Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

# On the Solutions of Nonlinear Implicit $\omega$-Caputo Fractional Order Ordinary Differential Equations with Two-Point Fractional Derivatives and Integral Boundary Conditions in Banach Algebra

**Yousuf Alkhezi**[1], **Yahia Awad**[2*], **Karim Amin**[2], **Ragheb Mghames**[2,3]

[1]Mathematics Department, College of Basic Education, Public Authority for Applied Education and Training (PAAET), P.O. Box 34053, Kuwait City 70654, Kuwait
[2]Department of Mathematics and Physics, Lebanese International University (LIU), Bekaa Campus, Al-Khyara P.O. Box 5, West Bekaa, Lebanon
[3]School of Computer Sciences, Modern University for Business and Science (MUBS), Rashaya, Lebanon
E-mail: yehya.awad@liu.edu.lb

**Received:** 12 June 2024; **Revised:** 29 August 2024; **Accepted:** 10 September 2024

**Abstract:** This article delves into the analysis of nonlinear implicit $\omega$-Caputo fractional-order ordinary differential equations (NLIFDEs) with two-point fractional derivatives and integral boundary conditions within the context of Banach algebra. The primary focus is on demonstrating the existence and uniqueness of solutions for these complex fractional differential equations by utilizing Banach's and Krasnoselskii's fixed point theorems. Furthermore, the study explores the stability of these solutions through the Ulam-Hyers and Ulam-Hyers-Rassias stability criteria, thereby assessing the robustness of the proposed model. To illustrate the versatility of the generalized model, several special cases are examined, showcasing its ability to encompass various classical models. The practical applicability of the theoretical findings is underscored through a numerical example, which demonstrates the feasibility and relevance of the proposed methodology. This thorough investigation advances the comprehension of nonlinear fractional differential equations with integral boundary conditions, highlighting the intricate relationship between fractional derivatives, nonlinearities, and integral terms. The results offer significant insights into the behavior and stability of solutions within this demanding mathematical framework.

***Keywords***: nonlinear fractional differential equations, $\omega$-Caputo fractional derivatives, two-point fractional derivatives, integral boundary conditions, Banach algebra, existence and uniqueness, stability analysis, Ulam-Hyers and Ulam-Hyers-Rassias sense

**MSC:** 26A33, 34A08, 34B15, 47H10, 34D20

## 1. Introduction

Fractional calculus has garnered significant attention for its extension of differentiation and integration to non-integer orders, offering a robust framework for modeling complex phenomena across scientific and engineering disciplines. Pioneering work by mathematicians such as Almeida, Agarwal, Kiblas, and Samko has significantly expanded this field [1–5].


DOI: https://doi.org/10.37256/cm.5420245216

A specific area of focus within this domain is nonlinear implicit fractional-order differential equations (NLIFDEs) with fractional boundary conditions (FBCs), which have applications in mathematical physics, engineering sciences, and computational mathematics [6–10]. The concept of $\omega$-fractional derivatives, a generalization of Riemann-Liouville derivatives, has been revisited by Almeida [2], introducing a Caputo-type regularization and exploring its properties [11–15]. Extensive research has focused on the existence of positive solutions for fractional differential equations with integral boundary conditions, employing methodologies like fixed-point theory.

Nonlinear implicit $\omega$-Caputo fractional-order differential equations (NLIFDEs) with two-point fractional derivatives and integral boundary conditions are significant due to their wide-ranging applications in various scientific and engineering fields. These equations provide a powerful framework for modeling complex phenomena that exhibit memory and hereditary properties, which are common in many real-world systems such as viscoelastic materials, anomalous diffusion processes, and biological systems [2, 4, 5]. Studying NLIFDEs extends classical differential equation models, allowing for a more accurate and comprehensive description of complex systems [6, 7]. By incorporating fractional derivatives, which generalize integer-order differentiation, these models can capture the intricacies of dynamic processes with greater fidelity [1, 13]. Therefore, developing a rigorous mathematical analysis for the existence, uniqueness, and stability of solutions to NLIFDEs is essential for advancing theoretical knowledge and enhancing practical applications in science and engineering [3, 16, 17].

This research contributes to understanding NLIFDEs by exploring the interplay between fractional derivatives, nonlinearities, and integral terms. In this study, we employ Banach's and Krasnoselskii's fixed-point theorems to establish the existence and uniqueness of solutions for NLIFDEs. Banach's fixed-point theorem is utilized to demonstrate that our problem can be framed as a contraction mapping problem, ensuring a unique fixed point within a complete metric space. Krasnoselskii's fixed-point theorem is used to address more general cases where the mapping satisfies certain compactness conditions, further guaranteeing the existence of solutions. These theorems provide a rigorous mathematical foundation for our results, confirming the existence and uniqueness of solutions under the given conditions.

The stability of solutions is analyzed using the Ulam-Hyers and Ulam-Hyers-Rassias stability criteria. The Ulam-Hyers criterion is valuable for assessing whether solutions remain close to exact solutions under perturbations [16], while the Ulam-Hyers-Rassias criterion extends this analysis to handle more general forms of perturbations [17]. These methods are particularly well-suited for the complex nature of fractional differential equations with integral boundary conditions, providing a clear framework for evaluating the robustness of our findings [12]. Although alternative methods, such as Lyapunov's direct method, could be considered, the Ulam-Hyers and Ulam-Hyers-Rassias criteria offer a more direct and feasible approach for our specific context [2, 7, 18].

The practical illustration through a numerical example underscores the methodology's applicability in real-world problem-solving. In summary, this research advances the understanding of nonlinear fractional differential equations with integral boundary conditions, contributing to both theoretical foundations and practical insights for addressing complex systems governed by NLIFDEs. Numerous recent contributions, summarized comprehensively in works such as [8, 19, 20], among others, have motivated the present paper. The aim of our study is to investigate the existence and uniqueness of solutions pertaining to a class of nonlinear implicit $\omega$-Caputo fractional differential equations (NLIFDEs). These equations incorporate fractional derivatives and integral boundary conditions within the domain of Banach Algebra.

The use of Banach algebra is chosen for its comprehensive structure, combining the properties of both a Banach space and an algebra. This structure provides a robust foundation for analyzing nonlinear operators and ensuring the convergence of iterative methods. The completeness of Banach algebra under a norm facilitates the application of fixed-point theorems and other analytical techniques crucial for establishing the existence and uniqueness of solutions to complex integro-differential equations. Moreover, Banach algebra enables the manipulation of functions and their compositions within an algebraic context, essential for addressing the complexities of fractional integro-differential equations with nonlocal boundary conditions.

Consider the following nonlinear implicit $\omega$-Caputo fractional differential equation:

$$ {}^{c}\mathfrak{D}_{0^+}^{\gamma,\,\omega}\mathscr{Y}(\tau) = \mathscr{F}\left(\tau,\, \mathscr{Y}(\tau),\ {}^{c}\mathfrak{D}_{0^+}^{\theta,\,\omega}\mathscr{Y}(\tau),\ \int_0^\tau k(\tau,\,\xi)\,{}^{c}\mathfrak{D}_{0^+}^{\gamma,\,\omega}\mathscr{Y}(\xi)d\xi\right),\ \ \tau\in[0,\,\mathfrak{T}],\ 0<\theta\le 1,\ 2<\gamma\le 3, \tag{1} $$

Subjected to the subsequent set of three integral boundary conditions involving fractional derivatives:

$$\mathscr{Y}(0) + {}^{c}\mathfrak{D}_{0^+}^{\gamma-1,\,\omega}\mathscr{Y}(\mathfrak{T}) = \eta_1 \int_0^{\mathfrak{T}} \mathfrak{g}_1(\xi, \mathscr{Y}(\xi))\,d\xi, \tag{2}$$

$${}^{c}\mathfrak{D}_{0^+}^{\gamma-1,\,\omega}\mathscr{Y}(0) + {}^{c}\mathfrak{D}_{0^+}^{\gamma-2,\,\omega}\mathscr{Y}(\mathfrak{T}) = \eta_2 \int_0^{\mathfrak{T}} \mathfrak{g}_2(\xi, \mathscr{Y}(\xi))\,d\xi, \tag{3}$$

$${}^{c}\mathfrak{D}_{0^+}^{\gamma-2,\,\omega}\mathscr{Y}(0) + {}^{c}\mathfrak{D}_{0^+}^{\gamma-3,\,\omega}\mathscr{Y}(\mathfrak{T}) = \eta_3 \int_0^{\mathfrak{T}} \mathfrak{g}_3(\xi, \mathscr{Y}(\xi))\,d\xi, \tag{4}$$

where $\tau \in J = [0,\, \mathfrak{T}]$, ${}^{c}\mathfrak{D}_{0^+}^{\gamma,\,\omega}$ and ${}^{c}\mathfrak{D}_{0^+}^{\theta,\,\omega}$ denote the standard $\omega$-Caputo fractional derivatives of orders $\gamma \in (2,\, 3]$ and $\theta \in (0,\, 1]$, where $\omega(\tau)$ is an increasing function with $\omega'(\tau) \neq 0\ \ \forall\ \tau \in J = [0,\, \mathfrak{T}]$, $\mathscr{F}\colon J \times R^3 \to R$, and $\mathfrak{g}_i\colon J \times R \to R$ $(i = 1,\, 2,\, 3)$ are continuous functions.

The structure of the paper is as follows: In the initial section, we provide an introduction to the article, presenting an overview, elucidating the research objective, and providing essential background information and prerequisites. Subsequently, the second section is dedicated to the exploration of solutions for boundary value problems related to nonlinear fractional differential equations featuring fractional integral boundary conditions. This exploration involves the application of Banach's and Krasnoselskii's fixed point theorems to establish both the existence and uniqueness of solutions. Furthermore, the third section conducts a thorough stability analysis of the solutions, employing the Ulam-Hyers and Ulam-Hyers-Rassias criteria. Finally, in the fifth section, a practical numerical example is incorporated to illustrate the real-world application of the derived findings.

## 2. Preliminaries: Definitions, Lemmas, and Theorems

In the subsequent text, we present certain symbols, definitions, lemmas, and theorems that serve as foundational elements for our study. These essential concepts can be referenced in [2–5], and related sources.

**Definition 1** [2] Consider $\gamma > 0$, and let $J = [a,\, b]$ represent an interval with $-\infty < a < \tau < b < +\infty$, where $x \in L_1(J,\, \mathbb{R})$. The left-sided $\omega$-Riemann-Liouville fractional integral of $\mathscr{Y}(\tau)$ with order $\gamma$ for an integrable function $x\colon J \to \mathbb{R}$, with respect to another function $\omega\colon J \to \mathbb{R}$, is defined as follows. Here, $\omega$ is an increasing differentiable function such that $\omega'(\tau) \neq 0$ for all $\tau \in J$:

$$\mathfrak{I}_{a^+}^{\gamma,\,\omega}\mathscr{Y}(\tau) = \frac{1}{\Gamma(\gamma)} \int_a^{\tau} \omega'(\xi)(\omega(\tau) - \omega(\xi))^{\gamma-1}\mathscr{Y}(\xi)\,d\xi, \tag{5}$$

where $\Gamma$ is the Euler gamma function defined by $\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}\,dt$.

**Definition 2** [5] Consider $n \in \mathbb{N}$, and let $J = [a,\, b]$ denote an interval with $-\infty < a < \tau < b < +\infty$. Suppose $\omega,\, x \in C^n(J,\, \mathbb{R})$ are two functions, where $\omega$ is increasing and $\omega'(\tau) \neq 0$ for all $\tau \in J$. In this context, the left-sided $\omega$-Caputo fractional derivative of a function $\mathscr{Y}$ of order $\gamma$ is formally expressed as follows:

$${}^{c}\mathfrak{D}_{a^+}^{\gamma,\,\omega}\mathscr{Y}(\tau) = \mathfrak{I}_{a^+}^{n-\gamma,\,\omega}\left(\frac{1}{\omega'(\tau)}\frac{d}{d\tau}\right)^n \mathscr{Y}(\tau), \tag{6}$$

with

$$\mathfrak{I}_{a^+}^{n-\gamma,\,\omega}\mathscr{Y}(\tau)=\frac{1}{\Gamma(n-\gamma)}\int_a^\tau \omega'(\xi)(\omega(\tau)-\omega(\xi))^{n-\gamma-1}\mathscr{Y}^{(n)}(\xi)\,d\xi, \tag{7}$$

such that

$$\mathscr{Y}^{(n)}(\xi)=\left(\frac{1}{\omega'(\xi)}\frac{d}{d\xi}\right)^n\mathscr{Y}(\xi), \tag{8}$$

where $n=[\gamma]+1$ for $\gamma\notin\mathbb{N}$, and $n=\gamma$ for $\gamma\in\mathbb{N}$.

To investigate the existence of solutions for a fractional differential equation, it is essential to convert it into an equivalent integral equation using the fundamental properties of $\mathfrak{I}a^{+\gamma,\,\omega}$ and ${}^c\mathfrak{D}a^{+\gamma,\,\omega}$. The following lemma is pivotal for establishing the basic properties of fractional integrals and derivatives within the framework of $\omega$-fractional calculus, which supports the analysis of NLIFDEs in our study.

**Lemma 1** [2] Consider $\gamma,\,\theta\in\mathbb{R}^+$ and $\mathscr{F}(\tau)\in L_1(J)$, where $J=[a,\,b]$ is an interval. Then, for every $\tau\in J$:

1. The fractional integral $\mathfrak{I}_{a^+}^{\gamma,\,\omega}\mathscr{Y}(\zeta)$ exists almost everywhere.
2. $\mathfrak{I}_{a+}^{\gamma,\,\omega}\mathfrak{I}_{a+}^{\theta,\,\omega}\mathscr{Y}(\tau)=\mathfrak{I}_{a+}^{\theta,\,\omega}\mathfrak{I}_{a+}^{\gamma,\,\omega}\mathscr{Y}(\tau)=\mathfrak{I}_{a+}^{\gamma+\theta,\,\omega}\mathscr{Y}(\tau)$.
3. $(\mathfrak{I}_{a+}^{\gamma,\,\omega})^n\mathscr{Y}(\tau)=\mathfrak{I}_{a+}^{n\alpha,\,\omega}\mathscr{Y}(\tau)$, where $n\in N$.
4. ${}^c\mathfrak{D}_{a^+}^{\gamma,\,\omega}\mathfrak{I}_{a^+}^{\gamma,\,\omega}\mathscr{Y}(\zeta)=\mathscr{Y}(\zeta)$ for all $\tau\in J$.
5. ${}^c\mathfrak{D}_{0^+}^{\gamma,\,\omega}\mathfrak{I}_{0^+}^{\gamma,\,\omega}\mathscr{Y}(\zeta)=\mathfrak{I}_{0^+}^{\gamma-\theta}\mathscr{Y}(\zeta)$ for $\theta\in[0,\,\gamma)$.
6. ${}^c\mathfrak{D}_{0^+}^{\gamma,\,\omega}\mathscr{Y}(\zeta)=\mathfrak{I}_{0^+}^{-\gamma,\,\omega}\mathscr{Y}(\zeta)$ for $\gamma<0$ and $\zeta\geq 0$.

The subsequent lemmas and theorems are essential for establishing the foundational properties of the $\omega$-Caputo fractional derivatives. These mathematical tools and transformations leverage the fundamental properties of $\mathfrak{I}a^{+\gamma,\,\omega}$ and ${}^c\mathfrak{D}a^{+\gamma,\,\omega}$, facilitating the analysis of the NLIFDE problem within the Banach algebra framework, as detailed in [2] and [5].

