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Research Article

New Solutions to the Fractional Perturbed Chen Lee Liu Model with Time-Dependent Coefficients: Applications to Complex Phenomena in Optical Fibers

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Abstract: In this paper, we consider the fractional perturbed Chen Lee Liu model with time-dependent coefficients (FPCLLM-TDCs). We apply the mapping method in order to get the exact solutions in the form of hyperbolic function, elliptic function, trigonometric function and rational function. These solutions are essential for comprehending certain fundamentally complex phenomena. The provided solutions will be extremely useful for applications including optical fibers. Furthermore, we show how the conformable fractional derivative order affect the exact solutions of the FPCLLM-TDCs. Finally, we examine the effects of time-dependent coefficients when these coefficients take on special cases such as random variables, polynomials, and hyperbolic functions.

***Keywords*:** conformable fractional derivative, mapping method, Chen Lee Liu model, time-dependent coefficients

MSC: 83C15, 35Q51

1. Introduction

Fractional differential equations (FDEs) find applications in a wide range of fields, including physics, biology, finance, and engineering [1–6]. They are especially useful when modeling processes involving long memory or non-local interactions. For example, in physics, FDEs have been successfully used to describe the behavior of viscoelastic materials, where the deformation response depends on the entire history of external forces. In biology, FDEs have been used to model population dynamics, where the movement and interaction of individuals depend on past experience.

On the other side, partial differential equations (PDEs) with variable coefficients present a more challenging problem compared to equations with constant coefficients due to the non-constant and often nonlinear nature of the coefficients. However, with the right techniques and methods, these equations can be solved and utilized to model a wide range of phenomena. Solving PDEs with variable coefficients is essential in many scientific and engineering applications, where the variability and nonlinearity of the coefficients play a crucial role in the dynamics and behavior of the systems under study. Recently, there are various helpful and practical methods for solving these equations, including sub-equation method [7], Hirota's bilinear approach [8], Jacobian elliptic functions and (G'/G^2)-expansion methods [9], solitary wave ansatz [10] and $(G^{'}/G)$-expansion method [11].

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To achieve a better degree of qualitative agreement, we look here at the fractional perturbed Chen Lee Liu model (FPCLLM) with time-dependent coefficients (TDCs) as follows [12]:

$$i\mathscr{U}_t + A(t)\mathscr{D}^{\alpha}_{xx}\mathscr{U} + iB(t)|\mathscr{U}|^2\mathscr{D}^{\alpha}_{x}\mathscr{U} = i[\ell_1(t)\mathscr{D}^{\alpha}_{x}\mathscr{U} + \ell_2(t)\mathscr{D}^{\alpha}_{x}(|\mathscr{U}|^2\mathscr{U}) + \ell_3(t)\mathscr{U}\mathscr{D}^{\alpha}_{x}(|\mathscr{U}|^2)], \tag{1}$$

where $\mathscr{U}$ is the the normalized electric-field envelope, $\mathscr{D}^{\alpha}$ is conformable fractional derivative operator for $\alpha \in (0,\ 1]$, $A(t)$, $B(t)$ $\ell_1(t)$, $\ell_2(t)$ and $\ell_3(t)$ are smooth real functions of the variable t such that $B(t) = 3\ell_2(t) + 2\ell_3(t)$.

The perturbed Chen Lee Liu model (1) is a mathematical framework that describes the interactions between particles in a system. It considers the effects of an external perturbation on the system, which can lead to the emergence of complex collective behaviors. The model is based on the Ising model, which describes the interactions between spins in a lattice. However, the perturbed Chen Lee Liu model goes beyond the Ising model by introducing an additional term in the Hamiltonian that accounts for the perturbation.

One of the key applications of the perturbed Chen Lee Liu model is in the study of phase transitions. Phase transitions are phenomena in which a system undergoes a sudden change in its properties, such as magnetization or conductivity. The perturbed Chen Lee Liu model can help elucidate the mechanisms behind phase transitions and predict the critical points at which they occur. By studying the behavior of the system as the perturbation is varied, researchers can gain insights into the nature of phase transitions and the critical exponents that characterize them.

The perturbed Chen Lee Liu model has been used to investigate a wide range of physical phenomena, including ferromagnetism, antiferromagnetism, and superconductivity. By studying the behavior of these systems using the perturbed Chen Lee Liu model, researchers can uncover new insights into the underlying mechanisms driving these phenomena. For example, the model has been used to study the effects of impurities on the magnetic properties of materials, shedding light on how disorder can impact the behavior of magnetic systems.

Because of the significance of the Chen Lee Liu model in optical fibers, many researchers have employed various techniques to get analytical solutions for this model, which include the modified Khater method [13], extended direct algebraic method [14], $(G'/G, 1/G)$-expansion approach [15], Riccati-Bernoulli and generalized tanh methods [16], Sardar sub-equation method [17], and modified extended tanh-expansion method [18].

This study aims to establish exact solutions for the FPCLLM-TDCs (1) using the mapping approach. The solutions include hyperbolic functions, elliptic functions, rational functions, and trigonometric functions. In addition, we utilize the Matlab software to construct 2D and 3D graphs for some of the analytical solutions created in this study to analyze the influence of the conformable fractional derivative and time-dependent coefficient on the obtained solutions of the FPCLLM-TDCs (1).

The organization of the paper is as follows: The CFD is defined and some of its properties are explained in Section 2. While in Section 3, we use a suitable wave transformation to obtain the wave equation of the FPCLLM-TDCs (1). Using the mapping approach, we find the exact solutions of the FPCLLM-TDCs (1) in Section 4. In Section 5, we discuss how the time-dependent coefficient and CFD affect the obtained solutions. In the end, the paper's conclusion is given.

