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Research Article

Computing Quadratic Eigenvalues and Solvent by a New Minimization Method and a Split-Linearization Technique

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Abstract: To solve quadratic eigenvalue problems (QEPs), especially the gyroscopic systems, two methods are proposed: an iterative direct detection method (DDM) of the complex eigenvalues of the original QEP, and a split-linearization method (SLM) for finding the solvent matrix, which results to a standard linear eigenvalue problem (LEP) being solved to compute all eigenvalues by the symmetry extension. Reducing the dimension to one-half, the LEP is recast in a simpler QEP involving the square of the solvent. We set up two new merit functions which are minimized to detect the complex eigenvalues from the original QEP and a simpler QEP. For each eigen-parameter the merit function consists of the Euclidean norm of each derived eigen-equation, whose vector variable is solved from a derived nonhomogeneous linear system. Then, the golden section search algorithm is employed to minimize the merit functions and locate the complex eigenvalue as a local minimal point. The results are compared with that computed by the cyclic-reduction-based solvent (CRS) method.

***Keywords*:** quadratic eigenvalue problem, gyroscopic system, solvent, nonhomogeneous linear system, split-linearization technique, new minimization methods

MSC: 15A18, 34L15, 34L16, 35P30

1. Introduction

There are two main sources of quadratic eigenvalue problems (QEPs): the vibration system of linear structure and the gyroscopic system. We first consider a mass-damping-spring q-degree system:

$$(\lambda^2\mathbf{M}+\lambda\mathbf{C}+\mathbf{K})\mathbf{u}=\mathbf{0}. \tag{1}$$

Equation (1) is a quadratic eigenvalue problem (QEP) to determine the eigen-pair $(\lambda, \mathbf{u})$.
Let

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$$\mathbf{v} = \lambda \mathbf{u} \tag{2}$$

be the generalized velocity of vibration mode. We can combine equations (2) and (1) together as

$$\begin{bmatrix} \mathbf{0}_q & \mathbf{I}_q \\ -\mathbf{K} & -\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{I}_q & \mathbf{0}_q \\ \mathbf{0}_q & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}. \tag{3}$$

Defining

$$\mathbf{x} := \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}, \ \mathbf{A} := \begin{bmatrix} \mathbf{0}_q & \mathbf{I}_q \\ -\mathbf{K} & -\mathbf{C} \end{bmatrix}, \ \mathbf{B} := \begin{bmatrix} \mathbf{I}_q & \mathbf{0}_q \\ \mathbf{0}_q & \mathbf{M} \end{bmatrix}, \tag{4}$$

equation (3) becomes a generalized eigenvalue problem for the n-vector $\mathbf{x}$:

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{B}\mathbf{x}, \tag{5}$$

where $\mathbf{A}$, $\mathbf{B} \in \mathbb{R}^{n\times n}$ with $n = 2q$. Equation (5) is used to determine the eigen-pair $(\lambda, \mathbf{x})$, which is a linear eigen-equation associated to the pencil $\mathbf{A} - \lambda\mathbf{B}$, where λ is an eigen-parameter. Notice that equation (1) can also be expressed by equation (5) with

$$\mathbf{A} := \begin{bmatrix} -\mathbf{K} & \mathbf{0}_q \\ \mathbf{0}_q & -\mathbf{I}_q \end{bmatrix}, \ \mathbf{B} := \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ -\mathbf{I}_q & \mathbf{0}_q \end{bmatrix}. \tag{6}$$

In the linearization from equation (1) to equation (5), an unsatisfactory aspect is that the dimension of the working space is doubled to $n = 2q$ and the transformation is not unique. Furthermore, if a complex eigenvalue is considered, the dimension is raised to $2n = 4q$. However, for the generalized eigenvalue problems many powerful numerical methods are available [1, 2]. The numerical computations in [3, 4] revealed that the methods based on the Krylov subspace could be very effective in the nonsymmetric eigenvalue problems by using the Lanczos bi-orthogonalization algorithm and Arnoldi's algorithm. The Arnoldi and nonsymmetric Lanczos methods are both of the Krylov subspace methods. Among the many algorithms to solve the matrix eigenvalue problems the Arnoldi method [4–7], the nonsymmetric Lanczos algorithm [8], and the subspace iteration method [9] are well known.

Next, we consider the eigenvalue problem of a q-degree gyroscopic system [10, 11]:

$$(\lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{K})\mathbf{u} = \mathbf{0}_q, \tag{7}$$

where $\mathbf{M}$ is a symmetric and positive definite matrix, $\mathbf{K}$ a symmetric and negative definite matrix, and $\mathbf{G} = -\mathbf{G}^{\mathrm{T}}$ a skew-symmetric matrix. Our purpose is to determine the eigen-pair $(\lambda, \mathbf{u})$ of equation (7), which belongs to the quadratic eigenvalue problems [12]. In [13] a novel exciting and maximizing method was developed to solve the quadratic eigenvalue problems for the structural free vibration frequencies.

The Bezout method [14] introduces a solvent matrix $\mathbf{S}$ to satisfy

$$\mathbf{M}\mathbf{S}^2+\mathbf{G}\mathbf{S}+\mathbf{K}=\mathbf{0}_q, \tag{8}$$

such that equation (7) is factorized as

$$(\lambda\mathbf{M}+\mathbf{M}\mathbf{S}+\mathbf{G})(\lambda\mathbf{I}_q-\mathbf{S})\mathbf{u}=\mathbf{0}_q. \tag{9}$$

Solving two q-dimensional eigenvalue problems:

$$(\mathbf{M}\mathbf{S}+\mathbf{G})\mathbf{u}=-\lambda\mathbf{M}\mathbf{u},\ \ \mathbf{S}\mathbf{u}=\lambda\mathbf{u}, \tag{10}$$

we can recover all eigenvalues of equation (7). Some techniques for the quadratic eigenvalue problems involving an accurate solution of the solvent $\mathbf{S}$ can refer to [10, 11, 15–24].

