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Research Article

Improved Solutions of OHAM Approximate Procedure for Classes of Nonlinear ODEs

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Abstract: The primary purpose of this study is to apply the Optimal Homotopy Asymptotic Method (OHAM) to various nonlinear initial value problems of different orders to evaluate its accuracy, convergence, and computational efficiency. OHAM is considered a highly effective technique for solving nonlinear differential equations and is commonly used in scientific and engineering disciplines. It combines the strengths of homotopy and asymptotic methods. OHAM offers a straightforward approach to controlling and adjusting the convergence of the series solution. This is achieved through the utilization of an auxiliary function that incorporates multiple convergent control parameters with one order of approximation, which are optimally determined. The OHAM approach heavily relies on the auxiliary function $H(p)$ which allows for the flexible and efficient solving of nonlinear differential equations. By carefully constructing and optimizing the parameters of $H(p)$, the convergence of the solution series can be effectively controlled. As a result, OHAM proves to be a versatile and effective approach for solving various mathematical and engineering problems. Several examples have been solved. Numerical comparisons, displayed in tables and shown graphically in figures, prove and confirm the capability, efficiency, and better accuracy with less computational work.

Keywords: OHAM, series solutions, nonlinear equations

MSC: 65A05, 65L05, 41A10, 65Y15

1. Introduction

In physics, engineering, biology, economics, and other disciplines, differential equations are essential mathematical tools for simulating dynamic systems. These equations define the relationship between a function and its derivatives, capturing the behavior of systems over time or across various spatial dimensions. While linear differential equations are well-understood and often have closed-form solutions, nonlinear differential equations pose substantial challenges due to their complex behavior and lack of universal solution techniques. Nonlinear differential equations are particularly significant in practical applications such as fluid dynamics, heat transfer, population dynamics, and financial modeling

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[1-5]. However, solving these equations can be difficult and requires advanced techniques to handle their inherent complexity. To solve nonlinear differential equations, numerous techniques have been developed, each having advantages and disadvantages of their own [5-10]. A variety of methods for dealing with perturbations includes the Homotopy perturbation method, Adomian decomposition method, variational approach, ultraspherical wavelets-Gauss collocation, Homotopy analysis method [10-15] and others [15-20], are used for solving these equations. Homotopy-based methods, such as the Homotopy Perturbation Method and the Homotopy Analysis Method, have grown in popularity recently because of their flexibility and ability to solve extremely nonlinear situations. These methods make use of a homotopy that progressively converges the given problem from an initial approximation to an exact solution. It is crucial to remember that these approaches may require careful parameter selection in order to achieve convergence [10].

OHAM is a powerful numerical technique for solving differential equations, which are fundamental in describing various physical phenomena and processes in science and engineering. Introduced as an improvement over traditional perturbation methods, OHAM combines the strengths of both homotopy and asymptotic approaches to provide more accurate and efficient solutions. Unlike classical methods that require small parameters, OHAM is versatile and can be applied to a wide range of nonlinear problems without the necessity of small or large parameter assumptions [21-25]. Several studies have shown the effectiveness of OHAM in solving various types of differential equations, including ordinary differential equations (ODEs), partial differential equations (PDEs), and integral equations. OHAM improves upon existing approaches by introducing an optimization process. This process systematically minimizes the residual error of the solution, enhancing both accuracy and convergence. As a result, OHAM becomes a powerful tool for solving a wide range of nonlinear differential equations [26]. Application of the Optimal Homotopy asymptotic method for solving nonlinear equations arising in heat transfer [27-28]. The development of OHAM has provided researchers and engineers with new possibilities for solving complex differential equations in areas such as fluid dynamics and thermal physics. This makes OHAM an invaluable tool, offering robust and accurate analytical methods.

In this study, we will enhance the OHAM solution procedure to achieve accurate results that closely approximate the exact solution within one order of approximation. The accuracy of the results depends on the construction of the auxiliary function. This work is organized as follows: Section 2 introduces the fundamental concept of OHAM. Section 3 provides numerical examples to demonstrate the effectiveness of the discussed procedure. The final section summarizes the importance and the conclusions of this work.

2. The fundamental idea of the OHAM procedure

According to multiple publications [29, 30], the fundamental concept of OHAM will be explained as follows.

$$L(u(x)) + g(x) + N(u(x)) = 0,\ B\left(u, \frac{dx}{du}\right) = 0. \tag{1}$$

L and N represent linear and nonlinear operators, respectively. $u(x)$ is the function to be found, x signifies an independent variable, $g(x)$ is a known function, and B represents the boundary conditions.