2. Conformable fractional derivative

Fractional calculus operators are useful for describing and assessing complex processes that cannot be adequately explained using standard integer-order calculus. The Hadamard derivative, Caputo derivative, Riemann-Liouville derivative, Katugampola derivative, Grünwald-Letnikov derivative, and Jumarie derivative [19–23], are the types of the fractional derivative operators proposed in the literature. Khalil et al. [24] recently suggested the conformable fractional derivative (CFD), that has similarities with the Newton derivative. Let us now define the CFD for the function $\mathscr{K} : (0,\ \infty) \to \mathbb{R}$ of order $\alpha \in (0,\ 1]$ as follows:

$$\mathscr{D}_x^{\alpha}\mathscr{K}(x) = \lim_{\varepsilon\to 0}\frac{\mathscr{K}(x+\varepsilon x^{1-\alpha}) - \mathscr{K}(x)}{\varepsilon}.$$

The CFD has the following characteristics for any constants a and b:
1. $\mathscr{D}_x^{\alpha}[a\mathscr{K}_1(x)+b\mathscr{K}_2(x)] = a\mathscr{D}_x^{\alpha}\mathscr{K}_1(x)+b\mathscr{D}_x^{\alpha}\mathscr{K}_2(x)$,
2. $\mathscr{D}_x^{\alpha}[\mathscr{K}_1(x)\mathscr{K}_2(x)] = \mathscr{K}_2(x)\mathscr{D}_x^{\alpha}\mathscr{K}_1(x)+\mathscr{K}_1(x)\mathscr{D}_x^{\alpha}\mathscr{K}_2(x)$,
3. $\mathscr{D}_x^{\alpha}[a] = 0$,
4. $\mathscr{D}_x^{\alpha}[x^b] = bx^{b-\alpha}$,
5. $\mathscr{D}_x^{\alpha}\mathscr{K}(x) = x^{1-\alpha}\dfrac{d\mathscr{K}}{dx}$,
6. $\mathscr{D}_x^{\alpha}(\mathscr{K}_1\circ\mathscr{K}_2)(x) = x^{1-\alpha}\mathscr{K}_2'(x)\mathscr{K}_1'(\mathscr{K}_2(x))$.

3. Wave equation for FPCLLM-TDCs

To attain the wave equation of the FPCLLM-TDCs (1), we employ

$$\mathscr{U}(x,\, t) = \mathscr{V}(\vartheta_{\alpha})e^{i\psi_{\alpha}},$$

$$\vartheta_{\alpha} = \frac{\vartheta}{\alpha}x^{\alpha}+\int_0^t f(\tau)d\tau,\ \text{and}\ \psi_{\alpha} = \frac{k}{\alpha}x^{\alpha}+\int_0^t g(\tau)d\tau, \tag{2}$$

where $\mathscr{V}$ is a real valued function, ϑ and k are real constants, f and g are real functions can be determined later. Plugging Eq. (2) into Eq. (1) and using

$$\frac{\partial\mathscr{U}}{\partial t} = [f(t)\mathscr{V}'+ig(t)\mathscr{V}]e^{i\psi_{\alpha}},\ \mathscr{D}_x^{\alpha}\mathscr{U} = (\vartheta\mathscr{V}'+ik\mathscr{V})e^{i\psi_{\alpha}},\ \mathscr{D}_x^{\alpha}\left(|\mathscr{U}|^2\right) = 2\vartheta\mathscr{V}\mathscr{V}',$$

$$\mathscr{D}_{xx}^{\alpha}\mathscr{U} = [\vartheta^2\mathscr{V}''+2ik\vartheta\mathscr{V}'-k^2\mathscr{V}]e^{i\psi_{\alpha}},\ \mathscr{D}_x^{\alpha}\left(|\mathscr{U}|^2\mathscr{U}\right) = (3\vartheta\mathscr{V}^2\mathscr{V}'+ik\mathscr{V}^3)e^{i\psi_{\alpha}}$$

we get for imaginary part

$$[f(t)+2k\vartheta A(t)-\vartheta\ell_1(t)]\mathscr{V}'+[\vartheta B(t)-3\vartheta\ell_2(t)-2\vartheta\ell_3(t)]\mathscr{V}^2\mathscr{V}' = 0, \tag{3}$$

and for real part

$$\vartheta^2 A(t)\mathscr{V}''-(g(t)+k^2A(t)-k\ell_1(t))\mathscr{V}+(k\ell_2(t)-kB(t))\mathscr{V}^3 = 0. \tag{4}$$

From (3), we have

$$f(t) = \vartheta\ell_1(t)-2k\vartheta A(t),\ \text{and}\ B(t) = 3\ell_2(t)+2\ell_3(t). \tag{5}$$

4. The solutions of the FPCLLM-TDCs

Here, the mapping method, which is reported in [25], is used. Let the solutions of Eq. (4) have the form

$$\mathscr{V}(\vartheta_\alpha)=\sum_{i=0}^{N}a_i(t)\mathscr{P}^i(\vartheta_\alpha), \tag{6}$$

where $a_i(t)$ for $i=0,\ 1,\ ....,\ N$ are unknown functions in t , and $\mathscr{P}$ is the solution of

$$\mathscr{P}'=\sqrt{b_1\mathscr{P}^4+b_2\mathscr{P}^2+b_3}, \tag{7}$$

where b_1, b_2 and b_3 are real constants.

By balancing $\mathscr{V}''$ with $\mathscr{V}^3$ in Eq. (4), we can determine N as

$$N+2=3N\Longrightarrow N=1.$$

With $N=1$, Eq. (6) turns into

$$\mathscr{V}(\vartheta_\alpha)=a_0(t)+a_1(t)\mathscr{P}(\vartheta_\alpha). \tag{8}$$

Differentiating Eq. (8) twice and using (7), we get

$$\mathscr{V}''=a_1(b_2\mathscr{P}+2b_1\mathscr{P}^3) \tag{9}$$

Substituting Eqs. (8) and (9) into Eq. (4) we have

$$[2\vartheta^2a_1b_1A(t)+a_1^3(k\ell_2(t)-kB(t))]\mathscr{P}^3+6kB(t)a_0a_1^2\mathscr{P}^2+[\vartheta^2a_1b_2A(t)+6ka_0^2a_1B(t)-a_1(g(t)$$

$$+k^2A(t)-k\ell_1(t))]\mathscr{P}+[2ka_0^3B(t)-a_0(g(t)+k^2A(t)-k\ell_1(t))]=0.$$