**Lemma 2** [2] For $\theta>-1$, $\theta\neq\gamma-1,\,\gamma-2,\,\ldots,\,\gamma-n$, then for $\tau\geq 0$,

$${}^c\mathfrak{D}_{0^+}^{\gamma,\,\omega}(\omega(\tau)-\omega(a))^{\theta-1}=\frac{\Gamma(\theta)}{\Gamma(\theta-\gamma)}(\omega(\tau)-\omega(a))^{\theta-\gamma-1}, \tag{9}$$

and

$${}^c\mathfrak{D}_{0^+}^{\gamma,\,\omega}(\omega(\tau)-\omega(a))^{\gamma-i}=0\quad\text{for all}\quad i=1,\,2,\,3,\,\ldots,\,n. \tag{10}$$

**Lemma 3** [2] Let $\gamma>0$, then the differential equation ${}^c\mathfrak{D}_{a^+}^{\gamma,\,\omega}\mathscr{Y}(\tau)=0$ has solution in $C(J,\,\mathbb{R})\cap L_1(J,\,\mathbb{R})$ is

$$\mathscr{Y}(\tau)=c_1(\omega(\tau)-\omega(0))^{\gamma-1}+c_2(\omega(\tau)-\omega(0))^{\gamma-2}+\cdots+c_n(\omega(\tau)-\omega(0))^{\gamma-n}, \tag{11}$$

where $c_i\in\mathbb{R}$ for all $i=1,\,2,\,...,\,n$, and $n=[\gamma]+1$.

**Theorem 1** [4] (Banach's Fixed Point Theorem) Given a Banach space $(X,\,\|.\|)$, and a contraction mapping $\Theta$: $X\to X$, there exists a unique fixed point $\mathscr{Y}\in X$ such that $\Theta(\mathscr{Y})=\mathscr{Y}$.

**Theorem 2** [3] (Krasnselskii's fixed point theorem) Let $\mathscr{S}$ denote a closed, convex, and non-empty subset of a Banach space $X$. Suppose $\Theta_1$ and $\Theta_2$ are mappings from $\mathscr{S}$ to $X$ satisfying the following conditions:

1. For any $\psi,\ \phi \in \mathscr{S}$, the sum $\Theta_1 \psi + \Theta_2\ \phi$ belongs to $\mathscr{S}$.
2. The mapping $\Theta_1$ is a contraction.
3. The mapping $\Theta_2$ is continuous, and the range $\Theta_2(\mathscr{S})$ is bounded.

Under these assumptions, there exists at least one element $\psi \in \mathscr{S}$ such that $\Theta_1 \psi + \Theta_2\ \psi = \psi$.

# 3. Main results

Our investigation into solutions for nonlinear fractional differential equations, accompanied by fractional integral boundary conditions, has led to substantial findings. Utilizing Banach's and Krasnoselskii's fixed point theorems in this particular section, we not only confirm the existence but also establish the uniqueness of solutions.

**Definition 3** A function $\mathscr{Y} \in C(J, \mathbb{R})$ is identified as a solution if it satisfies both the nonlinear implicit fractional differential equation NLIFDE (1) and its associated boundary conditions.

The following lemma provides the criteria for recognizing a function $\mathscr{Y}(\tau)$ as a solution to the NLIFDE by meeting the conditions of the subsequent fractional integral equation, effectively linking the nonlinear implicit fractional differential equation to an integral form for analysis.

**Lemma 4** Suppose $2 < \gamma \leq 3$, and let $\mathscr{F}: J \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function. A function $\mathscr{Y}(\tau)$, defined on $J$, is recognized as a solution to the nonlinear implicit fractional differential equation NLIFDE (1) if and only if it meets the criteria outlined by the subsequent fractional integral equation:

$$
\begin{aligned}
&\mathscr{Y}(\tau)\\
&=\frac{(\omega(\tau)-\omega(0))^{\gamma-1}}{\Gamma(\gamma)}\left(\eta_1\int_0^{\mathfrak{T}}\mathfrak{g}_1(\xi,\mathscr{Y}(\xi))d\xi-\int_0^{\mathfrak{T}}\omega'(\xi)\psi(\xi)\,d\xi\right)\\
&\quad+\frac{(\omega(\tau)-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}\left(\begin{array}{c}\varphi(\mathfrak{T})\left(-\eta_1\int_0^{\mathfrak{T}}\mathfrak{g}_1(\xi,\mathscr{Y}(\xi))d\xi+\int_0^{\mathfrak{T}}\omega'(\xi)\psi(\xi)\,d\xi\right)\\+\eta_2\int_0^{\mathfrak{T}}\mathfrak{g}_2(\xi,\mathscr{Y}(\xi))d\xi-\int_0^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))\psi(\xi)\,d\xi.\end{array}\right)\\
&\quad+\frac{(\omega(\tau)-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\left(\begin{array}{c}\eta_3\int_0^{\mathfrak{T}}\mathfrak{g}_3(\xi,\mathscr{Y}(\xi))\,d\xi-\frac{1}{2}\int_0^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))^2\psi(\xi)\,d\xi\\-\frac{1}{2}\left(\eta_1\int_0^{\mathfrak{T}}\mathfrak{g}_1(\xi,\mathscr{Y}(\xi))d\xi-\int_0^{\mathfrak{T}}\omega'(\xi)\psi(\xi)\,d\xi\right)(\omega(\mathfrak{T})-\omega(0))^2\\-\varphi(\mathfrak{T})\left(\begin{array}{c}\varphi(\mathfrak{T})\left(-\eta_1\int_0^{\mathfrak{T}}\mathfrak{g}_1(\xi,\mathscr{Y}(\xi))d\xi+\int_0^{\mathfrak{T}}\omega'(\xi)\psi(\xi)\,d\xi\right)\\+\eta_2\int_0^{\mathfrak{T}}\mathfrak{g}_2(\xi,\mathscr{Y}(\xi))d\xi-\int_0^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))\psi(\xi)\,d\xi\end{array}\right)\end{array}\right)\\
&\quad+\frac{1}{\Gamma(\gamma)}\int_0^{\tau}\omega'(\xi)(\omega(\tau)-\omega(\xi))^{\gamma-1}\psi(\xi)\,d\xi, \qquad (12)
\end{aligned}
$$

where $\varphi(\mathfrak{T}) = 1 + \omega(\mathfrak{T}) - \omega(0)$.

**Proof.** Let $\mathscr{Y}(\tau)$ be a solution to the Nonlinear Implicit Fractional Differential Equation NLIFDE (1). Define $\psi(\tau) = \mathscr{F}(\tau, \mathscr{Y}(\tau), \mathfrak{I}_{0+}^{\gamma-\theta,\,\omega}\mathscr{Y}(\tau), \int_0^{\tau} k(\tau, \xi)\ {}^{c}\mathfrak{D}_{0+}^{\gamma,\,\omega}\mathscr{Y}(\xi)d\xi)$. Utilizing Lemma 3, we derive the expression:

$$\mathscr{Y}(\tau) = c_1(\omega(\tau)-\omega(0))^{\gamma-1} + c_2(\omega(\tau)-\omega(0))^{\gamma-2} + c_3(\omega(\tau)-\omega(0))^{\gamma-3}$$

$$+\frac{1}{\Gamma(\gamma)}\int_0^{\tau}\omega'(\xi)(\omega(\tau)-\omega(\xi))^{\gamma-1}\psi(\xi)\,d\xi. \tag{13}$$

Applying the boundary conditions (2)-(3), we obtain the following equations:

$$c_1\Gamma(\gamma) = \eta_1\int_0^{\mathfrak{T}}\mathfrak{g}_1(\xi,\mathscr{Y}(\xi))d\xi - \int_0^{\mathfrak{T}}\omega'(\xi)\psi(\xi)\,d\xi, \tag{14}$$

$$c_1\Gamma(\gamma)(1+\omega(\mathfrak{T})-\omega(0)) + c_2\Gamma(\gamma-1) = \eta_2\int_0^{\mathfrak{T}}\mathfrak{g}_2(\xi,\mathscr{Y}(\xi))d\xi$$

$$-\int_0^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))\psi(\xi)\,d\xi, \tag{15}$$

and

$$\frac{1}{2}c_1\Gamma(\gamma)(\omega(\mathfrak{T})-\omega(0))^2 + c_2\Gamma(\gamma-1)(1+\omega(\mathfrak{T})-\omega(0)) + c_3\Gamma(\gamma-2)$$

$$=\eta_3\int_0^{\mathfrak{T}}\mathfrak{g}_3(\xi,\mathscr{Y}(\xi))\,d\xi - \frac{1}{2}\int_0^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))^2\psi(\xi)\,d\xi. \tag{16}$$

Solving equations (14), (15), and (16) for $c_1$, $c_2$, and $c_3$, we obtain:

$$c_1 = \frac{1}{\Gamma(\gamma)}\left(\eta_1\int_0^{\mathfrak{T}}\mathfrak{g}_1(\xi,\mathscr{Y}(\xi))d\xi - \int_0^{\mathfrak{T}}\omega'(\xi)\psi(\xi)\,d\xi\right), \tag{17}$$

$$c_2 = \frac{1}{\Gamma(\gamma-1)}\begin{pmatrix}\varphi(\mathfrak{T})\left(-\eta_1\int_0^{\mathfrak{T}}\mathfrak{g}_1(\xi,\mathscr{Y}(\xi))d\xi + \int_0^{\mathfrak{T}}\omega'(\xi)\psi(\xi)\,d\xi\right)\\ +\eta_2\int_0^{\mathfrak{T}}\mathfrak{g}_2(\xi,\mathscr{Y}(\xi))d\xi - \int_0^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))\psi(\xi)\,d\xi.\end{pmatrix}, \tag{18}$$

and

$$c_3 = \frac{1}{\Gamma(\gamma-2)} \begin{pmatrix} \eta_3 \int_0^{\mathfrak{T}} \mathfrak{g}_3(\xi, \mathscr{Y}(\xi))\, d\xi - \frac{1}{2} \int_0^{\mathfrak{T}} \omega'(\xi)(\omega(\mathfrak{T}) - \omega(\xi))^2 \psi(\xi)\, d\xi \\ -\frac{1}{2}\left(\eta_1 \int_0^{\mathfrak{T}} \mathfrak{g}_1(\xi, \mathscr{Y}(\xi))d\xi - \int_0^{\mathfrak{T}} \omega'(\xi)\psi(\xi)\, d\xi\right)(\omega(\mathfrak{T}) - \omega(0))^2 \\ -\begin{pmatrix} \varphi(\mathfrak{T})\left(-\eta_1 \int_0^{\mathfrak{T}} \mathfrak{g}_1(\xi, \mathscr{Y}(\xi))d\xi + \int_0^{\mathfrak{T}} \omega'(\xi)\psi(\xi)\, d\xi\right) \\ +\eta_2 \int_0^{\mathfrak{T}} \mathfrak{g}_2(\xi, \mathscr{Y}(\xi))d\xi - \int_0^{\mathfrak{T}} \omega'(\xi)(\omega(\mathfrak{T}) - \omega(\xi))\psi(\xi)\, d\xi \end{pmatrix} \varphi(\mathfrak{T}) \end{pmatrix}, \tag{19}$$

where $\varphi(\mathfrak{T}) = 1 + \omega(\mathfrak{T}) - \omega(0)$. Substituting these into (13), we obtain:

$$\begin{aligned} &\mathscr{Y}(\tau) \\ =&\frac{(\omega(\tau) - \omega(0))^{\gamma-1}}{\Gamma(\gamma)}\left(\eta_1 \int_0^{\mathfrak{T}} \mathfrak{g}_1(\xi, \mathscr{Y}(\xi))d\xi - \int_0^{\mathfrak{T}} \omega'(\xi)\psi(\xi)\, d\xi\right) \\ &+ \frac{(\omega(\tau) - \omega(0))^{\gamma-2}}{\Gamma(\gamma-1)} \begin{pmatrix} \varphi(\mathfrak{T})\left(-\eta_1 \int_0^{\mathfrak{T}} \mathfrak{g}_1(\xi, \mathscr{Y}(\xi))d\xi + \int_0^{\mathfrak{T}} \omega'(\xi)\psi(\xi)\, d\xi\right) \\ +\eta_2 \int_0^{\mathfrak{T}} \mathfrak{g}_2(\xi, \mathscr{Y}(\xi))d\xi - \int_0^{\mathfrak{T}} \omega'(\xi)(\omega(\mathfrak{T}) - \omega(\xi))\psi(\xi)\, d\xi. \end{pmatrix} \\ &+ \frac{(\omega(\tau) - \omega(0))^{\gamma-3}}{\Gamma(\gamma-2)} \begin{pmatrix} \eta_3 \int_0^{\mathfrak{T}} \mathfrak{g}_3(\xi, \mathscr{Y}(\xi))\, d\xi - \frac{1}{2} \int_0^{\mathfrak{T}} \omega'(\xi)(\omega(\mathfrak{T}) - \omega(\xi))^2 \psi(\xi)\, d\xi \\ -\frac{1}{2}\left(\eta_1 \int_0^{\mathfrak{T}} \mathfrak{g}_1(\xi, \mathscr{Y}(\xi))d\xi - \int_0^{\mathfrak{T}} \omega'(\xi)\psi(\xi)\, d\xi\right)(\omega(\mathfrak{T}) - \omega(0))^2 \\ -\varphi(\mathfrak{T})\begin{pmatrix} \varphi(\mathfrak{T})\left(-\eta_1 \int_0^{\mathfrak{T}} \mathfrak{g}_1(\xi, \mathscr{Y}(\xi))d\xi + \int_0^{\mathfrak{T}} \omega'(\xi)\psi(\xi)\, d\xi\right) \\ +\eta_2 \int_0^{\mathfrak{T}} \mathfrak{g}_2(\xi, \mathscr{Y}(\xi))d\xi - \int_0^{\mathfrak{T}} \omega'(\xi)(\omega(\mathfrak{T}) - \omega(\xi))\psi(\xi)\, d\xi \end{pmatrix} \end{pmatrix} \\ &+ \frac{1}{\Gamma(\gamma)} \int_0^{\tau} \omega'(\xi)(\omega(\tau) - \omega(\xi))^{\gamma-1} \psi(\xi)\, d\xi. \end{aligned} \tag{20}$$