Based on the homotopy, we construct the following function of homotopy equations

$$h(v(x, p), p) : R \times [0, 1] \to R$$

which satisfies

$$(1-p)\left[L(v(x, p)) - u_0(x)\right] = H(p)[L(v(x, p)) + g(x) + (v(x, p))],$$

$$B\left(v(x, p), \frac{\partial v(x, P)}{\partial x}\right) = 0, \tag{2}$$

where $x \in R$ and $p \in [0, 1]$ is an embedding parameter, $H(p)$ is a nonzero auxiliary function for $p \neq 0$, $H(0) = 0$, and $V(x, p)$ is an unknown function. Obviously, when $p = 0$ and $p = 1$ it holds that $v(x, 0) = u_0(x)$ and $v(x, 1) = u(x)$, respectively. Thus, as p varies from 0 to 1, the solution $v(x, p)$ approaches from $u_0(x)$ to $u(x)$ where $u_0(x)$ is the initial guess that satisfies the linear operator and the boundary conditions

$$L\left(u_0(x)\right) = 0,\ B\left(u_0, \frac{du_0}{dx}\right) = 0. \tag{3}$$

We will now select the auxiliary function $H(p)$ which plays an important role in adjusting the homotopy series, so that it converges more rapidly and accurately to the exact solution and provides the flexibility needed to overcome the limitations of other numerical methods, the convergence of the OHAM procedures was proved in [16].

$$H(p) = C_1 + C_2 x + C_3 x^2 + \ldots = \sum\nolimits_{m=1}^{i} c_m x^{m-1} \tag{4}$$

where C_1, C_2, C_3, ..., are constants (convergent control parameters) which can be determined later.

To find the approximate solution, we use Taylor series to expand $v(x, p, C_i)$ about p as following

$$v\left(x, p, C_i\right) = u_0(x) + \sum\nolimits_{K=1}^{\infty} U_k\left(x, C_1, C_2, \ldots, C_k\right) P^k. \tag{5}$$

Substituting (5) into (2) and equating the coefficient of like powers of p, we obtain the following linear equations. The zeroth-order problem is given by (3); the first- and second order problems are given as

$$L\left(u_1(x)\right) + g(x) = C_1 N_0\left(u_0(x)\right),\ B\left(u_1, \frac{du_1}{dX}\right) = 0 \tag{6}$$

$$L\left(u_2(x)\right) - L\left(u_1(x)\right) = C_2 N_0\left(u_0(x)\right) + C_1\left[L\left(u_1(x) + N_1\left(u_0(x), u_1(x)\right)\right],\right.$$

$$B\left(u_2, \frac{du_2}{dX}\right) = 0. \tag{7}$$

The general governing equations for (x) are

$$L\left(u_K(x)\right) - L\left(u_{K-1}(x)\right) = C_K N_0\left(u_0(x)\right)$$

$$+\sum_{i=1}^{K-1} C_i\left[L\left(u_{k-i}(x)\right) + N_{k-i} \times \left(u_0(x), u_1(x), \ldots, u_{K-1}(x)\right)\right],\ B\left(u_K, \frac{du_k}{dx}\right) = 0, \tag{8}$$

where $k = 2, 3, \ldots$, and $N_m(u_0(x), u_1(x), \ldots, u_m(x))$ is the coefficient of P^m in the expansion of $(v(x, p))$ about the embedding parameter p:

$$N\left(v\left(x, p, C_i\right)\right) = N_0\left(u_0(x)\right) + \sum_{m=1}^{\infty} N_m\left(u_0(x), u_1(x), \ldots, u_m(x)\right) \text{ of } P^m. \tag{9}$$

Notify that the convergence of (5) depends upon the auxiliary constants C_1, C_2, C_3. If it is convergent at $p = 1$, we have

$$v(x, C_i) = u_0(x) + \sum_{k=1}^{\infty} u_K(x, C_1, C_2, \ldots, C_k). \tag{10}$$

The mth-order approximation is

$$\tilde{u}(x, C_1, C_2, C_3, \ldots, C_m) = u_0(x) + \sum_{i=1}^{m} u_i(x, C_1, C_2, \ldots, C_i). \tag{11}$$