Seting all coefficient of $\mathscr{P}^i$ equal zero for $i=3,\ 2,\ 1,\ 0$, to attain

$$2\vartheta^2a_1b_1A(t)+ka_1^3(\ell_2(t)-B(t))=0,$$

$$3ka_0a_1^2(\ell_2(t)-B(t))=0,$$

$$\vartheta^2a_1b_2A(t)+3ka_0^2a_1(\ell_2(t)-B(t))-a_1(g(t)+k^2A(t)-k\ell_1(t))=0,$$

and

$$ka_0^3(\ell_2(t)-B(t))-a_0(g(t)+k^2A(t)-k\ell_1(t))=0,$$

Solving these equations yields:

$$a_0(t)=0,\ \ a_1(t)=\pm\sqrt{\frac{-2\vartheta^2 b_1 A(t)}{k(\ell_2(t)-B(t))}},\ g(t)=(\vartheta^2 b_2-k^2)A(t)+k\ell_1(t), \tag{10}$$

where $B(t)$ and $f(t)$ are defined in Eq. (5). By utilizing Eqs (2), (8) and (10), the solution of FPCLLM-TDCs (1) is

$$\mathscr{U}(x,\ t)=\pm\sqrt{\frac{\vartheta^2 b_1 A(t)}{k(\ell_2(t)+\ell_3(t))}}\mathscr{P}(\vartheta_\alpha)e^{i\psi_\alpha}, \tag{11}$$

where

$$\vartheta_\alpha=\frac{\vartheta}{\alpha}x^\alpha+\vartheta\int_0^t \ell_1(\tau)d\tau-2k\vartheta\int_0^t A(\tau)d\tau,$$

and

$$\psi_\alpha=\frac{k}{\alpha}x+(\vartheta^2 b_2-k^2)\int_0^t A(\tau)d\tau+k\int_0^t \ell_1(\tau)d\tau.$$

There are many sets depending on $b_1,\ b_2$ and b_3 :

Set 1: If $b_1=\varpi^2,\ b_2=-(1+\varpi^2)$ and $b_3=1$, then $\mathscr{P}(\xi)=sn(\vartheta_\alpha)$. Therefore, the solution of FPCLLM-TDCs (1), by using Eq. (11), is

$$\mathscr{U}(x,\ t)=\pm\varpi\sqrt{\frac{\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}sn(\vartheta_\alpha)e^{i\psi_\alpha}\ \text{If}\ \frac{A(t)}{k(\ell_2(t)+\ell_3(t))}>0. \tag{12}$$

At $\varpi\to 1$, Eq. (12) becomes

$$\mathscr{U}(x,\ t)=\pm\sqrt{\frac{\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}\tanh(\vartheta_\alpha)e^{i\psi_\alpha}\ \ \text{If}\ \frac{A(t)}{k(\ell_2(t)+\ell_3(t))}>0. \tag{13}$$

Set 2: If $b_1=1,\ b_2=2\varpi^2-1$ and $b_3=-\varpi^2(1-\varpi^2)$, then $\mathscr{P}(\vartheta_\alpha)=ds(\vartheta_\alpha)$. Consequently, the FPCLLM-TDCs (1) has the solution

$$\mathscr{U}(x,\ t) = \pm\sqrt{\frac{\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}ds(\vartheta_\alpha)e^{i\psi_\alpha}\ \text{If}\ \frac{A(t)}{k(\ell_2(t)+\ell_3(t))} > 0. \tag{14}$$

When $\varpi \to 1$, Eq. (14) becomes

$$\mathscr{U}(x,\ t) = \pm\sqrt{\frac{\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}\operatorname{csch}(\vartheta_\alpha)e^{i\psi_\alpha}\ \text{If}\ \frac{A(t)}{k(\ell_2(t)+\ell_3(t))} > 0. \tag{15}$$

At $\varpi \to 0$, Eq. (14) tends to

$$\mathscr{U}(x,\ t) = \pm\sqrt{\frac{\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}\csc(\vartheta_\alpha)e^{i\psi_\alpha}\ \text{If}\ \frac{A(t)}{k(\ell_2(t)+\ell_3(t))} > 0. \tag{16}$$

Set 3: If $b_1 = -\varpi^2,\ b_2 = 2\varpi^2 - 1$ and $b_3 = 1 - \varpi^2$, then $\mathscr{P}(\vartheta_\alpha) = cn(\vartheta_\alpha)$. Consequently, the solution of FPCLLM-TDCs (1) is

$$\mathscr{U}(x,\ t) = \pm\varpi\sqrt{\frac{-\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}[cn(\vartheta_\alpha)]e^{i\psi_\alpha}\ \text{If}\ \frac{A(t)}{k(\ell_2(t)+\ell_3(t))} < 0. \tag{17}$$

When $\varpi \to 1$, Eq. (17) becomes

$$\mathscr{U}(x,\ t) = \pm\sqrt{\frac{-\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}[\operatorname{sech}(\vartheta_\alpha)]e^{i\psi_\alpha}\ \text{If}\ \frac{A(t)}{k(\ell_2(t)+\ell_3(t))} < 0. \tag{18}$$

Set 4: If $b_1 = \dfrac{\varpi^2}{4},\ b_2 = \dfrac{(\varpi^2-2)}{2}$ and $b_3 = \dfrac{1}{4}$, then $\mathscr{P}(\vartheta_\alpha) = \dfrac{sn(\vartheta_\alpha)}{1+dn(\vartheta_\alpha)}$. As a result, the solution of FPCLLM-TDCs (1) is

$$\mathscr{U}(x,\ t) = \pm\frac{\varpi}{2}\sqrt{\frac{\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}\left[\frac{sn(\vartheta_\alpha)}{1+dn(\vartheta_\alpha)}\right]e^{i\psi_\alpha}\ \text{If}\ \frac{A(t)}{k(\ell_2(t)+\ell_3(t))} > 0. \tag{19}$$

At $\varpi \to 1$, Eq. (19) tends to

$$\mathscr{U}(x,\ t) = \pm\frac{1}{2}\sqrt{\frac{\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}\left[\frac{\tanh(\vartheta_\alpha)}{1+\operatorname{sech}(\vartheta_\alpha)}\right]e^{i\psi_\alpha}\ \text{If}\ \frac{A(t)}{k(\ell_2(t)+\ell_3(t))} > 0. \tag{20}$$

Set 5: If $b_1 = \frac{(1-\varpi^2)^2}{4}$, $b_2 = \frac{(1-\varpi^2)^2}{2}$ and $b_3 = \frac{1}{4}$, then $\mathscr{P}(\vartheta_\alpha) = \frac{sn(\vartheta_\alpha)}{dn(\vartheta_\alpha)+cn(\vartheta_\alpha)}$. Therefore, the solution of FPCLLM-TDCs (1) is

$$\mathscr{U}(x,\ t) = \pm\frac{(1-\varpi^2)}{2}\sqrt{\frac{\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}\left[\frac{sn(\vartheta_\alpha)}{dn(\vartheta_\alpha)+cn(\vartheta_\alpha)}\right]e^{i\psi_\alpha} \text{ If } \frac{A(t)}{k(\ell_2(t)+\ell_3(t))} > 0. \tag{21}$$

