Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Geometric Features of a Multivalent Function Pertaining to Fractional Operators

K. Divya Priya, K. Thilagavathi*

Department of Mathematics, School of Advanced Science, Vellore Institute of Technology, Vellore, 632014, India
E-mail: kthilagavathi@vit.ac.in

Received: 2 July 2024; **Revised:** 18 September 2024; **Accepted:** 27 September 2024

Abstract: The Prabhakar fractional operator is commonly acclaimed as the queen model of fractional calculus. The distinction between univalent and multivalent functions became more formalized as part of the broader field of geometric function theory. This area of mathematics focuses on the study of analytic functions with specific geometric properties, such as injectivity, and their applications in various domains, including conformal mapping and potential theory. This paper's goal is to discover new results of the harmonic multivalent functions $f = k + \bar{l}$ defined in the open unit disc $U = \{z: |z| < 1\}$. Let present $\mathfrak{v}^*(p, \varepsilon, \nu, \upsilon, \vartheta) \subset \mathfrak{w}^*(p, \varepsilon, \nu, \upsilon, \vartheta)$, the class of multivalent harmonic functions of the form $f = k + \bar{l}$ in the open unit disc. Analyzing convolution with prabhahar fractional differential operator ${}^{i}_{j}K^{\upsilon,\vartheta}_{\varepsilon,\nu}$ with multivalent harmonic function to be in the class $\mathfrak{w}^*(p, \varepsilon, \nu, \upsilon, \vartheta)$. The coefficient inequality, growth rates, distortion properties, closure characteristics, neighborhood behaviors, and extreme points, all pertinent to this class $\mathfrak{w}^*(p, \varepsilon, \nu, \upsilon, \vartheta)$ were explored.

Keywords: harmonic function, multivalent function, open unit disc, convolution, Prabhakar fractional operator

MSC: 30C50, 30F15, 30B40, 30R11, 30R32, 30B20

1. Introduction

Recent mathematical research has emphasized complex fractional differential operators, notably the Prabhakar operator within the open unit disc. These operators are pivotal in formulating advanced differential equations and have broad applications in fields such as mathematical physics and engineering. Geometric function theory is integral to these studies, focusing on the geometric properties of analytic and harmonic functions within complex domains like the open unit disc. Through geometric function theory, researchers aim to uncover the nuanced behaviors and structural characteristics of these fractional differential operators, enriching both theoretical insights and practical applications. Central to this study is the integral operator ${}^{i}_{j}M^{\upsilon,\vartheta}_{\varepsilon,\nu}$ associated with the fractional differential operator, highlighted in reference [1] for its detailed series expansion and foundational role in further analysis. Researchers investigate its spectral properties, functional spaces, and diverse applications across disciplines. Integrating geometric function theory enhances our grasp of these operators' principles and broadens avenues for future research. This approach aims to advance mathematical analysis and its interdisciplinary applications by bridging theoretical constructs with practical implications. Let $\mathfrak{v}^*(p, \varepsilon, \nu, \upsilon, \vartheta)$ be the class of multivalent harmonic functions of the form $f = k + \bar{l}$

DOI: https://doi.org/10.37256/cm.5420245227

$$k(z)=z^p+\sum_{\psi=2}^{\infty}a_{\psi+p}z^{\psi+p},\ (\psi,\ p\in N), \tag{1}$$

and

$$\bar{l}(z)=\sum_{\psi=1}^{\infty}b_{\psi+p}z^{\psi+p}.$$

"which are analytic and multivalent in the unit disc"

$$U=\{z\in C\colon |z|<1\}. \tag{2}$$

Let $\mathfrak{w}^*(p)$ denotes the subclass of $\mathfrak{v}^*(p,\ \varepsilon,\ \nu,\ \upsilon,\ \vartheta)$ containing functions of the form

$$f(z)=z^p+\sum_{\psi=2}^{\infty}a_{\psi+p}z^{\psi+p}+\sum_{\psi=1}^{\infty}b_{\psi+p}z^{\psi+p},\ \ (a_\psi\geq 0,\ \ \psi,\ \ p\in N).$$

The function as a whole was introduced in 1971 by Tilak R. Prabhakar [2]. The Prabhakar function $\Upsilon^{\upsilon}_{q,\ \gamma}(z)$ [3, 4], is also known as the three-parameter Mittag-Leffler function. The generalised Mittag-Leffler function is described in [5, 6].

$$\Upsilon^{\upsilon}_{q,\ \gamma}(z)=\sum_{\psi=0}^{\infty}\frac{(\upsilon)_\psi}{\Gamma(q\psi+\gamma)}\frac{z^\psi}{\psi!}, \tag{3}$$

where $Re(\upsilon)>0$; if $q,\ \gamma,\ \upsilon\in C$, $(\upsilon)_\psi$ is the pochammer symbol, which is defined above the equation

$$(\upsilon)_\psi=\begin{cases}1,\ \psi=0.\\ \\ \upsilon(\upsilon+1)(\upsilon+2)...(\upsilon+\psi-1),\ \psi\in N;\ p\in C.\end{cases} \tag{4}$$

The generalised hypergeometric function ${}_\delta G_\mu$ is derived as follows.

$$\sum_{m=0}^{\infty}\frac{(\varepsilon_1)_m...(\varepsilon_\delta)_m}{(\nu_1)_m...(\nu_\delta)_m}\frac{z^\psi}{\psi!}={}_\delta\ G_\mu(\varepsilon_1,\ ...\varepsilon_\delta;\ \nu_1,\ ...\nu_\delta;\ z). \tag{5}$$

In order to keep things simple, we'll suppose that the variable z is the numerator parameters ε_1, ..., ε_δ and the denominator parameters ν_1, ..., ν_δ, both of which can have complex values as long as ν and δ are positive integers or zero (interpreting their product as one). Equation (4) refers to the special case ${}_\delta G_\mu$ of (Gauss) hypergeometric series.

Geometrically studying the class of complex fractional operators (differential and integral), Srivastava et al. [7] extended their discoveries for fractional operators in two dimensions. Ibrahim's criteria pertain to a collection of analytical functions situated within the open unit's disc [3]. Our group employs equations [8, 9] to represent a variety of complex differential equations and analytical functions, dubbed ulam stability and differential variables with fractional algebras, utilizing the fractions of these operators [10].

The investigation continues in the field of complex fractional differential operators. This investigation uses the well-known Prabhakar fractional differential operator as the foundation for formulating the fractional differential operator in the open unit disc. As a consequence, let use geometric function theory to analyse the classes. The notation ${}^{i}_{j}K^{\upsilon,\vartheta}_{\varepsilon,\nu}$ represent the normalised complex Prabhakar operator in the open unit disc. Given that ${}^{i}_{j}M^{\upsilon,\vartheta}_{\varepsilon,\nu} \in \mathfrak{v}^*(p, \varepsilon, \nu, \upsilon, \vartheta)$, we can examine it via the lens of geometric function theory.

