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Research Article

On Controllability and Approximate Sturm-Liouville Problems in Time-Varying Second-Order Differential Equations

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Abstract: This paper considers a time-varying second-order Sturm-Liouville boundary value problem on a finite time interval as a nominal version of another perturbed differential equation that is not related to a Sturm-Liouville problem. It is proven that, under controllability conditions of the nominal system, there is a bounded continuous control function, for each real constant eigenvalue of the nominal Sturm-Liouville system, which plays its role in the current perturbed one such that the same two-point boundary values as those of the nominal Sturm-Liouville system are matched by the trajectory solution. In a more general context, it is possible to fix arbitrary finite two point boundary values for the perturbed current one which are not necessarily coincident with those of the nominal Sturm-Liouville system by an appropriate synthesis of such a function. This property is a direct consequence of the controllability property. Since the mentioned control function is not constant, the perturbed system is not a Sturm-Liouville one even if the two-boundary values are identical to those of the nominal Sturm-Liouville system. It is also characterized, in terms of norms, the worst-case errors of the two-point boundary values of the current perturbed differential system with respect to those of the nominal Sturm-Liouville system provided that the nominal constant eigenvalue is not replaced by the appropriate time-varying function based on nominal controllability conditions. Under small deviations of the parameterization of the perturbed system with respect to the nominal one, such a worst-error characterization makes the current perturbed system to be an approximate Sturm-Liouville system of the nominal one.

***Keywords*:** controllability, ordinary differential equations, Sturm-Liouville system, approximate Sturm-Liouville system

MSC: 34B24, 34A30, 34H05, 93B05

1. Introduction

Some problems in Physics result to be described by second-order differential equations with prefixed boundary values in the definition domains. Such problems are referred to as Sturm-Liouville systems or as Sturm-Liouville problems. Some of the problems within this class are, for instance, the Bessel's and Legendre's equations and some typical second-order ordinary differential equations with periodic solutions. One of the main properties of Sturm-Liouville systems is that the solution trajectory achieves the given two-boundary given under an infinite countable set of values of a constant real

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parameter λ which parameterizes the differential equation. That set of values conforms the eigenvalues of the Sturm-Liouville system [1, 2]. On the other hand, controllability is a known important property of controlled dynamic systems of interest in a wide set of applications where it is needed to fix the state-trajectory solution to arbitrary prescribed values at certain finite time instants or within certain finite time intervals [3–10]. In this context, a controllable dynamic system is the one which is able to match any arbitrary prefixed finite value of its state solution in finite time for any given finite initial conditions under an existing control function [3–6]. The potential extension of the results to differential equations subject to solution trajectory positivity, impulsive dynamics, impulsive forcing terms, fractional dynamics and internally delayed dynamics [11–16], may be also of interest. Some more recent works are briefly sketched as follows. In [11], the eigenvalues of generalized Sturm-Liouville problems are calculated while in [18] some more recent results on Sturm-Liouville theory are presented. In [19], a methodology of solution of the direct and inverse Sturm-Liouville problems is addressed. On the other hand, the transformation of the boundary value problem of the Sturm-Liouville canonical form to the Liouville normal form and vice-versa are considered in [20]. It is discussed, in particular, how the difficult implementation of the inverse Liouville's transformation can be nearly impossible in some cases.

This paper identifies and discusses some relationships between Sturm-Liouville systems which are widely studied in differential equations and some controllability properties and its associated control synthesis is borrowed from control theory of dynamic systems. The dynamic systems of first-order differential equations considered through this paper are of second order being exactly equivalent to the primary stated second-order differential equations. Basically, the constant parameter, whose set of values in the differential equation is the set of eigenvalues of the Sturm-Liouville system, is replaced by a time-varying function to be synthesized which plays the role of a feedback control function. Such a control is designed in such a way that, for given arbitrary and finite initial conditions of the differential system, the prescribed final conditions along a finite length time interval are achieved by the injection of the control law as a result from the controllability property. Those sets of conditions at the boundary points of the finite time interval under study conform the two-point boundary set of conditions assigned to the Sturm-Liouville system. Thus, the solution of the dynamic system, and then that of its differential equation counterpart, satisfies prefixed two-point boundary values at the initial and final time instants.

The paper is organized as follows. Section 2 discusses two linear time-varying second-order differential equations which will be later on in Section 3 associated with the two-point boundary values Sturm-Liouville-type problems. The first one plays the role of a nominal differential system while the second one is the current perturbed version of the above one. Both differential equations are equivalently reformulated by convenience for analysis as second-order differential systems of differential equations of first-order. Their solutions are presented in closed forms through the use of 'ad hoc' evolution operators and the comparison relations between such solutions and those of their respective evolution operators are characterized in an explicit way. The worst-case errors between both solutions are also characterized by using the appropriate vector and matrix norms. The Sturm-Liouville two-point boundary value problem is not specifically relevant through this section which mainly focuses on the closed forms of the solutions of both differential equations, the relations among them and the characterization and properties of the respective evolution operators. Section 3 considers the case when the nominal differential equation is a Sturm-Liouville controllable system. The real constant eigenvalues of the nominal Sturm-Liouville problem are replaced in the current perturbed one by an appropriate bounded continuous control function which makes such a current system to match the same two-point boundary values as those of the nominal Sturm-Liouville one. The differential equations are equivalently expressed as second-order differential systems of first-order equations to facilitate the analysis purposes. Section 4 is devoted to characterize the errors of the two-point boundary values between those of the nominal differential system and those of perturbed current one without driving the current differential system by a monitored real function playing the role of the nominal eigenvalues. In this way, the current system is interpreted as an approximate Sturm-Liouville version of the nominal one under small parameterization deviations between both of them including eventual small errors in the nominal eigenvalues. Finally, the paper ends with some concluding remarks.

1.1 *Notation*

I_n is the n-th identity matrix.

The set $\overline{n} = \{1, 2, \cdots, n\}$ denotes the set of natural numbers from one to n.

The superscript T denotes transposition.

The two dimensional canonical Euclidean two-dimensional vectors are $\boldsymbol{e}_1 = (1, 0)^T$, $\boldsymbol{e}_2 = (0, 1)^T$.

If $M \in \boldsymbol{R}^{n\times n}$ then $M \succ 0$ (respectively, $M \succeq 0$) denotes that M is positive definite (respectively, semidefinite positive). Also, $M \prec 0$ (respectively, $M \preceq 0$) denotes that M is negative definite (respectively, semidefinite negative).

The symbol $M^{\uparrow}$ denotes the Moore-Penrose pseudoinverse of $M \in \boldsymbol{R}^{n\times m}$. In particular, $M^{\uparrow} = M^{-1}$ if $n = m$ and M is non-singular.

$\mathbf{R}(M)$ denotes the Range (or Image) subspace of the matrix M.

