Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

UNIVERSAL WISER PUBLISHER

Research Article

An Inverse Inequality for Fractional Sobolev Norms in Unbounded Domains

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Received: 25 June 2024; **Revised:** 9 October 2024; **Accepted:** 8 November 2024

Abstract: The nonlocal operators have found applications in various areas of contemporary science. The anomalous diffusion phenomena have been modeled by the fractional Poisson boundary-value problem. Electromagnetic fluids have been described by fractional differential equations. The fractional differential operators have found applications in material sciences, planar and space elasticity, probabilistic theory, harmonic analysis, and even in finance. The inverse inequality plays an important role in Numerical Analysis. The well-known results on inverse inequalities have been obtained in bounded domains and finite-dimensional spaces. Naturally, a new challenge arises to obtain inverse inequalities in the fractional Sobolev spaces. This paper is devoted to differential inequalities between fractional Sobolev norms. We expand the notion of a monotone function into a new notion supermonotone function and rigorously prove an inverse inequality for a class of differentiable functions in unbounded domains. Examples that demonstrate the theory are presented.

Keywords: fractional Poisson boundary-value problem, inverse inequality, weak fractional Laplacian, supermonotone function, nonlocal fractional differential operator

MSC: 26A33, 34A08

1. Introduction

The nonlocal operators defined in fractional Sobolev spaces are basic tools for solving real-life problems in various applied areas:

- electromagnetic fluids;
- image processing;
- material sciences;
- planar and space elasticity;
- probabilistic theory;
- harmonic analysis [1, 2];
- the anomalous diffusion phenomena.

Fractional boundary-value problems have been an object of great interest in the last decades [3–7]. The fractional Poisson boundary-value problem emerges as a pivotal mathematical framework, combining the principles of fractional

DOI: https://doi.org/10.37256/cm.5420245192

calculus and the classical Poisson equation. This mathematical confluence not only presents intriguing theoretical challenges but also finds profound applications across various scientific and engineering areas [8]. The fractional Poisson boundary-value problem finds applications in modeling anomalous diffusion processes. Traditional diffusion models assume a constant rate, while fractional derivatives allow for the inclusion of memory effects, enabling the accurate portrayal of phenomena such as subdiffusion and superdiffusion. These features have implications for understanding the transport of particles in porous media, biological systems, and other intricate environments [9]. The Poisson problem have found applications in environmental science, aiding in the modeling of pollutant dispersion and contaminant transport in heterogeneous media. In biology, the problem contributes to understanding processes with memory, such as the diffusion of substances within living tissues [10, 11].

Except in some rare cases the fractional boundary-value problems have been solved numerically [3]. Such problems have been presented in weak forms by nonlocal fractional differential operators. The fractional embedding theorems declare estimating of given Sobolev norms by norms from higher Sobolev spaces [12]. The theory of Numerical Analysis requires plenty of results on various kinds of differential inequalities in bounded [13–15] and unbounded domains [16, 17].

A relation between Sobolev norms is called an inverse inequality if a given Sobolev norm (seminorm) is estimated by a lower-order Sobolev norm (seminorm). Fractional Sobolev embedding inequalities [18, 19], Poincaré-type inequalities [20, 21], and fractional inverse inequalities [22] have been widely used to prove optimal error estimates for finite element approximations of weak solutions of elliptic boundary value problems. However, there is a significant difference between Poincaré inequalities and the inverse inequalities for fractional Sobolev norms. The Poincaré inequalities have been proved in fractional Sobolev spaces but all known inverse inequalities have been validated in finite-dimensional spaces [22, 23]. Traditionally, inverse inequalities have been applied in finite element spaces [24], which involve piecewise polynomial functions. The inverse inequalities in finite-dimensional spaces [14, 15] are true due to the fact that all norms in these spaces are equivalent. The real problem of inverse inequality in finite element spaces is with the corresponding constant. This constant is independent of the estimating function. Still, it depends on the measure of the supporting domain and tends to infinity when the measure of the domain approaches zero [15].