If $\varpi \to 0$, then Eq. (21) is typically

$$\mathscr{U}(x,\ t) = \pm\frac{1}{2}\sqrt{\frac{\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}\left[\frac{\sin(\vartheta_\alpha)}{1+\cos(\vartheta_\alpha)}\right]e^{i\psi_\alpha} \text{ If } \frac{A(t)}{k(\ell_2(t)+\ell_3(t))} > 0. \tag{22}$$

Set 6: If $b_1 = \frac{1-\varpi^2}{4}$, $b_2 = \frac{(1-\varpi^2)}{2}$ and $b_3 = \frac{1-\varpi^2}{4}$, then $\mathscr{P}(\vartheta_\alpha) = \frac{cn(\vartheta_\alpha)}{1+sn(\vartheta_\alpha)}$. As a result, the solution of FPCLLM-TDCs (1) is

$$\mathscr{U}(x,\ t) = \pm\frac{1}{2}\sqrt{\frac{\vartheta^2(1-\varpi^2)A(t)}{k(\ell_2(t)+\ell_3(t))}}\left[\frac{cn(\vartheta_\alpha)}{1+sn(\vartheta_\alpha)}\right]e^{i\psi_\alpha} \text{ If } \frac{A(t)}{k(\ell_2(t)+\ell_3(t))} > 0. \tag{23}$$

At $\varpi \to 0$, Eq. (23) turns to

$$\mathscr{U}(x,\ t) = \pm\frac{1}{2}\sqrt{\frac{\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}\left[\frac{\cos(\vartheta_\alpha)}{1+\sin(\vartheta_\alpha)}\right]e^{i\psi_\alpha} \text{ If } \frac{A(t)}{k(\ell_2(t)+\ell_3(t))} > 0. \tag{24}$$

Set 7: If $b_1 = 1,\ b_2 = 0$ and $b_3 = 0$, then $\mathscr{P}(\vartheta_\alpha) = \frac{c}{\vartheta_\alpha}$. Therefore, the solution of FPCLLM-TDCs (1) is

$$\mathscr{U}(x,\ t) = \pm\sqrt{\frac{\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}\left[\frac{c}{\vartheta_\alpha}\right]e^{i\psi_\alpha} \text{ If } \frac{A(t)}{k(\ell_2(t)+\ell_3(t))} > 0. \tag{25}$$

Set 8: If $b_1 = 1,\ b_2 = 2-\varpi^2$ and $b_3 = (1-\varpi^2)$, then $\mathscr{P}(\vartheta_\alpha) = cs(\vartheta_\alpha)$. Thus, the solution of FPCLLM-TDCs (1) is

$$\mathscr{U}(x,\ t) = \pm\sqrt{\frac{\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}cs(\vartheta_\alpha)e^{i\psi_\alpha} \text{ If } \frac{A(t)}{k(\ell_2(t)+\ell_3(t))} > 0. \tag{26}$$

At $\varpi \to 1$, Eq. (26) is turns into

$$\mathscr{U}(x,\ t) = \pm\sqrt{\frac{\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}\operatorname{csch}(\vartheta_\alpha)e^{i\psi_\alpha} \text{ If } \frac{A(t)}{k(\ell_2(t)+\ell_3(t))} > 0. \tag{27}$$

If $\varpi \to 0$, then Eq. (26) becomes

$$\mathscr{U}(x,\,t) = \pm\sqrt{\frac{\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}\cot(\vartheta_\alpha)e^{i\psi_\alpha} \text{ If } \frac{A(t)}{k(\ell_2(t)+\ell_3(t))} > 0. \tag{28}$$

Set 9: If $b_1 = \dfrac{-1}{4}$, $b_2 = \dfrac{\varpi^2+1}{2}$ and $b_3 = \dfrac{-(1-\varpi^2)^2}{2}$, then $\mathscr{P}(\vartheta_\alpha) = \varpi cn(\vartheta_\alpha) + dn(\vartheta_\alpha)$. Therefore, the solution of FPCLLM-TDCs (1) is

$$\mathscr{U}(x,\,t) = \pm\frac{1}{2}\sqrt{\frac{-\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}\left[\varpi cn(\vartheta_\alpha)+dn(\vartheta_\alpha)\right]e^{i\psi_\alpha} \text{ If } \frac{A(t)}{k(\ell_2(t)+\ell_3(t))} < 0. \tag{29}$$

If $\varpi \to 1$, then Eq. (29) tends to Eq. (18).

Set 10: If $b_1 = \dfrac{\varpi^2-1}{4}$, $b_2 = \dfrac{\varpi^2+1}{2}$ and $b_3 = \dfrac{\varpi^2-1}{4}$, then $\mathscr{P}(\vartheta_\alpha) = \dfrac{dn(\vartheta_\alpha)}{1+sn(\vartheta_\alpha)}$. Hence, the solution of FPCLLM-TDCs (1) is

$$\mathscr{U}(x,\,t) = \pm\frac{1}{2}\sqrt{\frac{(\varpi^2-1)\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}\left[\frac{dn(\vartheta_\alpha)}{1+sn(\vartheta_\alpha)}\right]e^{i\psi_\alpha} \text{ If } \frac{A(t)}{k(\ell_2(t)+\ell_3(t))} < 0. \tag{30}$$

When $\varpi \to 0$, Eq. (30) is tends to

$$\mathscr{U}(x,\,t) = \pm\frac{1}{2}\sqrt{\frac{-\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}\left[\frac{1}{1+\sin(\vartheta_\alpha)}\right]e^{i\psi_\alpha} \text{ If } \frac{A(t)}{k(\ell_2(t)+\ell_3(t))} < 0. \tag{31}$$

Set 11: If $b_1 = -1$, $b_2 = 2-\varpi^2$ and $b_3 = \varpi^2-1$, then $\mathscr{P}(\vartheta_\alpha) = dn(\vartheta_\alpha)$. Therefore, the solution of FPCLLM-TDCs (1) is

$$\mathscr{U}(x,\,t) = \pm\sqrt{\frac{-\vartheta^2 A(t)}{k(\ell_2(t)+\ell_3(t))}}[dn(\vartheta_\alpha)]e^{i\psi_\alpha} \text{ If } \frac{A(t)}{k(\ell_2(t)+\ell_3(t))} < 0. \tag{32}$$

If $\varpi \to 1$, then Eq. (32) turns to Eq. (18).