Substituting (10) into (1) gives the following residual

$$R(x, C_1, C_2, C_3, \ldots, C_m) = L(\tilde{u}(x, C_1, C_2, C_3, \ldots, C_m) + g(x)$$

$$+N(\tilde{u}(x, C_1, C_2, C_3, \ldots, C_m)). \tag{12}$$

If $R = 0$, then $(\tilde{u})$ will be the exact solution. In general, this does not arise especially in nonlinear differential equations, but we can minimize the functional

$$J(C_1, C_2, C_3, \ldots, C_m) = \int_a^b R^2(x, C_1, C_2, C_3, \ldots, C_m)\,dx, \tag{13}$$

where a and b are the endpoints of the given problem. The unknown constants $C_i(i = 1, 2, 3, ..., m)$ can be obtained by using the conditions

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \ldots = \frac{\partial J}{\partial C_m} = 0. \tag{14}$$

Therefore, the approximate solution of order will be obtained.

3. Applications of OHAM

In the following section, we will showcase several examples of classes of nonlinear ODEs of initial value problems type. The purpose of presenting these examples is to demonstrate the effectiveness and reliability of the procedure being discussed above using Mathematica 14.

Example 1 Consider the following nonlinear intial value problem of first-order with its initial conditions [31].

$$u'(x) - u(x) + u^2(x) = e^{2x},\ u(0) = 1. \tag{15}$$

We will now derive the set of homotopy equations using the algorithm described in Section 2.

$$(1-p)\left[L\left(\frac{dv(x;p)}{dx}\right)\right] = \left(\sum_{m=0}^{i} c_m x^m\right) * p\left(\frac{dv(x;p)}{dx} - v(x;q) + v^2(x;q) - e^{2x}\right). \tag{16}$$

The zeroth-order deformation problem is given in Eq. (17) obtained based on Eq. (16) at $p = 0$, which give us the linear operator $\left[L\left(\frac{dv(x;p)}{dx}\right)\right]=0$ as follows:

$$u_0'(x)=0,\ u_0(0)=1, \tag{17}$$

which has the following solution

$$u_0(x)=1. \tag{18}$$

To utilize OHAM, the first order deformation problem is

$$u_1'(x)=-e^{2x}\left(c_1+x\left(c_2+x\left(c_3+x\left(c_4+x\left(c_5+x\left(c_6+xc_7\right)\right)\right)\right)\right)\right),\ u_1'(0)=0, \tag{19}$$

by solving it we have

$$\begin{aligned}u_1(x)=\frac{1}{8}\Big(&-4\left(-1+e^{2x}\right)c_1+\left(-2+e^{2x}(2-4x)\right)c_2+2c_3-2e^{2x}c_3\\&+4e^{2x}xc_3-4e^{2x}x^2c_3-3c_4+3e^{2x}c_4-6e^{2x}xc_4+6e^{2x}x^2c_4\\&-4e^{2x}x^3c_4+6c_5-6e^{2x}c_5+12e^{2x}xc_5-12e^{2x}x^2c_5+8e^{2x}x^3c_5\\&-4e^{2x}x^4c_5-15c_6+15e^{2x}c_6-30e^{2x}xc_6+30e^{2x}x^2c_6-20e^{2x}x^3c_6\\&+10e^{2x}x^4c_6-4e^{2x}x^5c_6+45c_7-45e^{2x}c_7+90e^{2x}xc_7\\&-90e^{2x}x^2c_7+60e^{2x}x^3c_7-30e^{2x}x^4c_7+12e^{2x}x^5c_7-4e^{2x}x^6c_7\Big).\end{aligned} \tag{20}$$

Therefore, the OHAM approximate solutions of first order is

$$\tilde{u}\left(x,c_1,\ldots,c_7\right)=u_0(x)+u_1(x), \tag{21}$$

now, using Eqs. (18) and (20) into Eq. (21), we have OHAM solution of first order

$$\begin{aligned}\tilde{u}\left(x,c_1,\ldots,c_7\right)=1&+\frac{c_1}{2}-\frac{1}{2}e^{2x}c_1-\frac{c_2}{4}+\frac{1}{4}e^{2x}c_2-\frac{1}{2}e^{2x}xc_2+\frac{c_3}{4}-\frac{1}{4}e^{2x}c_3\\&+\frac{1}{2}e^{2x}xc_3-\frac{1}{2}e^{2x}x^2c_3-\frac{3c_4}{8}+\frac{3}{8}e^{2x}c_4-\frac{3}{4}e^{2x}xc_4+\frac{3}{4}e^{2x}x^2c_4\\&-\frac{1}{2}e^{2x}x^3c_4+\frac{3c_5}{4}-\frac{3}{4}e^{2x}c_5+\frac{3}{2}e^{2x}xc_5-\frac{3}{2}e^{2x}x^2c_5+e^{2x}x^3c_5-\frac{1}{2}e^{2x}x^4c_5\end{aligned}$$