This paper is devoted to differential inequalities in unbounded domains. The inverse inequalities play an important role in obtaining the asymptotic rate of convergence of the approximate solutions [25]. We emphasize that the classical inverse inequalities are regarding norms in finite-dimensional spaces and bounded subdomains of n-dimensional Euclidean spaces [23]. Our goal in this investigation is to obtain an inverse inequality between fractional Sobolev norms in unbounded domains. The paper is organized as follows. Preliminary definitions and denotations are described in Section 2. Estimates between Sobolev norms and the inverse inequality are proved in Section 3. In this section, a new notion of supermonotone function is introduced. Examples of supermonotone functions are considered in Section 4. Section 5 deals with some concluding results.

2. Preliminary definitions and denotations

We denote the set of the real positive numbers by $\mathbf{R}^+$. The harmonic number H_σ for an arbitrary real σ is defined by

$$H_\sigma = \int_0^1 \frac{1-x^\sigma}{1-x} dx.$$

We define a list of special functions as follows:

• the gamma function of a real-valued argument

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt,$$

• the beta function

$$B(x,\ y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

• and the incomplete beta function

$$B(x,\ y,\ z)=\int_0^x t^{y-1}(1-t)^{z-1}dt.$$

Additionally, we use the generalized hypergeometric function

$${}_kF_l(p,\ q,\ x),\quad p\in\mathbf{R}^k,\quad q\in\mathbf{R}^l,\quad x\in\mathbf{R},\quad k,\ l\in\mathbf{N},$$

which is determined by:

$${}_kF_l(p,\ q,\ x)=\sum_{i=0}^{\infty}\frac{\prod_{j=1}^k(p_j)_i}{\prod_{j=1}^l(q_j)_i}\frac{x^i}{i!},$$

where

$$(x)_i=\frac{\Gamma(x+i)}{\Gamma(x)}\ \text{(the Pochhammer symbol).}$$

Let Ω_t be a simply connected domain in $\mathbf{R}^n$ with Lipschitz continuous boundary. The Lebesgue measure of Ω_t depends on the real parameter t that could tend to infinity. The set $\Omega_t^{\perp}$ is determined so that

$$\mathbf{R}^n=\overline{\Omega}_t\cup\Omega_t^{\perp}\text{ and }\overline{\Omega}_t\cap\Omega_t^{\perp}=\emptyset.$$

The fractional Sobolev space $W_p^s(\Omega_t)$ for whatever positive real t is defined by:

$$W_p^s(\Omega_t)=\left\{v\in L^p(\Omega_t)\mid |v|_{s,\ p,\ \Omega_t}<+\infty,\quad s\in(0,\ 1),\quad p\in[1,\ +\infty)\right\},$$

where

$$|v|_{s,\ p,\ \Omega_t}=\left(\int_{\Omega_t}\int_{\Omega_t}\frac{|v(y)-v(x)|^p}{|y-x|^{n+ps}}dxdy\right)^{\frac{1}{p}}. \tag{1}$$

The integral in (1) is well-known as the Gagliardo seminorm [12]. The norm in $W_p^s(\Omega_t)$ is defined by

$$||v||_{s,\ p,\ \Omega_t} = \left(\int_{\Omega_t} |v(x)|^p dx + \int_{\Omega_t} \int_{\Omega_t} \frac{|v(y) - v(x)|^p}{|y - x|^{n+ps}} dxdy \right)^{\frac{1}{p}}.$$

For more details regarding fractional Sobolev norms and embedding theorems, we refer the reader to [12].

We concentrate our considerations on the case $p = 2$. That is why we write $H^s(\Omega_t)$ instead of $W_2^s(\Omega_t)$ and $|v|_{s,\ \Omega_t}$, instead of $|v|_{s,\ 2,\ \Omega_t}$. The norm $||\cdot||_{s,\ \Omega_t}$ and the seminorm $|\cdot|_{s,\ \Omega_t}$ in the Sobolev space $H^s(\Omega_t)$ are related [4] by:

$$||\cdot||^2_{s,\ \Omega_t} = ||\cdot||^2_{0,\ \Omega_t} + |\cdot|^2_{s,\ \Omega_t}.$$

We define the Hilbert space

$$\mathbf{V}_t = \left\{ v \in H^s(\mathbf{R}^n) \mid v = 0 \text{ in } \Omega_t^{\perp} \right\}.$$