5. Impacts of CFD and TDCs

Discussion: In this paper, the exact solutions of the FPCLLM-TDCs (1) were acquired. We applied the mapping method which provided many kind solutions such as kink solutions, periodic solutions, bright solutions, dark optical solution, singular solution and etc. The obtained solutions of the FPCLLM-TDCs (1) can be used to study an enormous variety of important physical phenomena, such as optical fiber wave propagation in a magnetized plasma, oceanic rogue waves, and ion-acoustic waves [13–16, 26].

Impacts of CFD: Now, we address the impact of CFD on the acquired solutions of the FPCLLM-TDCs (1). A series of two-dimensional and three-dimensional graphs are generated by assigning suitable values to the unknown variables.

Figures 1-2 inroduce the behavior solutions of (17) and (18), respectively. Figure 1 shows the periodic solutions $|\mathscr{U}(x,\ t)|$ stated in Eq. (17) for $\vartheta = k = -1,\ \ell_2(t) = \ell_3(t) = 0.5,\ A(t) = \ell_1(t) = 1$, and for $\alpha = 1,\ 0.8,\ 0.6$. While, Figure 2 shows the dark solutions $|\mathscr{U}(x,\ t)|$ stated in Eq. (13) for $\vartheta = k = -1,\ \ell_2(t) = \ell_3(t) = 0.5,\ A(t) = \ell_1(t) = 1,\ t \in [0,\ 3],\ x \in [0,\ 4]$, and for $\alpha = 1,\ 0.8,\ 0.6$. The figures show that the surface extends as the derivative order α of MTD decreases.

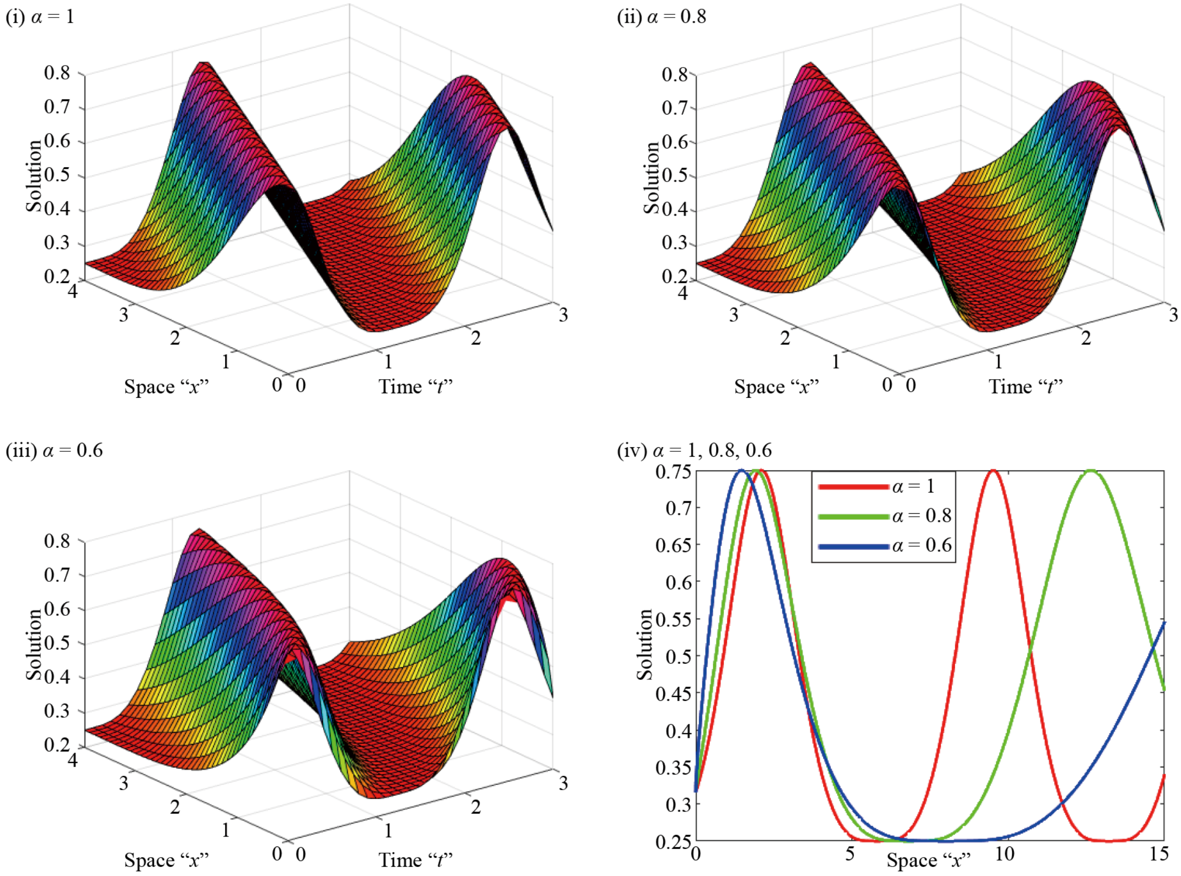
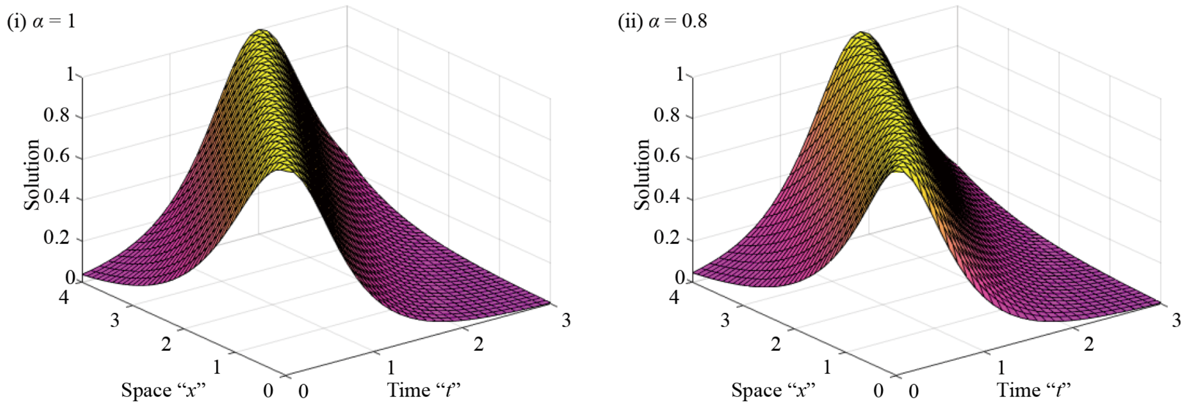