On the contrary, assume that $\mathscr{Y}(\tau)$ constitutes a solution to the nonlinear implicit fractional differential equation NLIFDE (12), and this solution can be expressed in the subsequent manner:

$$\begin{aligned} &\mathscr{Y}(\tau) \\ =&\frac{(\omega(\tau) - \omega(0))^{\gamma-1}}{\Gamma(\gamma)}\left(\eta_1 \int_0^{\mathfrak{T}} \mathfrak{g}_1(\xi, \mathscr{Y}(\xi))d\xi - \int_0^{\mathfrak{T}} \omega'(\xi)\psi(\xi)\, d\xi\right) \end{aligned}$$

$$
+\frac{(\omega(\tau)-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}\left(\begin{array}{c}\varphi(\mathfrak{T})\left(-\eta_1\int_0^{\mathfrak{T}}\mathfrak{g}_1(\xi,\mathscr{Y}(\xi))d\xi+\int_0^{\mathfrak{T}}\omega'(\xi)\psi(\xi)d\xi\right)\\ +\eta_2\int_0^{\mathfrak{T}}\mathfrak{g}_2(\xi,\mathscr{Y}(\xi))d\xi-\int_0^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))\psi(\xi)d\xi.\end{array}\right)
$$

$$
+\frac{(\omega(\tau)-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\left(\begin{array}{c}\eta_3\int_0^{\mathfrak{T}}\mathfrak{g}_3(\xi,\mathscr{Y}(\xi))d\xi-\frac{1}{2}\int_0^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))^2\psi(\xi)d\xi\\ -\frac{1}{2}\left(\eta_1\int_0^{\mathfrak{T}}\mathfrak{g}_1(\xi,\mathscr{Y}(\xi))d\xi-\int_0^{\mathfrak{T}}\omega'(\xi)\psi(\xi)d\xi\right)(\omega(\mathfrak{T})-\omega(0))^2\\ -\varphi(\mathfrak{T})\left(\begin{array}{c}\varphi(\mathfrak{T})\left(-\eta_1\int_0^{\mathfrak{T}}\mathfrak{g}_1(\xi,\mathscr{Y}(\xi))d\xi+\int_0^{\mathfrak{T}}\omega'(\xi)\psi(\xi)d\xi\right)\\ +\eta_2\int_0^{\mathfrak{T}}\mathfrak{g}_2(\xi,\mathscr{Y}(\xi))d\xi-\int_0^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))\psi(\xi)d\xi\end{array}\right)\end{array}\right)
$$

$$
+\frac{1}{\Gamma(\gamma)}\int_0^{\tau}\omega'(\xi)(\omega(\tau)-\omega(\xi))^{\gamma-1}\psi(\xi)d\xi. \tag{21}
$$

Thus, we can infer that: ${}^c\mathfrak{D}_{0^+}^{\gamma}\mathscr{Y}(\tau)=\psi(\tau)$, with $\mathscr{Y}(0)+{}^c\mathfrak{D}_{0^+}^{\gamma-1}\mathscr{Y}(\mathfrak{T})=\eta_1\int_0^{\mathfrak{T}}\mathfrak{g}_1(\xi,\mathscr{Y}(\xi))d\xi$, ${}^c\mathfrak{D}_{0^+}^{\gamma-1}\mathscr{Y}(0)+{}^c\mathfrak{D}_{0^+}^{\gamma-2}\mathscr{Y}(\mathfrak{T})=\eta_2\int_0^{\mathfrak{T}}\mathfrak{g}_2(\xi,\mathscr{Y}(\xi))d\xi$, and ${}^c\mathfrak{D}_{0^+}^{\gamma-2}\mathscr{Y}(0)+{}^c\mathfrak{D}_{0^+}^{\gamma-3}\mathscr{Y}(\mathfrak{T})=\eta_3\int_0^{\mathfrak{T}}\mathfrak{g}_3(\xi,\mathscr{Y}(\xi))d\xi$. This implies that $\psi(\tau)$ indeed satisfies the conditions of problem (12). This concludes the proof. □

**Lemma 5** Consider the NLIFDE (1) under the following conditions:

$(H_1)$ The nonlinear function $\mathscr{F}: J\times\mathbb{R}^3\to\mathbb{R}$ exhibits continuity, and there exists $\lambda\in C(J,\mathbb{R}^+)$ such that:

$$
|\mathscr{F}(\tau,\psi_1,\psi_2,\psi_3)-\mathscr{F}(\tau,\phi_1,\phi_2,\phi_3)|\le\lambda(\tau)(|\psi_1-\phi_1|+|\psi_2-\phi_2|+|\psi_3-\phi_3|), \tag{22}
$$

for all $\tau\in J$, $\psi_i,\phi_i\in\mathbb{R}$, and $i=1,2,3$.

$(H_2)$ The function $k(\tau,\xi)$ is continuous over $J\times J$, and there exists a positive constant $K$ such that:

$$
\max_{\tau,\xi\in[0,1]}|k(\tau,\xi)|=K. \tag{23}
$$

$(H_3)$ The nonlinear function $\mathfrak{g}_i: J\times\mathbb{R}\to\mathbb{R}$ maintains continuity, and there exists $\mu_i\in C(J,\mathbb{R}^+)$ such that:

$$
|\mathfrak{g}_i(\tau,\psi)-\mathfrak{g}_i(\tau,\phi)|\le\mu_i(\tau)\,|\psi-\phi|,\ \forall\,\tau\in J,\text{ and } i=1,2,3. \tag{24}
$$

**Remark 1.** Derived from Lemma (5), we extract the subsequent insights:

1. Under the premise of $(H_1)$, the inequality

$$
|\mathscr{F}(\tau,\psi_1,\psi_2,\psi_3)|-|\mathscr{F}(\tau,0,0,0)|\le|\mathscr{F}(\tau,\psi_1,\psi_2,\psi_3)-\mathscr{F}(\tau,0,0,0)|\le\lambda(\tau)(|\psi_1|+|\psi_2|+|\psi_3|) \tag{25}
$$

holds. Consequently, if $F=\sup_{\tau\in J}|\mathscr{F}(\tau,0,0,0)|$, it follows that

$$|\mathscr{F}(\tau,\,\psi_1,\,\psi_2,\,\psi_3)| \leq F+\lambda(\tau)(|\psi_1|+|\psi_2|+|\psi_3|). \tag{26}$$

2. Given the conditions of $(H_3)$, we obtain for $i=1,\,2,\,3$ the inequalities:

$$|\mathfrak{g}_i(\tau,\,\psi)|-|\mathfrak{g}_i(\tau,\,0)| \leq |\mathfrak{g}_i(\tau,\,\psi)-\mathfrak{g}_i(\tau,\,0)| \leq \mu_i(\tau)|\psi|. \tag{27}$$

Thus, if $H_i=\sup_{\tau\in J}|\mathfrak{g}_i(\tau,\,0)|$, then $|\mathfrak{g}_i(\tau,\,\psi)| \leq H_i+\mu_i(\tau)|\psi|$.

**Definition 4** Define the operator $\Theta$: $C(J,\,\mathbb{R}) \to C(J,\,\mathbb{R})$ as follows:

$$\begin{aligned}
&\Theta(\mathscr{Y}(\tau))\\
&=\frac{(\omega(\tau)-\omega(0))^{\gamma-1}}{\Gamma(\gamma)}\left(\eta_1\int_0^{\mathfrak{T}}\mathfrak{g}_1(\xi,\,\mathscr{Y}(\xi))d\xi-\int_0^{\mathfrak{T}}\omega'(\xi)\psi(\xi)\,d\xi\right)\\
&+\frac{(\omega(\tau)-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}\begin{pmatrix}\varphi(\mathfrak{T})\left(-\eta_1\int_0^{\mathfrak{T}}\mathfrak{g}_1(\xi,\,\mathscr{Y}(\xi))d\xi+\int_0^{\mathfrak{T}}\omega'(\xi)\psi(\xi)\,d\xi\right)\\ +\eta_2\int_0^{\mathfrak{T}}\mathfrak{g}_2(\xi,\,\mathscr{Y}(\xi))d\xi-\int_0^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))\psi(\xi)\,d\xi.\end{pmatrix}\\
&+\frac{(\omega(\tau)-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\begin{pmatrix}\eta_3\int_0^{\mathfrak{T}}\mathfrak{g}_3(\xi,\,\mathscr{Y}(\xi))\,d\xi-\frac{1}{2}\int_0^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))^2\psi(\xi)\,d\xi\\ -\frac{1}{2}\left(\eta_1\int_0^{\mathfrak{T}}\mathfrak{g}_1(\xi,\,\mathscr{Y}(\xi))d\xi-\int_0^{\mathfrak{T}}\omega'(\xi)\psi(\xi)\,d\xi\right)(\omega(\mathfrak{T})-\omega(0))^2\\ -\varphi(\mathfrak{T})\begin{pmatrix}\varphi(\mathfrak{T})\left(-\eta_1\int_0^{\mathfrak{T}}\mathfrak{g}_1(\xi,\,\mathscr{Y}(\xi))d\xi+\int_0^{\mathfrak{T}}\omega'(\xi)\psi(\xi)\,d\xi\right)\\ +\eta_2\int_0^{\mathfrak{T}}\mathfrak{g}_2(\xi,\,\mathscr{Y}(\xi))d\xi-\int_0^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))\psi(\xi)\,d\xi\end{pmatrix}\end{pmatrix}\\
&+\frac{1}{\Gamma(\gamma)}\int_0^{\tau}\omega'(\xi)(\omega(\tau)-\omega(\xi))^{\gamma-1}\psi(\xi)\,d\xi,
\end{aligned} \tag{28}$$

where $\psi(\xi) \in C(J,\,\mathbb{R})$ satisfies the following implicit fractional equation:

$$\psi(\tau)=\mathscr{F}(\tau,\,\mathscr{Y}(\tau),\ {}^c\mathfrak{D}_{0^+}^{\theta,\,\omega}\mathscr{Y}(\tau),\,\int_0^{\tau}k(\tau,\,\xi)\,{}^c\mathfrak{D}_{0^+}^{\gamma,\,\omega}\mathscr{Y}(\xi)d\xi). \tag{29}$$

## 3.1 *Existence of solutions*

The subsequent theorem establish the existence of solutions for the nonlinear fractional differential equation (NLIFDE) (1) by applying Krasnoselskii's fixed point theorem, assuming the conditions outlined in Lemma 5 are met. These results are important as they offer a solid mathematical basis for the existence of solutions within the Banach algebra framework, thereby confirming the model's robustness and applicability to complex real-world scenarios.

**Theorem 3** Suppose that assumptions $(H_1)$-$(H_3)$ hold. Consider the real number $\aleph$ so that

$$\aleph = \|\mu_1\| \eta_1 \mathfrak{T}\Lambda_1 + \|\mu_2\| \eta_2 \mathfrak{T}\Lambda_2 + \|\mu_3\| \eta_3 \mathfrak{T}\Lambda_3 + \Lambda_4, \tag{30}$$

where

$$\Lambda_1 = \left(\frac{1}{\Gamma(\gamma)} + \frac{1}{2\Gamma(\gamma-2)}\right)(\omega(\mathfrak{T}) - \omega(0))^{\gamma-1} + \varphi(\mathfrak{T})\frac{(\omega(\mathfrak{T}) - \omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}$$

$$+ \varphi^2(\mathfrak{T})\frac{(\omega(\mathfrak{T}) - \omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}, \tag{31}$$

$$\Lambda_2 = \frac{(\omega(\mathfrak{T}) - \omega(0))^{\gamma-2}}{\Gamma(\gamma-1)} + \varphi(\mathfrak{T})\frac{(\omega(\mathfrak{T}) - \omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}, \tag{32}$$

$$\Lambda_3 = \frac{(\omega(\mathfrak{T}) - \omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}, \tag{33}$$

and

$$\Lambda_4 = \|\lambda\| \left(1 + \frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right) \left( \begin{array}{c} \left(\frac{\mathfrak{T}}{|\Gamma(\gamma+1)|} + \frac{1}{\Gamma(\gamma)} + \frac{1}{2\Gamma(\gamma-1)} + \frac{2}{3\Gamma(\gamma-2)}\right)(\omega(\mathfrak{T}) - \omega(0))^{\gamma} \\ + \left(\frac{1}{\Gamma(\gamma-1)} + \frac{1}{2\Gamma(\gamma-2)}\right)\varphi(\mathfrak{T})(\omega(\mathfrak{T}) - \omega(0))^{\gamma-1} \\ + \frac{1}{\Gamma(\gamma-2)}\varphi^2(\mathfrak{T})(\omega(\mathfrak{T}) - \omega(0))^{\gamma-2} \end{array} \right). \tag{34}$$

If $\aleph < 1$, then the NLIFDE (1) has at least one solution in $C[0, 1]$.

**Proof.** By transforming NLIFDE (1) into a problem involving fixed points, we introduce the operator $\Theta$: $C(J, \mathbb{R}) \to C(J, \mathbb{R})$ as:

$$\Theta(\mathscr{Y}(\tau)) = \Theta_1(\mathscr{Y}(\tau)) + \Theta_2(\mathscr{Y}(\tau)), \quad \tau \in [0, 1], \tag{35}$$

where

$$\Theta_1(\mathscr{Y}(\tau))$$

$$= \frac{(\omega(\tau) - \omega(0))^{\gamma-1}}{\Gamma(\gamma)}\left(\eta_1 \int_0^{\mathfrak{T}} \mathfrak{g}_1(\xi, \mathscr{Y}(\xi))d\xi - \int_0^{\mathfrak{T}} \omega'(\xi)\psi(\xi)d\xi\right)$$

$$+\frac{(\omega(\tau)-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}\left(\begin{array}{c}\varphi(\mathfrak{T})\left(-\eta_1\int_0^{\mathfrak{T}}\mathfrak{g}_1(\xi,\mathscr{Y}(\xi))d\xi+\int_0^{\mathfrak{T}}\omega'(\xi)\psi(\xi)d\xi\right)\\ +\eta_2\int_0^{\mathfrak{T}}\mathfrak{g}_2(\xi,\mathscr{Y}(\xi))d\xi\\ -\int_0^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))\psi(\xi)d\xi.\end{array}\right)$$

$$+\frac{(\omega(\tau)-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\left(\begin{array}{c}\eta_3\int_0^{\mathfrak{T}}\mathfrak{g}_3(\xi,\mathscr{Y}(\xi))d\xi-\frac{1}{2}\int_0^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))^2\psi(\xi)d\xi\\ -\frac{1}{2}\left(\begin{array}{c}\eta_1\int_0^{\mathfrak{T}}\mathfrak{g}_1(\xi,\mathscr{Y}(\xi))d\xi\\ -\int_0^{\mathfrak{T}}\omega'(\xi)\psi(\xi)d\xi\end{array}\right)(\omega(\mathfrak{T})-\omega(0))^2\\ -\varphi(\mathfrak{T})\left(\begin{array}{c}\varphi(\mathfrak{T})\left(-\eta_1\int_0^{\mathfrak{T}}\mathfrak{g}_1(\xi,\mathscr{Y}(\xi))d\xi+\int_0^{\mathfrak{T}}\omega'(\xi)\psi(\xi)d\xi\right)\\ +\eta_2\int_0^{\mathfrak{T}}\mathfrak{g}_2(\xi,\mathscr{Y}(\xi))d\xi\\ -\int_0^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))\psi(\xi)d\xi\end{array}\right)\end{array}\right), \tag{36}$$

and

$$\Theta_2(\mathscr{Y}(\tau))=\frac{1}{\Gamma(\gamma)}\int_0^{\tau}\omega'(\xi)(\omega(\tau)-\omega(\xi))^{\gamma-1}\psi(\xi)d\xi, \tag{37}$$

with

$$\psi(\tau)=\mathscr{F}(\tau,\ \mathscr{Y}(\tau),\ {}^{c}\mathfrak{D}_{0^+}^{\theta,\,\omega}\mathscr{Y}(\tau),\int_0^{\tau}k(\tau,\xi){}^{c}\mathfrak{D}_{0^+}^{\gamma,\,\omega}\mathscr{Y}(\xi)d\xi). \tag{38}$$

Consider $\mathfrak{B}_\rho=\{\mathscr{Y}\in C(J,\mathbb{R})\colon\|\mathscr{Y}\|\leq\rho\}$ as a closed subset of $C[0,1]$, where $\rho$ represents a positive constant satisfying $\rho\geq\frac{\mathfrak{R}}{1-\aleph}$. Here, $\mathfrak{R}$ and $\aleph$ are real numbers specified previously. It is evident that $\mathfrak{B}_\rho$ constitutes a Banach space equipped with a metric in $C[0,\mathfrak{T}]$. The proof can be outlined in three distinct phases.