The notation $f(t) = O(g(t))$ (Landau's big-O notation) means that $|f(t)| \leq M\left|g(t)\right|$ for some real constant. $M > 0$ with the limit definition $\limsup\limits_{t\to\infty}\left|\dfrac{f(t)}{g(t)}\right| < +\infty$. It follows that $O(|g(t)|) + O(|h(t)|) = O(|g(t)| + |h(t)|)$.

The notation $f(t) = o(g(t))$ (Landau's little-o notation) means that $\lim\limits_{t\to\infty}\dfrac{f(t)}{g(t)} = 0$.

However, in the context of this paper, the interest relies on the behaviour on finite intervals $[a, b]$ rather that at the limit as $t \to \infty$. Thus, the interpretation of $f(t) = o\left(\varepsilon(t)\right)$ is that $\lim\limits_{\varepsilon(t)\to 0}\dfrac{f(t)}{\varepsilon(t)} = 0$. It follows that $O\left(|\varepsilon_1(t) + |\varepsilon_2(t)|\right) = O\left(|\varepsilon_1(t)| + O\left(|\varepsilon_2(t)|\right)\right.$.

2. Second-order nominal and current time-varying linear ordinary differential equations

Through this section, we discuss two ordinary linear second-order differential equations. One of them will be considered in the next section of this paper as a nominal version of the Sturm-Liouville two-point boundary value problem while the other differential equation is its current version which is considered to be a perturbation of the former one. In this section, we pay attention to the evolution operators of the differential equations and to the respective solutions under finite initial conditions without involving specific prescribed boundary values for the respective solutions. Now, assume that the following linear time-varying differential equation:

$$p_0(t)y_0''(t) + (q_0(t) + \lambda_0\omega_0(t))\,y_0(t) = 0 \tag{1}$$

is a nominal version in the time interval $[a, b]$ of the perturbed current differential equation:

$$\left(p(t)y'(t)\right)' + (q(t) + \lambda\,\omega(t))y(t) = 0 \tag{2}$$

In the same time interval, where:

$p_0(t) > 0$, $\omega_0(t) > 0$, $q_0(t)$, λ_0 and $\lambda = \lambda_0 + \widetilde{\lambda}$ are real numbers; $p(t) = p_0(t) + \widetilde{p}(t) > 0$, $q(t) = q_0(t) + \widetilde{q}(t)$ and $\omega(t) = \omega_0(t) + \widetilde{\omega}(t) > 0$ are bounded real functions defined in the real interval $[a, b]$ and piecewise-continuous in (a, b), $p_0(t)$ and $p(t)$ are differentiable in $\left(a, b\right)$ for $t \in (a, b)$.

Assumption 1 It is assumed that the current differential Equation (2) is distinct from the nominal one (Equation (1)) if and only if $\left|\widetilde{p}(t)\right| + \left|\widetilde{q}(t)\right| + \left|\widetilde{\omega}(t)\right| \neq 0$ for $t \in \left[a', b'\right] \subset \left[a\ b\right]$ where $\left[a', b'\right]$ is not necessary connected while it has nonzero Lebesgue measure.

The reason of formulating Assumption 1 is to consider that Equation (2) is not distinct from Equation (1) if they only differ by $\lambda \neq \lambda_0$ (i.e. $\widetilde{\lambda} \neq 0$), since, in this case, they would be properly identical for two distinct common eigenvalues λ and λ_0.

It turns out that

$$p^{-1}(t) = (p_0(t) + \widetilde{p}(t))^{-1} = p_0^{-1}(t) + \widetilde{p}_I(t) \tag{3}$$

where

$$\widetilde{p}_I(t) = p^{-1}(t) - p_0^{-1}(t) = -p_0^{-1}(t)\widetilde{p}(t)\,(p_0(t) + \widetilde{p}(t))^{-1} \tag{4}$$

under the constraint that $\widetilde{p}(t) = 0$ implies $\widetilde{p}_I(t) = 0$. Note that Equation (1) can be rewritten, equivalently, as a second-order linear time-invariant differential system of first-order differential equations by defining $x_0(t) = \left(y_0(t),\ y_0'(t)\right)^T$ as follows:

$$x_0'(t) = A_0(t)x_0(t);\ t \in [a,\ b] \tag{5}$$

where

$$A_0(t) = \begin{bmatrix} 0 & 1 \\ -p_0^{-1}(t)\,(q_0(t) + \lambda_0\omega_0(t)) & 0 \end{bmatrix} \tag{6}$$

and Equation (2) becomes, in the same way, to be equivalent to the second-order differential system:

$$x'(t) = A(t)x(t);\ t \in [a,\ b] \tag{7}$$

where

$$A(t) = A_0(t) + \tilde{A}(t) = \begin{bmatrix} 0 & 1 \\ -p^{-1}(t)(q(t) + \lambda\,\omega(t)) & 0 \end{bmatrix} \tag{8}$$

where

$$\begin{aligned} \widetilde{A}(t) &= A(t) - A_0(t) \\ &= \begin{bmatrix} 0 & 0 \\ p_0^{-1}(t)\,(q_0(t) + \lambda_0\omega_0(t)) - p^{-1}(t)\left[(q_0(t) + \widetilde{q}(t)) + \left(\lambda_0 + \widetilde{\lambda}\right)(\omega_0(t) + \widetilde{\omega}(t))\right] & 0 \end{bmatrix} \\ &= -\begin{bmatrix} 0 & \\ \widetilde{p}_I(t)\,(q_0(t) + \lambda_0\omega_0(t)) + \left(p_0^{-1}(t) + \widetilde{p}_I(t)\right)\left[\widetilde{q}(t) + \lambda_0\widetilde{\omega}(t) + \widetilde{\lambda}\,\omega_0(t) + \widetilde{\lambda}\,\widetilde{\omega}(t)\right] & 0 \end{bmatrix} \end{aligned} \tag{9}$$