The integral operator corresponding to the fractional differential operator ${}^{i}_{j}M^{\upsilon,\vartheta}_{\varepsilon,\nu} \in \mathfrak{v}^*(p, \varepsilon, \nu, \upsilon, \vartheta)$ is explained in [3]:

$$ {}^{i}_{j}K^{\upsilon,\vartheta}_{\varepsilon,\nu} f(z) = z^p + \Big[\sum_{\psi=2}^{\infty} \frac{\phi_k \Upsilon^{-\upsilon}_{\varepsilon,2-\nu}(\vartheta z^{\varepsilon})}{\Gamma(\psi+1)\Upsilon^{-\upsilon}_{\varepsilon,\psi+1-\nu}(\vartheta z^{\varepsilon})}\Big] z^{\psi+p} + \Big[\sum_{\psi=1}^{\infty} \frac{\phi_k \Upsilon^{-\upsilon}_{\varepsilon,2-\nu}(\vartheta z^{\varepsilon})}{\Gamma(\psi+1)\Upsilon^{-\upsilon}_{\varepsilon,\psi+1-\nu}(\vartheta z^{\varepsilon})}\Big] \overline{z}^{\psi+p} \tag{6} $$

It is a given that

$$ \left({}^{i}_{j}K^{\upsilon,\vartheta}_{\varepsilon,\nu} * {}^{i}_{j}M^{\upsilon,\vartheta}_{\varepsilon,\nu}\right) f(z) = \left({}^{i}_{j}M^{\upsilon,\vartheta}_{\varepsilon,\nu} * {}^{i}_{j}K^{\upsilon,\vartheta}_{\varepsilon,\nu}\right) f(z) = f(z). $$

The operators ${}^{i}_{j}K^{\upsilon,\vartheta}_{\varepsilon,\nu}$ and ${}^{i}_{j}M^{\upsilon,\vartheta}_{\varepsilon,\nu} \in \mathfrak{v}^*(p, \varepsilon, \nu, \upsilon, \vartheta)$ are combined to from a linear convex combination:

$$ {}^{i}_{j}\mathfrak{X}^{\upsilon,\vartheta}_{\varepsilon,\nu} = \mathfrak{C}\,{}^{i}_{j}K^{\upsilon,\vartheta}_{\varepsilon,\nu} f(z) + (1-\mathfrak{C})\,{}^{i}_{j}M^{\upsilon,\vartheta}_{\varepsilon,\nu} f(z), $$

$\mathfrak{C}$ within the interval [0, 1]. Obviously ${}^{i}_{j}\mathfrak{X}^{\upsilon,\vartheta}_{\varepsilon,\nu} \in \mathfrak{v}^*(p, \varepsilon, \nu, \upsilon, \vartheta)$. Let consider that Equation (1) specifies the function $f(z)$, Equation (6) shows us that

$$ {}^{i}_{j}K^{\upsilon,\vartheta}_{\varepsilon,\nu} f(z) = z^p + \Big[\sum_{\psi=2}^{\infty} \frac{\phi_k \Upsilon^{-\upsilon}_{\varepsilon,2-\nu}(\vartheta z^{\varepsilon})}{\Gamma(\psi+1)\Upsilon^{-\upsilon}_{\varepsilon,\psi+1-\nu}(\vartheta z^{\varepsilon})}\Big] z^{\psi+p} a_{\psi+p} + \Big[\sum_{\psi=1}^{\infty} \frac{\phi_k \Upsilon^{-\upsilon}_{\varepsilon,2-\nu}(\vartheta z^{\varepsilon})}{\Gamma(\psi+1)\Upsilon^{-\upsilon}_{\varepsilon,\psi+1-\nu}(\vartheta z^{\varepsilon})}\Big] \overline{z}^{\psi+p} b_{\psi+p}. \tag{7} $$

The above format is used to the full document for the boldness.

$$ {}^{i}_{j}K^{\upsilon,\vartheta}_{\varepsilon,\nu} f(z) = z^p + \Big[\sum_{\psi=2}^{\infty} [Q]^{-\upsilon,\varepsilon}_{\varepsilon,\psi+1-\nu,2-\nu}\Big] z^{\psi+p} a_{\psi+p} + \Big[\sum_{\psi=1}^{\infty} [Q]^{-\upsilon,\varepsilon}_{\varepsilon,\psi+1-\nu,2-\nu}\Big] \overline{z}^{\psi+p} b_{\psi+p}. $$

The statement k is subordinate to l is expressed as follows: $k(z) \prec l(z)$ for two analytic functions $k, l \in w$. This occurs when there is a Schwarz function $w(z)$, which is by definition analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for every $z \in U$ such that $k(z) = l(w(z))$, $z \in U$.

Likewise, the function $l(z)$ is multivalent in U, let get the following equivalency: $k(U) \subset l(U)$, $k(z) \prec l(z) \leftrightarrow k(0) = l(0)$.

Definition 1 A function of the form (1) is said to be in the class $\mathfrak{w}^*(p, \varepsilon, \nu, \upsilon, \vartheta)$; if satisfies the following condition:

$$\left(\frac{\frac{z^2({}^i_jK^{\upsilon,\,\vartheta}_{\varepsilon,\,\nu}f(z))''+z(p+1)({}^i_jK^{\upsilon,\,\vartheta}_{\varepsilon,\,\nu}f(z))'}{{}^i_jK^{\upsilon,\,\vartheta}_{\varepsilon,\,\nu}f(z)}+1}{\frac{z^2({}^i_jK^{\upsilon,\,\vartheta}_{\varepsilon,\,\nu}f(z))''+z({}^i_jK^{\upsilon,\,\vartheta}_{\varepsilon,\,\nu}f(z))'}{{}^i_jK^{\upsilon,\,\vartheta}_{\varepsilon,\,\nu}f(z)}+\zeta}\right)\leq\tau, \tag{8}$$

Where $\frac{1}{2}\leq\tau\leq 1$, $\zeta\geq 1$, $p=1, 2, 3, \ldots$ and $(\infty<\upsilon<1,\ \vartheta+\varepsilon+\nu>\upsilon,\ \vartheta>-\nu,\ \nu>0)$.

Some other authors (Ibrahim [11], Aouf [12], AL-khafaji and Atshan [13–15], Raina and Srivastava [16] and Atshan and Kulkarni [17], see also [18, 19]) were studies different classes of meromorphic multivalent functions with other operators.

In this paper, let's study and discuss the new class $\mathfrak{v}^*(p,\ \varepsilon,\ \nu,\ \upsilon,\ \vartheta)$ of harmonic multivalent functions defined by fractional calculus operator in the unit disc U. The coefficient inequality, growth rates, distortion properties, closure characteristics, neighborhood behaviors, and extreme points, are all pertinent to the class $\mathfrak{w}^*(p,\ \varepsilon,\ \nu,\ \upsilon,\ \vartheta)$.