Let

$$\mathbf{v}=\lambda\mathbf{M}\mathbf{u}. \tag{11}$$

We can combine equations (7) and (11) with $\mathbf{M}^{-1}\mathbf{v}=\lambda\mathbf{u}$ together as

$$\begin{bmatrix}\mathbf{0}_q & -\mathbf{K}\\ \mathbf{M}^{-1} & \mathbf{0}_q\end{bmatrix}\begin{bmatrix}\mathbf{v}\\ \mathbf{u}\end{bmatrix}=\lambda\begin{bmatrix}\mathbf{I}_q & \mathbf{G}\\ \mathbf{0}_q & \mathbf{I}_q\end{bmatrix}\begin{bmatrix}\mathbf{v}\\ \mathbf{u}\end{bmatrix}. \tag{12}$$

Defining

$$\mathbf{x}:=\begin{bmatrix}\mathbf{v}\\ \mathbf{u}\end{bmatrix},\ \ \mathbf{A}:=\begin{bmatrix}\mathbf{0}_q & -\mathbf{K}\\ \mathbf{M}^{-1} & \mathbf{0}_q\end{bmatrix},\ \ \mathbf{B}:=\begin{bmatrix}\mathbf{I}_q & \mathbf{G}\\ \mathbf{0}_q & \mathbf{I}_q\end{bmatrix}, \tag{13}$$

equation (12) becomes a generalized linear eigenvalue problem (LEP) in equation (5).

Especially, because of

$$(\mathbf{A}\mathbf{J})^{\mathrm{T}}=\mathbf{A}\mathbf{J},\ \ (\mathbf{B}\mathbf{J})^{\mathrm{T}}=-\mathbf{B}\mathbf{J}, \tag{14}$$

where

$$\mathbf{J}:=\begin{bmatrix}\mathbf{0}_q & \mathbf{I}_q\\ -\mathbf{I}_q & \mathbf{0}_q\end{bmatrix}, \tag{15}$$

$\mathbf{A}-\lambda\mathbf{B}$ is a skew-Hamiltonian/Hamiltonian pencil, whose eigenvalue happens in the form of quadruplets: $(\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda})$ [25, 26]. Owing to this special symmetry property, we merely need to solve the standard eigenvalue problem of the second one in equation (10); however, the premise is that we have to find $\mathbf{S}$ very accurately.

Recently, many applications and solvers of quadratic eigenvalue problems have been proposed, e.g., the stability analysis of time-delay systems [27], the free vibrations of fluid-solids structures [28], a modified second-order Arnoldi method [29], the inexact residual iteration method [30], the homotopy perturbation technique [31], the electromagnetic wave propagation and the analysis of an acoustic fluid contained in a cavity with absorbing walls [32], a friction-induced vibration problem under variability [33]. In addition, several applications and solvers of generalized eigenvalue problems have been addressed, e.g., the block Arnoldi-type contour integral spectral projection method [34], the small-sample statistical condition estimation [35], the matrix perturbation methods [36], the overlapping finite element method [37], the complex HZ method [38], the context of sensor selection [39], a generalized Arnoldi method [40], a frequency isolation algorithms [41], the neural networks based on the power method and inverse power method [42], and a novel fifth-order iterative method [43]. Besides, comparing with the above-mentioned references, our proposed minimization method and a split-linearization technique is easier to implement and more accurate.

The contents of the present paper are briefly sketched. In Section 2, we introduce a new detection method of the eigenvalue by a minimization method for the quadratic eigenvalue problem and the numerical process is given, where the skew-Hamiltonian/Hamiltonian form for the gyroscopic system is further enlarged to the $2n$-dimensional LEP for complex eigenvalue problem. In Section 3, an iterative method is developed to directly solve the quadratic eigenvalue problem by a minimization method. In Section 4, the gyroscopic system and a split-linearization technique for the solvent matrix are discussed, and we apply a similar method in Section 2 to solve the LEP $\mathbf{Su}=\lambda\mathbf{u}$ with dimension $n=2q$. Then we reduce the linear eigenvalue problem of the solvent to a simpler quadratic eigenvalue problem with dimension q, and a minimization method is proposed to formulate a solvent QEP method. Five examples are demonstrated in Section 5. Finally, we conclude the main results in Section 6.

2. A new detection method

For equation (5) the eigenvalue may be a complex number:

$$\lambda=\lambda_R+i\lambda_I. \tag{16}$$

Correspondingly, we take

$$\mathbf{x}=\mathbf{z}+i\mathbf{w}. \tag{17}$$

Inserting equations (16) and (17) into equation (5), yields

$$\begin{bmatrix} \mathbf{A}-\lambda_R\mathbf{B} & \lambda_I\mathbf{B} \\ -\lambda_I\mathbf{B} & \mathbf{A}-\lambda_R\mathbf{B} \end{bmatrix}\begin{bmatrix} \mathbf{z} \\ \mathbf{w} \end{bmatrix}=\mathbf{0}. \tag{18}$$

Letting

$$\mathbf{x} := \begin{bmatrix} \mathbf{z} \\ \mathbf{w} \end{bmatrix}, \ \mathbf{D}(\lambda_R, \ \lambda_I) := \begin{bmatrix} \mathbf{A} - \lambda_R \mathbf{B} & \lambda_I \mathbf{B} \\ -\lambda_I \mathbf{B} & \mathbf{A} - \lambda_R \mathbf{B} \end{bmatrix}, \tag{19}$$

equation (18) becomes

$$\mathbf{D}\mathbf{x} = \mathbf{0}, \tag{20}$$

where $\mathbf{x}$ is a $2n$-dimensional vector and $\mathbf{D}$ is a $2n \times 2n$ matrix. For saving notation we still use $\mathbf{x}$ in equation (19).

First we specify the theoretical foundation of the minimization method to be developed. For equation (20), $\mathbf{x}$ is not unique. For $\mathbf{x}$ to be a nontrivial solution of equation (20), at least one component of $\mathbf{x}$ is not zero; we can normalize $\mathbf{x}$ such that the value of the nonzero component, say the j_0-th component, is a unit number. On the other hand, taking the norm of equation (20) yields

$$\|\mathbf{D}\mathbf{x}\| = 0. \tag{21}$$

To determine the complex eigenvalue, we consider

$$\min_{(\lambda_R, \ \lambda_I) \in [a, \ b] \times [c, \ d]} f(\lambda_R, \ \lambda_I) := \|\mathbf{D}(\lambda_R, \ \lambda_I)\mathbf{x}\| \geq 0. \tag{22}$$

Obviously, $\mathbf{x} = \mathbf{0}$ is a trivial solution of equation (22) because of $f = 0$. However, when we demand that $\mathbf{x}$ is a nontrivial solution of equation (20), of which at least one component of $\mathbf{x}$ is not zero, $\mathbf{x} = \mathbf{0}$ can be excluded. Below we develop a new technique to find a nontrivial solution of $\mathbf{x}$, and use equation (22) to determine the complex eigenvalue.