Figure 1. (i-iii) Represent 3D-shape of the periodic solution $|\mathscr{U}(x,\ t)|$ stated in Eq. (17) with $\alpha = 1,\ 0.8,\ 0.6$ (iv) shows 2D-shape of Eq. (18) with various value of α



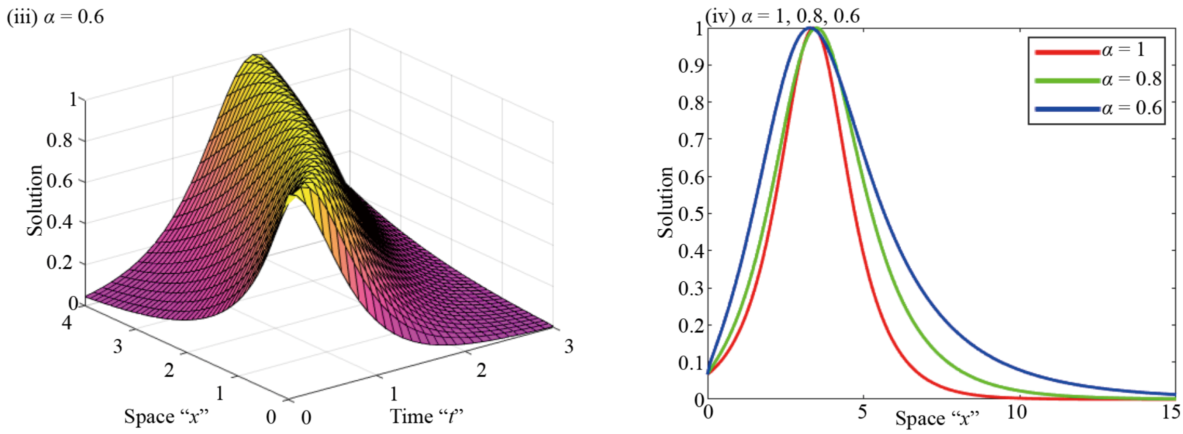
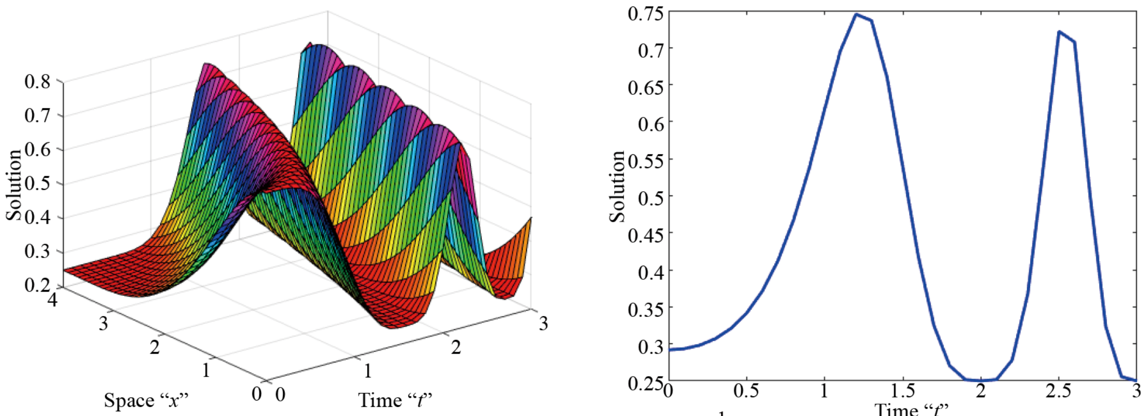
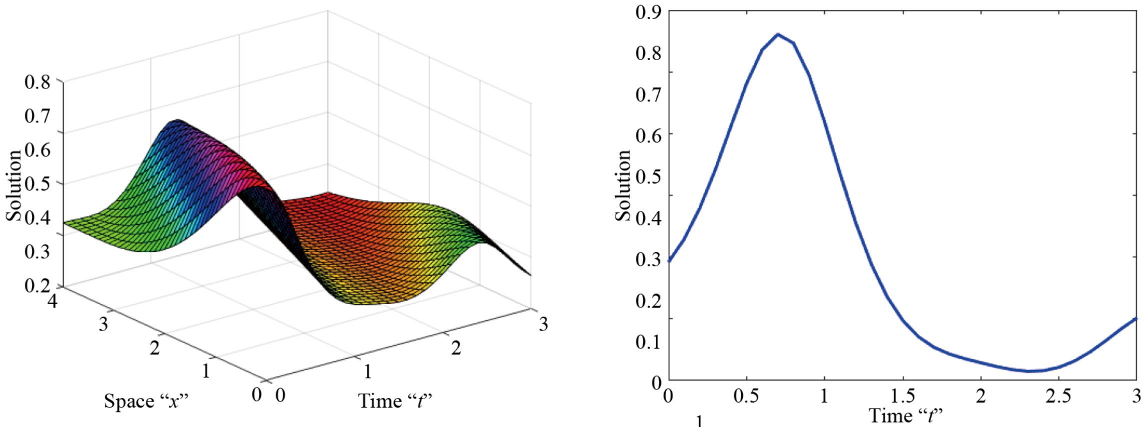


Figure 2. (i-iii) Represent 3D-shape of the dark solution $|\mathscr{U}(x,\ t)|$ stated in Eq. (18) with $\alpha = 1,\ 0.8,\ 0.6$ (iv) shows 2D-shape of Eq. (29) with various α