$$-\frac{15c_6}{8}+\frac{15}{8}e^{2x}c_6-\frac{15}{4}e^{2x}xc_6+\frac{15}{4}e^{2x}x^2c_6-\frac{5}{2}e^{2x}x^3c_6+\frac{5}{4}e^{2x}x^4c_6$$

$$-\frac{1}{2}e^{2x}x^5c_6+\frac{45c_7}{8}-\frac{45}{8}e^{2x}c_7+\frac{45}{4}e^{2x}xc_7-\frac{45}{4}e^{2x}x^2c_7$$

$$+\frac{15}{2}e^{2x}x^3c_7-\frac{15}{4}e^{2x}x^4c_7+\frac{3}{2}e^{2x}x^5c_7-\frac{1}{2}e^{2x}x^6c_7. \tag{22}$$

To find the values of the convergent control parameters $c_i's$ we follow the methodology that outlined in Section 2, we conducted our analysis within the specified domain of $a = 0$ and $b = 1$. During this process, we carefully examined the residual error to ensure accurate results.

$$R=\tilde{u}\left(x,c_1,\ldots,c_7\right)-\tilde{u}\left(x,c_1,\ldots,c_7\right)+\tilde{u}^2\left(x,c_1,\ldots,c_7\right)-e^{2x}, \tag{23}$$

then, we use the least-square method

$$J\left(c_1,c_2,\ldots,c_7\right)=\int_0^1 R^2dx, \tag{24}$$

which resulted the following system of algebraic equations

$$\frac{dJ}{dc_1}=\frac{dJ}{dc_2}=\ldots=\frac{dJ}{dc_7}, \tag{25}$$

and by solving it, we obtain the following optimal values of $c_i's$

$$c_1=-0.9999999097807601,\ c_2=0.9999959990308249,\ c_3=-0.49995436435807633,$$

$$c_4=0.16644262864487178,\ c_5=-0.041101653249229,\ c_6=0.0075570745718040,$$

$$c_7=-0.0008192290067115186.$$

By considering these values into Eq. (22), we get our OHAM approximate solution of first order in the following form

$$\tilde{u}(x)=0.012992550598761254+0.9870074494012389e^{2x}-0.9740149890217175e^{2x}x$$

$$+0.47401698950630505e^{2x}x^2-0.14935987155151126e^{2x}x^3+0.03306927861453769e^{2x}x^4$$

$$-0.005007380795969291e^{2x}x^5+0.0004096145033557593e^{2x}x^6. \tag{26}$$

Example 2 Consider the following nonlinear intial value problem of fourth order [32].

$$u^{(4)}(x)+u(x)u'(x)+u'^2(x)=0, \tag{27}$$

along with following initial conditions

$$u(0)=0,\ u'(0)=0,\ u''(0)=1,\ u'''(0)=1.$$

According to OHAM description, we will now define the following set of homotoy equation

$$(1-q)\left[L\left(\frac{d^4v(x;p)}{dx^4}\right)\right]=\left(\sum_{m=0}^{i}c_m x^m\right)*q\left(\frac{d^4v(x;p)}{dx^4}-v(x;q)\frac{dv(x;q)}{dx}+\left(\frac{dv(x;q)}{dx}\right)^2\right). \tag{28}$$

The zeroth-order deformation problem is given in Eq. (29) based on Eq. (28) at p = 0, which give us the linear operator $L\left(\frac{d^4v(x;p)}{dx^4}\right)=0$, subject to the given conditions as follows:

$$u^{(4)}(x)=0,\ u(0)=0,\ u'(0)=0,\ u''(0)=1,\ u'''(0)=1, \tag{29}$$

which has the solution

$$u_0(x)=\frac{1}{6}\left(6x+3x^2+x^3\right). \tag{30}$$

According to OHAM procedure, the first order deformation problem is

$$u_1^{(4)}(x)=\frac{1}{12}(-12-12x-6x^2+2x^3+2x^4+x^5)$$

$$\left(c_1+x\left(c_2+x\left(c_3+x\left(c_4+x\left(c_5+x\left(c_6+xc_7\right)\right)\right)\right)\right)\right), \tag{31}$$