In order to create a nontrivial and unique solution of equation (20), of which at least one component of $\mathbf{x}$ is not zero, we can normalize a certain nonzero j_0-th component of $\mathbf{x}$ by $x_{j_0} = 1$. Then, from equation (20) an $n_0 = (2n - 1)$-dimensional nonhomogeneous linear system can be obtained by moving the j_0-th column of the eigen-equation to the right-hand side using the following processes.

Let d_{ij} be the components of $\mathbf{D}(\lambda_R, \ \lambda_I)$ for each specified $(\lambda_R, \ \lambda_I)$, and then the components of equation (20) are

$$\begin{aligned} & d_{11}x_1 + \ldots + d_{1, \ j_0}x_{j_0} + \ldots + d_{1, \ 2n}x_{2n} = 0, \\ & \vdots \\ & d_{2n-1, \ 1}x_1 + \ldots + d_{2n-1, \ j_0}x_{j_0} + \ldots + d_{2n-1, \ 2n}x_{2n} = 0, \\ & d_{2n, \ 1}x_1 + \ldots + d_{2n, \ j_0}x_{j_0} + \ldots + d_{2n, \ 2n}x_{2n} = 0. \end{aligned} \tag{23}$$

Move the j_0-th column to the right-hand side by

$$d_{11}x_1 + \ldots + d_{1,\ j_0-1}x_{j_0-1} + d_{1,\ j_0+1}x_{j_0+1} + \ldots + d_{1,\ 2n}x_{2n} = -d_{1,\ j_0}x_{j_0},$$

$$\vdots$$

$$d_{2n-1,\ 1}x_1 + \ldots + d_{2n-1,\ j_0-1}x_{j_0-1} + d_{2n-1,\ j_0+1}x_{j_0+1} + \ldots + d_{2n-1,\ 2n}x_{2n} = -d_{2n-1,\ j_0}x_{j_0},$$

$$d_{2n,\ 1}x_1 + \ldots + d_{2n,\ j_0-1}x_{j_0-1} + d_{2n,\ j_0+1}x_{j_0+1} + \ldots + d_{2n,\ 2n}x_{2n} = -d_{2n,\ j_0}x_{j_0}. \tag{24}$$

By defining

$$e_i = -d_{ij_0},\ i = 1,\ \ldots,\ n_0, \tag{25}$$

where d_{ij_0} is the j_0-th column of the matrix **D**, and taking $x_{j_0} = 1$ in equation (24), we can obtain the first $n_0 = 2n-1$ equations as follows:

$$c_{ij}y_j = e_i,\ i,\ j = 1, \ldots,\ n_0, \tag{26}$$

where the coefficient matrix $[c_{ij}]$ is constructed from $[d_{ij}]$ by

Do $i = 1 :\ n_0$,

$k = 0$,

Do $j = 1 :\ 2n$,

If $j = j_0$ next j,

$k = k+1$,

$c_{ik} = d_{ij}$,

End do of j,

End do of i. (27)

The nonhomogeneous linear system in equation (26) was obtained by reducing the number of eigen-equation one less with dimension $2n-1$, and with its certain nonzero component being normalized to the unit and moving the column containing this component to the right-hand side. We can apply the Gaussian elimination method to solve $(y_1,\ \ldots,\ y_{n_0})$ in equation (26), and then $\mathbf{x}=(x_1,\ \ldots,\ x_{2n})^{\mathrm{T}}$ is computed from $\mathbf{y}=(y_1,\ \ldots,\ y_{n_0})^{\mathrm{T}}$ by

$$
\begin{aligned}
&k=0,\\
&\text{Do } j=1:\ 2n,\\
&\text{If } j=j_0\ \ x_j=1,\ \ \text{next } j,\\
&k=k+1,\\
&x_j=y_k,\\
&\text{End do of } j.
\end{aligned}
\tag{28}
$$

Therefore, the numerical procedures based on equation (22) for determining the complex eigenvalue are depicted as follows to be a new detection method (NDM) in Table 1.

Table 1. Algorithm: new detection method (NDM)

Algorithm: NDM
1: Give $a,\ b,\ c,\ d,\ j_0$, and ε
2: For each $(\lambda_R,\ \lambda_I)\in[a,\ b]\times[c,\ d]$, solve equations (25)-(28) to obtain $\mathbf{x}$
3: Apply golden section search algorithm (GSSA) to equation (22)

It can be seen that the Algorithm NDM is very simple. The convergence criterion of golden section search algorithm (GSSA) is fixed to be 10^{-15}. About the two-dimensional golden section search algorithm, one may refer to [44] and also the Appendix. When λ_R and λ_I take values inside a rectangle by $(\lambda_R,\ \lambda_I)\in[a,\ b]\times[c,\ d]$, we can plot $\|\mathbf{Dx}(\lambda_R,\ \lambda_I)\|$ vs. $(\lambda_R,\ \lambda_I)$ over the eigen-parametric plane, whose minimal point locates the complex eigenvalue.

The role of GSSA is helped us to locate the minimal point withing the given range $[a,\ b]\times[c,\ d]$.

3. An iterative method for the direct detection of complex eigenvalue

We directly solve equation (7), rather than equation (5) by the method NDM in Section 2. Inserting

$$\mathbf{u}=\mathbf{a}+i\mathbf{b},\tag{29}$$

and equation (16) into equation (7), yields

$$\begin{bmatrix} (\lambda_R^2-\lambda_I^2)\mathbf{M}+\lambda_R\mathbf{G}+\mathbf{K} & -2\lambda_R\lambda_I\mathbf{M}-\lambda_I\mathbf{G} \\ 2\lambda_R\lambda_I\mathbf{M}+\lambda_I\mathbf{G} & (\lambda_R^2-\lambda_I^2)\mathbf{M}+\lambda_R\mathbf{G}+\mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \mathbf{0}. \tag{30}$$

Letting

$$\mathbf{u} := \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \quad \mathbf{E} := \begin{bmatrix} (\lambda_R^2-\lambda_I^2)\mathbf{M}+\lambda_R\mathbf{G}+\mathbf{K} & -2\lambda_R\lambda_I\mathbf{M}-\lambda_I\mathbf{G} \\ 2\lambda_R\lambda_I\mathbf{M}+\lambda_I\mathbf{G} & (\lambda_R^2-\lambda_I^2)\mathbf{M}+\lambda_R\mathbf{G}+\mathbf{K} \end{bmatrix}, \tag{31}$$

equation (30) becomes

$$\mathbf{E}\mathbf{u} = \mathbf{0}, \tag{32}$$

which is an $n = 2q$ dimensional homogeneous linear system. For saving notation we still use $\mathbf{u}$ in equation (31).

