_1), {}_{gH}^{C}D^{\ell-1}h^{\beta}(\theta_1))d\delta_1 d\theta_1 \to f(\theta_1, h(\theta_1), {}_{gH}^{C}D^{\ell-1}h(\theta_1))d\delta_1 d\theta_1, \quad \text{as} \quad \beta \to 0.$$

Furthermore, approaching the controllability of the system (1), we obtain $\varpi R(\varpi, \Gamma_0^b) \to 0$ as $\varpi \to 0^+$ in the topology of strong continuity. Consequently, $\varpi \to 0^+,$ can be obtained.

$$d_\infty(h^\beta(b)-h_b,\hat{0}) \leq d_\infty\big(\varpi R(\varpi,\Gamma_0^b)(w)+\varpi R(\varpi,\Gamma_0^b)(p(h^\beta)-w),\hat{0}\big)$$

$$\leq d_\infty\big(\varpi R(\varpi,\Gamma_0^b)w+(p(h^\beta)-w),\hat{0}\big)\to 0.$$

Therefore, the system (1) is approximately controllable for type 1. □

Theorem 6 The fuzzy fractional evolution system (1) is approximately controllable on $\mathbb{P}$ in type 2 if the hypotheses (H1)-(H5) are true.

Proof. By theorem 4, for each fixed point in B_{k_0}, let h^β be a fixed point of $\mho_1\ominus(-1)\odot\mho_2$. For the system (1), h^γ is a mild solution in which

$$h^\beta(\wp)=E_{\ell,1}(\Lambda\wp^\ell)\odot h_0+\wp\odot E_{\ell,2}(\Lambda\wp^\ell)\odot h_1$$

$$\ominus(-1)\int_0^\wp\int_\theta^\wp\frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta-\theta)^\ell)\odot[Bu^\beta(\wp)\oplus f(\theta,h^\beta(\theta),{}_{gH}^{C}D^{\ell-1}h^\beta(\theta))]d\delta d\theta,$$

where

$$u^\beta(\wp)=B^*E_{\ell,1}^*(\Lambda(\delta-\theta)^\ell)\mathscr{R}(\varpi,\Gamma_0^b)\Big[h_b-E_{\ell,1}(\Lambda\wp^\ell)\odot h_0+\wp\odot E_{\ell,2}(\Lambda\wp^\ell)\odot h_1$$

$$\ominus(-1)\int_0^\wp\int_{\theta_1}^\wp\frac{(\wp-\delta_1)^{\ell-2}}{\Gamma(\ell-1)}E_{\ell,1}(\Lambda(\delta_1-\theta_1)^\ell)\odot f(\theta_1,h^\beta(\theta_1),{}_{gH}^{C}D^{\ell-1}h^\beta(\theta_1))d\delta_1 d\theta_1\Big].$$

Using a similar method as before with the Caputo $[(1)-gH]$ derivative, we obtain $f(\theta,h^\beta(\theta),{}_{gH}^{C}D^{\ell-1}h^\beta(\theta))$ that convergent weakly to $f(\theta,h(\theta),{}_{gH}^{C}D^{\ell-1}h(\theta))$ $L^1[\mathbb{P},J]$ and we know that from 5 (i.e)

$$d_\infty(p(h^\beta)\oplus(-1)w,\hat{0})\to 0,\quad \text{as}\quad \beta\to 0.$$

Furthermore, we derive $\varpi R(\varpi,\Gamma_0^b)\to 0$, as $\varpi\to 0^+$, in the strong continuous topology based on the approximate controllability of the system (1). Consequently, $\varpi\to 0^+$, can be obtained.

$$d_\infty(h^\beta(b)\oplus(-1)h_b,\hat{0}) \leq d_\infty\big(\varpi R(\varpi,\Gamma_0^b)(w)+\varpi R(\varpi,\Gamma_0^b)(p(h^\beta)\oplus(-1)w),\hat{0}\big)$$

$$\leq d_\infty\big(\varpi R(\varpi,\Gamma_0^b)w+(p(h^\beta)\oplus(-1)w),\hat{0}\big)\to 0.$$

This shows that an operator from B_{k_0} into B_{k_0} is $\mho_1\ominus(-1)\odot\mho_2$. In other words, the system (1) is approximately controllable of type 2 on $\mathbb{P}$. □

6. Application

This section present two example for the proposed solution for the better understanding.