3. Estimates between fractional Sobolev norms

In this section we suppose that Ω_t is a simply connected subset of $\mathbf{R}$. So, the norm (1) is reduced to

$$|v|_{s,\ \Omega_t} = \left(\int_{\Omega_t} \int_{\Omega_t} \frac{|v(y) - v(x)|^2}{|y - x|^{1+2s}} dxdy \right)^{\frac{1}{2}}. \tag{2}$$

For the sake of simplicity, we assume that

$$\Omega_t = \{ x \in (1,\ t) \mid t > 1 \}$$

but all results for

$$\Omega_t = \{ x \in (a,\ t) \mid a \in \mathbf{R}^+,\ \ t > a \}$$

can be obtained in the same way. The set $H^s(\Omega_t)$ is a Hilbert space with the scalar product

$$\langle u,\ v \rangle_{s,\ \Omega_t} = \int_{X_t} \frac{(u(x) - u(y))\,(v(x) - v(y))}{|x - y|^{1+2s}} dxdy,$$

where $X_t = \Omega_t \times \Omega_t$. Additionally, we denote by

$$X_t^+ = \{ (x,\ y) \in X_t \mid x > y \}.$$

The application of the finite element method for solving fractional boundary problems requires a weak representation of the original problem. The operator Lu in weak form is called the weak fractional Laplacian [6]. We define the operator Lu related to the space $H^s(\Omega_t)$ by

$$(Lu)(v) = \frac{c(s)}{2}\int_{\mathbf{R}}\int_{\mathbf{R}}\frac{(u(x)-u(y))\,(v(x)-v(y))}{|x-y|^{1+2s}}dxdy$$

$$= \frac{c(s)}{2}\left(\int_{\Omega_t}\int_{\Omega_t}\frac{(u(y)-u(x))\,(v(y)-v(x))}{|y-x|^{1+2s}}dxdy\right.$$

$$\left.+2\int_{\Omega_t}\int_{\Omega_t^{\perp}}\frac{(u(y)-u(x))\,(v(y)-v(x))}{|y-x|^{1+2s}}dxdy,\ \ u,\ v\in\mathbf{V}_t.\right.$$

The constant

$$c(s) = \frac{4^s\Gamma\left(s+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(1-s)}$$

depends only on the index of the fractional Sobolev space. The operator L should be considered in the sense of distribution. The theoretical properties of $(Lv)(v)$, $v\in\mathbf{V}_t$, $t\to+\infty$ are of considerable practical importance. That is why the main goal of this investigation is to present a sufficient condition for the existence of the fractional Sobolev norms in unbounded domains.

The lack of monotonicity for specific classes of functions provokes various researchers to find weak versions of this notion. We emphasize that the property quasi-monotonicity has been independently defined by Shi and Xiao [18], and Tantardini and Verfürth [26] in two different ways.

Definition 1 Let the function $v:\mathbf{R}\to\mathbf{R}$ have a finite support $\overline{\Omega}_t$ and satisfy:

• v is strictly increasing, square summable, and positive in Ω_t;

• v fulfills

$$\frac{v(x)-v(y)}{|x-y|}\leq\beta\frac{v(x)}{|y|},\ \ y<x,\ \ \forall x,\ y\in\overline{\Omega}_t,\ \ \beta\in\mathbf{R}^+. \tag{3}$$

Then we say that v is superincreasing.

Analogously.

Definition 2 Suppose that the function $v:\mathbf{R}\to\mathbf{R}$ has a finite support $\overline{\Omega}_t$ and satisfy:

• v is strictly decreasing, square summable, and positive in Ω_t;

• v fulfills

$$\frac{v(y)-v(x)}{|x-y|}\leq\beta\frac{v(y)}{|y|},\ \ y<x,\ \ \forall x,\ y\in\overline{\Omega}_t,\ \ \beta\in\mathbf{R}^+. \tag{4}$$

Then v is called superdecreasing.

Definition 3 A function v is called supermonotone if it is superincreasing or superdecreasing.

Remark 1 The notion supermonotone requires the function v to be bounded from below, i.e. there is a positive constant v_0 so that

$$0 < v_0 \le v(x),\ \forall x \in \overline{\Omega}_t.$$

The next theorem declares conditions for the supermonotone functions to belong to the fractional Sobolev spaces in unbounded domains. This theorem is a crucial point for proving the inverse inequality between fractional Sobolev norms.