When $\mathbf{u}$ is an n-dimensional vector, $\mathbf{E}$ is an $n \times n$ matrix. Compared to the linearization in equation (20), the direct formulation with equation (32) has one advantage with the dimension being reduced to one-half of equation (20). The procedures to find the nontrivial solution of Y are similar to that in equations (25)-(28), but with $\mathbf{x}$ replaced by $\mathbf{u}$, and $\mathbf{D}$ replaced by $\mathbf{E}$.

Like that in equation (22), we can employ the following merit function:

$$\min_{(\lambda_R,\ \lambda_I)\in[a,\ b]\times[c,\ d]} f(\lambda_R,\ \lambda_I) := \|\mathbf{E}(\lambda_R,\ \lambda_I)\mathbf{u}\| \tag{33}$$

to determine the complex eigenvalue, where $\mathbf{E}$ is given by equation (31). This detection method in a half-dimension n of that in Section 2 is named a direct detection method (DDM). Therefore, the numerical procedures based on equation (33) for determining the complex eigenvalue are depicted as follows to be a new direct detection method (DDM) in Table 2. It is remarkable that the complex eigenvector can be computed simultaneously.

Table 2. Algorithm: direct detection method (DDM)

Algorithm: DDM
1: Give a, b, c, d, j_0, and ε
2: For each $(\lambda_R,\ \lambda_I) \in [a,\ b] \times [c,\ d]$, solve equation (32) via equations (25)-(28) to obtain $\mathbf{u}$
3: Apply golden section search algorithm (GSSA) to equation (33)

4. An iterative method for the direct detection of complex eigenvalue

It is known that the eigenvalues of equation (7) occur in quadruplets for the gyroscopic system: $(\lambda,\ \bar{\lambda},\ -\lambda,\ -\bar{\lambda})$. We merely require to solve the eigenvalue λ of the solvent $\mathbf{S}$ in equation (8), and other eigenvalues are obtained by the symmetric extensions. In [24], inserting

$$\mathbf{S} = (\mathbf{I}_q + \mathbf{Y})(\mathbf{I}_q - \mathbf{Y})^{-1} \tag{34}$$

into equation (8), results to

$$\mathbf{P}^{\mathrm{T}}\mathbf{Y}^2 + \mathbf{Q}\mathbf{Y} + \mathbf{P} = \mathbf{0}_q, \tag{35}$$

where

$$\mathbf{P} = \mathbf{M} + \mathbf{G} + \mathbf{K}, \quad \mathbf{Q} = 2(\mathbf{M} - \mathbf{K}). \tag{36}$$

Further use

$$\mathbf{Y} = -\mathbf{Z}^{-1}\mathbf{P} \tag{37}$$

in equation (35), a nonlinear matrix equation is available:

$$\mathbf{Z} + \mathbf{P}^{\mathrm{T}}\mathbf{Z}^{-1}\mathbf{P} = \mathbf{Q}. \tag{38}$$

For equation (38) the method of cyclic-reduction-based solvent (CRS) is very efficient [11, 24].

4.1 *A split-linearization method (SLM)*

Rather than the CRS method, we develop a split-linearization method to solve equation (35) directly to find $\mathbf{Y}$, such that the solvent $\mathbf{S}$ can be computed from equation (34).

Inserting a weight factor γ in equation (35), we have

$$(1-\gamma)\mathbf{P}^{\mathrm{T}}\mathbf{Y}^2 + \mathbf{Q}\mathbf{Y} = -\mathbf{P} - \gamma\mathbf{P}^{\mathrm{T}}\mathbf{Y}^2. \tag{39}$$

We give an initial guess $\mathbf{Y}_0$ to initialize the iteration, in general $\mathbf{Y}_0 = \mathbf{0}$. At the kth step of the iteration, $\mathbf{Y}_k$ is known and equation (39) is linearized around $\mathbf{Y}_k$ to

$$[(1-\gamma)\mathbf{P}^{\mathrm{T}}\mathbf{Y}_k + \mathbf{Q}]\mathbf{Y} = -\mathbf{P} - \gamma\mathbf{P}^{\mathrm{T}}\mathbf{Y}_k^2. \tag{40}$$

For $\mathbf{Y}$ it is a linear matrix equation system, since the coefficient $(1-\gamma)\mathbf{P}^{\mathrm{T}}\mathbf{Y}_k + \mathbf{Q}$ is a constant matrix, and meanwhile, the right-hand side is a constant matrix. We can solve next $\mathbf{Y}_{k+1}$ by

$$\mathbf{Y}_{k+1} = -[(1-\gamma)\mathbf{P}^{\mathrm{T}}\mathbf{Y}_k + \mathbf{Q}]^{-1}[\mathbf{P} + \gamma\mathbf{P}^{\mathrm{T}}\mathbf{Y}_k^2]. \tag{41}$$

The above process is a novel split-linearization method (SLM) for obtaining $\mathbf{Y}$ until convergence and thus to obtain the solvent $\mathbf{S}$ by equation (34).

Compared to CRS, which needs to solve a nonlinear matrix equation (38), SLM is simpler to solve a linear matrix equation (40). The numerical procedure of SLM is simpler than that of CRS.

