Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Maximal Convergence by Faber Series in Morrey-Smirnov Classes with Variable Exponents

Burcin Oktay

Department of Mathematics, Faculty of Science and Letters, Balikesir University, Balikesir, Turkey
E-mail: burcinokt@gmail.com

Received: 2 July 2024; **Revised:** 21 August 2024; **Accepted:** 25 September 2024

Abstract: In this paper, we assume that G is a domain bounded by Γ Dini-smooth curve and $R > 1$ is the largest number such that a function f is analytic inside the level curve Γ_R in the exterior of Γ. By taking the function f in the Morrey-Smirnov classes with variable exponents $E^{p(\cdot),\ \alpha(\cdot)}(G_R)$, we obtain a rate of maximal convergence of the nth partial sums of the Faber series of the function f in the uniform norm on the closure of G. Here the rate of maximal convergence depends on the best approximation number $E_n^{p(\cdot),\ \alpha(\cdot)}(f,\ G_R)$.

***Keywords*:** rate of convergence, morrey-smirnov classes with variable exponents, faber series, dini-smooth curves

MSC: 30E10, 41A10, 41A25, 41A58

1. Introduction

The Lebesgue spaces with variable exponents are used in the solving of different problems in mechanic, especially in fluid dynamic, and also in the study of image processing. Many authors have obtained analogues of classical results in Lebesgue spaces with variable exponents motivated by their applications [1–7]. The Morrey spaces with variable exponents are natural generalizations of the variable exponent Lebesgue spaces. These spaces were introduced in [8] in the case of an open set $\Omega \subset \mathbb{R}^n$. For information about classical Morrey spaces and variable exponent Morrey spaces, you can see [8–16]. It is known that Smirnov classes are special classes of analytic functions. In this paper we consider the Morrey-Smirnov classes with variable exponents. These classes are natural generalizations of Smirnov classes with variable exponents. In [4, 17–21] the results on approximation to the function of Smirnov classes with variable exponents were obtained. There are the results on approximation to the functions of classical Morrey-Smirnov classes in [22–25]. Only a few studies on Morrey-Smirnov classes with variable exponents are available in the literature so far (see [26, 27]). While the standard definition of the norm of the class in the proofs of these studies is used, equivalent but more intriguing definition of the norm is used in the proofs of this paper, with the help of Lemma 5 in [8], and also [22, 23]. For given a domain G bounded by Γ, by considering that $R > 1$ is the largest number such that f is analytic inside the level curve Γ_R in the exterior of Γ, we obtain a rate of maximal convergence of the nth partial sums of the Faber series of the function f in the Morrey-Smirnov classes with variable exponents in the uniform norm on the closure of G. In the problems of maximal convergence, the rate of maximal convergence depends on the number $R > 1$ and the best approximation number

DOI: https://doi.org/10.37256/cm.6120255234

to the function, defined in the given space. There are some results related to maximal convergence in literature. Firstly, Bernstein and Walsh studied on the maximal convergence of polynomials. They also obtained direct and inverse theorems when the function f is analytic on canonical domain G_R, $R > 1$ (see [28] and also [29]). Many results about maximal convergence of Faber series were proved by P. K. Suetin in [28]. Suetin obtained some results on maximal convergence of Faber series of the function f in the case which is analytic on the canonical domain G_R, continuous on $\overline{G}_R$ and in the case which belongs to Smirnov class $E^p(G_R)$, $p \geq 1$. Suetin assumed the case the boundary of the domain G belongs to the class of Al'per curves. He also proved some results on the maximal convergence for the case a continuum K. The maximal convergence by Faber series was investigated in Smirnov-Orlicz classes in [30] in the case of continuum and in [31] in the case of the domain bounded by a special smooth curve. When f is a function in the Smirnov class with variable exponents, maximal convergence of Faber series in the case of continuum was studied in [19]. If f is a function in the weighted Smirnov class with variable exponents, maximal convergence of Faber series in the case of domain bounded by a special smooth curve was studied by us in [32].

2. Preliminaries

Let G be a simply connected domain bounded by a rectifiable curve Γ in the complex plane $\mathbb{C}$, and let also the complement of the closed domain $\overline{G}$ is a simply connected domain G' containing the point of infinity $z = \infty$. By the Riemann conformal mapping theorem there exists a unique function $w = \varphi(z)$ meromorphic in G' which maps the domain G' conformally and univalently onto the exterior of the unit circle $\mathbb{T}$ and satisfies the conditions

$$\varphi(\infty) = \infty,\ \varphi'(\infty) = \gamma > 0,$$

where γ is the capacity of G. Let ψ be the inverse to φ and let ψ_0 be the mapping which maps unit disk onto the domain G under the conditions $\psi_0(0) = 0$ and $\psi_0'(0) > 0$. Let Γ_r be the image of the circle $|w| = r$, $0 < r < 1$, under the mapping ψ_0. If a function $f(z)$, analytic on a domain G, satisfies the inequality

$$\int_{\Gamma_r} |f(z)|^p \, |dz| \leq M,\ p > 0$$

for any r such that $0 < r < 1$, then f belongs to the Smirnov class $E^p(G)$. In this definition one can replace the set of the images of the circles $\{\Gamma_r\}$ by an arbitrary sequence of rectifiable curves $\{\Gamma_n\}$, which converge from inside the domain G to the curve Γ (see [28]). If $f \in E^p(G)$, then it has a nontangential limit almost everywhere on Γ and the boundary function belongs to $L^p(\Gamma)$ [33]. Let J denote the interval $I_0 := [0,\ 2\pi]$ or the Jordan rectifiable curve $\Gamma \subset \mathbb{C}$ and let $p(\cdot) : J \to [0,\ \infty)$ be a Lebesgue measurable function such that

$$1 < p_- := \underset{z \in J}{essinf}\, p(z) \leq \underset{z \in J}{esssup}\, p(z) =: p_+ < \infty \tag{1}$$

Let $|J|$ is the Lebesgue measure of J. We say that $p(\cdot) \in P^{log}(J)$, if $p(.)$ satisfies the condition

$$|p(z_1) - p(z_2)|\, ln\left(\frac{|J|}{|z_1 - z_2|}\right) \leq c,\ \forall z_1,\ z_2 \in J,$$

with a positive constant c independent of z_1 and z_2. We note that the condition above is usually called the log-Hölder continuity or the Dini-Lipschitz condition.

Definition 1 Let $f:\Gamma\to\mathbb{R}$ be a Lebesgue measurable function. For a given $p(\cdot)\in P^{log}(\Gamma)$, if the condition

$$\int_{\Gamma}|f(z)|^{p(z)}\,|dz|<\infty.$$

holds, the set of Lebesgue measurable functions f is defined the Lebesgue class $L^{p(\cdot)}(\Gamma)$ with variable exponents.

If $p(\cdot):\Gamma\to[1,\,\infty)$, $L^{p(\cdot)}(\Gamma)$ becomes a Banach spaces with the norm

$$\|f\|_{L^{p(\cdot)}(\Gamma)}:=inf\left\{\lambda>0:\ \int_{\Gamma}\left|\frac{f(z)}{\lambda}\right|^{p(z)}|dz|\leq 1\right\}$$

If $p(\cdot):=constant$, then $L^{p(\cdot)}(\Gamma)$ coincides with the Lebesgue space $L^p(\Gamma)$.