2. Main results

2.1 *Coefficient property*

This section aims to establish precise criteria for the function f defined in equation (1) within the framework of the class $\mathfrak{w}^*(p,\ \varepsilon,\ \nu,\ \upsilon,\ \vartheta)$. The primary objective is to define sufficient conditions that ensure f belongs to this specific class of functions. The class $\mathfrak{w}^*(p,\ \varepsilon,\ \nu,\ \upsilon,\ \vartheta)$ is characterized by certain properties and parameters: p, ε, ν, υ, and ϑ. These parameters play crucial roles in determining the nature and behavior of functions within this class, often relating to their growth, decay, or specific functional forms.

In summary, this section is dedicated to formulating and validating the necessary specifications that govern the function f, thereby establishing its membership in the class $\mathfrak{w}^*(p,\ \varepsilon,\ \nu,\ \upsilon,\ \vartheta)$ and emphasizing the critical role of these specifications for functions within this class.

Theorem 1 If $f=k+\bar{l}$ with where k and $\bar{l}$ are given by (1) and $f(z)\in\mathfrak{w}^*(p,\ \varepsilon,\ \nu,\ \upsilon,\ \vartheta)$, then

$$\begin{aligned}&\sum_{\psi=2}^{\infty}[Q]^{-\upsilon,\,\varepsilon}_{\varepsilon,\,\psi+1-\nu,\,2-\nu}\left((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta)\right)(\psi+p)a_{\psi+p}\\&+\sum_{\psi=1}^{\infty}[Q]^{-\upsilon,\,\varepsilon}_{\varepsilon,\,\psi+1-\nu,\,2-\nu}\left((\psi+p)^2(1-\tau)+(1-\tau\zeta)\right)(\psi+p)b_{\psi+p}\\&\leq\tau(p^2+\zeta)-2p^2.\end{aligned}\tag{9}$$

Proof.

$$\frac{z^2({}_j^iK_{\varepsilon,\,v}^{\upsilon,\,\vartheta}f(z))''+z(p+1)({}_j^iK_{\varepsilon,\,v}^{\upsilon,\,\vartheta}f(z))'+{}_j^iK_{\varepsilon,\,v}^{\upsilon,\,\vartheta}f(z)}{z^2({}_j^iK_{\varepsilon,\,v}^{\upsilon,\,\vartheta}f(z))''+z({}_j^iK_{\varepsilon,\,v}^{\upsilon,\,\vartheta}f(z))'+\zeta{}_j^iK_{\varepsilon,\,v}^{\upsilon,\,\vartheta}f(z)}\leq\tau,\,0\leq\tau\leq1.$$

$$\frac{\begin{array}{l}(2p^2+1)z^p+\Big[\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-v,\,2-v}^{-\upsilon,\,\varepsilon}(\psi^2+2p^2+3p\psi)+1\Big]a_{\psi+p}z^{\psi+p}\\ +\Big[\sum_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-v,\,2-v}^{-\upsilon,\,\varepsilon}(\psi^2+2p^2+3p\psi)+1\Big]b_{\psi+p}z^{\psi+p}\end{array}}{\begin{array}{l}(p^2+\zeta)z^p+\Big[\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-v,\,2-v}^{-\upsilon,\,\varepsilon}(\psi^2+2p^2+2p\psi)+\zeta\Big]a_{\psi+p}z^{\psi+p}\\ +\Big[\sum_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-v,\,2-v}^{-v,\,\varepsilon}\left(\psi^2+2p^2+2p\psi\right)+\zeta\Big]b_{\psi+p}z^{\psi+p}\end{array}}$$

$$\leq\tau,$$

where,

$$[Q]_{\varepsilon,\,\psi+1-v,\,2-v}^{-\upsilon,\,\varepsilon}=\frac{\phi_k\Upsilon^{-\upsilon}(\vartheta z^{\varepsilon})}{\Gamma(\psi+1)\Upsilon_{\varepsilon,\,m+1-v}^{-\upsilon}(\vartheta z^{\varepsilon})}.$$

$$(2p^2+1)z^p+\Big[\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-v,\,2-v}^{-\upsilon,\,\varepsilon}(\psi^2+2p^2+3p\psi)+1\Big]a_{\psi+p}z^{\psi+p}$$

$$+\Big[\sum_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-v,\,2-v}^{-\upsilon,\,\varepsilon}(\psi^2+2p^2+3p\psi)+1\Big]b_{\psi+p}z^{\psi+p}$$

$$\leq\tau\Big((p^2+\zeta)z^p+\Big[\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-v,\,2-v}^{-\upsilon,\,\varepsilon}(\psi^2+2p^2+2p\psi)+\zeta\Big]a_{\psi+p}z^{\psi+p}$$

$$+\Big[\sum_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-v,\,2-v}^{-\upsilon,\,\varepsilon}(\psi^2+2p^2+2p\psi)+\zeta\Big]b_{\psi+p}z^{\psi+p}\Big),$$

$$(2p^2+1)z^p+\Big[\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}(\psi^2+2p^2+3p\psi)+1\Big]a_{\psi+p}z^{\psi+p}$$

$$+\Big[\sum_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}(\psi^2+2p^2+3p\psi)+1\Big]b_{\psi+p}z^{\psi+p}$$

$$-\tau\Big((p^2+\zeta)z^p+\Big[\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}(\psi^2+2p^2+2p\psi)+\zeta\Big]a_{\psi+p}z^{\psi+p}$$

$$+\Big[\sum_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}(\psi^2+2p^2+2p\psi)+\zeta\Big]b_{\psi+p}z^{\psi+p}\Big)$$

$$\leq 1.$$

Since $f(z)$ is analytic for $|z|=1$, let assume $|z|=r,$ for a suitable small r value with $0\leq r\leq 1,$ and the inequality produces

$$[(2p^2+1)-\tau(p^2+\zeta)]r^p+\Big[\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2+p(\psi+p)+1)-\tau((\psi+p)^2+\zeta)\Big]a_{\psi+p}r^{\psi+p}$$

$$+\Big[\sum_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2+p(\psi+p)+1)-\tau((\psi+p)^2+\zeta)\Big]b_{\psi+p}r^{\psi+p}\leq 1.$$

$$\Big[\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2+p(\psi+p)+1)-\tau((\psi+p)^2+\zeta)\Big]a_{\psi+p}$$

$$+\Big[\sum_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2+p(\psi+p)+1)-\tau((\psi+p)^2+\zeta)\Big]b_{\psi+p}\leq 1-(2p^2+1)+\tau(p^2+\zeta).$$

$$\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))(\psi+p)a_{\psi+p}$$

$$+\sum_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+(1-\tau\zeta))(\psi+p)b_{\psi+p}\leq\tau(p^2+\zeta)-2p^2.$$

In order to preserve brevity in this piece of work, let will assume,

$$\varpi_{\psi+p}=[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta)).$$

Theorem 2 Let $f(z) = k(z) + \bar{l}(z)$ by theorem (1) f is harmonic, sense-preserving univalent functions in U and $f(z) \in \mathfrak{w}^*(p, \varepsilon, \nu, \upsilon, \vartheta)$, then

$$a_{\psi+p} \leq \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}}, \ \psi \geq 2, \tag{10}$$

$$b_{\psi+p} \leq \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}}, \ \psi \geq 1. \tag{11}$$