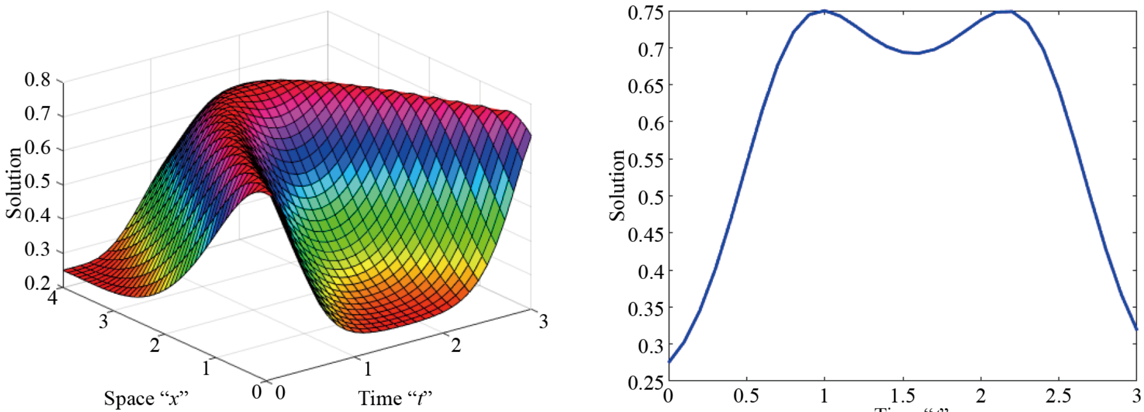
Impacts of TDCs: Now, we investigate the effect of the time-dependent coefficients on the obtained solutions of the FPCLLM-TDCs (1). Figures 3 and 4 show the solutions $|\mathscr{U}(x,\ t)|$ reported in Eqs (17) and (18) with $\vartheta = k = -1$ and for different time-dependent coefficients as follows: In Figures 3(i) and 4(i), we assume $A(t) = \ell_1(t) = t,\ \ell_2(t) = \ell_3(t) = \dfrac{1}{2}t$, this choice makes the surface twist from the left. In Figures 3(ii) and 4(ii), we assume $A(t) = \ell_1(t) = 1,\ \ell_2(t) = \ell_3(t) = \frac{1}{2}\sinh(t)$, this option makes the surface a little flat from the right. In Figures 3(iii) and 4(iii), we assume $A(t) = \ell_1(t) = \cos(t),\ \ell_2(t) = \ell_3(t) = 1$, this choice effects on the surface sides. While in Figures 3(v) and 4(v), we assume $A(t) = \ell_1(t) = 1, \ell_2(t) = \ell_3(t) = \dfrac{1}{2}\beta_t(t)$, where $\beta_t(t)$ is the derivative of Brownian motion $\beta(t)$, this option causes the surface to oscillate.



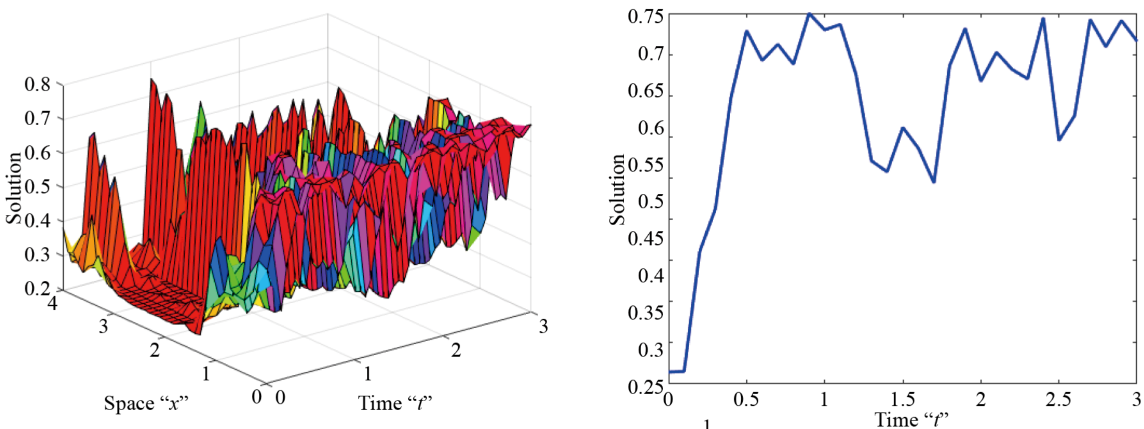
(i) $A(t) = \ell_1(t) = t,\ \ell_2(t) = \ell_3(t) = \dfrac{1}{2}t,$