Definition 2 Let f be an analytic function in G. For $p(\cdot):\Gamma\to[1,\,\infty)$, the set

$$E^{p(\cdot)}(G):=\left\{f\in E^1(G):\ f\in L^{p(\cdot)}(\Gamma)\right\}$$

is called Smirnov class with variable exponent $p(\cdot)$.

$E^{p(\cdot)}(G)$ becomes a Banach space equipping with the norm

$$\|f\|_{E^{p(\cdot)}(G)}:=\|f\|_{L^{p(\cdot)}(\Gamma)}.$$

We define the classical Morrey space $L^{p,\,\alpha}(\Gamma)$ such that $p\geq 1$ and $0\leq\alpha\leq 2$. For the definition you can see for example [23]. For $z\in\Gamma$ and $r>0$, we denote

$$B:=B(z,\,r):=\{\xi\in\mathbb{C}:|\xi-z|<r\}.$$

Definition 3 $L^{p,\,\alpha}(\Gamma)$ is defined as the set of all the functions $f\in L^p_{loc}(\Gamma)$ such that

$$\|f\|_{L^{p,\,\alpha}(\Gamma)}:=\left\{\sup_B\frac{1}{|B\cap\Gamma|^{1-\frac{\alpha}{2}}}\int_{B\cap\Gamma}|f(z)|^p\,|dz|\right\}^{\frac{1}{p}}<\infty$$

where the supremum is taken over all disks B centered on Γ.

$L^{p,\,\alpha}(\Gamma)$ becomes a Banach space; for $\alpha=2$ coincides with $L^p(\Gamma)$ and for $\alpha=0$ with $L^\infty(\Gamma)$. Moreover $L^{p,\,\alpha_1}(\Gamma)\subset L^{p,\,\alpha_2}(\Gamma)$ for $0\leq\alpha_1\leq\alpha_2\leq 2$. If $f\in L^{p,\,\alpha}(\Gamma)$ then $f\in L^p(\Gamma)$ and hence $f\in L^1(\Gamma)$. One could prefer to write the denominator of the fraction in front of the integral in the norm $L^{p,\,\alpha}(\Gamma)$ with respect to r. But we prefer to write it based on $|B\cap\Gamma|$. Because we can write

$$diam\{B\cap\Gamma\}\leq c\,|B\cap\Gamma|$$

since $\{B\cap\Gamma\}$ is a quasi-circle. This makes the fraction in front of the integral in the definition of the norm smaller.

Definition 4 Let f be an analytic function in G. For $p\geq 1$ and $0\leq\alpha\leq 2$, the set

$$E^{p,\,\alpha}(G):=\left\{f\in E^1(G):\ f\in L^{p,\,\alpha}(\Gamma)\right\}$$

is called Morrey-Smirnov class [23].

$E^{p,\,\alpha}(G)$ becomes a Banach space equipping with the norm

$$\|f\|_{E^{p,\,\alpha}(G)}:=\|f\|_{L^{p,\,\alpha}(\Gamma)}.$$

For $\alpha=2$, $E^{p,\,\alpha}(G)$ coincides with $E^p(G)$ and for $\alpha=0$ with $E^{\infty}(G)$. Moreover $E^{p,\,\alpha_1}(G)\subset E^{p,\,\alpha_2}(G)$ for $0\leq\alpha_1\leq\alpha_2\leq 2$. If $f\in E^{p,\,\alpha}(G)$ then $f\in E^p(G)$ and hence $f\in E^1(G)$.

We now define the Morrey space $L^{p(\cdot),\,\alpha(\cdot)}(\Gamma)$ with variable exponents as the following.

Definition 5 Let $p(\cdot):\Gamma\to[1,\,\infty)$ be a measurable function satisfying (1) and let $\alpha(\cdot):\Gamma\to[0,\,2]$ be a measurable function. The set of Lebesgue measurable functions f such that

$$I_{p(\cdot),\,\alpha(\cdot)}(f):=\sup_B\frac{1}{|B\cap\Gamma|^{1-\frac{\alpha(\cdot)}{2}}}\int_{B\cap\Gamma}|f(z)|^{p(z)}\,dz<\infty \tag{2}$$

is defined as the Morrey space $L^{p(\cdot),\,\alpha(\cdot)}(\Gamma)$ with variable exponents.

For the definition you can see [8, 14]. The norm in the space $L^{p(\cdot),\,\alpha(\cdot)}(\Gamma)$ is defined by

$$\|f\|_{L^{p(\cdot),\,\alpha(\cdot)}(\Gamma)}:=\|f\|_1=\inf\left\{\eta>0:\ I_{p(\cdot),\,\alpha(\cdot)}(\frac{f}{\eta})<1\right\}. \tag{3}$$

It is known that $L^{p(\cdot),\,\alpha(\cdot)}(\Gamma)$ is Banach space. If $\alpha(\cdot)=2$ then $L^{p(\cdot),\,\alpha(\cdot)}(\Gamma)$ coincides with the variable exponent Lebesgue space $L^{p(\cdot)}(\Gamma)$. In the cases $\alpha(\cdot)=const$ and $p(\cdot)=const$, the variable exponent Morrey space coincides with the classical Morrey space.

There is more intriguing definition of the norm: We may define the Morrey norm with variable exponents by:

$$\|f\|_{1^*}:=\sup_B\left\|\frac{1}{|B\cap\Gamma|^{(1-\frac{\alpha(\cdot)}{2})\frac{1}{p(\cdot)}}}f\chi_{B\cap\Gamma}\right\|_{L^{p(\cdot)}(\Gamma)}. \tag{4}$$

Let $\alpha(\cdot)$ be of the class $P^{log}(\Gamma)$. In this case, there is no difference in taking the parameter α depending on which variables (see [8]). By taking $1-\dfrac{\alpha(\cdot)}{2}$ instead of $\lambda(\cdot)$ in Lemma 5 in [8], we can deduce the following lemma which show the equivalence of the norms $\|f\|_1$ and $\|f\|_{1^*}$ in $L^{p(\cdot),\,\alpha(\cdot)}(\Gamma)$.

Lemma 1 If $\alpha(\cdot) \in P^{log}(\Gamma)$, for every $f \in L^{p(\cdot),\ \alpha(\cdot)}(\Gamma)$,

$$\|f\|_{L^{p(\cdot),\ \alpha(\cdot)}(\Gamma)} = \|f\|_1 = \|f\|_{1^*}.$$

Definition 6 The variable exponent Morrey-Smirnov class $E^{p(\cdot),\ \alpha(\cdot)}(G)$ is defined as

$$E^{p(\cdot),\ \alpha(\cdot)}(G) := \left\{ f \in E^1(G) :\ f \in L^{p(\cdot),\ \alpha(\cdot)}(\Gamma) \right\}.$$

Note that $E^{p(\cdot),\ \alpha(\cdot)}(G)$ is a Banach space with respect to the norm

$$\|f\|_{E^{p(\cdot),\ \alpha(\cdot)}(G)} := \|f\|_{L^{p(\cdot),\ \alpha(\cdot)}(\Gamma)}$$

$E^{p(\cdot),\ \alpha(\cdot)}(G)$ coincides with the class $E^{p(\cdot)}(G)$ for $\alpha(\cdot) = 2$.
We define the best approximation to the function $f \in E^{p(\cdot),\ \alpha(\cdot)}(G)$ as:

$$E_n^{p(\cdot),\ \alpha(\cdot)}(f,\ G) := inf\, \|f - p_n\|_{L^{p(\cdot),\ \alpha(\cdot)}(\Gamma)} \tag{5}$$

where inf is taken over the polynomials p_n of degree at most n.