For the function $f(z) \in \mathfrak{v}^*(p, \varepsilon, \nu, \upsilon, \vartheta)$ given by, the outcome is given above,

$$f(z) = z + \sum_{\psi=2}^{\infty} \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}} z^{\psi+p} + \sum_{\psi=1}^{\infty} \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}} \bar{z}^{\psi+p}. \tag{12}$$

Proof. If $f(z) \in \mathfrak{v}^*(p, \varepsilon, \nu, \upsilon, \vartheta)$, then let have for each $(\psi+p)$,

$$(\psi+p)\varpi_{\psi+p}a_{\psi+p} \leq (\psi+p)\varpi_{\psi+p}b_{\psi+p} \leq \sum_{\psi=2}^{\infty}(\psi+p)\varpi_{\psi+p}a_{\psi+p} \leq \sum_{\psi=1}^{\infty}(\psi+p)\varpi_{\psi+p}b_{\psi+p} \leq \tau(p^2+\zeta)-2p^2.$$

Hence,

$$a_{\psi+p} \leq \sum_{\psi=2}^{\infty} \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}},$$

$$b_{\psi+p} \leq \sum_{\psi=1}^{\infty} \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}}.$$

Therefore

$$f(z) = z + \sum_{\psi=2}^{\infty} \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}} z^{\psi+p} + \sum_{\psi=1}^{\infty} \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}} \bar{z}^{\psi+p}.$$

accomplishes the specifications of Theorem 1, $f(z) \in \mathfrak{w}^*(p, \varepsilon, \nu, \upsilon, \vartheta)$ and for this function, equality is achieved. Where $\sum_{\psi=2}^{\infty}|a_{\psi+p}| + \sum_{\psi=1}^{\infty}|b_{\psi+p}| = 1$ Demonstrate the sharpness of the coefficient bound provided by (9). The functions of the form (12) are in $\mathfrak{v}^*(p)$ because

$$\sum_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))(\psi+p)(a_{\psi+p}+b_{\psi+p})$$

$$=\tau(p^2+\zeta)-2p^2(\sum_{\psi=2}^{\infty}|a_{\psi+p}|+\sum_{\psi=1}^{\infty}|b_{\psi+p}|)=\tau(p^2+\zeta)-2p^2.$$

Theorem 3 Let $f(z)=k(z)+\bar{l}(z)$ be given in (1), then $f(z)\in\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$ if and only if

$$\sum_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))(\psi+p)(a_{\psi+p}+b_{\psi+p})$$

$$=\tau(p^2+\zeta)-2p^2(\sum_{\psi=2}^{\infty}|a_{\psi+p}|+\sum_{\psi=1}^{\infty}|b_{\psi+p}|)=\tau(p^2+\zeta)-2p^2, \tag{13}$$

Where $a_1=1,\ \frac{1}{2}\leq\tau\leq 1,\ \zeta\geq 1$.

Proof. The 'if part' follows from Theorem (1) upon noting that the functions $k(z)$ and $\bar{l}(z)$ in $f(z)\in\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$ are of the form (1), then $f(z)\in\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$. For the only if part, show that if $f(z)\in\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$ then the condition (13) holds. Take note that an essential and sufficient requirement for $f(z)=k(z)+\bar{l}(z)$ given by (1) be in $\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$ is that

$$Re\Big(\frac{z^2({}_j^iK_{\varepsilon,\,\nu}^{\upsilon,\,\vartheta}f(z))''+z(p+1)({}_j^iK_{\varepsilon,\,\nu}^{\upsilon,\,\vartheta}f(z))'+{}_j^iK_{\varepsilon,\,\nu}^{\upsilon,\,\vartheta}f(z)}{z^2({}_j^iK_{\varepsilon,\,\nu}^{\upsilon,\,\vartheta}f(z))''+z({}_j^iK_{\varepsilon,\,\nu}^{\upsilon,\,\vartheta}f(z))'+\zeta{}_j^iK_{\varepsilon,\,\nu}^{\upsilon,\,\vartheta}f(z)}\Big)\geq\tau,$$

or equivalently

$$\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))(\psi+p)a_{\psi+p}z^{\psi+p}+\sum_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}$$

$$((\psi+p)^2(1-\tau)+(1-\tau\zeta))(\psi+p)b_{\psi+p}z^{\psi+p}\geq(\tau(p^2+\zeta)-2p^2)z^p.$$

If let assume that z is real and that $z\to 1^-$,

$$\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))(\psi+p)a_{\psi+p}+\sum_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}$$

$$((\psi+p)^2(1-\tau)+(1-\tau\zeta))(\psi+p)b_{\psi+p}\geq(\tau(p^2+\zeta)-2p^2).$$

this is precisely the assertion of if part result.

2.2 *Distortion bounds*

Theorem 4 Let $f(z) \in \mathfrak{w}^*(p, \varepsilon, \nu, \upsilon, \vartheta)$, we obtain

$$r^p - \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}} r^{p+\psi} - \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}} \bar{r}^{p+\psi} \le |f(z)|$$

$$\le r^p + \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}} r^{p+\psi} + \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}} \bar{r}^{p+\psi}. \tag{14}$$

The outcome is accurate for

$$f(z) = z^p + \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}} z^{p+\psi} + \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}} \bar{z}^{p+\psi}.$$

Proof. If $f(z) = z^p + \sum_{\psi=2}^{\infty} \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}} z^{p+\psi} + \sum_{\psi=1}^{\infty} \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}} \bar{z}^{p+\psi}$, we get

$$|f(z)| \le z^p + \sum_{\psi=2}^{\infty} \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}} z^{p+\psi} + \sum_{\psi=1}^{\infty} \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}} \bar{z}^{p+\psi} \le r^p + r^{p+\psi} \sum_{\psi=2}^{\infty} a_{\psi+p} + \bar{r}^{p+\psi} \sum_{\psi=1}^{\infty} b_{\psi+p}.$$

Then,

$$a_{\psi+p} \le \sum_{\psi=2}^{\infty} \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}},$$

$$b_{\psi+p} \le \sum_{\psi=1}^{\infty} \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}}.$$

This contributed to us to,

$$|f(z)| \le r^p + \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}} r^{p+\psi} + \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}} \bar{r}^{p+\psi}.$$

similarly

$$|f(z)| \ge r^p - \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}} r^{p+\psi} - \frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}} \bar{r}^{p+\psi}.$$

The outcome is accurate for

$$f(z)=z^p+\frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}}z^{p+\psi}+\frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}}\bar{z}^{p+\psi}.$$

Exectly like this theorem proof also applicable for the following:
Theorem 5 Let $f(z)\in\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$, there exists

$$pr^{p-1}-\frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}}pr^{\psi+p-1}-\frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}}p\bar{r}^{\psi+p-1}\leq|f'(z)|$$

$$\leq pr^{p-1}+\frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}}pr^{\psi+p-1}+\frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}}p\bar{r}^{\psi+p-1}. \tag{15}$$