Defining the error $\widetilde{x}(t) = x(t) - x_0(t)$ between the solutions of both differential systems, one gets from Equations (5), (6) and (7), (8) that:

$$\widetilde{x}'(t) = x'(t) - x_0'(t) = A(t)x(t) - A_0(t)x_0(t) = A_0(t)\widetilde{x}(t) + \widetilde{A}(t)x(t), \quad t \in [a, b] \tag{10}$$

whose solution is:

$$\begin{aligned}
\widetilde{x}(t) &= x(t) - x_0(t) \\
&= \Psi(t,\ a)x(a) - \Psi_0(t,\ a)x_0(a) \\
&= \Psi_0(t,\ a)\widetilde{x}(a) + \int_a^t \Psi_0(t,\ \tau)\widetilde{A}(\tau)x(\tau)d\tau \\
&= \Psi_0(t,\ a)\widetilde{x}(a) + \int_a^t \Psi_0(t,\ \tau)\widetilde{A}(\tau)\Psi'(\tau,\ a)d\tau,\ x(a) \\
&= \Psi_0(t,\ a)\widetilde{x}(a) + \left(\int_a^t \Psi_0(t,\ \tau)\widetilde{A}(\tau)\Psi'(\tau,\ a)d\tau\right)x(a), \quad t \in [a, b]
\end{aligned} \tag{11}$$

since $x_0(t)$=$\Psi_0(t,\ a)x_0(a)$ and $x(t)$=$\Psi(t,\ a)x(a)$, where $\Psi_0 : [a,\ b] \times [a,\ b] \to R^2$ and $\Psi : [a,\ b] \times [a,\ b] \to R^2$ are the continuous, with respect to t and τ, evolution operators of (5), subject to (6), and (7), subject to (8)-(9), which are differentiable at $t \in (a,\ b)$ and satisfy, respectively, $\Psi_0'(t,\ \tau)$= $A(t)\Psi'(t,\ \tau)$ and $\Psi''(t,\ \tau)$= $A(t)\Psi'(t,\ \tau)$ with $\Psi_0'(t,\ \tau)$=$\Psi'(\tau,\ \tau)$= I_2 for $\tau \in [a,\ b]$. Also, define $\tilde{\Psi}'(t,\ \tau)$=$\Psi'(t,\ \tau)$-$\Psi_0'(t,\ \tau)$ for $t,\ \tau(\leq t) \in [a,\ b]$ with $\tilde{\Psi}'(\tau,\ \tau)$= 0 for $\tau \in [a,\ b]$.

The following results follow directly from Equations (5) to (12):

Proposition 1 The following properties hold:

(1)

$$\begin{aligned}
\Psi'(t,\ \tau) &= A_0(t)\Psi'(t,\ \tau) + \widetilde{A}(t)\Psi(t,\ \tau) \\
&= A_0(t)\Psi_0(t,\ \tau) + A_0(t)\widetilde{\Psi}(t,\ \tau) + \widetilde{A}(t)\Psi(t,\ \tau) \\
&= \Psi_0'(t,\ \tau) + A_0(t)\widetilde{\Psi}(t,\ \tau) + \widetilde{A}(t)\Psi(t,\ \tau) \\
&= \Psi_0'(t,\ \tau) + A(t)\widetilde{\Psi}(t,\ \tau) + \widetilde{A}(t)\Psi_0(t,\ \tau); \quad t,\ \tau\ (\leq t) \in [a,\ b]
\end{aligned} \tag{12}$$

(2)

$$\Psi(t,\ \tau) = \Psi_0(t,\ \tau) + \int_a^t \Big(A(\sigma)\widetilde{\Psi}(\sigma,\ \tau) + \widetilde{A}(\sigma)\Psi_0(\sigma,\ \tau)\Big)d\sigma \tag{13}$$

(3)

$$\widetilde{\Psi}'(t,\ \tau) = \Psi'(t,\ \tau) - \Psi_0'(t,\ \tau)$$

$$= A(t)\Psi(t,\ \tau) - A_0(t)\Psi_0(t,\ \tau)$$

$$= A(t)\widetilde{\Psi}(t,\ \tau) + \widetilde{A}(t)\Psi_0(t,\ \tau)$$

$$= A_0(t)\widetilde{\Psi}(t,\ \tau) + \widetilde{A}(t)\widetilde{\Psi}(t,\ \tau) + \widetilde{A}(t)\Psi_0(t,\ \tau); \quad t,\ \tau\ (\leq t) \in [a,\ b] \tag{14}$$

(4)

$$\widetilde{\Psi}(t,\ \tau) = \int_{\tau}^{t} \Big(A_0(\sigma)\widetilde{\Psi}(\sigma,\ \tau) + \widetilde{A}(\sigma)\widetilde{\Psi}(\sigma,\ \tau) + \widetilde{A}(\sigma)\Psi_0'(\sigma,\ \tau)\Big) d\sigma$$

$$= \int_{\tau}^{t} \Psi_0'(t,\ \tau)\widetilde{A}(\tau)\Psi'(\tau,\ a)d\tau; \quad t,\ \tau\ (\leq t) \in [a,\ b] \tag{15}$$

(5)

$$\widetilde{x}(t) = \psi(t,\ a)x(a) - \psi_0(t,\ a)x_0(a)$$

$$= \psi_0(t,\ a)\widetilde{x}(a) + \left(\int_a^t \psi_0(t,\ \tau)\widetilde{A}(\tau)\psi(\tau,\ a)d\tau\right)x(a), \quad t \in [a,\ b] \tag{16}$$

The following result follows directly from Equation (16) and the well-known Rouche-Capelli theorem from Linear Algebra:

Theorem 1 The following properties hold:

(i) A necessary condition for

$$\tilde{x}(t) = \hat{\tilde{x}}_t = M(a,\ t)x(a) \tag{17}$$

to hold at some $t \in [a,\ b]$ for some $x(a) \in R^2$ and for a given prefixed $\hat{\tilde{x}}_t \in R^2$ is that

$$\operatorname{rank} M(a,\ t) = \operatorname{rank}\left[\hat{\tilde{x}}_t,\ M(a,\ t)\right] \tag{18}$$

where

$$M(a,\ t) = \int_a^t \psi_0(t,\ \tau)\widetilde{A}(\tau)\psi(\tau,\ a)\,d\tau \tag{19}$$

An existing solution $x(a)$ for Equation (18), if Equation (19) holds, is:

$$x(a) = M(a,\ t)^{\dagger}\hat{\hat{x}}_t \tag{20}$$

In particular, $x(a)$ is unique and given by $x(a) = M(a,\ t)^{-1}\hat{\hat{x}}_t$ if and only if $M(a,\ t)$ is non-singular.

(ii) The rank condition (18) holds if and only if $\hat{\hat{x}}_t \in \mathrm{R}(M(a,\ t))$, equivalently, if and only if $\hat{\hat{x}}_t = M(a,\ t)M(a,\ t)^{\dagger}\hat{\hat{x}}_t$, so that the linear algebraic system Equation (17) is compatible indeterminate, and then the infinite set of solutions of Equation (17) are given by the equation:

$$x(a) = M(a,\ t)^{\dagger}\hat{\hat{x}}_t + \left(I_2 - M(a,\ t)^{\dagger}M(a,\ t)\right)\xi \tag{21}$$

for any arbitrary $\xi \in R^2$. On the other hand, Equation (20) is the unique solution from Equation (21) of the compatible determinate linear algebraic system Equation (17) if and only if $M(a,\ t)$ is non-singular.