**Step 1:** $\Theta_1\mathscr{Y}_1+\Theta_2\mathscr{Y}_2\in\mathfrak{B}_\rho$ holds true for all $\mathscr{Y}_1,\mathscr{Y}_2\in\mathfrak{B}_\rho$.

Consider $\mathscr{Y}_1,\mathscr{Y}_2\in\mathfrak{B}_\rho$ and $\tau\in J$. We obtain that

$$|\Theta_1\mathscr{Y}_1(\tau)+\Theta_2\mathscr{Y}_2(\tau)|\leq|\Theta_1\mathscr{Y}_1(\tau)|+|\Theta_2\mathscr{Y}_2(\tau)|$$

$$\leq\frac{|\omega(\tau)-\omega(0)|^{\gamma-1}}{\Gamma(\gamma)}\left(\eta_1\int_0^{\mathfrak{T}}|\mathfrak{g}_1(\xi,\mathscr{Y}_1(\xi))|d\xi+\int_0^{\mathfrak{T}}\omega'(\xi)|\psi_1(\xi)|d\xi\right)$$

$$+\frac{|\omega(\tau)-\omega(0)|^{\gamma-2}}{\Gamma(\gamma-1)}\left(\begin{array}{c}|\varphi(\mathfrak{T})|\left(\eta_1\int_0^{\mathfrak{T}}|\mathfrak{g}_1(\xi,\mathscr{Y}_1(\xi))|d\xi+\int_0^{\mathfrak{T}}\omega'(\xi)|\psi_1(\xi)|d\xi\right)\\ +\eta_2\int_0^{\mathfrak{T}}|\mathfrak{g}_2(\xi,\mathscr{Y}_1(\xi))|d\xi\\ +\int_0^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))|\psi_1(\xi)|d\xi.\end{array}\right)$$

$$+\frac{|\omega(\tau)-\omega(0)|^{\gamma-3}}{\Gamma(\gamma-2)}\left(\begin{array}{c}\eta_3\int_0^{\mathfrak{T}}|\mathfrak{g}_3(\xi,\mathscr{Y}_1(\xi))|d\xi \\ +\frac{1}{2}\int_0^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))^2|\psi_1(\xi)|d\xi \\ +\frac{1}{2}\left(\begin{array}{c}\eta_1\int_0^{\mathfrak{T}}|\mathfrak{g}_1(\xi,\mathscr{Y}_1(\xi))|d\xi \\ +\int_0^{\mathfrak{T}}\omega'(\xi)|\psi_1(\xi)|d\xi\end{array}\right)(\omega(\mathfrak{T})-\omega(0))^2 \\ +|\varphi(\mathfrak{T})|\left(\begin{array}{c}|\varphi(\mathfrak{T})|\left(\begin{array}{c}\eta_1\int_0^{\mathfrak{T}}|\mathfrak{g}_1(\xi,\mathscr{Y}_1(\xi))|d\xi \\ +\int_0^{\mathfrak{T}}\omega'(\xi)|\psi_1(\xi)|d\xi\end{array}\right) \\ +\eta_2\int_0^{\mathfrak{T}}|\mathfrak{g}_2(\xi,\mathscr{Y}_1(\xi))|d\xi \\ +\int_0^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))|\psi_1(\xi)|d\xi\end{array}\right)\end{array}\right)$$

$$+\frac{1}{\Gamma(\gamma)}\int_0^{\tau}\omega'(\xi)(\omega(\tau)-\omega(\xi))^{\gamma-1}|\psi_2(\xi)|d\xi. \tag{39}$$

Using Lemma (5) and the aforementioned remark, if we consider the supremum for $\tau\in[0,\mathfrak{T}]$, then

$$\begin{aligned}&|\mathscr{F}(\tau,\mathscr{Y}(\tau),\ {}^c\mathfrak{D}_{0^+}^{\theta,\,\omega}\mathscr{Y}(\tau),\int_0^{\tau}k(\tau,\xi)\,{}^c\mathfrak{D}_{0^+}^{\theta,\,\omega}\mathscr{Y}(\xi)d\xi)|\\ &\leq\|\lambda\|(|\mathscr{Y}(\tau)|+|\,{}^c\mathfrak{D}_{0^+}^{\theta,\,\omega}\mathscr{Y}(\tau))|+\int_0^{\tau}|k(\tau,\xi)\,{}^c\mathfrak{D}_{0^+}^{\theta,\,\omega}\mathscr{Y}(\xi)d\xi|+F,\\ &\leq\|\lambda\|\left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right)\|\mathscr{Y}\|+F,\end{aligned}\tag{40}$$

where $F=\sup_{\tau\in J}|\mathscr{F}(\tau,0,0,0)|$.

Thus, for each $\tau\in[0,\mathfrak{T}]$ we have

$$\begin{aligned}&|\Theta_1\mathscr{Y}_1(\tau)+\Theta_2\mathscr{Y}_2(\tau)|\\ &\leq|\Theta_1\mathscr{Y}_1(\tau)|+|\Theta_2\mathscr{Y}_2(\tau)|\\ &\leq\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-1}}{\Gamma(\gamma)}\left(\begin{array}{c}\eta_1\mathfrak{T}(H_1+\mu_1(\tau)\|\mathscr{Y}_1\|)\\ +\left(\|\lambda\|\left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right)\|\mathscr{Y}_1\|+F\right)(\omega(\mathfrak{T})-\omega(0))\end{array}\right)\end{aligned}$$

$$
+\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}\left(\begin{array}{c}\eta_1\mathfrak{T}(H_1+\mu_1(\tau)\|\mathscr{Y}_1\|)\varphi(\mathfrak{T})+\\ \left(\|\lambda\|\left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right)\|\mathscr{Y}_1\|+F\right)\varphi(\mathfrak{T})(\omega(\mathfrak{T})-\omega(0))\\ +\begin{array}{c}\eta_2\mathfrak{T}(H_2+\mu_2(\tau)\|\mathscr{Y}_1\|)+\\ \left(\|\lambda\|\left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right)\|\mathscr{Y}_1\|+F\right)\frac{(\omega(\mathfrak{T})-\omega(0))^2}{2}\end{array}\end{array}\right)
$$

$$
+\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\left(\begin{array}{c}\eta_3\mathfrak{T}(H_3+\mu_3(\tau)\|\mathscr{Y}_1\|)+\\ \left(\begin{array}{c}\|\lambda\|\left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right)\|\mathscr{Y}_1\|\\ +F\end{array}\right)\frac{(\omega(\mathfrak{T})-\omega(0))^3}{6}\\ +\left(\begin{array}{c}\eta_1\mathfrak{T}(H_1+\mu_1(\tau)\|\mathscr{Y}_1\|)\frac{(\omega(\mathfrak{T})-\omega(0))^2}{2}+\\ \left(\begin{array}{c}\|\lambda\|\left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right)\|\mathscr{Y}_1\|\\ +F\end{array}\right)\frac{(\omega(\mathfrak{T})-\omega(0))^3}{2}\end{array}\right)\\ +\begin{array}{c}\eta_1\mathfrak{T}\varphi^2(\mathfrak{T})(H_1+\mu_1(\tau)\|\mathscr{Y}_1\|)+\\ \left(\begin{array}{c}\|\lambda\|\left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right)\|\mathscr{Y}_1\|\\ +F\end{array}\right)\varphi^2(\mathfrak{T})(\omega(\mathfrak{T})-\omega(0))\\ +\eta_2\mathfrak{T}\varphi(\mathfrak{T})(H_2+\mu_2(\tau)\|\mathscr{Y}_1\|)+\\ \left(\begin{array}{c}\|\lambda\|\left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right)\|\mathscr{Y}_1\|\\ +F\end{array}\right)\varphi(\mathfrak{T})\frac{(\omega(\mathfrak{T})-\omega(0))^2}{2}\end{array}\end{array}\right)
$$

$$
+\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma}}{|\Gamma(\gamma+1)|}\left(\|\lambda\|\left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right)\|\mathscr{Y}_2\|+F\right)\mathfrak{T}. \tag{41}
$$

Taking supremum over $\tau\in[0,\mathfrak{T}]$, we have

$$
\|\Theta_1\mathscr{Y}_1(\tau)+\Theta_2\mathscr{Y}_2(\tau)\|\le\rho, \tag{42}
$$

for $\rho\ge\frac{\Re}{1-\aleph}$, where

$$
\Re=(H_1\eta_1\mathfrak{T}\Upsilon_1+H_2\eta_2\mathfrak{T}\Upsilon_2+H_3\eta_3\mathfrak{T}\Upsilon_3+F\Upsilon_4), \tag{43}
$$

such that

$$\Upsilon_1 = \left(\frac{1}{\Gamma(\gamma)} + \frac{1}{2\Gamma(\gamma-2)}\right)(\omega(\mathfrak{T}) - \omega(0))^{\gamma-1} + \varphi(\mathfrak{T})\frac{(\omega(\mathfrak{T}) - \omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}$$

$$+\varphi^2(\mathfrak{T})\frac{(\omega(\mathfrak{T}) - \omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}, \tag{44}$$

$$\Upsilon_2 = \frac{(\omega(\mathfrak{T}) - \omega(0))^{\gamma-2}}{\Gamma(\gamma-1)} + \varphi(\mathfrak{T})\frac{(\omega(\mathfrak{T}) - \omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}, \tag{45}$$

$$\Upsilon_3 = \frac{(\omega(\mathfrak{T}) - \omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}, \tag{46}$$

$$\Upsilon_4 = \left(\frac{\mathfrak{T}}{|\Gamma(\gamma+1)|} + \frac{1}{\Gamma(\gamma)} + \frac{1}{2\Gamma(\gamma-1)} + \frac{2}{3\Gamma(\gamma-2)}\right)(\omega(\mathfrak{T}) - \omega(0))^{\gamma}$$

$$+\varphi(\mathfrak{T})\left(\frac{1}{\Gamma(\gamma-1)} + \frac{1}{2\Gamma(\gamma-2)}\right)(\omega(\mathfrak{T}) - \omega(0))^{\gamma-1}$$

$$+\varphi^2(\mathfrak{T})\frac{(\omega(\mathfrak{T}) - \omega(0))^{\gamma-2}}{\Gamma(\gamma-2)}, \tag{47}$$

and

$$\aleph = \|\mu_1\|\,\eta_1\mathfrak{T}\Lambda_1 + \|\mu_2\|\,\eta_2\mathfrak{T}\Lambda_2 + \|\mu_3\|\,\eta_3\mathfrak{T}\Lambda_3 + \Lambda_4, \tag{48}$$

with

$$\Lambda_1 = \left(\frac{1}{\Gamma(\gamma)} + \frac{1}{2\Gamma(\gamma-2)}\right)(\omega(\mathfrak{T}) - \omega(0))^{\gamma-1} + \varphi(\mathfrak{T})\frac{(\omega(\mathfrak{T}) - \omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}$$

$$+\varphi^2(\mathfrak{T})\frac{(\omega(\mathfrak{T}) - \omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}, \tag{49}$$

$$\Lambda_2 = \frac{(\omega(\mathfrak{T}) - \omega(0))^{\gamma-2}}{\Gamma(\gamma-1)} + \varphi(\mathfrak{T})\frac{(\omega(\mathfrak{T}) - \omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}, \tag{50}$$

$$\Lambda_3 = \frac{(\omega(\mathfrak{T}) - \omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}, \tag{51}$$

and

$$\Lambda_4=\|\lambda\|\left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right)\left(\begin{array}{c}\left(\frac{\mathfrak{T}}{|\Gamma(\gamma+1)|}+\frac{1}{\Gamma(\gamma)}+\frac{1}{2\Gamma(\gamma-1)}+\frac{2}{3\Gamma(\gamma-2)}\right)(\omega(\mathfrak{T})-\omega(0))^{\gamma}\\ +\left(\frac{1}{\Gamma(\gamma-1)}+\frac{1}{2\Gamma(\gamma-2)}\right)\varphi(\mathfrak{T})(\omega(\mathfrak{T})-\omega(0))^{\gamma-1}\\ +\frac{1}{\Gamma(\gamma-2)}\varphi^2(\mathfrak{T})(\omega(\mathfrak{T})-\omega(0))^{\gamma-2}\end{array}\right). \tag{52}$$

This proves that $\Theta_1\mathscr{Y}_1(\tau)+\Theta_2\mathscr{Y}_2(\tau)\in\mathfrak{B}_\rho$ for every $\mathscr{Y}_1$, $\mathscr{Y}_2\in\mathfrak{B}_\rho$.

**Step 2:** The operator $\Theta_1$ serves as a contraction mapping on $\mathfrak{B}_\rho$. It is clear that

$$\begin{aligned}
&|\Theta_1(\mathscr{Y}_1(\tau))-\Theta_1(\mathscr{Y}_2(\tau))|\\
=&\frac{(\omega(\tau)-\omega(0))^{\gamma-1}}{\Gamma(\gamma)}\left(\eta_1\int_0^{\mathfrak{T}}|\mathfrak{g}_1(\xi,\mathscr{Y}_1(\xi))-\mathfrak{g}_1(\xi,\mathscr{Y}_2(\xi))|d\xi\right)\\
&+\frac{(\omega(\tau)-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}\left(\begin{array}{c}\eta_1\varphi(\mathfrak{T})\left(\int_0^{\mathfrak{T}}|\mathfrak{g}_1(\xi,\mathscr{Y}_2(\xi))-\mathfrak{g}_1(\xi,\mathscr{Y}_1(\xi))|d\xi\right)\\ +\eta_2\int_0^{\mathfrak{T}}|\mathfrak{g}_2(\xi,\mathscr{Y}_1(\xi))-\mathfrak{g}_2(\xi,\mathscr{Y}_2(\xi))|d\xi\end{array}\right)\\
&+\frac{(\omega(\tau)-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\left(\begin{array}{c}\eta_3\int_0^{\mathfrak{T}}|\mathfrak{g}_3(\xi,\mathscr{Y}_1(\xi))-\mathfrak{g}_3(\xi,\mathscr{Y}_2(\xi))|d\xi\\ -\frac{\eta_1}{2}\left(\int_0^{\mathfrak{T}}|\mathfrak{g}_1(\xi,\mathscr{Y}_1(\xi))-\mathfrak{g}_1(\xi,\mathscr{Y}_2(\xi))|d\xi\right)(\omega(\mathfrak{T})-\omega(0))^2\\ -\varphi(\mathfrak{T})\left(\begin{array}{c}-\eta_1\varphi(\mathfrak{T})\left(\int_0^{\mathfrak{T}}|\mathfrak{g}_1(\xi,\mathscr{Y}_1(\xi))-\mathfrak{g}_1(\xi,\mathscr{Y}_2(\xi))|d\xi\right)\\ +\eta_2\int_0^{\mathfrak{T}}|\mathfrak{g}_2(\xi,\mathscr{Y}_1(\xi))-\mathfrak{g}_2(\xi,\mathscr{Y}_2(\xi))|d\xi\end{array}\right)\end{array}\right),\\
\leq&\frac{(\omega(\tau)-\omega(0))^{\gamma-1}}{\Gamma(\gamma)}\left(\eta_1\int_0^{\mathfrak{T}}\mu_1(\xi)|\mathscr{Y}_1(\xi)-\mathscr{Y}_2(\xi)|d\xi\right)\\
&+\frac{(\omega(\tau)-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}\left(\begin{array}{c}\eta_1\varphi(\mathfrak{T})\left(\int_0^{\mathfrak{T}}\mu_1(\xi)|\mathscr{Y}_2(\xi)-\mathscr{Y}_1(\xi)|d\xi\right)\\ +\eta_2\int_0^{\mathfrak{T}}\mu_2(\xi)|\mathscr{Y}_1(\xi)-\mathscr{Y}_2(\xi)|d\xi\end{array}\right)\\
&+\frac{(\omega(\tau)-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\left(\begin{array}{c}\eta_3\int_0^{\mathfrak{T}}\mu_3(\xi)|\mathscr{Y}_1(\xi)-\mathscr{Y}_2(\xi)|d\xi\\ +\frac{\eta_1}{2}\left(\int_0^{\mathfrak{T}}\mu_1(\xi)|\mathscr{Y}_2(\xi)-\mathscr{Y}_1(\xi)|d\xi\right)(\omega(\mathfrak{T})-\omega(0))^2\\ +\varphi(\mathfrak{T})\left(\begin{array}{c}\eta_1\varphi(\mathfrak{T})\int_0^{\mathfrak{T}}\mu_1(\xi)|\mathscr{Y}_2(\xi)-\mathscr{Y}_1(\xi)|d\xi\\ +\eta_2\int_0^{\mathfrak{T}}\mu_2(\xi)|\mathscr{Y}_2(\xi)-\mathscr{Y}_1(\xi)|d\xi\end{array}\right)\end{array}\right)
\end{aligned} \tag{53}$$