The outcome is accurate for

$$f(z)=z^p+\frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}}z^{\psi+p}+\frac{\tau(p^2+\zeta)-2p^2}{(\psi+p)\varpi_{\psi+p}}\bar{z}^{\psi+p}.$$

2.3 *Convolution and convex combination*

In this section, we demonstrate the invariance of the class $\mathfrak{G}(z)\in\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$ under convolution and convex combination. Let multivalent harmonic functions f have the structure

$$f(z)=z^p+\sum_{\psi=2}^{\infty}|a_{\psi+p}|z^{\psi+p}+\sum_{\psi=1}|b_{\psi+p}|\bar{z}^{\psi+p},$$

$$F(z)=z^p+\sum_{\psi=2}^{\infty}|A_{\psi+p}|z^{\psi+p}+\sum_{\psi=1}|B_{\psi+p}|\bar{z}^{\psi+p}.$$

The definition of the convolution of $f(z)$ and $F(z)$ is

$$(f*F)(z)=f(z)*F(z)=\sum_{\psi=2}^{\infty}|a_{\psi+p}||A_{\psi+p}|z^{\psi+p}+\sum_{\psi=1}^{\infty}|b_{\psi+p}||B_{\psi+p}|\bar{z}^{\psi+p}. \tag{16}$$

Theorem 6 If $0\leq q<1$, Let $\mathfrak{w}^*(p)$ and $f\in\mathfrak{w}^*(q)$ when the convolution

$$(f*F)\in\mathfrak{w}^*(p)\subset\mathfrak{w}^*(q). \tag{17}$$

Proof. Assume that (17) provides the convolution $(f*F)$. Let deduce to demonstrate that, for $\mathfrak{w}^*(q)$, the coefficients $(f*F)$ of fulfil the necessary condition stated in Theorem (6). Let observe that $|A_{\psi+p}|\leq 1$ and $|B_{\psi+p}|\leq 1$. At this point, let get the convolution function $(f*F)$.

$$\left(\frac{\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}\left((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta)\right)}{\tau(p^2+\zeta)-2p^2}\right)\sum_{\psi=2}^{\infty}|a_{\psi+p}||A_{\psi+p}|$$

$$+\left(\frac{\sum_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}\left((\psi+p)^2(1-\tau)+(1-\tau\zeta)\right)}{\tau(p^2+\zeta)-2p^2}\right)\sum_{\psi=1}^{\infty}|b_{\psi+p}||B_{\psi+p}|$$

$$\leq\left(\frac{\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}\left((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta)\right)}{\tau(p^2+\zeta)-2p^2}\right)\sum_{\psi=2}^{\infty}|a_{\psi+p}|$$

$$+\left(\frac{\sum_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}\left((\psi+p)^2(1-\tau)+(1-\tau\zeta)\right)}{\tau(p^2+\zeta)-2p^2}\right)\sum_{\psi=1}^{\infty}|b_{\psi+p}|.$$

Therefore,

$$\left(\left(\frac{\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}{\tau(p^2+\zeta)-2p^2}\right)\sum_{\psi=2}^{\infty}|a_{\psi+p}|\right.$$

$$\left.+\left(\frac{\sum_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+(1-\tau\zeta))}{\tau(p^2+\zeta)-2p^2}\right)\sum_{\psi=1}^{\infty}|b_{\psi+p}|\right)\leq 1.$$

Then $0\leq p\leq q<1$, and $F\in\mathfrak{w}^*(q)$, $(f*F)\in\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)\subset\mathfrak{w}^*(q,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$.
Theorem 7 In the instance of a convex linear combination, class $\mathfrak{w}^*(q,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$ is closed.
Proof. Let $k=1,\,2,\,...,$ assume that $f_k\in\mathfrak{w}^*(q,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$

$$f_k(z)=z^p+\sum_{\psi=2}^{\infty}|a_{(\psi+p),\,k}|z^{\psi+p}+\sum_{\psi=1}^{\infty}|b_{(\psi+p),\,k}|\overline{z}^{\psi+p}.$$

$$\left(\left(\frac{\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}{\tau(p^2+\zeta)-2p^2}\right)\sum_{\psi=2}^{\infty}|a_{(\psi+p),\,k}|\right.$$

$$\left.+\left(\frac{\sum_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+(1-\tau\zeta))}{\tau(p^2+\zeta)-2p^2}\right)\sum_{\psi=1}^{\infty}|b_{(\psi+p),\,k}|\right)\leq 1. \tag{18}$$

Let assume that $\sum\limits_{\psi=1}^{\infty}\mathfrak{T}_k=1$, $0\le\mathfrak{T}_k<1$, the convex combination of f_k, which can be expressed as

$$\sum_{\psi=1}^{\infty}\mathfrak{T}_k f_k(z)=z^p+\sum_{\psi=2}^{\infty}\mathfrak{T}_k|a_{(\psi+p),\,k}|z^{\psi+p}+\sum_{\psi=1}^{\infty}\mathfrak{T}_k|b_{(\psi+p),\,k}|\bar{z}^{\psi+p}.$$

$$\left(\frac{\sum\limits_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}{\tau(p^2+\zeta)-2p^2}\right)\sum_{\psi=1}^{\infty}\mathfrak{T}_k a_{(\psi+p),\,k}$$

$$+\left(\frac{\sum\limits_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+(1-\tau\zeta))}{\tau(p^2+\zeta)-2p^2}\right)\sum_{\psi=1}^{\infty}\mathfrak{T}_k b_{(\psi+p),\,k},$$

$$=\sum_{\psi=1}^{\infty}\mathfrak{T}_k\left(\left(\frac{\sum\limits_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}{\tau(p^2+\zeta)-2p^2}\right)a_{(\psi+p),\,k}\right.$$

$$\left.+\left(\frac{\sum\limits_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+(1-\tau\zeta))}{\tau(p^2+\zeta)-2p^2}\right)b_{(\psi+p),\,k}\right)\le 1.$$

$\sum\limits_{\psi=1}^{\infty}\mathfrak{T}_k=1$, Therefore $\sum\limits_{\psi=1}^{\infty}\mathfrak{T}_k,\ f_k(z)\in\mathfrak{w}^*(q,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$.