(iii) The rank condition Equation (18) fails if and only if $\hat{\hat{x}}_t \notin \mathrm{R}(M(a,\ t))$, equivalently, if and only if $\hat{\hat{x}}_t \neq M(a,\ t)M(a,\ t)^{\dagger}\hat{\hat{x}}_t$, so that the linear algebraic system Equation (17) is incompatible, and then Equation (20) is the approximate solution to Equation (17) such that $x(a) = \left\{x \in \boldsymbol{R}^2 :\ \left\|x - M(a,\ t)^{\dagger}\ \hat{\hat{x}}_t\right\|_2^2 \leq \left\|z - M(a,\ t)^{\dagger}\ \hat{\hat{x}}_t\right\|_2^2;\ \forall z \in \boldsymbol{R}^2\right\}$.

Remark 1 Note that since Equation (17) is a second-order linear algebraic system then Equation (18) is equivalent to $\hat{\hat{x}}_t$ to be a linear combination of the columns of $M(a,\ t)$. Also, since $M(a,\ t) \in R^{2\times 2}$:

(a) *rank* $M(a,\ t) = 2$, equivalently, $M(a,\ t)$ non-singular implies that Equation (20) is the unique solution to Equation (17);

(b) *rank* $M(a,\ t) = 1$, then (18) implies that $\hat{\hat{x}}_t$ to be a linear combination of the columns of $M(a,\ t)$ and there are infinitely many solutions Equation (21) to Equation (17);

(c) If Equation (18) fails then *rank* $M(a,\ t) = 1 <$ *rank* $\left[\hat{\hat{x}}_t,\ M(a,\ t)\right] = 2$ ($\hat{\hat{x}}_t$ is not a linear combination of the columns of $M(a,\ t)$) and there is no exact solution to Equation (17) so that Equation (20) is the approximate solution in the quadratic error ℓ_2 - norm sense.

In view of Equation (10), the potential smallness of $\left\|\widetilde{A}(t)\right\|$ is characterized in the subsequent result from the smallness of the upper-bounds of $|\widetilde{p}(t)|$, $|\widetilde{q}(t)|$ and $|\omega(t)|$. The proof is direct by inspection of Equation (10).

Proposition 2 Assume that for some bounded real non-negative functions $\varepsilon_p,\ \varepsilon_q,\ \varepsilon_\omega : [a,\ b] \to \boldsymbol{R}_{0+}$ and some real non-negative constants $\varepsilon_{0p} > 0$, ε_{0q} and $\varepsilon_{0\omega}$, one has $|\widetilde{p}(t)| \leq \min\left(\varepsilon_p(t),\ p_0(t)\right) \leq \min\left(\varepsilon_{0p},\ p_0(t)\right)$, $|\widetilde{q}(t)| \leq \varepsilon_q(t) \leq \varepsilon_{0q}$, and $|\widetilde{\omega}(t)| \leq \varepsilon_\omega(t) \leq \varepsilon_{0\omega}$ for $t \in [a,\ b]$. Thus, the following properties hold:

(i)

$$\left|\widetilde{p}_j(t)\right| \leq \frac{P_0^{-1}(t)\,|\widetilde{p}(t)|}{P_0(t) - |\widetilde{p}(t)|} \leq \frac{\varepsilon_p(t)}{p_0(t)\left(p_0(t) - \max(\varepsilon_p(t),\ p_0(t))\right)} < \frac{\varepsilon_{0p}}{p_0^2(t)};\quad t \in [a,\ b] \tag{22}$$

(ii)

$$\|\tilde{A}(t)\| < \frac{1}{p_0(t)}\left[\frac{\varepsilon_{0p}}{p_0(t)}\left(|q_0(t)| + |\lambda_0|\,\omega_0(t)\right) + \left(\left(1 + \frac{\varepsilon_{0p}}{p_0(t)}\right)\left(\varepsilon_{0q} + |\lambda_0|\,\varepsilon_{0\omega} + |\widetilde{\lambda}|\left(\omega_0(t) + \varepsilon_{0\omega}\right)\right)\right)\right];\ t \in [a,\ b] \tag{23}$$

being valid for the ℓ_1, ℓ_∞ and ℓ_2 matrix norms.

(iii) If

$$\varepsilon(t) = \max\left(\varepsilon_p(t),\ \varepsilon_q(t),\ \varepsilon_\omega(t)\right) \leq \varepsilon = \max\left(\varepsilon_{0p},\ \varepsilon_{0q},\ \varepsilon_{0\omega}\right);\ t \in [a,\ b] \tag{24}$$

for some non-negative real constant ε then one has from Equation (23) that $\|\tilde{A}(t)\| = O(\varepsilon) + O(|\tilde{\lambda}|)$, and

$$\|\tilde{A}(t)\| \leq 3o(\varepsilon(t)) + o(\varepsilon(t)) + \frac{|\tilde{\lambda}|\omega_0(t)}{p_0(t)};\ t \in [a,\ b] \tag{25}$$

$$\limsup_{\varepsilon(t)\to 0}\left(\|\widetilde{A}(t)\| - \frac{|\widetilde{\lambda}|\omega_0(t)}{p_0(t)}\right) \leq \limsup_{\varepsilon\to 0}\left(\|\widetilde{A}(t)\| - \frac{|\widetilde{\lambda}|\omega_0(t)}{p_0(t)}\right) = 0;\ t \in [a,\ b] \tag{26}$$

3. Nominal Sturm-Liouville system and controllability related issues

A problem that arises is the characterization of the conditions under the current differential equation with two-point boundary values is a Sturm-Liouville system provided that the associated nominal one is a Sturm-Liouville system. Another problem to be addressed is the characterization of the worst-case errors of the solution deviation of the current differential equation from an achievable nominal Sturm-Liouville system.

It is now of interest to consider explicitly the case when nominal differential Equation (1) is a regular Sturm-Liouville system [1, 2], for the infinite countable set of possible values of λ_0 which generate non-trivial solutions in a time interval $[a,\ b]$ under the respective sets of two-point boundary value conditions, which satisfy:

$$\alpha_{01}y_0(a) + \beta_{01}y_0'(a) = 0 \tag{27}$$

$$\alpha_{02}y_0(b) + \beta_{02}y_0'(b) = 0 \tag{28}$$

where α_{0i}, β_{0i} are arbitrary real constants satisfying the constraints $|\alpha_{0i}| + |\beta_{0i}| \neq 0;\ i = 1,\ 2$.