To explore the stability and convergence, let us focus on the linear matrix equation (40), which is expressed as

$$\mathbf{AY} = \mathbf{B}. \tag{42}$$

In terms of residual matrix and descent matrix:

$$\mathbf{R}_k = \mathbf{B} - \mathbf{AY}_k, \tag{43}$$

$$\mathbf{AU}_k = \mathbf{R}_k, \tag{44}$$

we have

$$\mathbf{R}_{k+1} = \mathbf{R}_k - \mathbf{AU}_k. \tag{45}$$

The iteration in equation (41) is equivalent to take the kth step descent matrix by

$$\mathbf{U}_k = \mathbf{A}^{-1}\mathbf{R}_k, \tag{46}$$

such that the following identity holds:

$$\frac{\mathbf{R}_k \cdot (\mathbf{AU}_k)}{\|\mathbf{AU}_k\|^2} = 1. \tag{47}$$

Taking the squared norm of equation (45) yields

$$\|\mathbf{R}_{k+1}\|^2 = \|\mathbf{R}_k\|^2 - 2\mathbf{R}_k \cdot (\mathbf{AU}_k) + \|\mathbf{AU}_k\|^2, \tag{48}$$

which by equation (47) changes to

$$\|\mathbf{R}_{k+1}\|^2 = \|\mathbf{R}_k\|^2 - \|\mathbf{AU}_k\|^2 < \|\mathbf{R}_k\|^2, \tag{49}$$

owing to $\|\mathbf{AU}_k\|^2 > 0$. The inequality $\|\mathbf{R}_{k+1}\| < \|\mathbf{R}_k\|$ obtained from the above demonstrates that the residuals are decreased step-by-step, such that the split-linearization method (SLM) for obtaining $\mathbf{Y}$ is absolutely convergent and the numerical scheme is unconditionally stable.

When $\mathbf{S}$ is available, we can apply the method in Section 2 with $\mathbf{A}$ replaced by $\mathbf{S}$ and $\mathbf{B}$ by $\mathbf{I}_q$ to find the eigenvalues:

$$\begin{bmatrix} \mathbf{S}-\lambda_R\mathbf{I}_q & \lambda_I\mathbf{I}_q \\ -\lambda_I\mathbf{I}_q & \mathbf{S}-\lambda_R\mathbf{I}_q \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{w} \end{bmatrix} = \mathbf{0}, \tag{50}$$

which is an $n = 2q$ dimensional LEP.

We can apply the techniques in equations (25)-(28) to solve equation (50). Solving this LEP with dimension $n = 2q$, it is more time saving and can obtain more accurate eigenvalues when $\mathbf{S}$ is accurately computed.

In summary the pseudo-code of the split-linearization method (SLM) to compute the complex eigenvalue can be depicted by the following algorithm in Table 3.

Table 3. Algorithm: split-linearization method (SLM)

Algorithm: SLM
1: Give $\mathbf{M}$, $\mathbf{G}$, $\mathbf{K}$, $\mathbf{Y}_0$, a, b, c, d, γ, and ε
2: Compute $\mathbf{P} = \mathbf{M}+\mathbf{G}+\mathbf{K}$, $\mathbf{Q} = 2(\mathbf{M}-\mathbf{K})$
3: Do $k = 0,\ 1,\ \ldots$
4: $\mathbf{Y}_{k+1} = -[(1-\gamma)\mathbf{P}^{\mathrm{T}}\mathbf{Y}_k+\mathbf{Q}]^{-1}[\mathbf{P}+\gamma\mathbf{P}^{\mathrm{T}}\mathbf{Y}_k^2]$
5: If $\|\mathbf{Y}_{k+1}-\mathbf{Y}_k\| < \varepsilon$, go to 6; otherwise, go to 3
6: Compute $\mathbf{S} = (\mathbf{I}_q+\mathbf{Y}_{k+1})(\mathbf{I}_q-\mathbf{Y}_{k+1})^{-1}$
7: Apply equations (25)-(28) to solve equation (50)
8: Apply golden section search algorithm (GSSA) to $\min_{(\lambda_R,\ \lambda_I)\in[a,\ b]\times[c,\ d]} \|\mathbf{Dx}\|$

No matter which method is employed, we can measure the error by accounting for the value of the residual $\|\mathbf{Dx}\|$ in Sections 2 and 4.1, and $\|\mathbf{Eu}\|$ in Section 3. The above split-linearization technique was first employed by Liu et al. [45] to solve the system of nonlinear algebraic equations.

4.2 *A simpler quadratic eigenvalue problem of solvent*

For the q-dimensional standard eigenvalue problem of the solvent:

$$\mathbf{Su} = \lambda\mathbf{u}, \tag{51}$$

of which after inserting equations (16) and (29), we can obtain

$$\mathbf{S}(\mathbf{a}+i\mathbf{b}) = (\lambda_R+i\lambda_I)(\mathbf{a}+i\mathbf{b}). \tag{52}$$

Equating the real and imaginary parts generates

$$\mathbf{Sa} = \lambda_R\mathbf{a}-\lambda_I\mathbf{b}, \tag{53}$$

$$\mathbf{Sb} = \lambda_R\mathbf{b}+\lambda_I\mathbf{a} \tag{54}$$

To delete $\mathbf{b}$, applying $\mathbf{S}$ to equation (53) and using equation (54) yields

$$\mathbf{S}^2\mathbf{a} = \lambda_R\mathbf{S}\mathbf{a} - \lambda_R\lambda_I\mathbf{b} - \lambda_I^2\mathbf{a}. \tag{55}$$

Inserting $\lambda_I\mathbf{b} = \lambda_R\mathbf{a} - \mathbf{S}\mathbf{a}$ obtained from equation (53), we have

$$\mathbf{S}^2\mathbf{a} = \lambda_R\mathbf{S}\mathbf{a} - \lambda_R^2\mathbf{a} + \lambda_R\mathbf{S}\mathbf{a} - \lambda_I^2\mathbf{a}, \tag{56}$$

which can be arranged to

$$[\mathbf{S}^2 - 2\lambda_R\mathbf{S} + (\lambda_R^2 + \lambda_I^2)\mathbf{I}_q]\mathbf{a} = \mathbf{0}_q. \tag{57}$$

The advantage of this simpler quadratic eigenvalue problem (QEP) is that its dimension is q.

We can apply the techniques in equations (25)-(28) to solve equation (57) and search $(\lambda_R,\ \lambda_I)$ by minimizing

$$\min_{(\lambda_R,\ \lambda_I)\in[a,\ b]\times[c,\ d]} f(\lambda_R,\ \lambda_I) := \|[\mathbf{S}^2 - 2\lambda_R\mathbf{S} + (\lambda_R^2 + \lambda_I^2)\mathbf{I}_q]\mathbf{a}\| \geq 0. \tag{58}$$

Solving this simpler QEP with dimension q, is more time-saving and can obtain more accurate eigenvalues when $\mathbf{S}$ is accurately computed.