Theorem 1 Let the function $v \in \mathbf{V}_t,\ t \in \mathbf{R}^+$ with a finite support $\overline{\Omega}_t$ be supermonotone, and normed by

$$\hat{v} = \frac{v}{||v||_{0,\ \Omega_t}}.$$

Then $\hat{v} \in \mathbf{V}_{+\infty}$ for all $s \in \left[\frac{1}{2},\ 1\right)$.

Proof. For the sake of simplicity, we prove the theorem when $\beta = 1$. First, we assume that v is superincreasing. We estimate the limit

$$\Lambda = \lim_{t\to+\infty} \int_{\Omega_t}\int_{\Omega_t} \frac{(\hat{v}(x) - \hat{v}(y))^2}{|x-y|^{1+2s}} dxdy$$

from above. Having in mind the definition of $\hat{v}(x)$, we obtain

$$\Lambda = \lim_{t\to+\infty} \frac{1}{||v||^2_{0,\ \Omega_t}} \int_{X_t} \frac{(v(x) - v(y))^2}{|x-y|^{1+2s}} dxdy$$

$$= \lim_{t\to+\infty} \frac{2}{||v||^2_{0,\ \Omega_t}} \int_{X_t^+} \frac{(v(x) - v(y))^2}{|x-y|^{1+2s}} dxdy$$

$$=2 \lim_{t\to+\infty} \frac{1}{||v||^2_{0,\ \Omega_t}} \int_1^t \int_1^x \frac{(v(x) - v(y))^2}{|x-y|^{1+2s}} dxdy.$$

By applying (3) and L'Hospital's rule, we continue with

$$\Lambda \le 2 \lim_{t\to+\infty} \frac{\int_1^t \frac{v^2(t)(t-y)^{1-2s}}{y^2} dy}{v^2(t)} = 2 \lim_{t\to+\infty} \int_1^t \frac{(t-y)^{1-2s}}{y^2} dy. \tag{5}$$

We consider two cases to calculate the right-hand side of (5).

The first one is when $s = \frac{1}{2}$. Then

$$\Lambda \le 2 \lim_{t\to+\infty} \int_1^t \frac{1}{y^2} dy = 2.$$

In the second case, we assume that $s \in (0,\ 1) \backslash \left\{ \frac{1}{2} \right\}$. By direct straight forward computations, we obtain

$$\Lambda \leq 2 \lim_{t \to +\infty} \frac{1}{t^{1+2s}} \left[t(t-1+2s) + (1-2s) \left(tH_{-2s} + {}_3F_2 \left((1,\ 1,\ 1+2s),\ (2,\ 3), \frac{1}{t} \right) s - t \ln t \right) \right]$$

$$= 2 \lim_{t \to +\infty} \frac{1}{t^{1+2s}} \left[t(t-1+2s) + (2s-1) t \ln t \right].$$

The limit Λ is finite if $s \in \left[\frac{1}{2},\ 1 \right)$, i.e. the first part of the theorem is proved.

On the other hand, let v be superdecreasing. Then by applying (4) and L'Hospital's rule, we have

$$\Lambda = 2 \lim_{t \to +\infty} \frac{1}{||v||_{0,\ \Omega_t}^2} \int_1^t \int_1^x \frac{(v(x) - v(y))^2}{|x-y|^{1+2s}} dy dx$$

$$\leq 2 \lim_{t \to +\infty} \frac{\int_1^t \frac{v^2(y)(t-y)^{1-2s}}{y^2} dy}{v^2(t)} \leq 2 \lim_{t \to +\infty} \left(\frac{v(1)}{v(t)} \right)^2 \int_1^t \frac{(t-y)^{1-2s}}{y^2} dy.$$

Since the function v is supermonotone there exists a constant $\mu > 0$ so that $\frac{v(1)}{v(t)} < \mu$. Then

$$\Lambda \leq 2\mu^2 \lim_{t \to +\infty} \int_1^t (t-y)^{1-2s} y^{-2} dy.$$

As in the previous case, this integral converges if $s \in \left[\frac{1}{2},\ 1 \right)$. □