Now, assume that $\widetilde{\lambda} = \widetilde{\lambda}(t)$ is time-varying, in the most general case, and rewrite Equation (10) as follows:

$$\widetilde{A}(t) = \widetilde{A}_{\lambda_0}(t) + \widetilde{A}_{\widetilde{\lambda}}(t);\ t \in [a,\ b] \tag{29}$$

where

$$\widetilde{A}_{\lambda_0}(t) = -\begin{bmatrix} 0 & 0 \\ \widetilde{p}_I(t)\left(q_0(t) + \lambda_0\omega_0(t)\right) + \left(p_0^{-1}(t) + \widetilde{p}_I(t)\right)\left[\widetilde{q}(t) + \lambda_0\widetilde{\omega}(t) + \widetilde{\lambda}\,\widetilde{\omega}(t)\right] & 0 \end{bmatrix};\ t \in [a,\ b] \tag{30}$$

$$\widetilde{A}_{\widetilde{\lambda}}(t) = \widetilde{\lambda}(t)B(t);\ B(t) = \begin{bmatrix} 0 & 0 \\ \left(p_0^{-1}(t) + \widetilde{p}_I(t)\right)\omega_0(t) & 0 \end{bmatrix};\ t \in [a,\ b] \tag{31}$$

It can be noticed that $\widetilde{A}_{\widetilde{\lambda}}(t) = 0$ if and only if $\widetilde{\lambda}(t) = 0$ since $\omega_0(t) > 0$ and $p_0^{-1}(t) + \widetilde{p}_I(t) \neq 0$ (otherwise, $p(t) = +\infty$ from Equation (3) which would contradict its boundedness). Thus, $\widetilde{\lambda}(t) \neq 0 \Rightarrow B(t) \neq 0$. Since $A(t) = A_0(t) + \widetilde{A}_{\lambda_0}(t) + \widetilde{A}_{\widetilde{\lambda}}(t)$, the replacement of Equations (29)-(31) into Equations (10) and (16) yield:

$$\widetilde{x}(t) = A(t)x(t) - A_0(t)x_0(t) = A_0(t)\widetilde{x}(t) + \left(\widetilde{A}_{\lambda_0}(t) + \widetilde{\lambda}(t)B(t)\right)x(t);\ t \in [a,\ b] \tag{32}$$

$$\tilde{x}(t) = \Psi_0(t,\ a)\tilde{x}(a) + \left(\int_a^t \Psi_0(t,\ \tau)\left(\tilde{A}_{\lambda_0}(\tau) + \tilde{\lambda}(\tau)B(\tau)\right)\Psi(\tau,\ a)d\tau\right)x(a);\ t \in [a,\ b] \tag{33}$$

Theorem 2 Assume that:

(1) The nominal differential Equation (1) is a Sturm-Liouville system in $[a,\ b]$ with one of its eigenvalues being λ_0 and with two-point boundary values satisfying the conditions - Equations (27) and (28).

(2) The differential Equation (2) has boundary value conditions $y(a) = y_0(a) + \widetilde{y}(a)$, $y'(a) = y_0'(a) + \widetilde{y}'(a)$, $y(b) = y_0(b) + \widetilde{y}(b)$ and $y'(b) = y_0'(b) + \widetilde{y}'(b)$ such that the incremental values with respect to the boundary values of Equation (1) are arbitrary.

(3) For each $\xi \in R^2$, there is some $\tau = \tau(\xi) \in (a,\ b)$ such that

$$\xi^T \omega_0(\tau)y(\tau)\Psi_0(b,\ \tau)e_2 \neq 0 \tag{34}$$

Then, for any given arbitrary finite incremental values $\widetilde{y}(a)$ and $\widetilde{y}'(a)$, there is a bounded function $\lambda(t) = \lambda_0 + \widetilde{\lambda}(t)$ on $[a,\ b]$, which replaces the real constant real eigenvalue $\lambda = \lambda_0$ in Equation (2), for each given finite prefixed incremental values $\widetilde{y}(b)$ and $\widetilde{y}'(b)$ at $t = b$. In particular, for any given real finite (null or non-null) incremental values $\widetilde{y}(a)$ and $\widetilde{y}'(a)$, there is some $\lambda : [a,\ b] \to \boldsymbol{R}$ such that $\widetilde{y}(b) = \widetilde{y}'(b) = 0$.

Proof. First, note that $\widetilde{x}(t) = x(t) - x_0(t) = \left(y(a) - y_0(a),\ y'(a) - y_0'(a)\right)^T$ for $t \in [a,\ b]$. Note also that the following relation holds:

$$\widetilde{\lambda}(t)B(t)x(t) = \widetilde{\lambda}(t)\left(p_0^{-1}(t) + \widetilde{p}_I(t)\right)\omega_0(t)y(t)e_2 \quad t \in [a,\ b] \tag{35}$$

and, if

$$\tilde{\lambda}(\tau) = B^T(\tau)y(\tau)\Psi_0^T(b,\ \tau)\mu;\ \tau \in [a,\ b] \tag{36}$$

for some $\mu \in R^2$ then one has from Equation (31):

$$\begin{aligned}\Psi_0(b,\ \tau)B(\tau)\widetilde{\lambda}(\tau)x(\tau) &= \Psi_0(b,\ \tau)B(\tau)B^T(\tau)\Psi_0^T(b,\ \tau)\mu \\ &= \left(p_0^{-1}(\tau) + \widetilde{p}_I(\tau)\right)^2 \omega_0^2(\tau)y^2(\tau)\Psi_0(b,\ \tau)e_2e_2^T\Psi_0^T(b,\ \tau)\mu;\ t \in [a,\ b]\end{aligned} \tag{37}$$

Then, Equation (33) becomes for $t = b$ and any prefixed arbitrary $\hat{x}_b = \tilde{x}(b) \in R^2$:

$$\tilde{x}(b)-\Psi_0(b,\ a)\tilde{x}(a)-\left(\int_a^b \Psi_0(b,\ \tau)\tilde{A}_{\lambda_0}(\tau)\Psi(\tau,\ a)d\tau\right)x(a)$$

$$=\left(\int_a^b \left(p_0^{-1}(\tau)+\widetilde{p}_I(\tau)\right)^2 \omega_0^2(\tau)y^2(\tau)\Psi_0(b,\ \tau)e_2e_2^T\Psi_0^T(b,\ \tau)d\tau\right)\mu \tag{38}$$