In summary the pseudo-code of the split-linearization method together with a simpler QEP in equation (57), which is named the solvent QEP (SQEP), can be depicted by the following algorithm in Table 4.

Table 4. Algorithm: solvent quadratic eigenvalue problem (SQEP)

Algorithm: SQEP
1: Give $\mathbf{M}$, $\mathbf{G}$, $\mathbf{K}$, $\mathbf{Y}_0$, a, b, c, d, γ, and ε
2: Compute $\mathbf{P} = \mathbf{M} + \mathbf{G} + \mathbf{K}$, $\mathbf{Q} = 2(\mathbf{M} - \mathbf{K})$
3: Do $k = 0,\ 1,\ \ldots$
4: $\mathbf{Y}_{k+1} = -[(1-\gamma)\mathbf{P}^{\mathrm{T}}\mathbf{Y}_k + \mathbf{Q}]^{-1}[\mathbf{P} + \gamma\mathbf{P}^{\mathrm{T}}\mathbf{Y}_k^2]$
5: If $\|\mathbf{Y}_{k+1} - \mathbf{Y}_k\| < \varepsilon$, go to 6; otherwise, go to 3
6: Compute $\mathbf{S} = (\mathbf{I}_q + \mathbf{Y}_{k+1})(\mathbf{I}_q - \mathbf{Y}_{k+1})^{-1}$
7: Apply equations (25)-(28) to solve $[\mathbf{S}^2 - 2\lambda_R\mathbf{S} + (\lambda_R^2 + \lambda_I^2)\mathbf{I}_q]\mathbf{a} = \mathbf{0}_q$
8: Apply golden section search algorithm (GSSA) to $\min_{(\lambda_R,\ \lambda_I)\in[a,\ b]\times[c,\ d]} f(\lambda_R,\ \lambda_I) = \|[\mathbf{S}^2 - 2\lambda_R\mathbf{S} + (\lambda_R^2 + \lambda_I^2)\mathbf{I}_q]\mathbf{a}\|$

Notice that at each step in GSSA, we need to apply equations (25)-(28) to solve $[\mathbf{S}^2 - 2\lambda_R\mathbf{S} + (\lambda_R^2 + \lambda_I^2)\mathbf{I}_q]\mathbf{a} = \mathbf{0}_q$ for computing $\mathbf{a}$, which is an q-dimensional eigenvector.

5. Examples of quadratic eigenvalue problems

To demonstrate the efficiency and accuracy of the proposed iterative algorithms for computing the eigenvalues of QEPs, several examples will be examined. All the numerical computations are carried out by Fortran 77 in Microsoft

Developer Studio with Intel Core I7-3770, CPU 2.80GHz and 8GB memory. The precision is 10^{-16}. All the plots are produced by the Grapher system.

In the computation by an iteration method giving a suitable convergence criterion is required. In general we take a strict convergence criterion with $\varepsilon = 10^{-15}$ for obtaining highly accurate eigenvalue. For each example the value of ε is specified.

5.1 *Example 1*

We consider a free vibration problem of a q-degree linear structure [46]:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}, \tag{59}$$

where $\mathbf{q}(t)$ is a time-dependent q-dimensional vector to signify the generalized displacements of the system; the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ are symmetric and positive definite. In terms of the vibration mode $\mathbf{u}$, we can express the fundamental solution of equation (59) as

$$\mathbf{q}(t) = e^{\lambda t}\mathbf{u}, \tag{60}$$

which leads to a QEP for $(\lambda,\ \mathbf{u})$ in equation (1).

For simplicity we take $q = 3$ with the following mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{kip}\cdot\text{sec}^2/\text{in},\ \ \mathbf{K} = 600\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{bmatrix} \text{kip/in}, \tag{61}$$

and with a proportional damping in the system given by [46]

$$\mathbf{C} = a_0\mathbf{M} + a_1\mathbf{K}. \tag{62}$$

The real and imaginary parts of the eigenvalue are derived as follows:

$$\lambda_R = -\zeta\omega,\ \ \lambda_I = \sqrt{1-\zeta^2}\omega, \tag{63}$$

where the damping ratio is given by

$$\zeta = \frac{a_0}{2\omega} + \frac{a_1\omega}{2}. \tag{64}$$

When we take $a_0 = 0.01$ and $a_1 = 0.001$ and apply the method DDM in Section 3 to this eigenvalue problem, we obtain

$$\lambda_1 = -0.1104394223218184 + 14.52124787422935i,$$

$$\lambda_2 = -0.4869803136597109 + 31.04387711289919i,$$

$$\lambda_3 = -1.061803398874989 + 46.08711293657499i. \tag{65}$$

They are very close to that computed from the theoretical value in equation (63) with the error of $\|\mathbf{Eu}\| = 1.604999085892749 \times 10^{-7}$.

5.2 *Example 2*

We consider a modeling oscillation in an airplane wing with [19, 47]:

$$\mathbf{M} = \begin{bmatrix} 17.6 & 1.28 & 2.89 \\ 1.28 & 0.824 & 0.413 \\ 2.89 & 0.413 & 0.725 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 7.66 & 2.45 & 2.1 \\ 0.23 & 1.04 & 0.223 \\ 0.6 & 0.756 & 0.658 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 121 & 18.9 & 15.9 \\ 0 & 2.7 & 0.145 \\ 11.9 & 3.64 & 15.5 \end{bmatrix}. \tag{66}$$

By applying the method DDM in Section 3 to this eigenvalue problem, we obtain

$$\lambda_1 = -0.8848302276193057 \pm 8.441512059499651i,$$

$$\lambda_2 = 0.09472173815159608 \pm 2.52287655639731i,$$

$$\lambda_3 = -0.9179981428161036 \pm 1.760584228706396i. \tag{67}$$

They are very close to the exact value with the errors of $\|\mathbf{Eu}\| = 1.14795128039987 \times 10^{-13}$, $\|\mathbf{Eu}\| = 1.95051619919 4264 \times 10^{-13}$, and $\|\mathbf{Eu}\| = 1.834711592181303 \times 10^{-13}$. Very small values of $\|\mathbf{Eu}\|$ reflect that the DDM method is very accurate. In the literature a simpler method like the DDM in Section 3 is still deserved, because it can achieve highly accurate eigenvalues with low computational cost.