The best approximation number $E_n^{p(\cdot),\ \alpha(\cdot)}(f,\ G)$ exists and is unique in the case of $p(\cdot) \in P^{log}(\Gamma)$ (see [6]). Let $p(\cdot) : \mathbb{T} \to [1,\ \infty)$ and $\alpha(\cdot) : \mathbb{T} \to [0,\ 2]$ be measurable functions such that

$$0 < \alpha_- := \underset{z \in \mathbb{T}}{essinf}\, \alpha(z) \leq \underset{z \in \mathbb{T}}{esssup}\, \alpha(z) =: \alpha_+ \leq 2.$$

We also assume $p(\cdot) \in P^{log}(\mathbb{T})$. For $f \in L^{p(\cdot),\ \alpha(\cdot)}(\mathbb{T})$ we define the operator

$$(v_h f)(w) := \frac{1}{h} \int_0^h f(we^{it})dt,\ w \in \mathbb{T}\ 0 < h < \pi.$$

It is clear that the operator v_h is a bounded linear operator on $L^{p(\cdot),\ \alpha(\cdot)}(\mathbb{T})$ [14]:

$$\|v_h(f)\|_{L^{p(\cdot)}(\mathbb{T})} \leq c\, \|f\|_{L^{p(\cdot)}(\mathbb{T})}.$$

The function

$$\Omega(f,\ \delta)_{p(\cdot),\ \alpha(\cdot)} := \sup_{0<h\leq\delta} \|f(\cdot) - v_h(\cdot)\|_{L^{p(\cdot),\ \alpha(\cdot)}(\mathbb{T})},\ \delta > 0$$

is called the modulus of smoothness of $f \in L^{p(\cdot),\ \alpha(\cdot)}(\mathbb{T})$. It can easily be shown that $\Omega(f,\ \delta)_{p(\cdot),\ \alpha(\cdot)}$ is a continuous, non-negative and non-decreasing function satisfying the conditions

$$\lim_{\delta\to 0}\Omega(f,\ \delta)_{p(\cdot),\ \alpha(\cdot)}=0,$$

$$\Omega(f+g,\ \delta)_{p(\cdot),\ \alpha(\cdot)}\le\Omega(f,\ \delta)_{p(\cdot),\ \alpha(\cdot)}+\Omega(g,\ \delta)_{p(\cdot),\ \alpha(\cdot)},\ \delta>0,$$

for $f(\cdot),\ g(\cdot)\in L^{p(\cdot),\ \alpha(\cdot)}(\mathbb{T})$. Let $h(t)$ be a continuous function on $[0,\ 2\pi]$. Its modulus of continuity is defined by

$$\omega(h,\ \delta):=\sup_{\delta_1,\ \delta_2\in[0,\ 2\pi],\ |\delta_1-\delta_2|<\delta}|h(\delta_1)-h(\delta_2)|,\ \delta\ge 0.$$

Definition 7 A smooth Jordan curve Γ will be called Dini-smooth, if the function $\theta(s)$, the angle between the tangent line and the positive real axis expressed as a function of arclength s, has modulus of continuity $\omega(\theta,\ s)$ satisfying the Dini condition

$$\int_0^{\pi}\frac{\omega(\theta,\ s)}{s}ds<\infty.$$

Here the number π could be replaced by any positive constant [34]. If Γ is Dini-smooth, then the condition

$$0<c_1\le\left|\psi'(w)\right|\le c_2<\infty,\ |w|\ge 1 \tag{6}$$

holds for some positive constants c_1 and c_2 [28]. This inequality is also valid for $|\varphi'(\zeta)|$ for some positive constants c_3 and c_4 in $\zeta\in\overline{G'}$. Hence, for any disk $B\subset\mathbb{C}$ with sufficiently small radius, there exists a disc $B_0\subset\mathbb{C}$ such that

$$|B\cap\Gamma|\le c_5\,|B_0\cap\mathbb{T}|\le c_6\,|B\cap\Gamma|. \tag{7}$$

Indeed, let B is a disc such that the set $B\cap\Gamma$ is not empty. If we denote $\gamma_z:=B\cap\Gamma$ on Γ, there exists an arc $\gamma_w:=\varphi(\gamma_z)$ on $\mathbb{T}$. Let be B_0 a disc containing γ_w, with sufficiently small radius. Hence we have $\int_{\gamma_w}|dw|=|B_0\cap\mathbb{T}|$ and therefore

$$|B\cap\Gamma|=\int_{\gamma_z}|dz|=\int_{\gamma_w}\left|\psi'(w)\right||dw|\le c_7\int_{\gamma_w}|dw|=c_8\,|B_0\cap\mathbb{T}|$$

$$=c_8\int_{\gamma_z}\left|\varphi'(z)\right||dz|\le c_9\int_{\gamma_z}|dz|=c_9\,|B\cap\Gamma|. \tag{8}$$

holds. For construction of the approximation aggregates in $E^{p(\cdot),\ \alpha(\cdot)}(G)$, we use the Faber polynomials φ_k, $k=0,\ 1,\ 2,\ \ldots$. This is the polynomial part of the Laurent expansion at ∞ of $\varphi(z)^k$, and hence φ_k is of degree k, with leading coefficient γ^k. For the sum of the terms containing negative powers of z in the expansion of $\varphi(z)^k$, we use the notation $-E_k(z)$. Hence the identity

$$\varphi_k(z) = \varphi^k(z) + E_k(z),\ z \in \overline{G'} \tag{9}$$

holds in the sense of convergence. Now we denote the level curve by

$$\Gamma_R := \{\zeta \in ext\Gamma;\ |\varphi(\zeta)| = R,\ R > 1\},\ G_R := int\Gamma_R,$$

and

$$T_R := \{t \in ext(|w| = 1);\ |t| = R,\ R > 1\}.$$

Faber polynomials have the following integral representation

$$\varphi_k(z) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{\varphi^k(\varsigma)}{\varsigma - z} d\varsigma,\ z \in G_R$$

If a function f is analytic in the canonical domain G_R, then the expansion holds

$$f(z) = \sum_{k=0}^{\infty} a_k \varphi_k(z),\ z \in G_R,\ R > 1 \tag{10}$$

and the series converge absolutely and uniformly on G_R, where

$$a_k := \frac{1}{2\pi i}\int_{|t|=R} \frac{f(\psi(t))}{t^{k+1}} dt,\ k = 0,\ 1,\ 2,\ \ldots \tag{11}$$

Now we use the value

$$R_n(z,\ f) := f(z) - \sum_{k=0}^{n} a_k \varphi_k(z) = \sum_{k=n+1}^{\infty} a_k \varphi_k(z) \tag{12}$$

which is called remaining term.