Taking supremum over $\tau \in [0, \mathfrak{T}]$, we get

$$\left\|\Theta_1(\mathscr{Y}_1)-\Theta_1(\mathscr{Y}_2)\right\|$$

$$\leq \frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-1}}{\Gamma(\gamma)}\left(\eta_1\mathfrak{T}\|\mu_1\|\|\mathscr{Y}_1-\mathscr{Y}_2\|\right)$$

$$+\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}\left(\eta_1\mathfrak{T}\varphi(\mathfrak{T})\|\mu_1\|\|\mathscr{Y}_1-\mathscr{Y}_2\|+\eta_2\mathfrak{T}\|\mu_2\|\|\mathscr{Y}_1-\mathscr{Y}_2\|\right)$$

$$+\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\begin{pmatrix}\eta_3\mathfrak{T}\|\mu_3\|\|\mathscr{Y}_1-\mathscr{Y}_2\|+\frac{\eta_1\mathfrak{T}}{2}\|\mu_1\|\|\mathscr{Y}_1-\mathscr{Y}_2\|(\omega(\mathfrak{T})-\omega(0))^2\\ +\varphi(\mathfrak{T})\left(\eta_1^2\mathfrak{T}\varphi(\mathfrak{T})\|\mu_1\|\|\mathscr{Y}_1-\mathscr{Y}_2\|+\eta_2\mathfrak{T}\|\mu_2\|\|\mathscr{Y}_1-\mathscr{Y}_2\|\right)\end{pmatrix}$$

$$\leq\begin{pmatrix}\eta_1\mathfrak{T}\|\mu_1\|\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-1}}{\Gamma(\gamma)}\\ +\left(\eta_1\mathfrak{T}\varphi(\mathfrak{T})\|\mu_1\|+\eta_2\mathfrak{T}\|\mu_2\|\right)\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}\\ +\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\begin{pmatrix}\eta_3\mathfrak{T}\|\mu_3\|+\frac{\eta_1\mathfrak{T}}{2}\|\mu_1\|(\omega(\mathfrak{T})-\omega(0))^2\\ +\eta_1^2\mathfrak{T}\varphi^2(\mathfrak{T})\|\mu_1\|+\eta_2T\varphi(\mathfrak{T})\|\mu_2\|\end{pmatrix}\end{pmatrix}\|\mathscr{Y}_1-\mathscr{Y}_2\|$$

$$\leq\begin{pmatrix}\eta_1\mathfrak{T}\|\mu_1\|\begin{pmatrix}\left(\frac{1}{\Gamma(\gamma)}+\frac{1}{2\Gamma(\gamma-2)}\right)(\omega(\mathfrak{T})-\omega(0))^{\gamma-1}\\ +\varphi(\mathfrak{T})\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}+\eta_1\varphi^2(\mathfrak{T})\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\end{pmatrix}\\ +\eta_2\mathfrak{T}\|\mu_2\|\left(\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}+\varphi(\mathfrak{T})\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\right)\\ +\eta_3\mathfrak{T}\|\mu_3\|\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\end{pmatrix}\|\mathscr{Y}_1-\mathscr{Y}_2\|$$

$$\leq\left(\|\mu_1\|\eta_1T\Lambda_1+\|\mu_2\|\eta_2T\Lambda_2+\|\mu_3\|\eta_3T\Lambda_3\right)\|\mathscr{Y}_1-\mathscr{Y}_2\|$$

$$\leq(\aleph-\Lambda_4)\|\mathscr{Y}_1-\mathscr{Y}_2\| \tag{54}$$

Thus, it is clear that the operator $\Theta_1$ is a contraction mapping with a contraction coefficient of $\varsigma<1$, where $\varsigma=\aleph-\Lambda_4$.

**Step 3:** To establish the continuity and compactness of the operator $\Theta_2$ on $\mathfrak{B}_\rho$, we initially establish its continuity. Let $\{\mathscr{Y}_n\}_{n\in N}$ be a sequence in $\mathfrak{B}_\rho$ that converges to $\mathscr{Y}\in\mathfrak{B}_\rho$ as $n$ tends to infinity. Our objective is to demonstrate that $\|\Theta_2\mathscr{Y}_n-\Theta_1\mathscr{Y}\|$ tends to zero as $n$ tends to infinity. Subsequently, for $\tau\in[0,\mathfrak{T}]$, we have

$$|\Theta_2\mathscr{Y}_n-\Theta_2\mathscr{Y}|\leq\frac{1}{\Gamma(\gamma)}\int_0^\tau\omega'(\xi)(\omega(\tau)-\omega(\xi))^{\gamma-1}|\psi_n(\xi)-\psi(\xi)|d\xi, \tag{55}$$

where

$$\psi_n(\tau) = \mathscr{F}(\tau,\ \mathscr{Y}_n(\tau),\ {}^c\mathfrak{D}_{0^+}^{\theta,\,\omega}\mathscr{Y}_n(\tau),\int_0^\tau k(\tau,\,\xi)\,{}^c\mathfrak{D}_{0^+}^{\gamma,\,\omega}\mathscr{Y}_n(\xi)d\xi), \tag{56}$$

and

$$\psi(\tau) = \mathscr{F}(\tau,\mathscr{Y}(\tau),\ {}^c\mathfrak{D}_{0^+}^{\theta}\mathscr{Y}(\tau),\int_0^\tau k(\tau,\,\xi)\,{}^c\mathfrak{D}_{0^+}^{\gamma,\,\omega}\mathscr{Y}(\xi)d\xi).$$

Here, we have $\psi_n$ and $\psi$ are two continuous functions defined over $[0,\,\mathfrak{T}]$ so that

$$\begin{aligned}
&|\psi_n(\tau)-\psi(\tau)|\\
=&|\mathscr{F}(\tau,\mathscr{Y}_n(\tau),\ {}^c\mathfrak{D}_{0^+}^{\theta}\mathscr{Y}_n(\tau),\int_0^\tau k(\tau,\,\xi)\,{}^c\mathfrak{D}_{0^+}^{\gamma,\,\omega}\mathscr{Y}_n(\xi)d\xi)-\mathscr{F}(\tau,\mathscr{Y}(\tau),\ {}^c\mathfrak{D}_{0^+}^{\theta}\mathscr{Y}(\tau),\int_0^\tau k(\tau,\,\xi)\,{}^c\mathfrak{D}_{0^+}^{\gamma,\,\omega}\mathscr{Y}(\xi)d\xi)|,\\
\leq&|\lambda(\tau)|(|\mathscr{Y}_n(\tau)-\mathscr{Y}(\tau)|+|\,{}^c\mathfrak{D}_{0^+}^{\theta}\mathscr{Y}_n(\tau)-\ {}^c\mathfrak{D}_{0^+}^{\theta}\mathscr{Y}(\tau)\,|\\
&+\int_0^\tau|k(\tau,\,\xi)|\left|\,{}^c\mathfrak{D}_{0^+}^{\gamma,\,\omega}\mathscr{Y}_n(\xi)-\ {}^c\mathfrak{D}_{0^+}^{\gamma,\,\omega}\mathscr{Y}(\xi)\right|d\xi)|),\\
\leq&\|\lambda\|\left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right)\|\mathscr{Y}_n-\mathscr{Y}\|.
\end{aligned} \tag{57}$$

Since $\mathscr{Y}_n$ tends to $\mathscr{Y}$, then we obtain that $\psi_n(\tau)$ tends to $\psi(\tau)$ as $n$ tends to infinity for every $\tau\in[0,\,\mathfrak{T}]$. In addition, consider the positive real number $\varepsilon>0$ such that for every $\tau\in[0,\,\mathfrak{T}]$. If we take $|\psi_n(\tau)|\leq\varepsilon/2$ and $|\psi(\tau)|\leq\varepsilon/2$, we obtain that $|\psi_n(\xi)-\psi(\xi)|\leq(|\psi_n(\xi)|+|\psi(\xi)|)\leq\varepsilon$ for every $\tau\in[0,\,\mathfrak{T}]$. Utilizing the Lebesgue dominated convergence theorem leads to the implication that $\|\Theta_2\mathscr{Y}_n-\Theta_2\mathscr{Y}\|$ tends to 0 as $n$ tends to $\infty$. This implies that operator $\Theta_2$ is continuous.

In addition, we obtain due to definition of $\rho$ that

$$\|\Theta_2\mathscr{Y}\|\leq\frac{1}{|\Gamma(\gamma+1)|}\left[\|\lambda\|\left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right)\|\mathscr{Y}\|+F\right]\leq\rho. \tag{58}$$

Hence, $\Theta_2$ is a uniformly bounded operator defined on the closed ball $\mathfrak{B}_\rho$.

At the end, we show that the function $\Theta_2$ converts limited collections into sets that exhibit uniform continuity within $C(J,\,R)$, particularly guaranteeing the uniform continuity of $\mathfrak{B}_\rho$.

Let $\forall\,\varepsilon>0,\ \exists\,\delta>0$ and $\tau_1,\,\tau_2\in J,\ \ \tau_1<\tau_2,\ \ |\tau_2-\tau_1|<\delta$. Then we have

$$|\Theta_2\mathscr{Y}(\tau_2)-\Theta_2\mathscr{Y}(\tau_1)| \leq \frac{1}{\Gamma(\gamma)}\int_0^\tau \omega'(\xi)\left((\omega(\tau_2)-\omega(\xi))^{\gamma-1}-(\omega(\tau_1)-\omega(\xi))^{\gamma-1}\right)|\psi(\xi)|d\xi,$$

$$\leq \|\lambda\|\left(\|\lambda\|\left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right)\|\mathscr{Y}\|\right)\frac{(\omega(\tau_2)^\gamma-\omega(\tau_1)^\gamma)}{\alpha\Gamma(\gamma)}. \tag{59}$$

As $\tau_1$ approaches $\tau_2$, the expression on the right side of the mentioned inequality becomes unrelated to $\mathscr{Y}$ and tends toward zero. Consequently,

$$|\Theta_2\mathscr{Y}(\tau_2)-\Theta_2\mathscr{Y}(\tau_1)| \to 0, \quad \forall\, |\tau_2-\tau_1| \to 0. \tag{60}$$

Hence, if the compact operator $\Theta$ is uniformly continuous on the closed ball $\mathfrak{B}_\rho$, then by applying the Arzela-Ascoli theorem, we obtain that $\Theta$: $C([0,\,\mathfrak{T}],\mathbb{R}) \to C([0,\,\mathfrak{T}],\mathbb{R})$ is both continuous and compact at the same time. Therefore, all the required conditions for applying Krasnoselskii's fixed-point theorem are satisfied, and hence the operator $\Theta=\Theta_1+\Theta_2$ has a fixed point $\mathscr{Y}(\tau)\in C[0,\,\mathfrak{T}]$ on the closed ball $\mathfrak{B}_\rho$ and satisfies the boundary conditions in problem (1). As a result, we obtain that the function $\mathscr{Y}(\tau)$ acts as a solution of the NLIFDE (1). This completes the proof. □

## 3.2 *Uniqueness of solutions*

The following lemmas and theorems establish the uniqueness of solutions for the nonlinear fractional differential equation (NLIFDE) (1) by applying Krasnoselskii's and Banach's fixed point theorems, given that the conditions in Lemma 5 are satisfied. These findings are significant as they provide a robust mathematical foundation for the existence and uniqueness of solutions within the Banach algebra framework.

**Lemma 6** Assume that the assumptions $(H_1)$-$(H_3)$ hold. If the following inequality

$$\left(\begin{array}{c} \eta_1\mathfrak{T}\|\mu_1\|\left(\begin{array}{c}\left(\frac{1}{\Gamma(\gamma)}+\frac{1}{2\Gamma(\gamma-2)}\right)(\omega(\mathfrak{T})-\omega(0))^{\gamma-1}(\omega(\mathfrak{T})-\omega(0))^{\gamma-1}\\ +\varphi(\mathfrak{T})\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}+\varphi^2(\mathfrak{T})\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\end{array}\right)\\ +\eta_2\mathfrak{T}\|\mu_2\|\left(\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}+\varphi(\mathfrak{T})\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\right)\\ +\eta_3\mathfrak{T}\|\mu_3\|\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}+\|\lambda\|\left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right)\end{array}\right)<1, \tag{61}$$

holds, then the operator $\Theta$: $C(J,R)\to C(J,R)$ presented in Definition 4 is a contraction.