5.3 *Example 3*

We consider an MGK system with [24]:

$$\mathbf{M}_0 = \begin{bmatrix} 10^{-7} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{G}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & g \\ 0 & 0 & -g & 0 \end{bmatrix},$$

$$\mathbf{K}_0 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -10^{-7} & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} -0.43 & -1.15 & 0.33 & -0.59 \\ -1.67 & 1.19 & 0.17 & 2.18 \\ 0.13 & 1.19 & -0.19 & -0.14 \\ 0.29 & -0.04 & 0.73 & 0.11 \end{bmatrix},$$

$$\mathbf{M} = \mathbf{W}^{\mathrm{T}}\mathbf{M}_0\mathbf{W}, \quad \mathbf{G} = \mathbf{W}^{\mathrm{T}}\mathbf{G}_0\mathbf{W}, \quad \mathbf{K} = \mathbf{W}^{\mathrm{T}}\mathbf{K}_0\mathbf{W}, \tag{68}$$

where we take $g = 2.999999$.

By applying the method DDM in Section 3 to this eigenvalue problem, we obtain the eigenvalues:

$$\pm 0.7071067989967894 \pm 0.7071067621580626i,$$

$$\pm 0.001224744768419265 \pm 1.414213032041997i. \tag{69}$$

They are very accurate with the errors $\|\mathbf{Eu}\| = 1.605990467444744 \times 10^{-15}$ and $\|\mathbf{Eu}\| = 1.273284490727198 \times 10^{-14}$. The high accuracy of DDM is confirmed with very small values of $\|\mathbf{Eu}\|$.

When we apply the CRS to find the solvent $\mathbf{S}$, it is convergent with 17 iterations to satisfy the convergence criterion $\varepsilon = 10^{-15}$; however, the residual error to satisfy equation (8) is $1.245683658005252 \times 10^{-8}$. Obtained eigenvalues are

$$\pm 0.7097778691381612 \pm 0.7112949472245013i,$$

$$\pm 0.001224711667456435 \pm 1.414213032097788i. \tag{70}$$

The errors of equation (7) are $1.88126907801905 \times 10^{-9}$ and $5.091216337849501 \times 10^{-10}$. The CRS is less accurate than the iterative method DDM in Section 3. We must emphasize that the simpler method DDM can improve the method existed in the literature with the accuracy of the eigenvalue being raised about five orders.

When we apply the SLM in Section 4.1 with $\gamma = 0$ to solve this problem, we find that $\|\mathbf{Dx}\| = 3.546988563973862 \times 10^{-9}$ and $\|\|\mathbf{Dx}\| = 2.409392359237624 \times 10^{-9}$, which is convergent slower and less accurate than the CRS. The SLM does not satisfy the convergence criterion $\varepsilon = 10^{-5}$ within 100 iterations.

5.4 *Example 4*

This example is Example 6.1 of [25], and we have a quadratic eigenvalue problem (7) with dimension $q = 100$. When we apply the CRS to find the solvent $\mathbf{S}$, it is convergent with 8 iterations to satisfy the convergence criterion $\varepsilon = 10^{-15}$. One quadruplet of the eigenvalues is

$$\pm 1.700677132426048 \pm 0.03080172655141696i, \tag{71}$$

The error of equation (7) is $7.082854615565958 \times 10^{-16}$. Another isolated eigenvalues obtained is

$$\pm 0.2816529303211303 \pm 0.8724658668457154i, \tag{72}$$

whose error $1.514636773591871 \times 10^{-5}$ is quite large.

When we apply the SLM in Section 4.1 with $\gamma = 0.05$ to solve this problem we find that $\|\mathbf{Dx}\| = 2.81004183202572 \times 10^{-16}$, and the complex eigenvalues are

$$\pm 1.700678402616394 \pm 0.03080232464229823i. \tag{73}$$

For the isolated eigenvalues we obtained is

$$\pm 0.2816529303204782 \pm 0.8724658668457154i, \tag{74}$$

and $\|\mathbf{Dx}\| = 1.514637096517238 \times 10^{-5}$ is near to that obtained by the CRS.

By applying the method DDM in Section 3 to this eigenvalue problem, we obtain

$$\pm 1.700677891610475 \pm 0.03080221866527308i, \tag{75}$$

whose $\|\mathbf{Eu}\| = 2.635669273673072 \times 10^{-16}$ is very small. The isolated eigenvalues are

$$\pm 0.2816529533434994 \pm 0.8724658668457159i, \tag{76}$$

and $\|\mathbf{Eu}\| = 1.508995105798522 \times 10^{-5}$ is near to that obtained by the CRS and the SLM.

Solving this QEP by the method SQEP in Section 4.2 with dimension q is more time saving, and we can obtain more accurate eigenvalues. Indeed, we can obtain

$$\pm 1.700677891610475 \pm 0.03080387867252627i, \tag{77}$$

whose $f = 7.452286103066368 \times 10^{-17}$ and $\|\mathbf{Dx}\| = 9.598114601107021 \times 10^{-15}$ are very small. The isolated eigenvalues are

$$\pm 0.2816529369978221 \pm 0.8724654898473985i, \tag{78}$$

and $f = 1.139077578617813 \times 10^{-14}$, which corresponds to $\|\mathbf{Dx}\| = 2.119339196182723 \times 10^{-13}$ and is much smaller than that obtained by CRS. In Figure 1 the distribution of eigenvalues is displayed with respect to λ_R and λ_I.

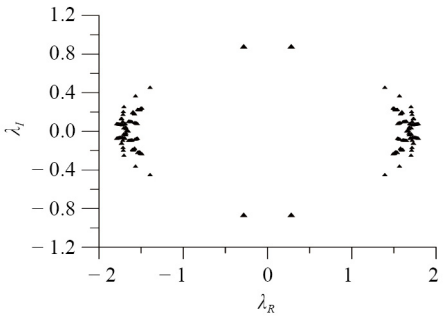

Figure 1. For example 4 the distribution of eigenvalues computed by the SLM for the solvent and an iterative detection method

5.5 *Example 5*

This example is Example 6.2 of [25], and we have a quadratic eigenvalue problem (7) with dimension $q = 25$. First, we apply the SLM in Section 4.1 with $\gamma = 0.005$ to solve this problem with 25 iterations to satisfy the convergence criterion $\varepsilon = 10^{-15}$. We plot the merit function f resulted from equation (57) with $\lambda_I = 0$ in Figure 2, which shows the first three real eigenvalues as the local minimums.