3. Auxiliary results

The formulas (11) and (12) represent that,

$$R_n(z,\ f) = \frac{1}{2\pi i}\int_{|t|=R} f(\psi(t)) \left[\sum_{k=n+1}^{\infty} \frac{\varphi_k(z)}{t^{k+1}}\right] dt. \tag{13}$$

From (9), we can write

$$\sum_{k=n+1}^{\infty}\frac{\varphi_k(z)}{t^{k+1}}=\sum_{k=n+1}^{\infty}\frac{\varphi^k(z)}{t^{k+1}}+\sum_{k=n+1}^{\infty}\frac{E_k(z)}{t^{k+1}},\ z=\psi(w) \tag{14}$$

It is seen from [35] and [28] that $E_k(\psi(w))$ is defined as the following

$$E_k(\psi(w))=\frac{1}{2\pi i}\int_{|\tau|=1}\tau^k F(\tau,\ w)d\tau,\ |w|\geq 1, \tag{15}$$

where

$$F(\tau,\ w):=\frac{\psi'(\tau)}{\psi(\tau)-\psi(w)}-\frac{1}{\tau-w},\ |\tau|\geq 1,\ |w|\geq 1. \tag{16}$$

It is very important to notice that if one wants to estimate remaining term $R_n(z,\ f)$, it is necessary to show that the integral

$$\int_{|\tau|=1}|F(\tau,\ \omega)|\,|d\tau| \tag{17}$$

is finite for all $|w|\geq 1$, according to the geometric properties of the boundary of the domain G. Hence if Γ is Dini-smooth, by (15) and Lemma 2, it can be written

$$\sum_{k=n+1}^{\infty}\frac{E_k(\psi(w))}{t^{k+1}}=\frac{1}{2\pi i}\int_{|\tau|=1}F(\tau,\ w)\sum_{k=n+1}^{\infty}\frac{\tau^k}{t^{k+1}}d\tau \tag{18}$$

for $|w|=1$ and $|t|=R$ (see [28] for Al'per curves).

By applying the similar methods of the proofs of Theorem 2 in [28] and Theorem 3 in [28] for Dini-smooth curves, we prove the following lemma.

Lemma 2 If G is a domain bounded by a Dini-smooth curve Γ, then there exists a constant $c>0$ such that for all $|w|\geq 1$ the following inequality occurs

$$\int_{|\tau|=1}|F(\tau,\ w)|\,|d\tau|=\int_{|\tau|=1}\left|\frac{\psi'(\tau)}{\psi(\tau)-\psi(w)}-\frac{1}{\tau-w}\right||d\tau|\leq c<\infty$$

and this integral converges uniformly with respect to $|w|\geq 1$.

Proof. $F(\tau,\ w)$ can be written in the following way:

$$F(\tau,\ w)=\left[\frac{\psi'(\tau)(\tau-w)-\psi(\tau)+\psi(w)}{(\tau-w)^2}\right]:\left[\frac{\psi(\tau)-\psi(w)}{\tau-w}\right].$$

By (6), the inequality

$$\left[\frac{\psi(\tau)-\psi(w)}{\tau-w}\right] \geq c_{11} > 0,\ |\tau| \geq 1,\ |w| \geq 1$$

holds. We estimate the integral

$$I(w) := \int_{|\tau|=1} \left|\frac{\psi'(\tau)(\tau-w)-\psi(\tau)+\psi(w)}{(\tau-w)^2}\right| |d\tau|$$

with the similar prosedure as that in [28] as the following:

$$\begin{aligned} I(w) &= \int_{|\tau|=1} \left|\frac{1}{(\tau-w)^2}\int_w^\tau [\psi'(\tau)-\psi'(t)]dt\right| |d\tau| \\ &\leq \int_{|\tau|=1} \frac{1}{(\tau-w)^2}\int_w^\tau \omega(\psi', |\tau-w|)\,|dt|\,|d\tau| \\ &\leq \int_{|\tau|=1} \frac{\omega(\psi', |\tau-w|)}{|\tau-w|}\,|d\tau| \end{aligned} \tag{19}$$

Since Γ is smooth, from the equality (3) in [34], we write that

$$arg\psi'(w) = \theta(s) - argw - \frac{\pi}{2},\ w = e^{it}.$$

By this equality, the inequality

$$\left|arg\psi'(e^{i(t+h)}) - arg\psi'(e^{it})\right| \leq |\theta(s+h)-\theta(s)| + |h|$$

and hence the inequality

$$\omega(arg\psi',\ \delta) \leq c_1\omega(\theta,\ \delta) + \delta. \tag{20}$$

holds. It is known that

$$ln\psi'(w) = ln\left|\psi'(w)\right| + iarg\psi'(w)$$

for $|w| = 1$. From this formula, it is written that

$$\left|ln\psi'(e^{i(t+h)})-ln\psi'(e^{it})\right|\leq\left|ln\mid\psi'(e^{i(t+h)})\mid-ln\mid\psi'(e^{it})\mid\right|$$

$$+\left|arg\psi'(e^{i(t+h)})-arg\psi'(e^{it})\right|. \tag{21}$$

On the other hand, the following inequality

$$\left|\psi'(e^{i(t+h)})-\psi'(e^{it})\right|\leq c_5\left|ln\psi'(e^{i(t+h)})-ln\psi'(e^{it})\right| \tag{22}$$

holds (see [28]). By (6), we have that

$$\left|ln\mid\psi'(e^{i(t+h)})\mid-ln\mid\psi'(e^{it})\mid\right|\leq lnc_2+lnc_3=c_4 \tag{23}$$

Hence, it implies that

$$\left|\psi'(e^{i(t+h)})-\psi'(e^{it})\right|\leq c_6\left|arg\psi'(e^{i(t+h)})-arg\psi'(e^{it})\right|$$

From this we have that

$$\omega(\psi',\ \delta)\leq c_6\omega(arg\psi',\ \delta) \tag{24}$$

Hence from (20) and (24)

$$\omega(\psi',\ \delta)\leq c_6\omega(arg\psi',\ \delta)\leq c_1\omega(\theta,\ \delta)+\delta,\ w=e^{it} \tag{25}$$

If a function $\lambda(w)$ is analytic in the domain $|w|>1$ and continuous in the closed domain $|w|\geq 1$, then its modulus of continuity $\omega(\lambda,\ \delta)$ is connected in $|w|\geq 1$ with the modulus of continuity $\omega(\lambda_1,\ \delta)$ of the function $\lambda_1(t)=\lambda(e^{it})$ by the inequality (see [28]),

$$c\omega(\lambda,\ \delta)\leq\omega(\lambda_1,\ \delta)\leq c\omega(\lambda,\ \delta). \tag{26}$$

Hence the inequality (26) holds for $|w|\geq 1$. Now we divide by δ both side of the inequality (26), and then, by taking $|\tau-w|$ instead of δ, integrate to the $|\tau|=1$, hence we can write that

$$I(w) \leq \int_{|\tau|=1} \frac{\omega(\psi', \ |\tau - w|)}{|\tau - w|} |d\tau|$$

$$\leq c \int_{|\tau|=1} \frac{\omega(arg\psi', \ |\tau - w|)}{|\tau - w|} |d\tau|$$

$$\leq c \int_{|\tau|=1} \frac{\omega(\theta, \ |\tau - w|)}{|\tau - w|} |d\tau| \tag{27}$$

Since Γ is a Dini-smooth, the last integral is finite. It means that

$$\int_{|\tau|=1} |F(\tau, \ w)| \, |d\tau| \leq c < \infty.$$

Hence the proof is completed.