In view that the left-hand-side of (38) is a continuous function in $[a,\ b]$, there is a neigborhood $\mathbf{B}\left(\tau(\xi),\ r_{\tau(\xi)}\right)\subset(a,\ b)$ of each such a $\tau=\tau(\xi)$ such that

$$\xi^T\left(p_0^{-1}(\theta)+p_I(\theta)\right)\omega_0(\theta)y(\theta)\Psi_0(b,\ \theta)e_2\neq 0;\ \forall\theta\in\mathbf{B}\left(\tau(\xi),\ r_{\tau(\xi)}\right),\ \forall\xi\in\boldsymbol{R}^2 \tag{39}$$

so that

$$\xi^T\left(\int_{\mathrm{B}\left(\tau(\xi),\ r_{\tau(\xi)}\right)}\left(p_0^{-1}(\theta)+p_I(\theta)\right)^2\omega_0^2(\theta)y^2(\theta)\Psi_0(b,\ \theta)e_2e_2^T\Psi_0^T(b,\ \theta)d\theta\right)\xi>0 \tag{40}$$

for some existing open ball $\mathbf{B}\left(\tau(\xi),\ r_{\tau(\xi)}\right)\subset(a,\ b)$ depending on each $\xi\in R^2$, and then Equation (40) implies that

$$\int_a^b\left(p_0^{-1}(\tau)+\widetilde{p}_I(\tau)\right)^2\omega_0^2(\tau)y^2(\tau)\Psi_0(b,\ \tau)e_2e_2^T\Psi_0^T(b,\ \tau)d\tau$$

$$\succeq\left(\int_{\mathbf{B}}\left(\tau(\xi),\ r_{\tau(\xi)}\right)\left(p_0^{-1}(\theta)+p_I(\theta)\right)^2\omega_0^2(\theta)y^2(\theta)\Psi_0(b,\ \theta)e_2e_2^T\Psi_0^T(b,\ \theta)d\theta\right)$$

$$\succeq\min\left(\left(p_0^{-1}(\theta)+p_I(\theta)\right)^2\right)\left(\int_{\mathbf{B}}\left(\tau(\xi),\ r_{\tau(\xi)}\right)\omega_0^2(\theta)y^2(\theta)\Psi_0(b,\ \theta)e_2e_2^T\Psi_0^T(b,\ \theta)d\theta\right)\succ 0;\ \theta\in[a,\ b] \tag{41}$$

provided that Equation (34) holds so that the coefficient matrix of the right-hand-side of Equation (38) is positive definite and then non-singular as well. As a result, one can fix

$$\mu=\left(\int_a^b\left(p_0^{-1}(\tau)+\widetilde{p}_I(\tau)\right)^2\omega_0^2(\tau)y^2(\tau)\Psi_0(b,\ \tau)e_2e_2^T\Psi_0^T(b,\ \tau)d\tau\right)^{-1}$$

$$\times\left(\widetilde{x}(b)-\Psi_0(b,\ a)\widetilde{x}(a)-\left(\int_a^b\Psi_0(b,\ \tau)\widetilde{A}_{\lambda_0}(\tau)\Psi(\tau,\ a)d\tau\right)\times(a)\right) \tag{42}$$

in Equation (36) to get a function $\widetilde{\lambda}(\tau)=\lambda(\tau)-\lambda_0;\ \tau\in[a,\ b]$ defined by

$$\tilde{\lambda}(\tau) = \left(p_0^{-1}(\tau) + \widetilde{p}_I(\tau)\right) \omega_0(\tau) y(\tau) e_2^T \Psi_0^T(b,\ \tau) \left(\int_a^b \left(p_0^{-1}(\tau) + \widetilde{p}_I(\tau)\right)^2 \omega_0^2(\tau) y^2(\tau) \Psi_0(b,\ \tau) e_2 e_2^T \Psi_0^T(b,\ \tau) d\tau \right)^{-1}$$

$$\times \left(\hat{\tilde{x}}_b - \Psi_0(b,\ a)\tilde{x}(a) - \left(\int_a^b \Psi_0(b,\ \tau) \tilde{A}_{\lambda_0}(\tau) \Psi(\tau,\ a) d\tau \right) x(a) \right); \quad \tau \in [a,\ b] \tag{43}$$

which fixes an arbitrary $\hat{x}_b = x(b) = x_0(b) + \hat{\tilde{x}}_b = x_0(b) + \tilde{x}_0(b) \in \boldsymbol{R}^2$ in Equation (2) being associated with the arbitrary prefixed incremental value $\hat{\tilde{x}}_b = \tilde{x}_0(b) = x(b) - x_0(b) \in \boldsymbol{R}^2$.

Remark 2 Note that Equation (34) holds, that is, the third condition of Theorem 2 is fulfilled if and only if the controllability gramian of the pair $(A_0(t),\ e_2)$ on $[a,\ b]$ is positive definite, that is, if and only if

$$Cg[a,\ b]\,(A_0(t),\ e_2) = \int_a^b \Psi_0(b,\ \tau) e_2 e_2^T \Psi_0^T(b,\ \tau) d\tau \succ 0 \tag{44}$$

which holds also trivially if the above integral constrained to any subset of nonzero measure of $[a,\ b]$ is positive definite. The equivalence between Equation (44) and Equation (34) follows from the fact that the evolution operator of the nominal differential system $\Psi_0(t,\ \tau)$ is non-singular, since it is a fundamental matrix of the differential system as well, and the continuity of the integrand conforming the controllability gramian $Cg\,[a,\ b]\,(A_0\,(t)\,,\ e_2)$ of the differential second-order system of the nominal differential system.

Remark 3 Note that if $\widetilde{A}_{\lambda_0}(t)$ is zero on $[a,\ b]$ (that is, if Assumption 1 is not fulfilled) and $\widetilde{x}(a) = \widetilde{x}(b) = \widetilde{x}(b) = 0$ (that is, if the two boundary values at $t = a$ and $t = b$ are identical for both the nominal and the current differential systems) then both the nominal and the current differential systems, Eqns. (1) and (2), are identical, so that $\widetilde{\lambda}\,(t) = 0$ from (43) and λ_0 is an eigenvalue for such a Sturm-Liouville system.

Remark 4 A main conclusion from Theorem 2, Remark 2 and Remark 3 is that, if the nominal differential system is a controllable regular Sturm-Liouville system on a finite interval $[a,\ b]$, then the current differential system can have the same two boundary values at $t = a$ and $t = b$ as those of the nominal one by the replacement of any nominal real eigenvalue by an appropriate bounded real function on the interval $[a,\ b]$.

Remark 5 It can be pointed out that the current system under the controllability condition of the nominal system can be subject even to a wider set of two-boundary values than that of the nominal one since controllability applies for arbitrary finite initial and final conditions on a finite interval. However, Sturm-Liouville systems have restrictions on the two point-boundary values and the conditions on the eigenvalues. See, for instance [13]. This assertion is visualized through the subsequent example.