which has the solution

$$u_1(x)=-\frac{1}{24}x^4c_1-\frac{x^5c_1}{120}-\frac{x^6c_1}{720}+\frac{x^7c_1}{5{,}040}+\frac{x^8c_1}{10{,}080}+\frac{x^9c_1}{36{,}288}-\frac{x^5c_2}{120}$$

$$-\frac{x^6c_2}{360}-\frac{x^7c_2}{1{,}680}+\frac{x^8c_2}{10{,}080}+\frac{x^9c_2}{18{,}144}+\frac{x^{10}c_2}{60{,}480}-\frac{x^6c_3}{360}-\frac{x^7c_3}{840}$$

$$-\frac{x^8c_3}{3{,}360}+\frac{x^9c_3}{18{,}144}+\frac{x^{10}c_3}{30{,}240}+\frac{x^{11}c_3}{95{,}040}-\frac{x^7c_4}{840}-\frac{x^8c_4}{1{,}680}$$

$$-\frac{x^9c_4}{6{,}048}+\frac{x^{10}c_4}{30{,}240}+\frac{x^{11}c_4}{47{,}520}+\frac{x^{12}c_4}{142{,}560}-\frac{x^8c_5}{1{,}680}-\frac{x^9c_5}{3{,}024}$$

$$-\frac{x^{10}c_5}{10{,}080}+\frac{x^{11}c_5}{47{,}520}+\frac{x^{12}c_5}{71{,}280}+\frac{x^{13}c_5}{205{,}920}-\frac{x^9c_6}{3{,}024}-\frac{x^{10}c_6}{5{,}040}$$

$$-\frac{x^{11}c_6}{15,840}+\frac{x^{12}c_6}{71,280}+\frac{x^{13}c_6}{102,960}+\frac{x^{14}c_6}{288,288}-\frac{x^{10}c_7}{5,040}-\frac{x^{11}c_7}{7,920}$$

$$-\frac{x^{12}c_7}{23,760}+\frac{x^{13}c_7}{102,960}+\frac{x^{14}c_7}{144,144}+\frac{x^{15}c_7}{393,120}. \tag{32}$$

Therefore, the OHAM approximate solutions of first order is

$$\tilde{u}\left(x, c_1, \ldots, c_7\right)=u_0(x)+u_1(x). \tag{33}$$

Now, using Eqs. (29) and Eq. (32) into Eq. (33), we have

$$\tilde{u}\left(x, c_1, \ldots, c_7\right)=x+\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4c_1}{24}-\frac{x^5c_1}{120}-\frac{x^6c_1}{720}+\frac{x^7c_1}{5,040}$$

$$+\frac{x^8c_1}{10,080}+\frac{x^9c_1}{36,288}-\frac{x^5c_2}{120}-\frac{x^6c_2}{360}-\frac{x^7c_2}{1,680}+\frac{x^8c_2}{10,080}+\frac{x^9c_2}{18,144}$$

$$+\frac{x^{10}c_2}{60,480}-\frac{x^6c_3}{360}-\frac{x^7c_3}{840}-\frac{x^8c_3}{3,360}+\frac{x^9c_3}{18,144}+\frac{x^{10}c_3}{30,240}$$

$$+\frac{x^{11}c_3}{95,040}-\frac{x^7c_4}{840}-\frac{x^8c_4}{1,680}-\frac{x^9c_4}{6,048}+\frac{x^{10}c_4}{30,240}+\frac{x^{11}c_4}{47,520}$$

$$+\frac{x^{12}c_4}{142,560}-\frac{x^8c_5}{1,680}-\frac{x^9c_5}{3,024}-\frac{x^{10}c_5}{10,080}+\frac{x^{11}c_5}{47,520}+\frac{x^{12}c_5}{71,280}$$

$$+\frac{x^{13}c_5}{205,920}-\frac{x^9c_6}{3,024}-\frac{x^{10}c_6}{5,040}-\frac{x^{11}c_6}{15,840}+\frac{x^{12}c_6}{71,280}$$

$$+\frac{x^{13}c_6}{102,960}+\frac{x^{14}c_6}{288,288}-\frac{x^{10}c_7}{5,040}-\frac{x^{11}c_7}{7,920}-\frac{x^{12}c_7}{23,760}$$

$$+\frac{x^{13}c_7}{102,960}+\frac{x^{14}c_7}{144,144}+\frac{x^{15}c_7}{393,120}. \tag{34}$$