Theorem 8 Assume that the function f_j, $(j=1,\,2,\,...)$ is defined in the class $\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$ by itself.

$$k(z)=z^p+\sum_{\psi=2}^{\infty}(|a_{(\psi+p),\,(k,\,1)}|^2+|a_{(\psi+p),\,(k,\,2)}|^2)z^{\psi+p}. \tag{19}$$

belong to the class $\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$,

$$\varphi\le 1-\frac{\tau(p^2+\zeta)-2p^2\left(\sum\limits_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}\left((\psi+p)^2(1-\tau)+p(\psi+p)\right)(1-\tau\zeta)\right)^2+\left(\sum\limits_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}p(\psi+p)\right)2\left(\tau(p^2+\zeta)-2p^2\right)^2}{\sum\limits_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}(p(\psi+p))\,2\left(\tau(p^2+\zeta)-2p^2\right)^2}. \tag{20}$$

Proof. Figuring out the maximum φ value is necessary to ensure

$$\sum_{\psi=2}^{\infty}\frac{[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}{\tau(p^2+\zeta)-2p^2}(|a_{(\psi+p),\,(k,\,1)}|^2+|a_{(\psi+p),\,(k,\,2)}|^2)\leq 1.$$

Then

$$\left(\frac{\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}{\tau(p^2+\zeta)-2p^2}\right)^2|a_{(\psi+p),\,(k,\,1)}|^2$$

$$\leq\left(\frac{\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}{\tau(p^2+\zeta)-2p^2}|a_{(\psi+p),\,(k,\,1)}|\right)^2\leq 1,$$

and

$$\left(\frac{\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}{\tau(p^2+\zeta)-2p^2}\right)^2|a_{(\psi+p),\,(k,\,2)}|^2$$

$$\leq\left(\frac{\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}{\tau(p^2+\zeta)-2p^2}|a_{(\psi+p),\,(k,\,2)}|\right)^2\leq 1.$$

therefore, by adding these inequality,

$$\frac{1}{2}\left(\frac{\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}{\tau(p^2+\zeta)-2p^2}\right)^2|a_{(\psi+p),\,(k,\,1)}|^2+|a_{(\psi+p),\,(k,\,2)}|^2\leq 1,$$

$$\frac{\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\varphi\zeta))}{\tau(p^2+\zeta)-2p^2}|a_{(\psi+p),\,(k,\,1)}|^2+|a_{(\psi+p),\,(k,\,2)}|^2.$$

This will satisfy the inequality if

$$\frac{\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\varphi\zeta))}{\tau(p^2+\zeta)-2p^2}$$

$$\leq \frac{(\sum\limits_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p))+(1-\tau\zeta))^2}{2(\tau(p^2+\zeta)-2p^2)^2}.$$

such that

$$2(\tau(p^2+\zeta)-2p^2)^2\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\varphi\zeta))$$

$$\leq \tau(p^2+\zeta)-2p^2(\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}(\psi+p)^2(1-\tau)+p(\psi+p))+(1-\tau\zeta))^2.$$

Then obtain Equation (20):

$$\varphi\leq 1-\frac{\begin{array}{c}\tau(p^2+\zeta)-2p^2\left(\sum\limits_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}\left((\psi+p)^2(1-\tau)+p(\psi+p)\right)(1-\tau\zeta)\right)^2\\ +\left(\sum\limits_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}p(\psi+p)\right)2\left(\tau(p^2+\zeta)-2p^2\right)^2\end{array}}{\sum\limits_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}\left(p(\psi+p)\right)2\left(\tau(p^2+\zeta)-2p^2\right)^2}.$$

Theorem 9 Assume that the function f_j, $(j=1,\,2,\,...)$ is defined in the class $\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$.

$$\bar{l}(z)=z^p+\sum_{\psi=1}^{\infty}(|b_{(\psi+p),\,(k,\,1)}|^2+|b_{(\psi+p),\,(k,\,2)}|^2)z^{\psi+p} \tag{21}$$

belong to the class $\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$,

$$\phi\leq 1-\frac{\begin{array}{c}\tau(p^2+\zeta)-2p^2\left(\sum\limits_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p))(1-\tau\zeta)\right)^2\\ +\left(\sum\limits_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}p(\psi+p)\right)2(\tau(p^2+\zeta)-2p^2)^2\end{array}}{\sum\limits_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}(p(\psi+p))2(\tau(p^2+\zeta)-2p^2)^2}.$$

Similarly let derived to get same result previous theorem (8).

Corollary 10 Convex linear combinations result in the class $\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$ is closed.

Proof. If $\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$ be the class that contains the functions $f_k(z)(k=1,\,2,\,...)$ described by (18). After that is finished, the function $\mathfrak{s}(z)$ defined by

$$\mathfrak{s}(z) = \Theta f_k(z) + (1-\Theta) f_k(z),\ 0 \leq \Theta \leq 1. \tag{22}$$

be in the class $\mathfrak{w}^*(p)$.

The following theorem explained the neighbourhood result for the class $\mathfrak{w}^*(p,\ \varepsilon,\ \nu,\ \upsilon,\ \vartheta)$

2.4 *Neighbourhood for the class* $\mathfrak{w}^*$*(p, ε, ν, υ, ϑ)*

The decision to include all of the neighborhoods related to the inclusion relation $(m,\ \rho)$ occurs next using Ruschewayh's approach. Our explanation of functions $(m,\ \rho)$ neighborhood $f(z) \in \mathfrak{w}^*(p,\ \varepsilon,\ \nu,\ \upsilon,\ \vartheta)$ as per earlier research on the neighbors of analytic functions by Goodman [20], which Ruscheweyh [21] generalized and other authors, such as Atshan [22] and Atshan and Kulkarni [23], investigated:

Theorem 11 If

$$\rho = \left(\left(\frac{\tau(p^2+\zeta) - 2p^2}{\sum\limits_{\psi=2}^{\infty} [Q]^{-\upsilon,\,\varepsilon}_{\varepsilon,\,\psi+1-\nu,\,2-\nu} ((\psi+p)^2(1-\tau) + p(\psi+p) + (1-\tau\zeta))} \right) \right.$$

$$\left. + \left(\frac{\tau(p^2+\zeta) - 2p^2}{\sum\limits_{\psi=2}^{\infty} [Q]^{-\upsilon,\,\varepsilon}_{\varepsilon,\,\psi+1-\nu,\,2-\nu} ((\psi+p)^2(1-\tau) + p(\psi+p) + (1-\tau\zeta))} \right) \right) \leq 1. \tag{23}$$

therefore $\mathfrak{w}^*(p,\ \varepsilon,\ \nu,\ \upsilon,\ \vartheta) \in N_{n,\,\rho}(e)$.

Proof. Let $f \in \mathfrak{w}^*(p,\ \varepsilon,\ \nu,\ \upsilon,\ \vartheta)$, then

$$\sum_{\psi=2}^{\infty} \left([Q]^{-\upsilon,\,\varepsilon}_{\varepsilon,\,\psi+1-\nu,\,2-\nu} ((\psi+p)^2(1-\tau) + p(\psi+p) + (1-\tau\zeta))(\psi+p) a_{\psi+p} \right) +$$

$$\sum_{\psi=1}^{\infty} \left([Q]^{-\upsilon,\,\varepsilon}_{\varepsilon,\,\psi+1-\nu,\,2-\nu} ((\psi+p)^2(1-\tau) + p(\psi+p) + (1-\tau\zeta))(\psi+p) b_{\psi+p} \right) \leq \tau(p^2+\zeta) - 2p^2.$$