Example 1 Consider, for instance, that the nominal matrix of dynamics $A_0\,(t)$ in (5) satisfies the constraint $\sup_{t\in[a,\ b]} \left(p_0^{-1}\,(t) + \lambda_0 \omega_0\,(t)\right) \leq 0$, equivalently $\alpha = \inf_{t\in[a,\ b]} \left(-p_0^{-1}\,(t) - \lambda_0 \omega_0\,(t)\right) \geq 0$, for any eigenvalue λ_0. Then, $A_0\,(t) = A_{00} + \widetilde{A}_0\,(t)\,,\ t \in [a,\ b]\,,$ where $A_{00_{12}} = 1,\ A_{00_{21}} = \alpha \geq 0,\ A_{00_{12}} = A_{00_{21}} = 0,\ \widetilde{A}_{00_{21}}\,(t) \geq \alpha \geq 0$ and $\widetilde{A}_{00_y}\,(t) = 0$ for $(i,\ j) \in \{(1,\ 1)\,,\ (1,\ 2)\,,\ (2,\ 2)\}$ and $t \in [a,\ b]$. It is obvious that A_{00} is a Metzler matrix (i.e. both of its off-diagonal entries are non-negative) what implies also that $\widetilde{A}_0\,(t)$ for any $t \in [a,\ b]$. As a result, the nominal evolution operator is non-singular and $T_{00_y}\,(t,\ \tau) = \left(e^{A_{00}(t-\tau)}\right)_{ij} \geq 0;\ i,\ j = 1,\ 2,\ \tau,\ t(\geq \tau) \in [a,\ b]$ since it is a fundamental matrix of the time-invariant part of the nominal differential system while its infinitesimal generator is a Metzler matrix A_{00}. It is also obvious that $T_0\,(t,\ \tau) = e^{A_{00}(t-\tau)}\left(I_2 + t_H' e^{A_{00}(t-\sigma)} d\sigma\right);\ \tau,\ t(\geq \tau) \in [a,\ b]$ is also non-singular since it is a fundamental matrix of the whole nominal system and, since $\widetilde{A}_0\,(t)$ has no negative entry on the interval $[a,\ b]$, $T_{0_y}\,(t,\ \tau) \geq T_{00_y}\,(t,\ \tau)\,;\ i,\ j = 1,\ 2;\ \tau,\ t(\geq\ \tau) \in [a,\ b]$. As a result, the nominal differential system is a positive linear dynamic system so that both components of its state-trajectory solution of the second-order differential system are non-negative (respectively, non-positive) on $[a,\ b]$ if they are non-negative (respectively, non-positive) at $t = a$. A conclusion which follows is that for the nominal system to be a Sturm-Liouville one it has to fulfil that: (1) $min\left(y_0(a),\ y_0(a)\right) \geq 0$ with $\left|y_0(a) + \left|y_0(a)\right|\right| > 0$ (this constraint excludes the trivial solution) implies that $min\left(y_0(b),\ y_0(b)\right) \geq 0$; (2) $min\left(y_0(a),\ y_0(a)\right) \leq 0$, with $\left|y_0(a) + \left|y_0(a)\right|\right| > 0$, implies that $min\left(y_0(b),\ y_0(b)\right) \leq 0$. But note that the above final boundary conditions at $t = b$ cannot be reversed in the conditions (1) and (2). Thus, there are specific conditions to be fulfilled by the nominal system in order to be a Sturm-Liouville one. However, under its controllability, there are no such restrictions on the two-point boundary value conditions of the current differential system under the appropriate synthesis of the control function on $[a,\ b]$ although it is not a Sturm-Liouville one since the control function is not a real constant eigenvalue of the nominal Sturm-Liouville system.

4. Worst-case error characterization of the two-point boundary values between the nominal and current differential systems

This section characterizes, under the support of Propositions 1 and 2, the errors of the two-point boundary values between those of the nominal differential system and those of perturbed current one without driving the current differential system by a monitored real function playing the role of the nominal eigenvalues. The subsequent result gives a related characterization of the perturbed current system as an approximate Sturm-Liouville version of the nominal one for small parametrical deviation between both of them.

Theorem 3 Assume that the upper-bound of Equation (23) of $\left\|\widetilde{A}(t)\right\|$ on $[a,\ b]$ is strictly upper-bounded as follows:

$$\frac{1}{p_0(t)}\left[\frac{\varepsilon_{0p}}{p_0(t)}\left(|q_0(t)| + |\lambda_0|\,\omega_0(t)\right) + \left(\left(1 + \frac{\varepsilon_{0p}}{p_0(t)}\right)\left(\varepsilon_{0q} + |\lambda_0|\,\varepsilon_{0\omega} + |\tilde{\lambda}|\left(\omega_0(t) + \varepsilon_{0\omega}\right)\right)\right)\right]$$

$$< \frac{1}{(b-a)\sup\limits_{\tau\in[a,\ t]}\|\Psi_0(\tau,\ a)\|};\ t \in [a,\ b] \tag{45}$$

Then,

$$\|\widetilde{x}(b)\| \leq (b-a) \times \left[\sup_{\tau\in[a,\ b]}\|\Psi_0(\tau,\ a)\|^2 \sup_{\tau\in[a,\ b]}\|\tilde{A}(\tau)\|\left(1 + \frac{(b-a)\sup\limits_{\tau\in[a,\ b]}\|\tilde{A}(\tau)\|\sup\limits_{\tau\in[a,\ b]}\|\Psi_0(\tau,\ a)\|}{1-(b-a)\sup\limits_{\tau\in[a,\ b]}\|\tilde{A}(\tau)\|\sup\limits_{\tau\in[a,\ b]}\|\Psi_0(\tau,\ a)\|}\right)\right]$$

$$(1 + \widetilde{\rho}_a)\,\|x_0(a)\| \tag{46}$$

which leads also to $\|\widetilde{x}(b)\| = o(\varepsilon + |\widetilde{\lambda}|)\,(1 + o\,(\widetilde{\rho}_a))\,\|x_0(a)\|$ where ε is defined in Proposition 2 and $\widetilde{\rho}_a \in R_{0+}$ is such that $\|\widetilde{x}(a)\| \leq \widetilde{\rho}_a\,\|x_0(a)\|$.

Proof. One gets from Equations (7)-(10) that:

$$
\begin{aligned}
x(t) &= \Psi(t,\ a)x(a) \\
&= \Psi_0(t,\ a)\left[x(a)+\int_a^t \Psi_0(a,\ \tau)\widetilde{A}(\tau)x(\tau)\right]d\tau \\
&= \Psi_0(t,\ a)x_0(a)+\Psi_0(t,\ a)\left(x(a)-x_0(a)\right)+\int_a^t \Psi_0(t,\ \tau)\widetilde{A}(\tau)x(\tau)d\tau \\
&= \Psi_0(t,\ a)x_0(a)+\left(\Psi_0(t,\ a)\widetilde{x}(a)+\int_a^t \Psi_0(t,\ \tau)\widetilde{A}(\tau)x_0(\tau)d\tau+\int_a^t \Psi_0(t,\ \tau)\widetilde{A}(\tau)\widetilde{x}(\tau)d\tau\right) \\
&= x_0(t)+\left(\Psi_0(t,\ a)\widetilde{x}(a)+\int_a^t \Psi_0(t,\ \tau)\widetilde{A}(\tau)x_0(\tau)d\tau+\int_a^t \Psi_0(t,\ \tau)\widetilde{A}(\tau)\widetilde{x}(\tau)d\tau\right);\ t\in[a,\ b]
\end{aligned} \tag{47}
$$