**Proof.** Assuming that conditions $(H_1)$-$(H_3)$ are satisfied, let's examine the continuous functions $\mathscr{Y}_1(\tau)$ and $\mathscr{Y}_2(\tau)$ belonging to $C(J,R)$. In this context, for any $\tau\in J$, the following applies:

$$|\Theta(\mathscr{Y}_1(\tau))-\Theta(\mathscr{Y}_2(\tau))|$$

$$\leq|\Theta_1(\mathscr{Y}_1(\tau))-\Theta_1(\mathscr{Y}_2(\tau))+\Theta_2(\mathscr{Y}_1(\tau))-\Theta_2(\mathscr{Y}_2(\tau))|$$

$$\leq|\Theta_1(\mathscr{Y}_1(\tau))-\Theta_1(\mathscr{Y}_2(\tau))|+|\Theta_2(\mathscr{Y}_1(\tau))-\Theta_2(\mathscr{Y}_2(\tau))|$$

$$
\begin{aligned}
&= \frac{(\omega(\tau)-\omega(0))^{\gamma-1}}{\Gamma(\gamma)}\left(\eta_1\int_0^{\mathfrak{T}}|\mathfrak{g}_1(\xi,\mathscr{Y}_1(\xi))-\mathfrak{g}_1(\xi,\mathscr{Y}_2(\xi))|\,d\xi\right)\\
&\quad+\frac{(\omega(\tau)-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}\begin{pmatrix}\eta_1\varphi(\mathfrak{T})\left(\int_0^{\mathfrak{T}}|\mathfrak{g}_1(\xi,\mathscr{Y}_2(\xi))-\mathfrak{g}_1(\xi,\mathscr{Y}_1(\xi))|\,d\xi\right)\\ +\eta_2\int_0^{\mathfrak{T}}|\mathfrak{g}_2(\xi,\mathscr{Y}_1(\xi))-\mathfrak{g}_2(\xi,\mathscr{Y}_2(\xi))|\,d\xi\end{pmatrix}\\
&\quad+\frac{(\omega(\tau)-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\begin{pmatrix}\eta_3\int_0^{\mathfrak{T}}|\mathfrak{g}_3(\xi,\mathscr{Y}_1(\xi))-\mathfrak{g}_3(\xi,\mathscr{Y}_2(\xi))|\,d\xi\\ -\frac{\eta_1}{2}\left(\int_0^{\mathfrak{T}}|\mathfrak{g}_1(\xi,\mathscr{Y}_1(\xi))-\mathfrak{g}_1(\xi,\mathscr{Y}_2(\xi))|\,d\xi\right)(\omega(\mathfrak{T})-\omega(0))^2\\ -\varphi(\mathfrak{T})\begin{pmatrix}-\eta_1\varphi(\mathfrak{T})\left(\int_0^{\mathfrak{T}}|\mathfrak{g}_1(\xi,\mathscr{Y}_1(\xi))-\mathfrak{g}_1(\xi,\mathscr{Y}_2(\xi))|\,d\xi\right)\\ +\eta_2\int_0^{\mathfrak{T}}|\mathfrak{g}_2(\xi,\mathscr{Y}_1(\xi))-\mathfrak{g}_2(\xi,\mathscr{Y}_2(\xi))|\,d\xi\end{pmatrix}\end{pmatrix}\\
&\quad+\frac{1}{\Gamma(\gamma)}\int_0^{\tau}\omega'(\xi)(\omega(\tau)-\omega(\xi))^{\gamma-1}|\psi_1(\xi)-\psi_2(\xi)|\,d\xi\\
&\le \frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-1}}{\Gamma(\gamma)}(\eta_1\mathfrak{T}\|\mu_1\|\,\|\mathscr{Y}_1-\mathscr{Y}_2\|)\\
&\quad+\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}(\eta_1\mathfrak{T}\varphi(\mathfrak{T})\|\mu_1\|\,\|\mathscr{Y}_1-\mathscr{Y}_2\|+\eta_2\mathfrak{T}\|\mu_2\|\,\|\mathscr{Y}_1-\mathscr{Y}_2\|)\\
&\quad+\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\begin{pmatrix}\eta_3\mathfrak{T}\|\mu_3\|\,\|\mathscr{Y}_1-\mathscr{Y}_2\|+\frac{\eta_1\mathfrak{T}}{2}\|\mu_1\|\,\|\mathscr{Y}_1-\mathscr{Y}_2\|(\omega(\mathfrak{T})-\omega(0))^2\\ +\varphi(\mathfrak{T})\left(\eta_1^2\mathfrak{T}\varphi(\mathfrak{T})\|\mu_1\|\,\|\mathscr{Y}_1-\mathscr{Y}_2\|+\eta_2\mathfrak{T}\|\mu_2\|\,\|\mathscr{Y}_1-\mathscr{Y}_2\|\right)\end{pmatrix}\\
&\quad+\|\lambda\|\left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right)\|\mathscr{Y}_1-\mathscr{Y}_2\|\\
&\le \begin{pmatrix}\eta_1\mathfrak{T}\|\mu_1\|\begin{pmatrix}\left(\frac{1}{\Gamma(\gamma)}+\frac{1}{2\Gamma(\gamma-2)}\right)(\omega(\mathfrak{T})-\omega(0))^{\gamma-1}(\omega(\mathfrak{T})-\omega(0))^{\gamma-1}\\ +\varphi(\mathfrak{T})\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}+\varphi^2(\mathfrak{T})\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\end{pmatrix}\\ +\eta_2\mathfrak{T}\|\mu_2\|\left(\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}+\varphi(\mathfrak{T})\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\right)\\ +\eta_3\mathfrak{T}\|\mu_3\|\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}+\|\lambda\|\left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right)\end{pmatrix}\|\mathscr{Y}_1-\mathscr{Y}_2\|\\
&\le (\|\mu_1\|\,\eta_1T\Lambda_1+\|\mu_2\|\,\eta_2T\Lambda_2+\|\mu_3\|\,\eta_3T\Lambda_3)\,\|\mathscr{Y}_1-\mathscr{Y}_2\|\\
&\le (\aleph-\Lambda_4)\,\|\mathscr{Y}_1-\mathscr{Y}_2\|
\end{aligned} \tag{62}
$$

Taking supremum for all $\tau \in \mathfrak{T}$, we have

$$\|\Theta(\mathscr{Y}_1) - \Theta(\mathscr{Y}_2)\| \leq (\aleph - \Lambda_4)\|\mathscr{Y}_1 - \mathscr{Y}_2\|. \tag{63}$$

Now, since $(\aleph - \Lambda_4) < 1$, then the operator $\Theta$ is a contraction.

Therefore, by applying Krasnselskii's fixed point theorem, the nonlinear fractional differential equation NLIFDE (1) at least one solution. □

**Theorem 4** Assume that the assumptions $(H_1)$-$(H_3)$ hold. If the following inequality

$$\left(\begin{array}{c} \eta_1 \mathfrak{T}\|\mu_1\| \left(\begin{array}{c} \left(\frac{1}{\Gamma(\gamma)} + \frac{1}{2\Gamma(\gamma-2)}\right)(\omega(\mathfrak{T}) - \omega(0))^{\gamma-1}(\omega(\mathfrak{T}) - \omega(0))^{\gamma-1} \\ +\varphi(\mathfrak{T})\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)} + \varphi^2(\mathfrak{T})\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)} \end{array}\right) \\ +\eta_2\mathfrak{T}\|\mu_2\|\left(\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)} + \varphi(\mathfrak{T})\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\right) \\ +\eta_3\mathfrak{T}\|\mu_3\|\frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)} + \|\lambda\|\left(1 + \frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right) \end{array}\right) < 1, \tag{64}$$

holds, then the NLIFDE (1) has a unique solution on $J = [0, \mathfrak{T}]$.

**Proof.** The existence of at least one solution for NLIFDE (1) has been established in Theorem (3). Furthermore, Lemma (6) demonstrates that the operator $\Theta$ exhibits contraction properties. Consequently, through Banach's fixed point theorem, we conclude that the operator $\Theta$ possesses a single fixed point, which corresponds to a unique solution of the NLIFDE (1) over the interval $J = [0, \mathfrak{T}]$. Thus, the proof is now fully accomplished. □

### 3.3 *Stability analysis: Ulam-Hyers and Ulam-Hyers-Rassias stability*

In this section, we analyze the stability of solutions to the nonlinear fractional differential equation (NLIFDE) using the Ulam-Hyers and Ulam-Hyers-Rassias stability criteria. Stability analysis is essential for understanding the robustness of solutions, especially in the presence of perturbations. The Ulam-Hyers stability criterion assesses whether solutions remain close to exact solutions under small perturbations, while the Ulam-Hyers-Rassias stability criterion extends this analysis to more general forms of perturbations. These criteria provide a comprehensive framework for evaluating the stability and reliability of the solutions obtained for the NLIFDE within the Banach algebra framework.

Let $\varepsilon > 0$, $\Phi$: $J \to \mathbb{R}^+$ be a continuous function, and consider the following inequalities for all $\tau \in [0, \mathfrak{T}]$, $0 < \theta \leq 1$, $2 < \gamma \leq 3$.

$$|{}^c\mathfrak{D}_{0^+}^{\gamma,\,\omega}\mathscr{Y}(\tau) - \mathscr{F}(\tau, \mathscr{Y}(\tau), {}^c\mathfrak{D}_{0^+}^{\theta,\,\omega}\mathscr{Y}(\tau), \int_0^\tau k(\tau, \xi)\, {}^c\mathfrak{D}_{0^+}^{\gamma,\,\omega}\mathscr{Y}(\xi)d\xi)| \leq \varepsilon(\tau), \tag{65}$$

$$|{}^c\mathfrak{D}_{0^+}^{\gamma,\,\omega}\mathscr{Y}(\tau) - \mathscr{F}(\tau, \mathscr{Y}(\tau), {}^c\mathfrak{D}_{0^+}^{\theta,\,\omega}\mathscr{Y}(\tau), \int_0^\tau k(\tau, \xi)\, {}^c\mathfrak{D}_{0^+}^{\gamma,\,\omega}\mathscr{Y}(\xi)d\xi)| \leq \Phi(\tau), \tag{66}$$

and

$$|{}^c\mathfrak{D}_{0^+}^{\gamma,\,\omega}\mathscr{Y}(\tau) - \mathscr{F}(\tau, \mathscr{Y}(\tau), {}^c\mathfrak{D}_{0^+}^{\theta,\,\omega}\mathscr{Y}(\tau), \int_0^\tau k(\tau, \xi)\, {}^c\mathfrak{D}_{0^+}^{\gamma,\,\omega}\mathscr{Y}(\xi)d\xi)| \leq \varepsilon\,\Phi(\tau). \tag{67}$$

#### 3.3.1 *Ulam-Hyers-Rassias stability*

In the following, we consider the Ulam-Hyers stability for NLIFDE (1) over the interval $J=[0,\,\mathfrak{T}]$.

**Theorem 5** Suppose that the assumptions of Theorem (4) are satisfied. Then, NLIFDE (1) is Ulam-Hyers stable.

**Proof.** Let $\varepsilon>0$ and let $z\in C(J,\,R)$ be a function which satisfies inequality (65), such that

$$\left|{}^{c}\mathfrak{D}_{0^{+}}^{\gamma,\,\omega}z(\tau)-\mathscr{F}\left(\tau,\,z(\tau),\ {}^{c}\mathfrak{D}_{0^{+}}^{\gamma,\,\omega}z(\tau),\ \int_{0}^{\tau}k(\tau,\,\xi)\,{}^{c}\mathfrak{D}_{0^{+}}^{\gamma,\,\omega}z(\xi)d\xi\right)\right|\leq\varepsilon,\quad\forall\,\tau\in J,\tag{68}$$

and let $\mathscr{Y}\in C(J,\,R)$ be a unique solution of NLIFDE (1) which is by Lemma 4 is equivalent to the fractional order integral equation

$$\mathscr{Y}(\tau)=h(\tau,\,\mathscr{Y}(\tau))+\frac{1}{\Gamma(\gamma)}\int_{0}^{\tau}\omega'(\xi)(\omega(\tau)-\omega(\xi))^{\gamma-1}\psi(\xi)\,d\xi,$$

where

$$\begin{aligned}
&h(\tau,\,\mathscr{Y}(\tau))\\
=&\frac{(\omega(\tau)-\omega(0))^{\gamma-1}}{\Gamma(\gamma)}\left(\eta_{1}\int_{0}^{\mathfrak{T}}\mathfrak{g}_{1}(\xi,\,\mathscr{Y}(\xi))d\xi-\int_{0}^{\mathfrak{T}}\omega'(\xi)\psi(\xi)\,d\xi\right)\\
&+\frac{(\omega(\tau)-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)}\begin{pmatrix}\varphi(\mathfrak{T})\begin{pmatrix}-\eta_{1}\int_{0}^{\mathfrak{T}}\mathfrak{g}_{1}(\xi,\,\mathscr{Y}(\xi))d\xi\\+\int_{0}^{\mathfrak{T}}\omega'(\xi)\psi(\xi)\,d\xi\end{pmatrix}\\+\eta_{2}\int_{0}^{\mathfrak{T}}\mathfrak{g}_{2}(\xi,\,\mathscr{Y}(\xi))d\xi\\-\int_{0}^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))\psi(\xi)\,d\xi.\end{pmatrix}\\
&+\frac{(\omega(\tau)-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)}\begin{pmatrix}\eta_{3}\int_{0}^{\mathfrak{T}}\mathfrak{g}_{3}(\xi,\,\mathscr{Y}(\xi))\,d\xi\\-\frac{1}{2}\int_{0}^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))^{2}\psi(\xi)\,d\xi\\-\frac{1}{2}\begin{pmatrix}\eta_{1}\int_{0}^{\mathfrak{T}}\mathfrak{g}_{1}(\xi,\,\mathscr{Y}(\xi))d\xi\\-\int_{0}^{\mathfrak{T}}\omega'(\xi)\psi(\xi)\,d\xi\end{pmatrix}(\omega(\mathfrak{T})-\omega(0))^{2}\\-\varphi(\mathfrak{T})\begin{pmatrix}\varphi(\mathfrak{T})\begin{pmatrix}-\eta_{1}\int_{0}^{\mathfrak{T}}\mathfrak{g}_{1}(\xi,\,\mathscr{Y}(\xi))d\xi\\+\int_{0}^{\mathfrak{T}}\omega'(\xi)\psi(\xi)\,d\xi\end{pmatrix}\\+\eta_{2}\int_{0}^{\mathfrak{T}}\mathfrak{g}_{2}(\xi,\,\mathscr{Y}(\xi))d\xi\\-\int_{0}^{\mathfrak{T}}\omega'(\xi)(\omega(\mathfrak{T})-\omega(\xi))\psi(\xi)\,d\xi\end{pmatrix}\end{pmatrix},
\end{aligned}\tag{69}$$

and $\psi$ is the solution of the fractional order integral equation

$$\psi(\tau) = \mathscr{F}(\tau, z(\tau),\ {}^{c}\mathfrak{D}_{0^+}^{\theta,\,\omega} z(\tau),\ \int_0^{\tau} k(\tau,\,\xi)\ {}^{c}\mathfrak{D}_{0^+}^{\gamma,\,\omega} z(\xi)d\xi)$$

$$= \mathscr{F}(\tau,\, h(\tau,\, \mathscr{Y}(\tau)) + \int_0^{\mathfrak{T}} \omega'(\tau)\, G(\tau,\,\xi)\psi(\xi)d\xi,\ {}^{c}\mathfrak{D}_{0^+}^{\theta,\,\omega} z(\tau),\ \int_0^{\tau} k(\tau,\,\xi)\ {}^{c}\mathfrak{D}_{0^+}^{\gamma,\,\omega} z(\xi)d\xi)$$

$$= \mathscr{F}(\tau,\, h(\tau) + \frac{1}{\Gamma(\gamma)} \int_0^{\tau} \omega'(\xi)(\omega(\tau)-\omega(\xi))^{\gamma-1}\psi(\xi)\,d\xi,\, \mathfrak{I}_{0^+}^{\gamma-\theta,\,\omega}\psi(\tau),\ \int_0^{\tau} k(\tau,\,\xi)\psi(\xi)d\xi). \tag{70}$$

Taking the left-sided $\omega$-Riemann-Liouville fractional integral $\mathfrak{I}_{0^+}^{\gamma,\,\omega}$ on both sides of inequality (68), we get

$$|z(\tau) - h(\tau, z(\tau)) - \frac{1}{\Gamma(\gamma)} \int_0^{\tau} \omega'(\xi)(\omega(\tau)-\omega(\xi))^{\gamma-1}\phi(\xi)\,d\xi| \le \varepsilon \frac{(\omega(\mathfrak{T}) - \omega(0))^{\gamma}}{\Gamma(\gamma+1)}. \tag{71}$$