Hence,

$$\sum_{\psi=2}^{\infty} \left([Q]^{-\upsilon,\,\varepsilon}_{\varepsilon,\,\psi+1-\nu,\,2-\nu} ((\psi+p)^2(1-\tau) + p(\psi+p) + (1-\tau\zeta)) \right) \sum_{\psi=2}^{\infty} (\psi+p) a_{\psi+p} +$$

$$\sum_{\psi=1}^{\infty} \left([Q]^{-\upsilon,\,\varepsilon}_{\varepsilon,\,\psi+1-\nu,\,2-\nu} ((\psi+p)^2(1-\tau) + p(\psi+p) + (1-\tau\zeta)) \right) \sum_{\psi=1}^{\infty} (\psi+p) b_{\psi+p} \leq \tau(p^2+\zeta) - 2p^2.$$

and which gives

$$\sum_{\psi=2}^{\infty} a_{\psi}+\sum_{\psi=1}^{\infty} b_{\psi} \leq \Bigg(\Big(\frac{\tau(p^2+\zeta)-2p^2}{\sum\limits_{\psi=2}^{\infty}\Big([Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))\Big)}\Big)+$$

$$\Big(\frac{\tau(p^2+\zeta)-2p^2}{\sum\limits_{\psi=2}^{\infty}\Big([Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))\Big)}\Big)\Bigg)=\rho.$$

2.5 *Extreme points*

After that, applying the expression clco $\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$, let derive the extreme points of the closed convex hulls of $\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$.

Theorem 12 Mention f to be established using (1). Then $f\in\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$ if and only if

$$f(z)=\sum_{\psi=1}^{\infty}[\mathfrak{H}_{\psi}k_z(z)+\mathfrak{I}_{\psi}\bar{l}_z(z)], \tag{24}$$

Where,

$$k_1(z)=z,\, k_z(z)=\frac{\tau(p^2+\zeta)-2p^2}{\sum\limits_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}z^{\psi+p},\ (\psi=2,\,3,\,...)$$

$$\bar{l}_z(z)=\frac{\tau(p^2+\zeta)-2p^2}{\sum\limits_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}\bar{z}^{\psi+p},\ (\psi=1,\,2,\,...)$$

$\mathfrak{H}_1=1,\ \sum\limits_{\psi=2}^{\infty}\mathfrak{H}_{\psi}+\sum\limits_{\psi=1}^{\infty}\mathfrak{I}_{\psi}\geq 0,\ \mathfrak{H}_{\psi}\geq 0,\ \mathfrak{I}_{\psi}\geq 0.$

The extreme points of $\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$ are especially k_z and $\bar{l}_z$.

Proof. Let have the following form f functions with form (24):

$$f(z)=\sum_{\psi=1}^{\infty}[\mathfrak{H}_{\psi}k_z(z)+\mathfrak{I}_{\psi}\bar{l}_z(z)],$$

$$k_z(z)=\frac{\tau(p^2+\zeta)-2p^2}{\sum\limits_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}z^{\psi+p},$$

$$\bar{l}_z(z)=\frac{\tau(p^2+\zeta)-2p^2}{\sum\limits_{\psi=1}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}\bar{z}^{\psi+p}.$$

Let get

$$\left(\frac{\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}{\tau(p^2+\zeta)-2p^2}\right)$$

$$\frac{\tau(p^2+\zeta)-2p^2}{\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}\mathfrak{H}_\psi+$$

$$\left(\frac{\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}{\tau(p^2+\zeta)-2p^2}\right)$$

$$\frac{\tau(p^2+\zeta)-2p^2}{\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}\mathfrak{I}_\psi$$

$\sum_{\psi=2}^{\infty}\mathfrak{H}_\psi+\sum_{\psi=1}^{\infty}\mathfrak{I}_\psi\le 1$ then $f\in colo\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$. Consider, however that $f\in colo\mathfrak{w}^*(p,\,\varepsilon,\,\nu,\,\upsilon,\,\vartheta)$ setting,

$$\mathfrak{H}_\psi=\frac{\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}{\tau(p^2+\zeta)-2p^2}|a_{\psi+p}|,\ 0\le\mathfrak{H}_\psi\le 1,\ \psi=2,\,3,\,...$$

$$\mathfrak{I}_\psi=\frac{\sum_{\psi=2}^{\infty}[Q]_{\varepsilon,\,\psi+1-\nu,\,2-\nu}^{-\upsilon,\,\varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}{\tau(p^2+\zeta)-2p^2}|b_{\psi+p}|,\ 0\le\mathfrak{I}_\psi\le 1,\ \psi=1,\,2,\,...$$

and $\mathfrak{H}_1=1,\ \sum_{\psi=2}^{\infty}\mathfrak{H}_\psi+\sum_{\psi=1}^{\infty}\mathfrak{I}_\psi\ge 0$ notice that, according to $\mathfrak{H}_1\ge 0$.

As a result, we acquire $f(z)=\sum_{\psi=1}^{\infty}[\mathfrak{H}_\psi k_z(z)+\mathfrak{I}_\psi\bar{l}_z(z)]$ as needed.

3. Examples of functions for class

Example 1 Let $f(z)=z^p+\frac{z^{p+1}}{p+1}$ for simplicity. Then, the fractional operator applied to $f(z)$ is:

$$D^\alpha f(z)=\frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}z^{p-\alpha}+\frac{1}{p+1}\frac{\Gamma(p+2)}{\Gamma(p+2-\alpha)}z^{p+1-\alpha}.$$

This function has the geometric property of being starlike if:

$$\operatorname{Re}\left(\frac{z(p-\alpha)}{p-\alpha}+\frac{(p+1-\alpha)z^{p+1-\alpha}}{(p-\alpha)z^{p-\alpha}}\right)>0.$$

indicating that the fractional operator has transformed the function into one that retains the multivalent nature with adjusted geometric properties.

Example 2 Mention f to be established using (1). Then $f \in \mathfrak{w}^*(p,\, \varepsilon,\, \nu,\, \upsilon,\, \vartheta)$ if and only if

$$f(z)=\sum_{\psi=1}^{\infty}[\mathfrak{H}_\psi k_z(\phi(x))^\mu+\mathfrak{I}_\psi \bar{l}_z(\phi(x))^\mu],$$

Where,

$$k_1(\phi(x)^\mu)=z,\, k_z(\phi(x)^\mu)=\frac{\tau(p^2+\zeta)-2p^2}{\sum\limits_{\psi=2}^{\infty}[Q]_{\varepsilon,\, \psi+1-\nu,\, 2-\nu}^{-\upsilon,\, \varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}\phi(x)^{\mu(\psi+p)},\, (\psi=2,\, 3,\, ...)$$

$$\bar{l}_z(\phi(x)^\mu)=\frac{\tau(p^2+\zeta)-2p^2}{\sum\limits_{\psi=2}^{\infty}[Q]_{\varepsilon,\, \psi+1-\nu,\, 2-\nu}^{-\upsilon,\, \varepsilon}((\psi+p)^2(1-\tau)+p(\psi+p)+(1-\tau\zeta))}\overline{\phi}(x)^{\mu(\psi+p)},\, (\psi=1,\, 2,\, ...)$$

$\mathfrak{H}_1=1,\ \sum\limits_{\psi=2}^{\infty}\mathfrak{H}_\psi+\sum\limits_{\psi=1}^{\infty}\mathfrak{I}_\psi\geq 0,\, \mathfrak{H}_\psi\geq 0,\, \mathfrak{I}_\psi\geq 0.$

The extreme points of $\mathfrak{w}^*(p,\, \varepsilon,\, \nu,\, \upsilon,\, \vartheta)$ are especially k_z and $\bar{l}_z$.