(ii) $A(t) = \ell_1(t) = 1$, $\ell_2(t) = \ell_3(t) = \frac{1}{2}\sinh(t)$,

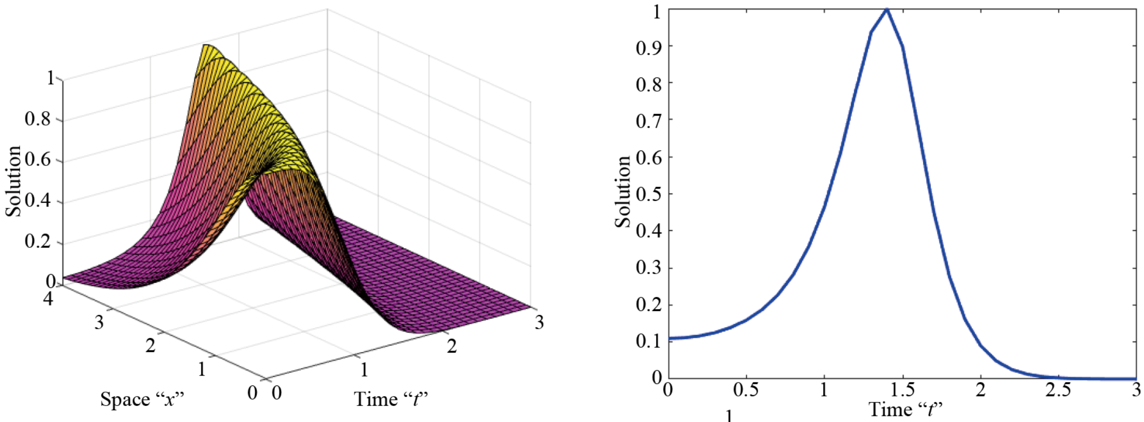


(iii) $A(t) = \ell_1(t) = \cos(t)$, $\ell_2(t) = \%\ell_3(t) = 1$,

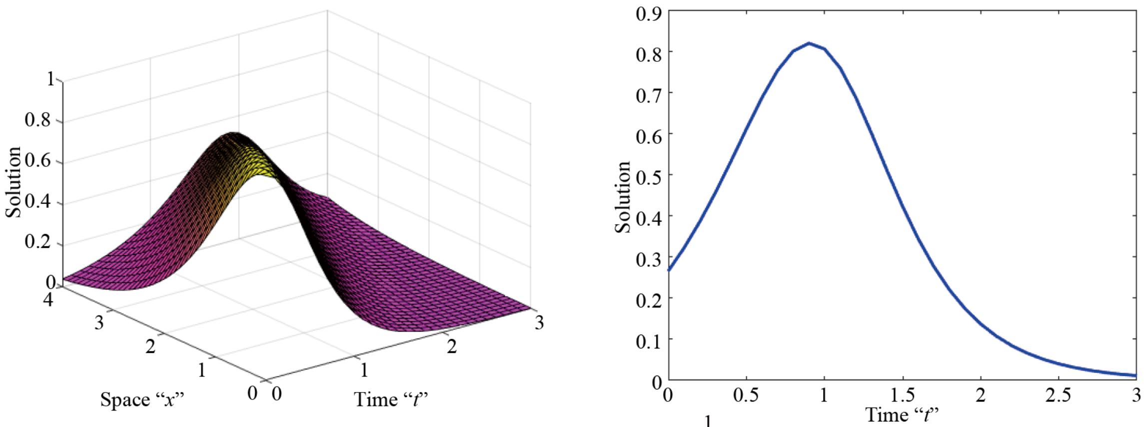


(v) $A(t) = \ell_1(t) = 1$, $\ell_2(t) = \ell_3(t) = \frac{1}{2}\beta_t(t)$,

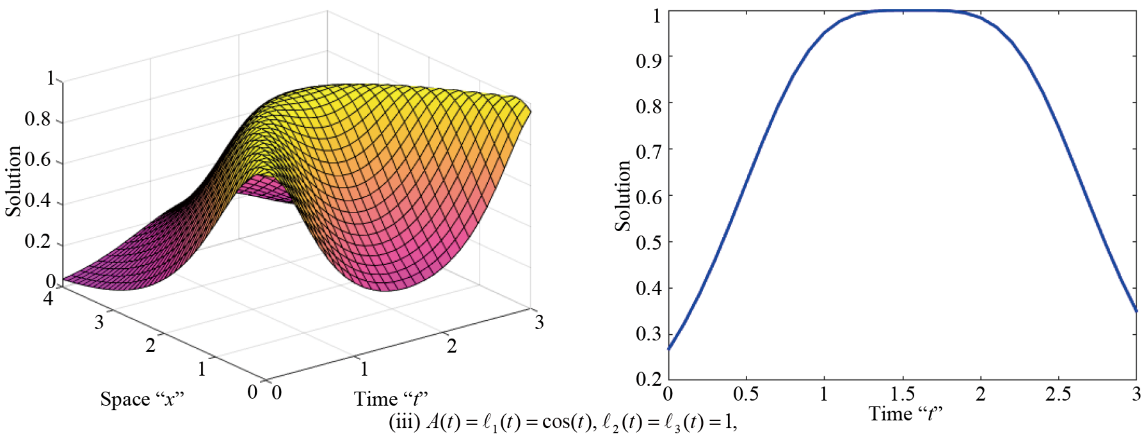
Figure 3. Represents 3D and 2D profile for the solution $|\mathscr{U}(x, t)|$ stated in Eq. (17) for $\vartheta = k = -1$, $\alpha = 1$, $t \in [0, 3]$, $x \in [0, 4]$, $\alpha = 1$ and for various time-dependent coefficients



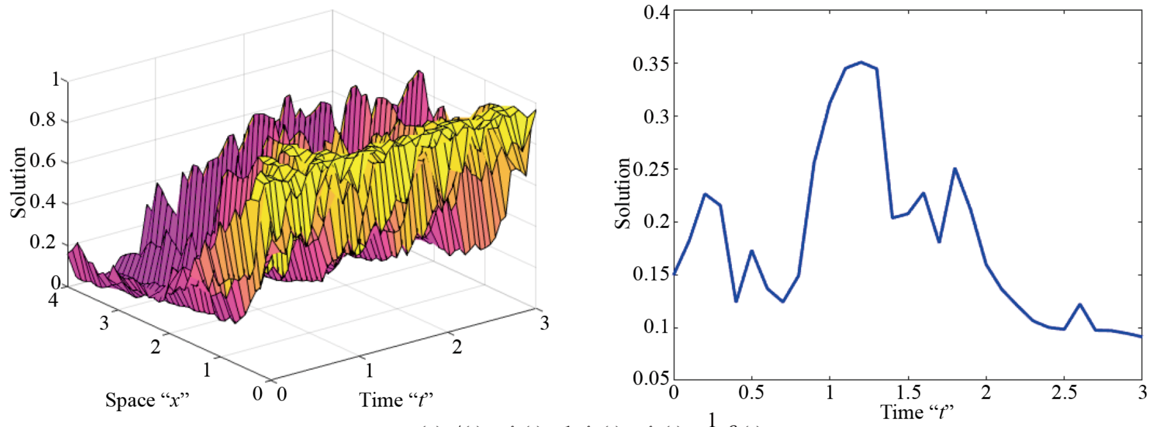
(i) $A(t) = \ell_1(t) = t,\ \ell_2(t) = \ell\%_3(t) = \dfrac{1}{2}t,$



(ii) $A(t) = \ell_1(t) = 1,\ \ell_2(t) = \%\ell_3(t) = \dfrac{1}{2}\sinh(t),$



(iii) $A(t) = \ell_1(t) = \cos(t),\ \ell_2(t) = \ell_3(t) = 1,$



(v) $A(t) = \ell_1(t) = 1,\ \ell_2(t) = \ell_3(t) = \frac{1}{2}\beta_t(t),$

Figure 4. Represents 3D and 2D profile for the solution $|\mathscr{U}(x,\ t)|$ stated in Eq. (17) for $\vartheta = k = -1,\ \alpha = 1,\ x \in [0,\ 4],\ t \in [0,\ 3],\ \alpha = 1$ and for different time-dependent coefficients

6. Conclusions

In this paper, we introduced a large variety of exact solutions to the fractional perturbed Chen Lee Liu model with time-dependent coefficients (FPCLLM-TDCs) (1). One of the key advantages of the FPCLLM-TDCs is its ability to capture the complex behavior of dynamic systems. By incorporating the time-dependent coefficient, the model can account for variations in system parameters that may arise due to external factors or internal dynamics. This flexibility allows researchers to study a wider range of scenarios and better understand the dynamics of the system under different conditions. Therefore, we acquired here the exact solutions in the form of elliptic functions, hyperbolic functions, trigonometric functions and rational functions by using the mapping method. The created solutions are extremely useful for applications which include optical fibers [27]. Finally, we addressed how the conformable fractional derivative order and the time-dependent coefficients affects the exact solution of the FPCLLM-TDCs (1). In the future work, further research is needed to understand the chaotic dynamics exhibited by the FPCLLM-TDCs such as conducting a comprehensive analysis of the Lyapunov exponents and sensitivity to initial conditions and investigating the physical implications of chaos in the context of optical fibers and plasma physics.

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Conflict of interest

The author declares no competing financial interest.

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