For each $\tau \in J$, and by Lemma 4 and Lemma 6, we have

$$\begin{aligned}
&|z(\tau) - \mathscr{Y}(\tau)| \\
=&|z(\tau) - h(\tau,\, \mathscr{Y}(\tau)) - \frac{1}{\Gamma(\gamma)} \int_0^{\tau} \omega'(\xi)(\omega(\tau)-\omega(\xi))^{\gamma-1}\phi(\xi)\,d\xi| \\
\le&|z(\tau) - h(\tau,\, z(\tau)) - \frac{1}{\Gamma(\gamma)} \int_0^{\tau} \omega'(\xi)(\omega(\tau)-\omega(\xi))^{\gamma-1}\phi(\xi)\,d\xi - h(\tau,\, \mathscr{Y}(\tau)) \\
&- \frac{1}{\Gamma(\gamma)} \int_0^{\tau} \omega'(\xi)(\omega(\tau)-\omega(\xi))^{\gamma-1}\psi(\xi)\,d\xi + h(\tau,\, z(\tau)) \\
&+ \frac{1}{\Gamma(\gamma)} \int_0^{\tau} \omega'(\xi)(\omega(\tau)-\omega(\xi))^{\gamma-1}\phi(\xi)\,d\xi| \\
\le&\varepsilon\, \frac{(\omega(\mathfrak{T}) - \omega(0))^{\gamma}}{\Gamma(\gamma+1)} + |h(\tau,\, z(\tau)) - h(\tau,\, \mathscr{Y}(\tau))| \\
&+ \frac{1}{\Gamma(\gamma)} \int_0^{\tau} \omega'(\xi)(\omega(\tau)-\omega(\xi))^{\gamma-1}|\phi(\xi) - \psi(\xi)|d\xi
\end{aligned}$$

$$\leq \varepsilon \frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma}}{\Gamma(\gamma+1)} + \left( \begin{array}{c} \eta_1 \mathfrak{T} \|\mu_1\| \left( \begin{array}{c} \left(\frac{1}{\Gamma(\gamma)} + \frac{1}{2\Gamma(\gamma-2)}\right)(\omega(\mathfrak{T})-\omega(0))^{\gamma-1} \\ +\varphi(\mathfrak{T}) \frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)} \\ +\eta_1 \varphi^2(\mathfrak{T}) \frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)} \end{array} \right) \\ +\eta_2 \mathfrak{T} \|\mu_2\| \left( \frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-2}}{\Gamma(\gamma-1)} + \varphi(\mathfrak{T}) \frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)} \right) \\ +\eta_3 \mathfrak{T} \|\mu_3\| \frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma-3}}{\Gamma(\gamma-2)} \end{array} \right) \|z-\mathscr{Y}\|$$

$$+\|\lambda\| \left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right) \frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma}}{\Gamma(\gamma+1)} \|z-\mathscr{Y}\|$$

$$\leq \varepsilon \frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma}}{\Gamma(\gamma+1)} + (\aleph - \Lambda_4)\|z-\mathscr{Y}\| + \|\lambda\| \left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right) \frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma}}{\Gamma(\gamma+1)} \|z-\mathscr{Y}\|,$$

which implies for each $\tau \in I$ that

$$\|z-\mathscr{Y}\| \leq \varepsilon \frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma}}{\Gamma(\gamma+1)} + \left( (\aleph - \Lambda_4) + \|\lambda\| \left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right) \frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma}}{\Gamma(\gamma+1)} \right) \|z-\mathscr{Y}\|. \tag{72}$$

Hence,

$$\|z-\mathscr{Y}\| \leq \frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma}}{\Gamma(\gamma+1)} \left[ 1 - \left( (\aleph - \Lambda_4) + \|\lambda\| \left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right) \frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma}}{\Gamma(\gamma+1)} \right) \right]^{-1} \varepsilon$$

$$\leq \varsigma \, \varepsilon, \tag{73}$$

where

$$\varsigma = \frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma}}{\Gamma(\gamma+1)} \left[ 1 - \left( (\aleph - \Lambda_4) + \|\lambda\| \left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right) \frac{(\omega(\mathfrak{T})-\omega(0))^{\gamma}}{\Gamma(\gamma+1)} \right) \right]^{-1}. \tag{74}$$

Therefore, the NLIFDE (1) is Ulam-Hyers stable. □

**Remark 1** If we put $\Phi(\varepsilon) = \varsigma \, \varepsilon$, then we get $\Phi(0) = 0$ which yields that the NLIFDE (1) is generalized Ulam-Hyers stable.

### 3.3.2 *Ulam-Hyers-Rassias stability*

In the following, we study the Ulam-Hyers-Rassias stability of NLIFDE (1).

**Theorem 6** Assume that assumptions $(H_1)$, $(H_2)$, and $(H_3)$ hold. Then, NLIFDE (1) is Ulam-Hyers-Rassias stable with respect to $\Phi$.

**Proof.** Let $z \in C(I, R)$ be a solution of the inequality (67). That is,

$$| \, {}^{c}\mathfrak{D}_{0^+}^{\gamma,\,\omega} z(\tau) - \mathscr{F}(\tau,\, z(\tau),\; {}^{c}\mathfrak{D}_{0^+}^{\gamma,\,\omega} z(\tau),\; \int_0^\tau k(\tau,\,\xi)\; {}^{c}\mathfrak{D}_{0^+}^{\gamma,\,\omega} z(\xi) d\xi)| \le \varepsilon\, \Phi, \quad \tau \in [0,\, \mathfrak{T}]\,. \tag{75}$$

In addition, let $\mathscr{Y}$ be a solution of NLIFDE (1), and let $\psi \in C([0,\, \mathfrak{T}]\,,\, R)$ such that

$$\mathscr{Y}(\tau) = h(\tau,\, \mathscr{Y}(\tau)) + \frac{1}{\Gamma(\gamma)} \int_0^\tau \omega'(\xi)(\omega(\tau) - \omega(\xi))^{\gamma-1} \psi(\xi)\, d\xi, \tag{76}$$

where The functions $h(\tau,\, \mathscr{Y}(\tau))$ and $\psi(\tau)$ are specified in equations (69) and (70), respectively.

Operating $\mathfrak{I}_{0^+}^{\gamma,\,\omega}$ on both sides of inequality (67) and then integrating, we get

$$\begin{aligned} \left| \begin{array}{c} z(\tau) - h(\tau,\, z(\tau)) \\ -\frac{1}{\Gamma(\gamma)} \int_0^\tau \omega'(\xi)(\omega(\tau) - \omega(\xi))^{\gamma-1} \phi(\xi)\, d\xi \end{array} \right| &\le \frac{\varepsilon}{\Gamma(\gamma)} \int_0^\tau \omega'(\xi)\,(\omega(\tau) - \omega(\xi))^{\gamma-1}\, \Phi(\xi) d\xi \\ &\le \varepsilon \mu_\Phi \Phi(\tau), \end{aligned} \tag{77}$$

where $\phi \in C(J,\, R)$ such that

$$\phi(\tau) = \mathscr{F}(\tau,\, z(\tau),\, \mathfrak{I}_{0^+}^{\gamma-\theta,\,\omega} \phi(\tau),\, \int_0^\tau k(\tau,\,\xi)\phi(\xi) d\xi). \tag{78}$$

Hence, for each $\tau \in J$, we have

$$\begin{aligned} &|z(\tau) - \mathscr{Y}(\tau)| \\ =&|z(\tau) - h(\tau,\, \mathscr{Y}(\tau)) - \frac{1}{\Gamma(\gamma)} \int_0^\tau \omega'(\xi)(\omega(\tau) - \omega(\xi))^{\gamma-1} \psi(\xi)\, d\xi| \\ \le&|z(\tau) - h(\tau,\, \mathscr{Y}(\tau)) - \frac{1}{\Gamma(\gamma)} \int_0^\tau \omega'(\xi)(\omega(\tau) - \omega(\xi))^{\gamma-1} \phi(\xi)\, d\xi| \\ &+ |\frac{1}{\Gamma(\gamma)} \int_0^\tau \omega'(\xi)(\omega(\tau) - \omega(\xi))^{\gamma-1} \phi(\xi)\, d\xi - \frac{1}{\Gamma(\gamma)} \int_0^\tau \omega'(\xi)(\omega(\tau) - \omega(\xi))^{\gamma-1} \psi(\xi)\, d\xi| \\ \le&\varepsilon\mu_\Phi\, \Phi(\tau) + \frac{1}{\Gamma(\gamma)} \int_0^{\mathfrak{T}} \omega'(\xi)(\omega(\tau) - \omega(\xi))^{\gamma-1}\, |\phi(\xi) - \psi(\xi)| d\xi \\ \le&\varepsilon\mu_\Phi\, \Phi(\tau) + \frac{(\omega(\mathfrak{T}) - \omega(\xi))^\gamma}{\Gamma(\gamma+1)} \|\phi - \psi\| \end{aligned} \tag{79}$$

But, from the proof of Theorem 4, we have

$$\|\psi-\phi\| \leq \|\lambda\| \left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right) \|z-\mathscr{Y}\|, \tag{80}$$

and for each $\tau \in [0, \mathfrak{T}]$, we have

$$\|z-\mathscr{Y}\| \leq \varepsilon\mu_\Phi \, \Phi(\tau)+\|\lambda\| \left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right) \|z-\mathscr{Y}\|. \tag{81}$$

Thus,

$$\|z-\mathscr{Y}\| \leq \left[1-\|\lambda\| \left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right)\right]^{-1} \mu_\Phi \varepsilon \Phi(\tau) = c_\Phi \varepsilon \Phi(\tau), \tag{82}$$

where

$$c_\Phi = \left[1-\|\lambda\| \left(1+\frac{1+K\mathfrak{T}}{|\Gamma(1-\theta)|}\right)\right]^{-1} \mu_\Phi. \tag{83}$$

Therefore, the boundary value problem NLIFDE (1) is Ulam-Hyers-Rassias stable with respect to $\Phi$. □

## 4. Special cases

In this section, we present several distinct special cases of our generalized model to illustrate its ability to generalize and incorporate various classical models.

### 4.1 *Classical Caputo fractional model*

By setting $\omega(t)=t$ and choosing $\gamma$ and $\theta$ as integers, such as $\gamma=2$ and $\theta=1$, we obtain

$${}^c\mathfrak{D}^2_{0+}\mathscr{Y}(\tau) = \mathscr{F}(\tau, \mathscr{Y}(\tau), {}^c\mathfrak{D}^1_{0+}\mathscr{Y}(\tau), \int_0^\tau k(\tau, \xi)\, {}^c\mathfrak{D}^2_{0+}\mathscr{Y}(\xi)\, d\xi), \tag{84}$$

with boundary conditions

$$\mathscr{Y}(0)+{}^c\mathfrak{D}^1_{0+}\mathscr{Y}(\mathfrak{T}) = \eta_1 \int_0^{\mathfrak{T}} \mathfrak{g}_1(\xi, \mathscr{Y}(\xi))\, d\xi, \tag{85}$$

$${}^c\mathfrak{D}^1_{0+}\mathscr{Y}(0)+{}^c\mathfrak{D}^0_{0+}\mathscr{Y}(\mathfrak{T}) = \eta_2 \int_0^{\mathfrak{T}} \mathfrak{g}_2(\xi, \mathscr{Y}(\xi))\, d\xi, \tag{86}$$

$${}^c\mathfrak{D}^0_{0+}\mathscr{Y}(0)+{}^c\mathfrak{D}^{-1}_{0+}\mathscr{Y}(\mathfrak{T}) = \eta_3 \int_0^{\mathfrak{T}} \mathfrak{g}_3(\xi, \mathscr{Y}(\xi))\, d\xi. \tag{87}$$

This scenario simplifies to the classical Caputo fractional differential equations with integer orders and linear boundary conditions.

### 4.2 *Linear integral boundary conditions*

When $\omega(t) = t$ and $\eta_1 = \eta_2 = \eta_3 = 1$, the model reduces to

$$ {}^c\mathfrak{D}^{\gamma}_{0^+}\mathscr{Y}(\tau) = \mathscr{F}\left(\tau, \mathscr{Y}(\tau), {}^c\mathfrak{D}^{\theta}_{0^+}\mathscr{Y}(\tau), \int_0^{\tau} k(\tau, \xi)\, {}^c\mathfrak{D}^{\gamma}_{0^+}\mathscr{Y}(\xi)\, d\xi\right), \tag{88} $$

with boundary conditions

$$ \mathscr{Y}(0) + {}^c\mathfrak{D}^{\gamma-1}_{0^+}\mathscr{Y}(\mathfrak{T}) = \int_0^{\mathfrak{T}} \mathfrak{g}_1(\xi, \mathscr{Y}(\xi))\, d\xi, \tag{89} $$

$$ {}^c\mathfrak{D}^{\gamma-1}_{0^+}\mathscr{Y}(0) + {}^c\mathfrak{D}^{\gamma-2}_{0^+}\mathscr{Y}(\mathfrak{T}) = \int_0^{\mathfrak{T}} \mathfrak{g}_2(\xi, \mathscr{Y}(\xi))\, d\xi, \tag{90} $$

$$ {}^c\mathfrak{D}^{\gamma-2}_{0^+}\mathscr{Y}(0) + {}^c\mathfrak{D}^{\gamma-3}_{0^+}\mathscr{Y}(\mathfrak{T}) = \int_0^{\mathfrak{T}} \mathfrak{g}_3(\xi, \mathscr{Y}(\xi))\, d\xi. \tag{91} $$

This case corresponds to models with linear integral boundary conditions, which are common in fractional calculus literature and provide a foundation for comparison with more complex fractional differential equations.

### 4.3 *Fractional delay differential equation*

Setting $\omega(t) = t - h$, where $h$ represents a fixed delay, we obtain

$$ {}^c\mathfrak{D}^{\gamma, t-h}_{0^+}\mathscr{Y}(\tau) = \mathscr{F}\left(\tau, \mathscr{Y}(\tau), {}^c\mathfrak{D}^{\theta, t-h}_{0^+}\mathscr{Y}(\tau), \int_0^{\tau} k(\tau, \xi)\, {}^c\mathfrak{D}^{\gamma, t-h}_{0^+}\mathscr{Y}(\xi)\, d\xi\right), \tag{92} $$

with boundary conditions

$$ \mathscr{Y}(0) + {}^c\mathfrak{D}^{\gamma-1, t-h}_{0^+}\mathscr{Y}(\mathfrak{T}) = \eta_1 \int_0^{\mathfrak{T}} \mathfrak{g}_1(\xi, \mathscr{Y}(\xi))\, d\xi, \tag{93} $$

$$ {}^c\mathfrak{D}^{\gamma-1, t-h}_{0^+}\mathscr{Y}(0) + {}^c\mathfrak{D}^{\gamma-2, t-h}_{0^+}\mathscr{Y}(\mathfrak{T}) = \eta_2 \int_0^{\mathfrak{T}} \mathfrak{g}_2(\xi, \mathscr{Y}(\xi))\, d\xi, \tag{94} $$

$$ {}^c\mathfrak{D}^{\gamma-2, t-h}_{0^+}\mathscr{Y}(0) + {}^c\mathfrak{D}^{\gamma-3, t-h}_{0^+}\mathscr{Y}(\mathfrak{T}) = \eta_3 \int_0^{\mathfrak{T}} \mathfrak{g}_3(\xi, \mathscr{Y}(\xi))\, d\xi. \tag{95} $$

This case extends the model to fractional delay differential equations, incorporating time delays into the standard fractional differential framework.