The following theorem characterizes Hölder's inequality in the variable exponent Lebesgue spaces (see [2] and [6]).

Theorem 1 Let $p(\cdot) : \Gamma \to [1, \ \infty]$ be a measurable function. If $f \in L^{p(\cdot)}(\Gamma)$ and $g \in L^{q(\cdot)}(\Gamma)$, such that $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$, then $fg \in L^1(\Gamma)$ and the inequality

$$\int_\Gamma |f(z)g(z)| \, dz \leq c(p) \, \|f\|_{L^{p(\cdot)}(\Gamma)} \, \|g\|_{L^{q(\cdot)}(\Gamma)}$$

holds for some positive constant $c(p)$ dependent on the function $p(\cdot)$.

Let $f_0(w) := f[\psi(w)]$ for $f(\cdot) \in L^{p(\cdot), \ \alpha(\cdot)}(\Gamma)$ and $p_0(w) := p(\psi(w))$. It can be shown that $f_0(\cdot) \in L^{p_0(\cdot), \ \alpha(\cdot)}(\mathbb{T})$ by using (6). Let $\lambda(\cdot) : \Gamma \to [0, \ 1], \ \lambda_- := \underset{z \in \Gamma}{essinf} \lambda(z)$ and $\lambda_+ := \underset{z \in \Gamma}{esssup} \lambda(z)$. The following theorem is used for Corollary 1 by taking $1 - \frac{\alpha(\cdot)}{2}$ instead of $\lambda(\cdot)$ and Γ_R instead of Γ.

Theorem 2 [26] Let Γ be a Dini-smooth curve. Let $p(\cdot) : \Gamma \to [1, \ \infty]$ and $\lambda(\cdot) : \Gamma \to [0, \ 1]$ be measurable functions such that $p(\cdot) \in P^{log}(\Gamma)$ and $0 \leq \lambda_- \leq \lambda_+ < 1$. If $f(z) \in E^{p(\cdot), \ \alpha(\cdot)}(G)$, then for every natural n the inequality

$$\left\| f(z) - \sum_{k=0}^{n} a_k \varphi_k(z) \right\|_{L^{p(\cdot), \ \lambda(\cdot)}(\Gamma)} \leq C\Omega(f_0, \ \frac{1}{n})_{p_0(\cdot), \ \lambda(\cdot)}$$

holds with a constant $C > 0$ independent of n.

4. New results and proofs

Our new results and their proofs are as the following.

Theorem 3 Let G be a domain bounded by Γ Dini-smooth curve, $R > 1$ is the largest number such that a function f is analytic inside the level curve Γ_R in the exterior of Γ and $G_R = int\Gamma_R$. If f belongs to the class $E^{p(\cdot), \ \alpha(\cdot)}(G_R)$ where $p(\cdot) : \ \Gamma_R \to [1, \ \infty)$, $1 < p_- \leq p_+ < \infty$, $p(\cdot) \in P^{log}(\Gamma_R)$ and $\alpha(\cdot) : \ \Gamma_R \to [0, \ 2]$, $\alpha(\cdot) \in P^{log}(\Gamma_R)$, then the value $R_n(z, \ f)$ satisfies the inequality

$$|R_n(z,\ f)| \leq \frac{c(p)}{2\pi \sup\limits_{\zeta\in\Gamma_R} R^{n+1+\frac{\alpha(\zeta)-2}{2p(\zeta)}}(R-1)} E_n^{p(\cdot),\ \alpha(\cdot)}(f,\ G_R),\ z\in\overline{G}$$

with some constant $c(p) > 0$ independent of n. Here $E_n^{p(\cdot),\ \alpha(\cdot)}(f,\ G_R)$ is the best approximation number to the function $f \in E^{p(\cdot),\ \alpha(\cdot)}(G_R)$.

Theorem 3 characterises the maximal convergence of the nth partial sums of Faber series of the functions belong to the Morrey-Smirnov class with variable exponents in the case of domains bounded by Dini-smooth curves. In the case of $\alpha(\cdot) = 2$, we obtained a rate of convergence when f belongs to the weighted Smirnov class with variable exponents for a domain bounded by a special smooth curve in [32]. We note that the main result in [32] could also be obtained for the case of the domains bounded by Dini-smooth curves. Hence Theorem 3 becomes a generalization of the main result in [32] when the weight function $\omega(\cdot) = 1$. On the other hand, if we compare Theorem 3 with the main result in [19] in the case of $\alpha(\cdot) = 2$, we see that Theorem 3 gives a better approximation error because of the Dini-smoothness property of the boundary of the domain G in this paper. Let f is analytic on G_R, $R > 1$. Let $f_0(w) := f(\psi(Rw))$ and $p_0(w) := p(\psi(Rw))$. Taking into account these conditions and applying Theorem 2 for Γ_R, we can estimate $R_n(z,\ f)$ by modulus of smoothness $\Omega(f_0,\ \frac{1}{n})_{p_0(\cdot),\ \alpha(\cdot)}$ from above in the uniform norm on $\overline{G}$ as the following.

Corollary 1 Let G be a domain bounded by Γ Dini-smooth curve, $R > 1$ is the largest number such that a function f is analytic inside the level curve Γ_R in the exterior of Γ and $G_R = int\Gamma_R$. If f belongs to the class $E^{p(\cdot),\ \alpha(\cdot)}(G_R)$ where $p(\cdot): \Gamma_R \to [1,\ \infty)$, $1 < p_- \leq p_+ < \infty$, $p(\cdot) \in P^{log}(\Gamma_R)$ and $\alpha(\cdot): \Gamma_R \to [0,\ 2]$, $0 < \alpha_- \leq \alpha_+ \leq 2$, $\alpha(\cdot) \in P^{log}(\Gamma_R)$, then the value $R_n(z,\ f)$ satisfies the inequality

$$|R_n(z,\ f)| \leq \frac{c(p)}{2\pi \sup\limits_{\zeta\in\Gamma_R} R^{n+1+\frac{\alpha(\zeta)-2}{2p(\zeta)}}(R-1)} \Omega(f_0,\ \frac{1}{n})_{p_0(\cdot),\ \alpha(\cdot)},\ z\in\overline{G}$$

with some constant $c(p) > 0$ independent of n.