Now, Equation (47) becomes after using Equations (33),

$$
\begin{aligned}
\widetilde{x}(t) =& x(t)-x_0(t)=\Psi_0(t,\ a)\widetilde{x}(a)+\int_a^t \Psi_0(t,\ \tau)\widetilde{A}(\tau)x_0(\tau)d\tau \\
&+\left(\int_a^t \Psi_0(t,\ \tau)\widetilde{A}(\tau)\Psi_0(\tau,\ a)d\tau\right)\widetilde{x}(a) \\
&+\left(\int_a^t\int_a^\tau \Psi_0(t,\ \tau)\widetilde{A}(\tau)\Psi_0(\tau,\ \sigma)\widetilde{A}(\sigma)\Psi(\sigma,\ a)d\sigma d\tau\right)x(a);\ t\in[a,\ b]
\end{aligned} \tag{48}
$$

then,

$$
\begin{aligned}
\tilde{x}(t) =& x(t)-x_0(t)=\Psi_0(t,\ a)\tilde{x}(a)+\left(\int_a^t \Psi_0(t,\ \tau)\tilde{A}(\tau)\Psi_0(\tau,\ a)d\tau\right)x_0(a) \\
&+\left(\int_a^t \Psi_0(t,\ \tau)\tilde{A}(\tau)\Psi_0(\tau,\ a)d\tau\right)\tilde{x}(a) \\
&+\left(\int_a^t\int_a^\tau \Psi_0(t,\ \tau)\tilde{A}(\tau)\Psi_0(\tau,\ \sigma)\tilde{A}(\sigma)\Psi(\sigma,\ a)d\sigma d\tau\right)x_0(a) \\
&+\left(\int_a^t\int_a^\tau \Psi_0(t,\ \tau)\tilde{A}(\tau)\Psi_0(\tau,\ \sigma)\tilde{A}(\sigma)\Psi(\sigma,\ a)d\sigma d\tau\right)\tilde{x}(a);\ t\in[a,\ b]
\end{aligned} \tag{49}
$$

so that, if for some real constant $\widetilde{\rho}_a\in\boldsymbol{R}_{0+}\|\tilde{x}(a)\|\le\widetilde{\rho}_a\|x_0(a)\|$, one has from Equation (49) that

$$\|\tilde{x}(t)\| \leq (t-a)\left[\sup_{\tau\in[a,\, t]}\|\Psi_0(t,\, a)\|^2 \sup_{\tau\in[a,\, t]}\|\tilde{A}(\tau)\|\left(1+(t-a)\sup_{\tau\in[a,\, t]}\|\tilde{A}(\tau)\|\sup_{\tau\in[a,\, t]}\|\Psi(\tau,\, a)\|\right)\right]$$

$$(1+\widetilde{\rho}_a)\,\|x_0(a)\|\,;\; t\in[a,\, b] \tag{50}$$

Also, one has from Equation (15) that:

$$\sup_{\tau\in[a,\, t]}\|\Psi(\tau,\, a)\| \leq \sup_{\tau\in[a,\, t]}\|\Psi_0(\tau,\, a)\| + \sup_{\tau\in[a,\, t]}\|\widetilde{\Psi}(\tau,\, a)\|$$

$$\leq \left(1+(t-a)\sup_{\tau\in[a,\, b]}\|\tilde{A}(\tau)\|\sup_{\tau\in[a,\, t]}\|\Psi(\tau,\, a)\|\right)\sup_{\tau\in[a,\, t]}\|\Psi_0(\tau,\, a)\| \tag{51}$$

which implies that

$$\sup_{\tau\in[a,\, t]}\|\Psi(\tau,\, a)\| \leq \frac{\sup_{\tau\in[a,\, t]}\|\Psi_0(\tau,\, a)\|}{1-(t-a)\sup_{\tau\in[a,\, b]}\|\tilde{A}(\tau)\|\sup_{\tau\in[a,\, t]}\|\Psi_0(\tau,\, a)\|} \tag{52}$$

provided that the subsequent constraint holds:

$$\sup_{\tau\in[a,\, b]}\|\tilde{A}(\tau)\| = O(\varepsilon+|\tilde{\lambda}|) < \frac{1}{(b-a)\sup_{\tau\in[a,\, t]}\|\Psi_0(\tau,\, a)\|} \tag{53}$$

see Equation (23) in Proposition 2 (iii), and note that Equation (53) is guaranteed by the constraint Equation (45) on $\|\tilde{A}(t)\|$ on $[a,b]$ of the theorem. The replacement of Equation (53) into Equation (50) leads to:

$$\|\widetilde{x}(t)\| \leq (t-a)\times\left[\sup_{\tau\in[a,\, t]}\|\Psi_0(\tau,\, a)\|^2 \sup_{\tau\in[a,\, t]}\|\tilde{A}(\tau)\|\left(1+\frac{(t-a)\sup_{\tau\in[a,\, t]}\|\tilde{A}(\tau)\|\sup_{\tau\in[a,\, t]}\|\Psi_0(\tau,\, a)\|}{1-(t-a)\sup_{\tau\in[a,\, t]}\|\tilde{A}(\tau)\|\sup_{\tau\in[a,\, t]}\|\Psi_0(\tau,\, a)\|}\right)\right]$$

$$(1+\widetilde{\rho}_a)\,\|x_0(a)\|\,;\; t\in[a,\, b] \tag{54}$$

then, in particular, Equation (46) holds from Equation (54) which leads to $\|\widetilde{x}(b)\| = o(\varepsilon+|\widetilde{\lambda}|)\,(1+o\,(\widetilde{\rho}_a))\,\|x_0(a)\|$.

5. Conclusions

This paper has discussed the relations between two linear time-varying second-order differential equations associated with the two-point boundary values. One of those differential equations is assumed to be a Sturm-Liouville system on a finite time interval while the other one is a perturbed current version of the first one which looses, in general, the Sturm-

Liouville property since the four two-point boundary values cannot be jointly to be kept identical for both systems. The Sturm-Liouville differential system plays the role of a nominal differential system and it is assumed to be controllable while the second one is viewed as a perturbation of the above one. Any constant real eigenvalue of the nominal Sturm-Liouville system is replaced in the current perturbed one by an explicit bounded continuous-time control function which makes such a current system to match the same two- point boundary values as those of the nominal Sturm-Liouville one. Such a control is synthesized by taking advantage of the controllability property of the nominal system. The differential equations are equivalently expressed and analyzed as second-order differential systems to facilitate the analysis methodology. Afterwards, the errors of the two-point boundary values between those of the nominal differential system and those of perturbed current one without are evaluated in terms of norm worst-case errors. In this way, the current system is interpreted as an approximate Sturm-Liouville problem version of the nominal Sturm-Liouville one under small enough parameterization deviations between both of them.

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Conflict of interests

The author declares that he has no competing interests.

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