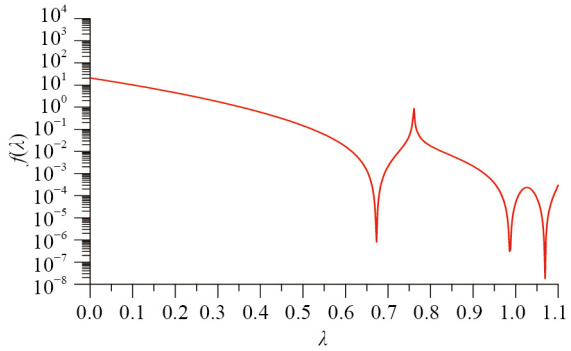

Figure 2. For the QEP of a gyroscopic system in Example 5, showing three local minimums of a merit function in the range [0, 1.1]

Precisely, by using the GSSA we can obtain the following eigenvalues:

$$\lambda_1 = 0.6726432664742701, \quad \lambda_2 = 0.98664429351828, \quad \lambda_3 = 1.068910182356432, \tag{79}$$

of which $\|\mathbf{Dx}\| = 5.156261683403625 \times 10^{-16}$, $\|\mathbf{Dx}\| = 1.12439767554822 \times 10^{-16}$ and $\|\|\mathbf{Dx}\| = 1.722994053665819 \times 10^{-16}$ are obtained respectively. Very small values of $\|\mathbf{Dx}\|$ confirm that the SLM method is very accurate.

In [25], one obtained

$$\lambda_1 = 0.6726432397672, \quad \lambda_2 = 0.9866442639296, \quad \lambda_3 = 1.0689101679903, \tag{80}$$

of which $\|\mathbf{Dx}\| = 9.396497532747466 \times 10^{-16}$, $\|\mathbf{Dx}\| = 3.343995064352587 \times 10^{-16}$ and $\|\mathbf{Dx}\| = 1.365687731249037 \times 10^{-15}$ are obtained respectively, which are slightly less accurate than that in equation (79).

6. Conclusions

In the paper we have developed fast iterative methods for finding solutions of quadratic eigenvalue problems, involving mass-spring-damper (MCK) and another mass-spring-damper (MGK) systems. In the constructed curve or surface the real and complex eigenvalues were being the local minimums of the constructed merit functions. In the merit function the vector variable is solved from the nonhomogeneous linear system, which is available by reducing the eigen-equation with one dimension less and by moving the normalized component to the right-hand side. We can quickly obtain eigenvalues by using the golden section search algorithm to solve the resultant minimization problems. Eigenvalue and eigenvector can be obtained merely through a few iterations with high precisions and the computations of the merit functions was saved. A new technique SLM was developed to quickly find an accurate solvent matrix for the MGK quadratic eigenvalue problems. Then we reduced it to a QEP with half dimension for computing all eigenvalues by using the symmetry. Upon comparing to the cyclic-reduction-based solvent method, the convergence speed of the SLM is slightly slower; however, its accuracy by combining with the minimization method is better.

In many engineering applications of MCK and MGK systems, the eigenvalues and their precise values are of utmost important. The practical significance of the obtained eigenvalues with high accuracy is that we can design a better method to handle the corresponding system for further understanding its stability behavior.

The proposed methods were based on GSSA to solve the derived minimization problems. We may encounter the complex eigenvalue problem whose complex eigenvalues are clustered in a narrow region. Limited by the resolution of GSSA, the detection methods might be difficult to find very accurate eigenvalues which are very close.

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Conflict of interest

The authors declare there is no conflict of interest at any point with reference to research findings.

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Appendix

In this appendix we give the two-dimensional golden section search algorithm (GSSA) to find the minimum of a give function $f(x,\ y),\ (x,\ y) \in [A,\ B] \times [C,\ D]$ with a given stopping criterion ε:

$$R = [\sqrt{5} - 1]/2$$

$$X_1 = A + (1 - R)(B - A)$$

$$X_2 = A + R(B - A)$$

$$Y_1 = C + (1 - R)(D - C)$$

$$Y_2 = C + R(D - C)$$

$$F_{11} = f(X_1, Y_1)$$

$$F_{12} = f(X_1, Y_2)$$

$$F_{21} = f(X_2, Y_1)$$

$$F_{22} = f(X_2, Y_2)$$

$$FMIN = \min(F_{11}, F_{11}, F_{21}, F_{22})$$

If $\sqrt{(B - A)^2 + (D - C)^2} < \varepsilon$ Then

If $(FMIN.EQ.F_{11})$ Then

$$f_{\min} = F_{11}$$

$$x_{\min} = X_1$$

$$y_{\min} = Y_1$$

End if

If $(FMIN.EQ.F_{12})$ Then

$f_{\min} = F_{12}$

$x_{\min} = X_1$

$y_{\min} = Y_2$

End if

If $(FMIN.EQ.F_{21})$ Then

$f_{\min} = F_{21}$

$x_{\min} = X_2$

$y_{\min} = Y_1$

End if

If $(FMIN.EQ.F_{22})$ Then

$f_{\min} = F_{22}$

$x_{\min} = X_2$

$y_{\min} = Y_2$

End if

Stop

End if

If $(FMIN.EQ.F_{11})$ Then

$B = X_2$

$D = Y_2$

End if

If $(FMIN.EQ.F_{12})$ Then

$B = X_2$

$C = Y_1$

End if

If $(FMIN.EQ.F_{22})$ Then

$A = X_1$

$C = Y_1$

End if

If $(FMIN.EQ.F_{21})$ Then

$A = X_1$

$D = Y_2$

End if

$X_1 = A + (1-R)(B-A)$

$X_2 = A + R(B-A)$

$Y_1 = C + (1-R)(D-C)$

$Y_2 = A2 + R(D-C)$

$F_{11} = f(X_1, Y_1)$

$$F_{12} = f(X_1, Y_2)$$

$$F_{21} = f(X_2, Y_1)$$

$$F_{22} = f(X_2, Y_2)$$

$$FMIN = \min(F_{11}, F_{12}, F_{21}, F_{22})$$