4.1 *Proof of Theorem 3*

Let Γ is Dini-smooth curve, $z \in \Gamma$ and $\zeta \in \Gamma_R$, $R > 1$. We assume that f is a function which belongs to the class $E^{p(\cdot),\ \alpha(\cdot)}(G_R)$ such that $p(\cdot):\ \Gamma_R \to [1,\ \infty)$, $1 < p_- \leq p_+ < \infty$, $p(\cdot) \in P^{log}(\Gamma_R)$ and $\alpha(\cdot):\ \Gamma_R \to [0,\ 2]$, $\alpha(\cdot) \in P^{log}(\Gamma_R)$. Let P_n be the best approximation polynomial to the function $f \in E^{p(\cdot),\ \alpha(\cdot)}(G_R)$. The formula (13) implies that

$$R_n(z,\ f) = \frac{1}{2\pi i}\int_{|t|=R} \{f(\psi(t)) - P_n(\psi(t))\} \sum_{k=n+1}^{\infty} \frac{\varphi_k(z)}{t^{k+1}} dt \tag{28}$$

From the relations (14), we write

$$|R_n(z,\ f)| \leq \frac{1}{2\pi}\int_{|t|=R} |f(\psi(t)) - P_n(\psi(t))| \left|\sum_{k=n+1}^{\infty} \frac{w^k}{t^{k+1}}\right| |dt|$$

$$+ \frac{1}{2\pi}\int_{|t|=R} |f(\psi(t)) - P_n(\psi(t))| \left|\sum_{k=n+1}^{\infty} \frac{E_k(\psi(w))}{t^{k+1}}\right| |dt|$$

where $w = \varphi(z)$ and $t = \varphi(\zeta)$.

Let's denote

$$I_1 := \frac{1}{2\pi}\int_{|t|=R} |f(\psi(t)) - P_n(\psi(t))| \left|\sum_{k=n+1}^{\infty} \frac{w^k}{t^{k+1}}\right| |dt|$$

and

$$I_2 := \frac{1}{2\pi}\int_{|t|=R} |f(\psi(t)) - P_n(\psi(t))| \left|\sum_{k=n+1}^{\infty} \frac{E_k(\psi(w))}{t^{k+1}}\right| |dt|$$

Since the level curves Γ_R are sufficiently smooth, we consider them as Dini-smooth curves in this paper. For this reason, we can use the inequalities (6) and hence (7) in the proof.

First we estimate I_1. If we change variables for $t = \varphi(\zeta)$ in I_1 and use (6), we have

$$\begin{aligned}
I_1 &= \frac{1}{2\pi}\int_{\Gamma_R} |f(\zeta) - P_n(\zeta)| \left|\sum_{k=n+1}^{\infty} \frac{[\varphi(z)]^k}{[\varphi(\zeta)]^{k+1}}\right| |\varphi'(\zeta)|\, |d\zeta| \\
&\leq \frac{c_1}{2\pi}\int_{\Gamma_R} |f(\zeta) - P_n(\zeta)| \frac{|\varphi(z)|^{n+1}}{|\varphi(\zeta)|^{n+1}\, |\varphi(\zeta) - \varphi(z)|} |d\zeta| \\
&\leq \frac{c_1}{2\pi R^{n+1}(R-1)}\int_{\Gamma_R} |f(\zeta) - P_n(\zeta)|\, |d\zeta| \qquad (29)
\end{aligned}$$

When B is a disc with centered any point ζ on Γ_R and the radius $r > 0$, it can be written that $\Gamma_R = (B \cap \Gamma_R) \cup (B' \cap \Gamma_R)$. According to this

$$I_1 \leq \frac{c_1}{2\pi R^{n+1}(R-1)} \left(\int_{B\cap\Gamma_R} |f(\zeta) - P_n(\zeta)|\, |d\zeta| + \int_{B'\cap\Gamma_R} |f(\zeta) - P_n(\zeta)|\, |d\zeta| \right)$$

Let's denote

$$J_1 := \int_{B\cap\Gamma_R} |f(\zeta)-P_n(\zeta)|\,|d\zeta|$$

$$J_2 := \int_{B'\cap\Gamma_R} |f(\zeta)-P_n(\zeta)|\,|d\zeta| \tag{30}$$

We first estimate J_1 by the following.

$$J_1 = \int_{\Gamma_R} \left(\frac{1}{|B\cap\Gamma_R|^{(1-\frac{\alpha(\cdot)}{2})\frac{1}{p(\cdot)}}} |f(\zeta)-P_n(\zeta)|\,\chi_{B\cap\Gamma_R}(\zeta) \right) |B\cap\Gamma_R|^{(1-\frac{\alpha(\cdot)}{2})\frac{1}{p(\cdot)}} |d\zeta|$$

Using Hölder's inequality for $\frac{1}{p(\cdot)}+\frac{1}{q(\cdot)}=1$, we have

$$J_1 \le c_2(p) \left\| \frac{1}{|B\cap\Gamma_R|^{(1-\frac{\alpha(\cdot)}{2})\frac{1}{p(\cdot)}}} (f(\zeta)-P_n(\zeta))\chi_{B\cap\Gamma_R}(\zeta) \right\|_{L^{p(\cdot)}(\Gamma_R)} \left\| |B\cap\Gamma_R|^{(1-\frac{\alpha(\cdot)}{2})\frac{1}{p(\cdot)}} \right\|_{L^{q(\cdot)}(\Gamma_R)}$$

If we take the supremum for all over B with centered on Γ_R and radius $r>0$, and infimum for all P_n, which degree at most n, on the right hand side of the last inequality, respectively, by (4), (5) and (7), we have

$$\begin{aligned} J_1 &\le c_2(p) E_n^{p(\cdot)\alpha(\cdot)}(f,\ G_R)\,|B\cap\Gamma_R|^{(1-\frac{\alpha(\cdot)}{2})\frac{1}{p(\cdot)}}\,\|1\|_{L^{q(\cdot)}(\Gamma_R)} \\ &\le c_3(p)\,|B_0\cap T_R|^{(1-\frac{\alpha(\cdot)}{2})\frac{1}{p(\cdot)}}\,E_n^{p(\cdot)\alpha(\cdot)}(f,\ G_R)\,\|1\|_{L^{q(\cdot)}(\Gamma_R)} \\ &\le c_4(p) R^{(1-\frac{\alpha(\cdot)}{2})\frac{1}{p(\cdot)}}\,E_n^{p(\cdot)\alpha(\cdot)}(f,\ G_R) \end{aligned} \tag{31}$$

where B_0 is a disc with centered t on T_R and radius $r_0>0$. We also estimate $\|1\|_{L^{q(\cdot)}(\Gamma_R)}$ by a constant positive in the last inequality. Now we estimate J_2. It can be written that

$$\chi_{B'\cap\Gamma_R} = \sum_{k=0}^{\infty} 2^{-2k}\chi_{(2^{k+1}B\setminus 2^kB)\cap\Gamma_R}.$$

For this equality, you can see [23]. Hence

$$
\begin{aligned}
J_2 &= \int_{\Gamma_R} |f(\zeta) - P_n(\zeta)| \, \chi_{B' \cap \Gamma_R}(\zeta) \, |d\zeta| \\
&= \int_{\Gamma_R} |f(\zeta) - P_n(\zeta)| \sum_{k=0}^{\infty} 2^{-2k} \chi_{(2^{k+1}B \setminus 2^k B) \cap \Gamma_R}(\zeta) \, |d\zeta| \\
&= \sum_{k=0}^{\infty} 2^{-2k} \int_{(2^{k+1}B \setminus 2^k B) \cap \Gamma_R} |f(\zeta) - P_n(\zeta)| \, |d\zeta| \\
&\leq \sum_{k=0}^{\infty} 2^{-2k} \int_{2^{k+1}B \cap \Gamma_R} |f(\zeta) - P_n(\zeta)| \, |d\zeta| \\
&= \sum_{k=0}^{\infty} 2^{-2k} 2^{k+1} \int_{B \cap \Gamma_R} |f(\zeta) - P_n(\zeta)| \, |d\zeta| \\
&= \sum_{k=0}^{\infty} 2^{-k+1} J_1 \\
&\leq c_5(p) R^{(1-\frac{\alpha(\cdot)}{2})\frac{1}{p(\cdot)}} E_n^{p(\cdot)\alpha(\cdot)}(f, \, G_R) \qquad (32)
\end{aligned}
$$

holds, where $\sum_{k=0}^{\infty} 2^{-k+1} < \infty$.