4. Applications and implications

Complex Dynamics:

• Geometric Dynamics: The geometric dynamics of multivalent functions, such as their iterative behavior and fractal patterns, can be explored using fractional operators. This leads to a better understanding of complex systems and chaos theory.

• Physical Applications: [24] In physics and engineering, the geometric features of multivalent functions pertaining to fractional operators are applied in modeling phenomena like viscoelasticity, anomalous diffusion, and signal processing.

Mathematical and Computational Methods:

• Numerical Techniques: Numerical methods for fractional calculus, including discretization techniques and algorithmic implementations, leverage geometric principles to solve problems involving multivalent functions.

• Geometric Algorithms: Algorithms that incorporate geometric features of multivalent functions enhance the accuracy and efficiency of computations in various applications.

5. Conclusion

This study investigates harmonic multivalent functions $f=k+\bar{l}$ in the open unit disc $U=\{z\colon |z|<1\}$, employing the Prabhakar fractional operator as a central model in fractional calculus. Let's introduce $\mathfrak{v}^*(p,\, \varepsilon,\, \nu,\, \upsilon,\, \vartheta)\subset \mathfrak{w}^*(p,\, \varepsilon,\, \nu,\, \upsilon,\, \vartheta)$, focusing on this subset as the class of interest. Our primary goal is to analyze the impact of convolutions with the Prabhakar fractional differential operator on multivalent harmonic functions within $\mathfrak{w}^*(p, \varepsilon, \nu, \upsilon, \vartheta)$. Our investigation has revealed significant insights into the properties of multivalent harmonic functions within the class $w^*(p, \varepsilon, \nu, \upsilon, \vartheta)$. Let's have explore coefficient inequality, growth rates, distortion properties, closure characteristics, neighborhood behaviors, and extreme points, all pertinent to this class. These findings underscore the intricate relationship between these functions and the Prabhakar fractional operator, enriching our understanding of their analytical properties in complex analysis. This study not only advances theoretical knowledge but also carries practical implications across disciplines like mathematical physics and engineering.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Srivastava HM, Darus M, Ibrahim RW. Classes of analytic functions with fractional powers defined by means of a certain linear operator. *Integral Transforms and Special Functions*. 2011; 22(1): 17-28.

[2] Prabhakar TR. A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Bulletin of Yokohama City University, Department D, Mathematics*. 1971; 19(1): 7-15.

[3] Alarifi NM, Ibrahim RW. A new class of analytic normalized functions structured by a fractional differential operator. *Journal of Function Spaces*. 2021; 2021(1): 6270711.

[4] Gorenflo R, Kilbas AA, Mainardi F, Rogosin SV. *Mittag-Leffler functions, related topics and applications*. Berlin: Springer; 2020.

[5] Salim TO. Some properties relating to the generalized Mittag-Leffler function. *Advances in Applied Mathematics and Analysis*. 2009; 4(1): 21-30.

[6] Salim TO, Faraj AW. A generalization of Mittag-Leffler function and integral operator associated with fractional calculus. *Journal of Fractional Calculus and Applications*. 2012; 3(5): 1-3.

[7] Srivastava HM. *Univalent functions, fractional calculus, and their applications*. Ellis Horwood Halsted Press; 1989.

[8] Atshan WG, Kulkarni SR, Raina RK. A class of multivalent harmonic functions involving a generalized Ruscheweyh type operator. *Mathematical Journal*. 2008; 60(233): 207-213.

[9] Ibrahim RW. Generalized Ulam-Hyers stability for fractional differential equations. *International Journal of Mathematics*. 2012; 23(5): 1250056.

[10] Ibrahim RW. On generalized Srivastava-Owa fractional operators in the unit disk. *Advances in Difference Equations*. 2011; 2011: 1-10.

[11] Ibrahim RW. Ulam stability for fractional differential equation in complex domain. In: *Abstract and Applied Analysis*. vol. 2012. Hindawi Publishing Corporation; 2012. p.649517.

[12] Aouf MK. A family of meromorphically P-valent functions with positive coefficient. *General Mathematics*. 2012; 20(1): 25-37.

[13] AL-khafaji AK, Atshan WG, Abed SS. On the generalization of a class of harmonic univalent functions defined by differential operator. *Mathematics*. 2018; 6(12): 312.

[14] AL-khafaji AK, Atshan WG, Abed SS. On a differential subordination of a certain subclass of univalent functions. *Journal of Kufa for Mathematics and Computer*. 2018; 5(3): 11-16.

[15] AL-khafaji AK, Atshan WG, Abed SS. Some properties of a class of harmonic multivalent functions defined by an integral operator. *Journal of University of Babylon for Pure and Applied Sciences*. 2019; 27(2): 261-268.

[16] Raina RK, Srivastava HM. A new class of meromorphically multivalent functions with applications to generalized hypergeometric functions. *Mathematical and Computer Modelling*. 2006; 43(3-4): 350-356.

[17] Atshan WG, Kulkarni SR. Some applications of generalized Ruscheweyh derivatives involving a general fractional derivative operator to a class of analytic functions with negative coefficients II. *General Mathematics*. 2020; 28(1): 85-103.

[18] AL-khafaji AK, Al-Haytham I, Atshan WG, Abed SS. Some interesting properties of a novel subclass of multivalent function with positive coefficients. *Journal of Kufa for Mathematics and Computer*. 2019; 6(1): 13-20.

[19] AL-khafaji AK, Atshan WG, Abed SS. Neighborhoods and partial sums of a new class of meromorphic multivalent functions defined by fractional calculus. *Karbala International Journal of Modern Science*. 2019; 5(2): 3.

[20] Goodman AW. Univalent functions and nonanalytic curves. *Proceedings of the American Mathematical Society*. 1957; 8(3): 598-601.

[21] Rusheweyh S. Neighborhoods of univalent functions. *Proceedings of the American Mathematical Society*. 1981; 81(4): 521-527.

[22] Atshan WG. Subclass of meromorphic functions with positive coefficients defined by Ruscheweyh derivative II. *Surveys in Mathematics and Its Applications*. 2008; 3: 67-77.

[23] Atshan WG, Kulkarni SR. Neighborhoods and partial sums of subclass of k-uniformly convex functions and related class of k-starlike functions with negative coefficients based on integral operator. *Southeast Asian Bulletin of Mathematics*. 2009; 33(4): 623-637.

[24] Pandey V. *Physical and geometrical interpretation of fractional derivatives in viscoelasticity and transport phenomena*. Reprosentralen, University of Oslo; 2016.