Theorem 2 Let the function $v \in \mathbf{V}_t$ have a finite support $\overline{\Omega}_t$ and it is monotone and positive in $\overline{\Omega}_t$. We suppose also that v satisfies:

$$\alpha \frac{v(y)}{x} \leq \frac{v(x) - v(y)}{x-y},\ (x,\ y) \in X_t^+ \text{ if } v \text{ is increasing in } \overline{\Omega}_t \tag{6}$$

or

$$\alpha \frac{v(x)}{x} \leq \frac{v(y) - v(x)}{x-y},\ (x,\ y) \in X_t^+ \text{ if } v \text{ is decreasing}$$

with $\alpha > 0$. Then we assert that $v \notin \mathbf{V}_{+\infty}$ if

$$v(x) \geq x^{s-\frac{1}{2}},\ \forall x \in \overline{\Omega}_t,\ s \in (0,\ 1). \tag{7}$$

Proof. We present a proof when $\alpha = 1$ and in the case when v is increasing. The proof when v is decreasing can be made in the same way. Let the estimates (6) and (7) be true. Then by applying (6), we obtain

$$\Lambda = \lim_{t\to+\infty}\int_{\Omega_t}\int_{\Omega_t}\frac{(v(x)-v(y))^2}{|x-y|^{1+2s}}dxdy = 2\lim_{t\to+\infty}\int_{X_t^+}\frac{(v(x)-v(y))^2}{|x-y|^{1+2s}}dxdy$$

$$= 2\lim_{t\to+\infty}\int_1^t\int_1^x\frac{(v(x)-v(y))^2}{|x-y|^{1+2s}}dydx \geq 2\lim_{t\to+\infty}\int_1^t\int_1^x\frac{v^2(y)}{x^2(x-y)^{2s-1}}dydx$$

$$\geq 2\lim_{t\to+\infty}\int_1^t\frac{1}{x^2}\int_1^x\frac{y^{2s-1}}{(x-y)^{2s-1}}dydx.$$

We denote

$$I(x) = \int_1^x\frac{y^{2s-1}}{(x-y)^{2s-1}}dy.$$

We change the variables in $I(x)$ by $p = \dfrac{x}{y}$ and obtain

$$I(x) = x\int_1^x\frac{dp}{p^2(p-1)^{2s-1}}.$$

Then

$$\Lambda \geq 2\lim_{t\to+\infty}\int_1^t\frac{I(x)}{x^2}dx = 2\lim_{t\to+\infty}\int_1^t\frac{1}{x}\int_1^x\frac{dp}{p^2(p-1)^{2s-1}}dx$$

$$= 2\lim_{t\to+\infty}\int_1^t\frac{1}{x}\left(\frac{\pi(1-2s)}{\sin 2\pi s} - B\left(\frac{1}{x},\ 2s,\ 2-2s\right)\right)dx \tag{8}$$

$$= 2\lim_{t\to+\infty}\frac{\pi(1-2s)}{\sin 2\pi s}\int_1^t\frac{dx}{x} - 2\lim_{t\to+\infty}\int_1^t\frac{1}{x}B\left(\frac{1}{x},\ 2s,\ 2-2s\right)dx.$$

We explain that the limit

$$\Lambda_1 = \lim_{t\to+\infty}\int_1^t\frac{1}{x}B\left(\frac{1}{x},\ 2s,\ 2-2s\right)dx < +\infty. \tag{9}$$

To this end, we estimate the incomplete beta function $B\left(\dfrac{1}{x},\ 2s,\ 2(1-s)\right)$ from above. We consider three different cases: $s \in \left(0,\ \dfrac{1}{2}\right)$, $s = \dfrac{1}{2}$ and $s \in \left(\dfrac{1}{2},\ 1\right)$.

If $s \in \left(0,\ \frac{1}{2}\right)$, then $1-2s>0$ and

$$B\left(\frac{1}{x},\ 2s,\ 2-2s\right) = \int_0^{\frac{1}{x}} \frac{(1-p)^{1-2s}}{p^{1-2s}} dp < \int_0^{\frac{1}{x}} \frac{dp}{p^{1-2s}} = \frac{1}{2sx^{2s}}.$$

Therefore

$$0 < \Lambda_1 \leq \lim_{t \to +\infty} \frac{1}{2s} \int_1^t \frac{dx}{x^{1+2s}} = \frac{1}{4s^2} < +\infty.$$

Thus, for any fixed $s \in \left(0,\ \frac{1}{2}\right)$, we have $\Lambda = +\infty$.