1. Consider the following equation;

$$\begin{cases} {}^{C}D_{0}^{\frac{3}{2}}\varpi(\wp, h) = \frac{\partial}{\partial \wp}\varpi(\wp, h) + Bu(\wp) + \frac{e^{-\wp}}{9+e^{\wp}}\left(\frac{|\varpi(\wp, h)|}{1+|\varpi(\wp, h)|}\right), \quad (\wp, h) \in (0, 1)\times(0, 1), \\ \varpi(\wp, h) = \varpi(\wp, 1) = 0, \quad \wp \in (0, 1), \\ \varpi(0, h) = \wp(h), \quad h \in (0, 1), \\ {}^{C}D_{0}^{\frac{3}{2}}\varpi(0, h) = \varphi(h), \quad h \in (0, 1). \end{cases} \tag{12}$$

We select $\mathbb{H} = C([0, 1]\times J, J)$, we have the function $f : (0, 1)\times J \to J$ and remember to include the operator $\Lambda : D(\Lambda) \subset \mathbb{H} \to \mathbb{H}$, which is indicated by

$$D(\Lambda) = \left\{\varpi \in \mathbb{H} : \frac{\partial}{\partial \wp}\varpi \in \mathbb{H} \quad \text{and } \varpi(0, 0) = \varpi(0, 1) = 0\right\},$$

$$\Lambda\varpi = \frac{\partial}{\partial \wp}\varpi \quad \text{and} \quad f(\wp, H(\wp)) = \frac{e^{-\wp}}{9+e^{\wp}}\left(\frac{|H(\wp)|}{1+|H(\wp)|}\right).$$

Then, we get

$$\overline{D(\Lambda)} = \{\varpi \in \mathbb{H} : \varpi(\wp, 0) = \varpi(\wp, 1) = 0\}.$$

This suggests that Λ fulfills every hypotheses.

As it is well known, Λ generates a compact C_0 semigroup $E_{\ell, n}(\Lambda\wp^{\ell})$ on $\overline{D(\Lambda)}$ and $E_{\ell, n}(\Lambda\wp^{\ell})$ compact operator. Thus, clearly $||E_{\ell, n}(\Lambda\wp^{\ell})|| = M_E$.

Assume that $H(\wp) = \varpi(\wp, \cdot)$, or $H(\wp)(h) = \varpi(\wp, h)$, for any $(\wp, h) \in (0, 1)\times(0, 1)$.

We know that

$$f(\wp, H(\wp)) = \frac{e^{-\wp}}{9+e^{\wp}}\left(\frac{|H(\wp)|}{1+|H(\wp)|}\right).$$

Clearly, for everyone $H, G \in \mathscr{C}([0, 1], B_r)$, we have

$$d_\infty(f(\wp, H(\wp)), f(\wp, G(\wp))) \le \rho(\wp)d_\infty(H(\wp), G(\wp)), \quad \text{where } \rho(\wp) = \frac{e^{-\wp}}{9+e^{\wp}} \quad \forall \wp \in (0, 1)$$

and that

$$d_\infty(f(\wp, H(\wp)), 0) = \leq m(\wp), \quad \text{where } m(\wp) = \frac{1}{9+e^{\wp}} \quad \forall \wp \in (0, 1).$$

Furthermore, it is also strongly measurable. Because f is continuous. Thus, the problem (12) admits two types of solutions, as stated below, it follows that 3 and the hypotheses.