Hence by combining the estimations for J_1 and J_2, we find that

$$
I_1 \leq \frac{c_6(p) R^{(1-\frac{\alpha(\cdot)}{2})\frac{1}{p(\cdot)}}}{2\pi R^{n+1}(R-1)} E_n^{p(\cdot), \, \alpha(\cdot)}(f, \, G_R)
$$

and hence

$$
I_1 \leq \frac{c_6(p)}{2\pi \sup\limits_{\zeta \in \Gamma_R} R^{n+1+\frac{\alpha(\zeta)-2}{2p(\zeta)}} (R-1)} E_n^{p(\cdot), \, \alpha(\cdot)}(f, \, G_R)
$$

holds. Now we estimate

$$
I_2 = \frac{1}{2\pi} \int_{|t|=R} |f(\psi(t)) - P_n(\psi(t))| \left| \sum_{k=n+1}^{\infty} \frac{E_k(\psi(w))}{t^{k+1}} \right| |dt|
$$

Using (18), Lemma 2 and changing variables for $t = \varphi(\zeta)$, we have

$$I_2 = \frac{1}{2\pi}\int_{|t|=R}|f(\psi(t))-P_n(\psi(t))|\left|\frac{1}{2\pi i}\int_{|\tau|=1}\left\{\sum_{k=n+1}^{\infty}\frac{\tau^k}{t^{k+1}}\right\}F(\tau,\ w)d\tau\right||dt|$$

$$\leq \frac{1}{4\pi^2}\int_{|\tau|=1}|F(\tau,\ w)|\left\{\int_{|t|=R}|f(\psi(t))-P_n(\psi(t))|\left|\frac{\tau^{n+1}}{t^{n+1}(t-\tau)}\right||dt|\right\}|d\tau|$$

$$\leq \frac{1}{4\pi^2 R^{n+1}(R-1)}\int_{|\tau|=1}|F(\tau,\ w)||d\tau|\left\{\int_{|t|=R}|f(\psi(t))-P_n(\psi(t))||dt|\right\}$$

$$\leq \frac{c_7}{4\pi^2 R^{n+1}(R-1)}\int_{\Gamma_R}|f(\zeta)-P_n(\zeta)|\left|\varphi'(\zeta)\right||d\zeta|$$

$$\leq \frac{c_8}{4\pi^2 R^{n+1}(R-1)}\int_{\Gamma_R}|f(\zeta)-P_n(\zeta)||d\zeta|$$

By using the estimations for J_1 and J_2 in the last integral, we have

$$I_2 \leq \frac{c_9(p)R^{\left(1-\frac{\alpha(\cdot)}{2}\right)\frac{1}{p(\cdot)}}}{4\pi^2 R^{n+1}(R-1)}E_n^{p(\cdot),\ \alpha(\cdot)}(f,\ G_R)$$

$$\leq \frac{c_9(p)}{4\pi^2 \sup\limits_{\zeta\in\Gamma_R} R^{n+1+\frac{\alpha(\zeta)-2}{2p(\zeta)}}(R-1)}E_n^{p(\cdot),\ \alpha(\cdot)}(f,\ G_R) \tag{33}$$

By the estimations obtained for I_1 and I_2, we finally conclude that

$$|R_n(z,\ f)| \leq \frac{c_{10}(p)}{2\pi \sup\limits_{\zeta\in\Gamma_R} R^{n+1+\frac{\alpha(\zeta)-2}{2p(\zeta)}}(R-1)}E_n^{p(\cdot),\ \alpha(\cdot)}(f,\ G_R),\ z\in\Gamma.$$

with some constant $c_{10}(p)>0$ independent of n. This estimation is valid also for $z\in\overline{G}$, by using the maximum modulus principle. The approximation error for $\alpha(\cdot)=2$ is better than for $\alpha(\cdot)=0$.

By taking $1-\dfrac{\alpha(\cdot)}{2}$ instead of $\lambda(\cdot)$ in Theorem 2, we have the following Corollary.

4.2 *Proof of Corollary 1*

If P_n is the best approximation polynomial to $f\in E^{p(\cdot),\ \alpha(\cdot)}(f,\ G_R)$. We write

$$\|f(z)-P_n(z)\|_{L^{p(\cdot),\ \alpha(\cdot)}(\Gamma_R)} \leq \left\|f(z)-\sum_{k=0}^{n}a_k\varphi_k(z)\right\|_{L^{p(\cdot),\ \alpha(\cdot)}(\Gamma_R)}$$

Applying Theorem 2 for Γ_R we have

$$E_n^{p(\cdot),\ \alpha(\cdot)}(f,\ G_R) \leq c_{11}\Omega(f_0,\ \frac{1}{n})_{p_0(\cdot),\ \alpha(\cdot)}$$

By Theorem 3 we conclude

$$|R_n(z,\ f)| \leq \frac{c_{12}(p)}{2\pi \sup\limits_{\zeta\in\Gamma_R} R^{n+1+\frac{\alpha(\zeta)-2}{2p(\zeta)}}(R-1)}\Omega(f_0,\ \frac{1}{n})_{p_0(\cdot),\ \alpha(\cdot)},\ z\in\overline{G}$$

with some constant $c_{12}(p) > 0$ independent of n.

5. Discussion

Only a few studies on Morrey-Smirnov classes with variable exponents are available in the literature so far (see [26, 27]). While the standard definition of the norm of the class in the proofs of these studies is used, equivalent but the more intriguing definition of the norm is used in the proofs of this paper, with the help of Lemma 5 in [8], and also [22, 23]. Moreover, the results in this paper give a point of view to the studies related to approximation of the functions in the classes of Smirnov type, since the function is accepted as analytic inside the level curve G_R in the exterior of G such that R is the largest number. The problem of maximal convergence can be investigated in the Smirnov type classes aside from Morrey-Smirnov classes with variable exponents and by the series aside from Faber series in the future studies.

6. Conclusion

In this work, for given a domain G bounded by Γ, by considering that $R > 1$ is the largest number such that f is analytic inside the level curve Γ_R in the exterior of Γ, we estimated the remaining term $R_n(z,\ f)$ with respect to the best approximation number $E_n^{p(\cdot),\ \alpha(\cdot)}(f,\ G_R)$. Hence, we obtained a rate of maximal convergence of the nth partial sums of Faber series of the function $f \in E^{p(\cdot),\ \alpha(\cdot)}(f,\ G_R)$ where $p(\cdot) : \Gamma \to [1,\ \infty)$, $p(\cdot) \in P^{log}(\Gamma)$, $1 < p_- \leq p_+ < \infty$ and $\alpha(\cdot) : \Gamma \to [0,\ 2]$, $\alpha(\cdot) \in P^{log}(\Gamma)$, in the uniform norm on the closure of G. We also give a corollary according to the modulus of smoothness $\Omega(f_0,\ \frac{1}{n})_{p_0(\cdot),\ \alpha(\cdot)}$ if $0 < \alpha_- \leq \alpha_+ \leq 2$.