In the second case $s = \frac{1}{2}$. Then

$$B\left(\frac{1}{x},\ 1,\ 1\right) = \int_0^{\frac{1}{x}} dp = \frac{1}{x}$$

and

$$\Lambda_1 = \lim_{t \to +\infty} \int_1^t \frac{1}{x^2} dx = 1.$$

Again, it fallows that $\Lambda = +\infty$.

In the third case, we have $s \in \left(\frac{1}{2},\ 1\right)$ and $2s-1>0$. We again estimate the incomplete beta function

$$B\left(\frac{1}{x},\ 2s,\ 2(1-s)\right) = \int_0^{\frac{1}{x}} p^{2s-1}(1-p)^{1-2s} dp$$

$$= \int_0^{\frac{1}{x}} \frac{p^{2s-1}}{(1-p)^{2s-1}} dp = \int_0^{\frac{1}{x}} \frac{1}{\left(\frac{1}{p}-1\right)^{2s-1}} dp < \int_0^{\frac{1}{x}} \frac{1}{(x-1)^{2s-1}} dp$$

$$= \frac{1}{(x-1)^{2s-1}} \cdot \frac{1}{x} \quad \left(0 < p < \frac{1}{x} => 1 < x < \frac{1}{p}\right).$$

Then

$$\Lambda_1 = 2\lim_{t\to+\infty}\int_1^t \frac{1}{x}B\left(\frac{1}{x},\ 2s,\ 2(1-s)\right)dx$$

$$\leq 2\lim_{t\to+\infty}\int_1^t \frac{1}{x^2}\cdot\frac{1}{(x-1)^{2s-1}}dx$$

$$= 2\lim_{t\to+\infty}\left(\frac{\pi(1-2s)}{\sin 2\pi s} - B\left(\frac{1}{t},\ 2s,\ 2(1-s)\right)\right) = \frac{\pi(1-2s)}{\sin 2\pi s} < +\infty.$$

So, we conclude that (9) is true for all $s\in(0,\ 1)$. Since the first limit in (8) is infinity but the second one is a finite number, we conclude that $v\notin \mathbf{V}_{+\infty}$. □

We denote the set of superincreasing functions with a finite support $\overline{\Omega}_t$ by Q_t. Definition 1 can be extended in $\mathbf{R}^n$. Let $x,\ y\in\mathbf{R}^n$. We say that $y<x$ if $|y|<|x|$.

Definition 4 The function $v:\mathbf{R}^n\to\mathbf{R}$ is said to be increasing in $\overline{\Omega}_t\subset\mathbf{R}^n$ if $v(y)\leq v(x),\ \forall x,\ y\in\Omega_t$ that $y<x$.

The decreasing function in Ω_t is defined in the same way.

Definition 5 Let the function $v:\mathbf{R}^n\to\mathbf{R}$ be with a finite support $\overline{\Omega}_t\subset\mathbf{R}^n$ and satisfy:

· v is strictly monotone, square summable, and positive in Ω_t,

· v fulfills

$$\begin{cases}\dfrac{v(x)-v(y)}{|x-y|}\leq\beta\dfrac{v(x)}{|y|},\ y<x,\ \forall x,\ y\in\Omega_t,\ \beta\in\mathbf{R}^+\\ \\ \text{if } v \text{ is increasing}\end{cases}$$

or

$$\begin{cases}\dfrac{v(y)-v(x)}{|x-y|}\leq\beta\dfrac{v(y)}{|y|},\ y<x,\ \forall x,\ y\in\Omega_t,\ \beta\in\mathbf{R}^+\\ \\ \text{if } v \text{ is decreasing.}\end{cases}$$

Then we say that v is a supermonotone function.

Obviously, the results in Theorem 1 can be extended in the n-dimensional case but this is beyond our considerations. Definition 3 can be strengthened so that the estimating terms do not depend on the estimated function.