$$\begin{aligned} H(\wp) =& E_{\ell,1}(\frac{\partial}{\partial \wp}\wp^{\ell}) \odot h_0 + \wp \odot E_{\ell,2}(\frac{\partial}{\partial \wp}\wp^{\ell}) \odot h_1 \\ & + \int_0^{\wp}\int_{\theta}^{\wp} \frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)} E_{\ell,1}(\frac{\partial}{\partial \wp}(\delta-\theta)^{\ell}) \odot \left[Bu(\wp) \oplus \frac{e^{-\wp}}{9+e^{\wp}}\left(\frac{|H(\wp)|}{1+|H(\wp)|}\right)\right] d\delta d\theta. \end{aligned}$$

and

$$\begin{aligned} H(\wp) =& E_{\ell,1}(\frac{\partial}{\partial \wp}\wp^{\ell}) \odot h_0 + \wp \odot E_{\ell,2}(\frac{\partial}{\partial \wp}\wp^{\ell}) \odot h_1 \\ & \ominus (-1) \odot \int_0^{\wp}\int_{\theta}^{\wp} \frac{(\wp-\delta)^{\ell-2}}{\Gamma(\ell-1)} E_{\ell,1}(\frac{\partial}{\partial \wp}(\delta-\theta)^{\ell}) \odot \left[Bu(\wp) \oplus \frac{e^{-\wp}}{9+e^{\wp}}\left(\frac{|H(\wp)|}{1+|H(\wp)|}\right)\right] d\delta ds\theta. \end{aligned}$$

2. Consider the following equation for another example;

$$\begin{cases} {}^C D_0^{\frac{4}{3}} \kappa(\wp, h) = \frac{\partial^2}{\partial^2 \wp}\kappa(\wp, h) + Bu(\wp, h) + \wp^{\frac{-1}{3}} \sin\kappa(\wp, h), \quad (\wp, h) \in (0, 1)\times(0, 1), \\ \kappa(\wp, h) = \kappa(\wp, 1) = 0, \quad \wp \in (0, 1), \\ \kappa(0, h) = \wp(h), \quad h \in (0, 1), \\ {}^C D_0^{\frac{4}{3}} \kappa(0, h) = \kappa(h), \quad h \in (0, 1). \end{cases} \tag{13}$$

where D_0^{ℓ} is a Caputo fractional derivative of order $1 < \ell < 2$. We have the function $f : (0, 1) \times J \to J$ and remember to include the operator $\Lambda : D(\Lambda) \subset C([0, 1] \times J, J) \to C([0, 1] \times J, J)$.

We define an operator Λ by $\Lambda\kappa = \kappa^{''}$ with the domain

$$D(\Lambda) = \{\kappa(\cdot) \in C([0, 1]\times J, J) : \kappa,\ \kappa^{'} \text{absolutely continuous},\ \kappa^{''} \in C([0, 1]\times J, J),\ \kappa(0, 0) = \kappa(0, 1) = 0\}.$$

Then, Λ generates a strongly continuous semigroup $E_{\ell,n}(\Lambda\wp^{\ell})$ which is compact. For any $(\wp, h) \in (0, 1) \times (0, 1)$

$$f(\wp, \kappa(\wp, h)) = \wp^{\frac{-1}{3}} \sin\kappa(\wp, h).$$

For every, $\kappa(\cdot), \mathfrak{h}(\cdot) \in \mathscr{C}([0, 1], B_r)$, we obtain

$$d_\infty(f(\wp, \kappa(\cdot)), f(\wp, \mathfrak{h}(\cdot))) \leq \rho(\wp) d_\infty(\kappa(\cdot), \mathfrak{h}(\cdot)), \quad \text{where } \rho(\wp) = \wp^{\frac{-1}{3}} \quad \forall \wp \in (0, 1).$$

Since f is continuous, it is strongly measurable. Thus, the problem (13) admits two types of solutions.

7. Conclusion

This work addresses the approximate controllability of the nonlocal Cauchy problem for fuzzy evolution equations with order $\ell \in (1, 2)$. We have defined a mild solution and presented the idea of approximate controllability first. Then, we list some conditions and assumptions important to our main approximate controllability theorem. Using Krasnoselskii's fixed point theorem, we rigorously established this primary controllability theorem. Finally, a framework has been developed to enhance theoretical applications. In the future, we will focus on investigating fuzzy fractional evolution systems with order $\ell \in (2, 3)$.

Acknowledgements

The authors expressed their gratitude to the editor and reviewer for their insightful feedback and suggestions on how to raise the standard of the article.

Authors contribution

To the research, thoughts and creation of the manuscript, writers R. Hariharan and R. Udhayakumar made equal contributions.

Data availability

For this article, data sharing is not applicable because no datasets were created or examined for this study.

Conflict of interest

The authors declare no competing financial interest.

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