Acknowledgement

The author would like to thank the anonymous referee for his/her comments that helped to improve this article.

Conflict of interest

The author declares no competing financial interest.

References

[1] Akgun R, Kokilashvili V. The refined direct and converse inequalities of trigonometric approximation in weighted variable exponent Lebesgue spaces. *Georgian Mathematical Journal*. 2011; 18(3): 399-423.

[2] Cruz-Uribe DV, Fiorenza A. *Variable Lebesgue Spaces, Foundations and Harmonic Analysis*. Heidelberg: Springer; 2013.

[3] Diening L, Harjulehto P, Hasto P, Ruzicka M. *Lebesgue and Sobolev Spaces with Variable Exponent*. Lecture Notes on Mathematics. New York: Springer; 2011.

[4] Israfilov DM, Kokilashvili V, Samko S. Approximation in weighted Lebesgue and Smirnov spaces with variable exponents. *Proceedings of A. Razmadze Mathematical Institute*. 2007; 143: 25-35.

[5] Izuki M, Nakai E, Sawano Y. Function spaces with variable exponents-an introduction. *Scientiae Mathematicae Japonicae*. 2014; 77(2): 187-315.

[6] Sharapudinov II. *Some Questions in Approximation Theory for Lebesgue Spaces With Variable Exponent*. Southern Mathematical Institute of the Vladikavkaz Scientific Center of the Russian Academy of Sciences and Republic of North Ossetia-Alania; 2012.

[7] Sharapudinov II. On direct and inverse theorems of approximation theory in variable Lebesgue space and Sobolev spaces. *Azerbaijan Journal of Mathematics*. 2014; 4(1): 55-72.

[8] Almeida A, Hasanov J, Samko S. Maximal and potential operators in variable exponent Morrey spaces. *Georgian Mathematical Journal*. 2008; 15(2): 195-208.

[9] Almeida A, Samko S. Approximation in Morrey spaces. *Journal of Functional Analysis*. 2007; 272(6): 2392-2411.

[10] Bandaliyev RA, Guliyev VS. Embedding theorems between variable-exponent morrey spaces. *Mathematical Notes*. 2019; 106(4): 488-500.

[11] Cakir Z, Aykol C, Soylemez D, Serbetci A. Approximation by trigonometric polynomials in Morrey spaces. *Transactions Issue Mathematics National Academy of Sciences of Azerbaijan Series of Physical-Technical and Mathematical Sciences*. 2019; 39(1): 24-37.

[12] Gong R, Vempati MN, Wu Q, Xie P. Boundedness and compactness of caucy-type integral commutator on weighted morrey spaces. *Journal of the Australian Mathematical Society*. 2022; 113(1): 36-56.

[13] Guliyev VS, Hasanov J, Samko S. Boundness of the maximal, potential and singular operators in the generalized variable exponent Morrey spaces. *Mathematica Scandinavica*. 2010; 107: 285-304.

[14] Guliyev VS, Ghorbanalizadeh A, Sawano Y. Approximation by trigonometric polynomials in variable exponent Morrey spaces. *Analysis and Mathematical Physics*. 2019; 9: 1265-1285.

[15] Fan X. Variable exponent Morrey and Campanato spaces. *Nonlinear Analysis*. 2010; 72: 4148-4161.

[16] Kokilashvili V, Meskhi A. Boundness of maximal and singular operators in Morrey spaces with variable exponent. *Armenian Journal of Mathematics*. 2008; 1(1): 18-28.

[17] Israfilov DM, Testici A. Approximation in Smirnov classes with variable exponent. *Complex Variables and Elliptic Equations*. 2015; 60(9): 1243-1253.

[18] Testici A. Some theorems of approximation theory in weighted smirnov classes with variable exponent. *Computational Methods and Function Theory*. 2020; 20: 39-61.

[19] Israfilov DM, Gursel E, Aydin E. Maximal convergence of faber series in smirnov classes with variable exponent. *Bulletin of the Brazilian Mathematical Society, New Series*. 2018; 49: 955-963.

[20] Kokilashvili V, Paatashvili V. The Dirichlet Problem for harmonic functions from variable exponent Smirnov classes in domains with piecewise smooth boundary. *Proceedings of A. Razmadze Mathematical Institute*. 2011; 155: 154-157.

[21] Kokilashvili V, Paatashvili V. Riemann boundary value problem in variable exponent Smirnov class of generalized analytic functions. *Proceedings of A. Razmadze Mathematical Institute*. 2015; 169: 105-118.

[22] Israfilov DM, Tozman NP. Approximation by polynomials in Morrey-Smirnov classes. *East Journal of Approximation*. 2000; 14(3): 255-269.

[23] Israfilov DM, Tozman NP. Approximation in Morrey-Smirnov classes. *Azerbeijan Journal of Mathematics*. 2011; 1(1): 99-113.

[24] Jafarov SZ. Estimates of the approximation by zygmund sums in morrey-smirnov classes of analytic functions. *Azerbaijan Journal of Mathematics*. 2020; 10(2): 110-124.

[25] Ali M, Mahmoud S, Kinj A. Approximation by rational functions in Morrey-Smirnov classes. *Kuwait Journal of Science*. 2018; 45(2): 1-7.

[26] Jafarov SZ. Approximation by faber-laurent rational functions in variable exponent morrey spaces. *Computational Methods of Function Theory*. 2022; 22: 629-643.

[27] Kinj A. Approximation by rational functions in variable exponent morrey-smirnov classes. *Journal of Mathematics*. 2022; 2022(1): 7200777.

[28] Suetin PK. *Series of Faber Polynomials*. Amsterdam: Gordon and Breach Science Publishers; 1998.

[29] Gaier D. *Lectures on Complex Approximation*. Translated from German by Renate McLaughlin. Boston: Basel: Stuttgart: Birkauser; 1985.

[30] Israfilov DM, Oktay B, Akgun R. Approximation in smirnov-orlicz classes. *Glasnik Matematicki*. 2005; 40(60): 87-102.

[31] Oktay B. Maximal convergence of faber series in smirnov-orlicz classes. *Journal of Mathematical Analysis*. 2019; 10(6): 23-31.

[32] Oktay B, Aydin E. Maximal convergence of faber series in weighted smirnov classes with variable exponent on the domains bounded by smooth curves. *Journal of Mathematical Analysis*. 2023; 14(5): 28-38.

[33] Goluzin GM. *Geometric Theory of Functions of a Complex Variable*. Providence, Rhode Island: American Mathematical Society; 1968.

[34] Pommerenke CH. *Boundary Behavior of Conformal Maps*. Berlin: Springer-Verlag; 1991.

[35] Suetin PK. *Polynomials Orthogonal over a Region and Bieberbach Polynomials*. Providence, Rhode Island: American Mathematical Society; 1975.