Definition 6 Let the function $v:\mathbf{R}^n\to\mathbf{R}$, $\text{supp}(v)=\overline{\Omega}_t$ be strictly monotone, square summable, and positive in $\overline{\Omega}_t$. If v fulfills:

$$\begin{cases}\dfrac{v(x)-v(y)}{|x-y|}\leq\dfrac{\beta}{|y|},\ y<x,\ \forall x,\ y\in\overline{\Omega}_t,\ \beta\in\mathbf{R}^+\\ \\ \text{when } v \text{ is increasing}\end{cases}$$

or

$$\begin{cases} \dfrac{v(y)-v(x)}{|x-y|} \leq \dfrac{\beta}{|y|},\ y < x,\ \forall x,\ y \in \overline{\Omega}_t,\ \beta \in \mathbf{R}^+ \\ \\ \text{when } v \text{ is decreasing} \end{cases}$$

then we call this function strongly supermonotone.

Theorem 3 If the function $v : \mathbf{R} \to \mathbf{R}$ belongs to Q_t then $\hat{v} \in \mathbf{V}_{+\infty}$, $\forall s \in \left[\frac{1}{2},\ 1\right)$ and the following inverse inequality holds

$$||v||_{s,\ \Omega_\infty} \leq C||v||_{0,\ \Omega_\infty},\ \forall s \in \left[\frac{1}{2},\ 1\right).$$

Proof. This theorem is a direct consequence of Theorem 1.

4. Examples of supermonotone functions

In this section, we consider functions that satisfy the inverse inequality.

Example 1 The function of interest is $v(x) = 1 + \ln x$. We prove that v is strong superincreasing.

Proof. Two-sided estimates can be obtained for this function. By applying the classical two-sided inequality

$$\frac{x}{x+1} \leq \ln x \leq x,\ \forall x > 0,$$

we estimate the quotient

$$Ev = \frac{v(x)-v(y)}{x-y},\ 1 \leq y < x,\ x,\ y \in \overline{\Omega}_t$$

as follows:

$$Ev = \frac{\ln x - \ln y}{x-y} = \frac{\ln \dfrac{x}{y}}{x-y} = \frac{\ln\left(1+\dfrac{x-y}{y}\right)}{x-y} \geq \frac{\dfrac{x-y}{y}}{(x-y)\left(1+\dfrac{x-y}{y}\right)} = \frac{1}{x},$$

$$Ev = \frac{\ln\left(1+\dfrac{x-y}{y}\right)}{x-y} \leq \frac{x-y}{y(x-y)} = \frac{1}{y}.$$

So, we have

$$\frac{1}{x} \leq Ev \leq \frac{1}{y}.$$

Having in mind that v is strictly increasing, square summable, and positive in $\overline{\Omega}_t$, we conclude that v is strongly superincreasing. □

Example 2 We choose the second function to be

$$v(x) = 1 + \arcsin \frac{2x}{1+x^2}.$$

We prove that this function is strongly superdecreasing.

Proof. Obviously, $v(x)$ is differentiable, strictly decreasing, and positive in Ω_t. It remains to prove that $-Ev \leq \frac{\beta}{y}$. By applying Lagrange's theorem, we have

$$v(y) - v(x) = -\frac{2}{1+\xi^2}(y-x),\ y < \xi < x,\ \forall x,\ y \in \overline{\Omega}_t.$$

Therefore

$$0 < -Ev = \frac{2}{1+\xi^2} \leq \frac{1}{\xi} < \frac{1}{y},$$

which indicates that v is strongly superdecreasing. □

5. Conclusion

Inequalities between fractional Sobolev norms in unbounded domains are the object of interest in this paper. A new notion of supermonotone function is introduced. An inverse inequality between fractional Sobolev norms is proved for a class of supermonotone functions. This result can be directly applied to numerical methods for solving fractional elliptic boundary-value problems. All analyses are done in the one-dimensional case generating double integrals in the nonlocal operators but the presented results can be extended in the n-dimensional case. By estimating the quotient $|Ev|$ we demonstrate two examples of supermonotone functions that support the theoretical results.

Author contributions

The arrangement of the authors is alphabetically according to their family names.

Conflict of interest

The authors declare no competing financial interest.

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