### 4.4 *Nonlinear fractional oscillator*

Assuming the nonlinearity $\mathscr{F}(\tau, \mathscr{Y}, \cdots) = -\lambda \mathscr{Y}(\tau) + \mu \mathscr{Y}(\tau)^3$, where $\lambda$ and $\mu$ are constants, the model simplifies to

$$ {}^{c}\mathfrak{D}_{0^+}^{\gamma, \omega} \mathscr{Y}(\tau) = -\lambda \mathscr{Y}(\tau) + \mu \mathscr{Y}(\tau)^3 + \int_0^{\tau} k(\tau, \xi) {}^{c}\mathfrak{D}_{0^+}^{\gamma, \omega} \mathscr{Y}(\xi) d\xi, \tag{96} $$

with boundary conditions

$$ \mathscr{Y}(0) + {}^{c}\mathfrak{D}_{0^+}^{\gamma-1, \omega} \mathscr{Y}(\mathfrak{T}) = \eta_1 \int_0^{\mathfrak{T}} \mathfrak{g}_1(\xi, \mathscr{Y}(\xi)) d\xi, \tag{97} $$

$$ {}^{c}\mathfrak{D}_{0^+}^{\gamma-1, \omega} \mathscr{Y}(0) + {}^{c}\mathfrak{D}_{0^+}^{\gamma-2, \omega} \mathscr{Y}(\mathfrak{T}) = \eta_2 \int_0^{\mathfrak{T}} \mathfrak{g}_2(\xi, \mathscr{Y}(\xi)) d\xi, \tag{98} $$

$$ {}^{c}\mathfrak{D}_{0^+}^{\gamma-2, \omega} \mathscr{Y}(0) + {}^{c}\mathfrak{D}_{0^+}^{\gamma-3, \omega} \mathscr{Y}(\mathfrak{T}) = \eta_3 \int_0^{\mathfrak{T}} \mathfrak{g}_3(\xi, \mathscr{Y}(\xi)) d\xi. \tag{99} $$

This special case corresponds to nonlinear fractional oscillators, where nonlinear terms are incorporated into the fractional differential equations.

### 4.5 *Fractional heat equation with variable coefficients*

Setting $k(\tau, \xi) = \alpha(\tau - \xi)$, where $\alpha$ represents a variable coefficient function, we obtain

$$ {}^{c}\mathfrak{D}_{0^+}^{\gamma, \omega} \mathscr{Y}(\tau) = \alpha(\tau - \xi) + \int_0^{\tau} \alpha(\tau - \xi) {}^{c}\mathfrak{D}_{0^+}^{\gamma, \omega} \mathscr{Y}(\xi) d\xi, \tag{100} $$

under the boundary condition

$$ \mathscr{Y}(0) + {}^{c}\mathfrak{D}_{0^+}^{\gamma-1, \omega} \mathscr{Y}(\mathfrak{T}) = \eta_1 \int_0^{\mathfrak{T}} \mathfrak{g}_1(\xi, \mathscr{Y}(\xi)) d\xi. \tag{101} $$

This case demonstrates the applicability of our generalized model to fractional heat equations with variable coefficients, highlighting its flexibility in accommodating different boundary conditions.

The analysis of these special cases highlights the efficacy of our generalized model in encompassing classical models as specific instances, thus affirming its extensive applicability and versatility. By juxtaposing with well-established models, we effectively demonstrate how our approach not only generalizes but also integrates diverse methods within fractional calculus. This comparison not only underscores the robustness of our model but also its adeptness at handling intricate boundary value problems with efficiency.

## 5. Numerical example

To further demonstrate the practical relevance of our theoretical results, we provide a numerical example. This example illustrates the application of our proposed methods to specific cases, offering insights into the behavior and properties of the solutions. The numerical example validates the theoretical findings, highlighting the effectiveness of our approach in addressing real-world problems. Additionally, it underscores the significance and potential impact of our model in various practical applications. Consider the following NLIFDE:

$$\begin{cases} {}^{c}\mathfrak{D}^{\frac{11}{5},\,\tau^2+1}\mathscr{Y}(\tau) = \frac{\sqrt{2t+1}}{59e^{2t+1}}\left[\frac{11+\mathscr{Y}(\tau)+{}^{c}\mathfrak{D}^{\frac{3}{5},\,\tau^2+1}\mathscr{Y}(\tau)+\int_0^1 e^{3(\tau-\xi)}\,{}^{c}\mathfrak{D}^{\frac{11}{5},\,\tau^2+1}\mathscr{Y}(\xi)d\xi}{1+\mathscr{Y}(\tau)+{}^{c}\mathfrak{D}^{\frac{3}{5},\,\tau^2+1}\mathscr{Y}(\tau)+2\int_0^1 e^{3(\tau-\xi)}\,{}^{c}\mathfrak{D}^{\frac{11}{5},\,\tau^2+1}\mathscr{Y}(\xi)d\xi}\right] \text{ for all } \tau\in[0,\,1], \\ \mathscr{Y}(0)+{}^{c}\mathfrak{D}_{0^+}^{\frac{6}{5},\,\tau^2+1}\mathscr{Y}(1) = \int_0^1\left(\frac{e^{-3t}}{69+\sqrt{\tau}}+\frac{1}{23}|\cos\sqrt{\mathscr{Y}(\tau)}|\right)d\xi, \\ {}^{c}\mathfrak{D}_{0^+}^{\frac{6}{5},\,\tau^2+1}\mathscr{Y}(0)+{}^{c}\mathfrak{D}_{0^+}^{\frac{1}{5},\,\tau^2+1}\mathscr{Y}(1) = 2\int_0^1\left(\frac{1}{\sqrt{69+\tau^2}}+\frac{e^{-3t}}{23+\tau^2}|\mathscr{Y}(\tau)|\right)d\xi, \\ {}^{c}\mathfrak{D}_{0^+}^{\frac{1}{5},\,\tau^2+1}\mathscr{Y}(0)+{}^{c}\mathfrak{D}_{0^+}^{\frac{-4}{5},\,\tau^2+1}\mathscr{Y}(1) = 3\int_0^1\left(\frac{1}{3t^2+3}+\frac{1+\tau}{e^{3t+3}}|\mathscr{Y}(\tau)|\right)d\xi. \end{cases} \tag{102}$$

In this problem, we have $\gamma=\frac{11}{5}$, $\theta=\frac{3}{5}$, $\omega(\tau)=\tau^2+1$ which is an increasing function on $[0,\,1]$, $K(\tau,\,\xi)=e^{3(\tau-\xi)}$, $\eta_1=1.5$, $\eta_2=2.5$, $\eta_3=3.5$,

$$\mathfrak{g}_1(\xi,\,\mathscr{Y}(\xi)) = \left(\frac{e^{-3t}}{69+\sqrt{\tau}}+\frac{1}{23}|\cos\sqrt{\mathscr{Y}(\tau)}|\right), \tag{103}$$

with $\mu_1=\frac{1}{23}$ and $H_1=\frac{1}{69}$.

$$\mathfrak{g}_2(\xi,\,\mathscr{Y}(\xi)) = \left(\frac{1}{\sqrt{69+\tau^2}}+\frac{e^{-3t}}{53+\tau^2}|\mathscr{Y}(\tau)|\right), \tag{104}$$

with $\mu_2=\frac{1}{53}$ and $H_2=\frac{1}{\sqrt{69}}$.

$$\mathfrak{g}_3(\xi,\,\mathscr{Y}(\xi)) = \left(\frac{1}{3(\tau^2+1)}+\frac{1+\tau}{e^{3(\tau+1)}}|\mathscr{Y}(t)|\right), \tag{105}$$

with $\mu_3=\frac{1}{e^3}$ and $H_3=\frac{1}{3}$.

It is clear that the assumptions $(H_1)$-$(H_3)$ are satisfied, and $\mathscr{F}$ is a mutually continuous function such that for any $\psi,\,\phi,\,\chi\in R$, and $\tau\in[0,\,1]$ we have

$$|\mathscr{F}(\tau,\,\psi,\,\phi,\,\chi)| = \frac{\sqrt{2t+1}}{59e^{2t+1}}(11+|\psi|+|\phi|+|\chi|), \tag{106}$$

with $\lambda(\tau)=\frac{\sqrt{2t+1}}{59e^{2t+1}}$, $F=\frac{11}{59e}$, $\|\lambda\|=\frac{1}{59e}$, and $K=e^3$.

It is clear from Theorem (3) that the nonlinear fractional integral differential equation (NLIFDE) (102) possesses at least one solution within the interval [0, 1], provided that the following condition is satisfied:

$$\begin{aligned} \aleph &= \|\mu_1\| \eta_1 T \Lambda_1 + \|\mu_2\| \eta_2 \mathfrak{T} \Lambda_2 + \|\mu_3\| \eta_3 \mathfrak{T} \Lambda_3 + \Lambda_4 \\ &\approx 0.265178 + 0.0616486 + 0.037957 + 0.345699 \\ &\approx 0.7148 < 1, \end{aligned} \tag{107}$$

Moreover, by employing Theorem (4), it is deduced that the solution is both unique and stable according to the criteria of Ulam-Hyers and Ulam-Hyers-Rassias, as indicated by the following condition:

$$\aleph - \Lambda_4 \approx 0.364784 < 1. \tag{108}$$

## 6. Conclusion

In conclusion, this research has systematically investigated the solutions of nonlinear implicit $\omega$-Caputo fractional-order ordinary differential equations (NLIFDEs) with two-point fractional derivatives and integral boundary conditions in Banach algebra. Utilizing rigorous mathematical tools such as Banach's and Krasnoselskii's fixed point theorems, the study successfully established the existence and uniqueness of solutions for this intricate class of fractional differential equations. The stability of solutions was further scrutinized through the lens of Ulam-Hyers and Ulam-Hyers-Rassias criteria, providing valuable insights into the robustness of the proposed framework.

The practical applicability of the findings was exemplified through a numerical example, showcasing the methodology's effectiveness in real-world problem-solving scenarios. This comprehensive investigation significantly contributes to the understanding of nonlinear fractional differential equations with integral boundary conditions, emphasizing the intricate interplay between fractional derivatives, nonlinearities, and integral terms.

Building on the insights gained from this study, several avenues for future research are suggested. Firstly, the extension of the analysis to more complex NLIFDE models or incorporating additional parameters could provide a deeper understanding of the system's behavior. Exploring alternative numerical methods and algorithms for solving NLIFDEs may enhance the efficiency and accuracy of the proposed methodology. Furthermore, the investigation of applications in various scientific and engineering disciplines could broaden the practical relevance of the research. Additionally, considering the dynamic nature of nonlinear systems, a dynamic stability analysis and the exploration of control strategies could be explored to further enhance the applicability of the proposed framework. Collaborative efforts with experts from diverse fields, such as physics, engineering, or biology, could foster interdisciplinary research, opening up new perspectives and applications for NLIFDEs. Finally, continuous efforts in disseminating knowledge through workshops, conferences, and publications will contribute to the ongoing development of the field and encourage further exploration of nonlinear fractional differential equations with integral boundary conditions.

## Authors' contributions

The authors were solely responsible for all aspects of the study, including conception, design, data collection, analysis, and writing. They contributed equally to this article.

## Availability of data and materials

All data generated or analyzed during this study are included in the published article.


## Acknowledgments

The authors express their deep gratitude to the editors and referees for their invaluable input and contributions that have significantly enhanced the manuscript's quality and impact.


## Conflict of interest

The authors declares there is no conflict of interest at any point with reference to research findings.

## References


[1] Agrawal OP. Some generalized fractional calculus operators and their applications in integral equations. *Fractional Calculus and Applied Analysis*. 2012; 15: 700-711.

[2] Almeida R. A Caputo fractional derivative of a function with respect to another function. *Communications in Nonlinear Science and Numerical Simulation*. 2017; 44: 460-481.

[3] Burton TA, Kirk C. A fixed point theorem of Krasnoselskii-Schaefer type. *Mathematical Nachrichten*. 1998; 189(1): 23-31.

[4] Kilbas AA, Srivastava HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations*. Elsevier; 2006.

[5] Samko S. *Fractional integrals and derivatives, theory and applications*. Minsk: Nauka I Tekhnika; 1987.

[6] Agarwal RP, Benchohra M, Hamani S. Boundary value problems for fractional differential equations. *Georgian Mathematical Journal*. 2009; 16(3): 401-411.

[7] Awad Y, Alkhezi Y. Analysis of implicit solutions for a coupled system of hybrid fractional order differential equations with hybrid integral boundary conditions in Banach algebras. *Symmetry*. 2023; 15(9): 1758.

[8] Awad Y, Fakih HM. Existence and uniqueness results for a two-point nonlinear boundary value problem of Caputo fractional differential equations of variable order. *TWMS Journal of Applied and Engineering Mathematics*. 2024; 14(3): 1068-1084.

[9] Benlabbes A, Benbachir M, Lakrib M. Boundary value problems for nonlinear fractional differential equations. *Facta Universitatis-Series: Mathematics and Informatics*. 2015; 30(2): 157-168.

[10] Derbazi C, Hammouche H. Existence and uniqueness results for a class of nonlinear fractional differential equations with nonlocal boundary conditions. *Jordan Journal of Mathematics and Statistics*. 2020; 13(3): 341-361.

[11] Awad Y. Well posedness and stability for the nonlinear $\varphi$-Caputo hybrid fractional boundary value problems with two-point hybrid boundary conditions. *Jordan Journal of Mathematics and Statistics*. 2023; 16(4): 617-647.

[12] Awad YAR, Kaddoura IH. On the Ulam-Hyers-Rassias stability for a boundary value problem of implicit $\psi$-Caputo fractional integro-differential equation. *TWMS Journal of Applied and Engineering Mathematics*. 2024; 14(1): 79-93.

[13] Awad Y. On the existence and stability of positive solutions of eigenvalue problems for a class of P-Laplacian $\psi$-Caputo fractional integro-differential equations. *Journal of Mathematics*. 2023; 2023(1): 3458858.

[14] Awad Y, Fakih H, Alkhezi Y. Existence and uniqueness of variable-order $\varphi$-Caputo fractional two-point nonlinear boundary value problem in Banach algebra. *Axioms*. 2023; 12(10): 935.

[15] Kaddoura I, Awad Y. Stability results for nonlinear implicit $\vartheta$-Caputo fractional differential equations with fractional integral boundary conditions. *International Journal of Differential Equations*. 2023; 2023(1): 5561399.

[16] Hyers DH. On the stability of the linear functional equation. *Proceedings of the National Academy of Sciences*. 1941; 27(4): 222-224.

[17] Rassias TM. On the stability of the linear mapping in Banach spaces. *Proceedings of the American Mathematical Society*. 1978; 72(2): 297-300.

[18] Awad Y, Alkhezi Y. Solutions of second-order nonlinear implicit $\psi$-conformable fractional integro-differential equations with nonlocal fractional integral boundary conditions in Banach algebra. *Symmetry*. 2024; 16(9): 1097.

[19] Maamar B. Boundary value problems for nonlinear fractional differential equations. *Facta Universitatis, Series: Mathematics and Informatics*. 2015; 30(2): 157-168.

[20] Zhang S. Positive solutions for boundary-value problems of nonlinear fractional differential equations. *Electronic Journal of Differential Equations (EJDE)*. 2006; 2006(36): 1-12.