To find the values of the convergent control parameters $c_i's$, we follow the methodology that outlined in Section 2, we conducted our analysis within the specified domain of $a = 0$ and $b = 1$.

$$R=\tilde{u}^{(4)}\left(x, c_1, \ldots, c_7\right)-\tilde{u}\left(x, c_1, \ldots, c_7\right)\tilde{u}'^{\left(x, c_1, \ldots, c_7\right)}+\tilde{u}'^2\left(x, c_1, \ldots, c_7\right), \tag{35}$$

then, we use the least-square method

$$J\left(c_1, c_2, \ldots, c_7\right) = \int_0^1 R^2 dx, \tag{36}$$

which resulted the following system of algebraic equations

$$\frac{dJ}{dc_1} = \frac{dJ}{dc_2} = \ldots = \frac{dJ}{dc_7}, \tag{37}$$

then by solving it, we obtain the following optimal values of the convergent control parameters $c_i's$

$c_1 = -1.0000480780251586,\ c_2 = 0.002290397007220993,\ c_3 = -0.02658607267843225,$

$c_4 = -0.20762616291599395,\ c_5 = -0.15755335777461987,\ c_6 = 0.2461502491339939,$

$c_7 = -0.1613646965058402.$

By considering these valuesinto Eq. (34) our approximate solution of one order of approximation is

$$\begin{aligned}\tilde{u}(x) = x + \frac{x^2}{2} + \frac{x^3}{6} &+ 0.041668669917714944x^4 \\ &+ 0.008314647341816148x^5 + 0.001456443651899418x^6 \\ &+ 0.00007903852073804277x^7 + 0.00012629738701994464x^8 \\ &- 0.000023865882610893564x^9 - 0.000008899480143382213x^{10} \\ &- 0.000003129947925016214x^{11} + 0.000006577972775237062x^{12} \\ &+ 5.836124456918959 \times 10^{-8} x^{13} - 2.656341709598962 \times 10^{-7} x^{14} \\ &- 4.104718572085883 \times 10^{-7} x^{15}.\end{aligned} \tag{38}$$

Example 3 Consider the following nonlinear boundary value problem of fifth order [33]

$$u^{(5)}(x) = e^{-x} u^2(x),\ 0 \le x \le 1, \tag{39}$$

along with following boundary conditions

$$u(0) = u'(0) = u''(0) = 1,\ u(1) = u'(1) = e.$$

We will now derive the set of homotopy equations using the algorithm described in Section 2.

$$(1-q)\left[L\left(\frac{d^5 v(x;p)}{dx^5}\right)\right] = H(q)\left(\frac{d^5 v(x;p)}{dx^5} - e^{-x}\left(\frac{dv(x;q)}{dx}\right)^2\right). \tag{40}$$

The zeroth-order deformation problem is given in Eq. (41) based on Eq. (40) at $p = 0$, which give us the linear operator $L\left(\frac{d^5 v(x;\, p)}{dx^5}\right) = 0$ as follows:

$$u_0^{(5)}(x) = u'(0) = u''(0) = 1,\ u(1) = u'(1) = e. \tag{41}$$

The solutions of Eq. (41) is given by

$$u_0(x) = 1 + x + \frac{x^2}{2} - 8x^3 + 3ex^3 + \frac{11x^4}{2} - 2ex^4. \tag{42}$$

According to OHAM description, the first order deformation problem is

$$u_1^{(5)}(x) = -2c_1 - 3xc_1 + \frac{93x^2c_1}{2} - 18ex^2c_1 + \frac{121x^3c_1}{6}$$

$$-8ex^3c_1 - \frac{15{,}143x^4c_1}{24} + 452ex^4c_1 - 81e^2x^4c_1 + \frac{126{,}721x^5c_1}{120} - 780ex^5c_1$$

$$+144e^2x^5c_1 - \frac{396{,}499x^6c_1}{720} + 401e^6c_1 - 73e^2x^6c_1 + \frac{785{,}471x^7c_1}{5{,}040}$$

$$-\frac{1{,}373}{12}ex^7c_1 + 21e^2x^7c_1 - \frac{170{,}721x^8c_1}{1{,}120} + \frac{559}{5}e^8c_1 - \frac{41}{2}e^2x^8c_1$$

$$+\frac{172{,}751x^9c_1}{2{,}016} - \frac{3{,}767}{60}e^9c_1 + \frac{23}{2}e^2x^9c_1 - \frac{658{,}403x^{10}c_1}{20{,}160} + \frac{12{,}049}{504}ex^{10}c_1$$

$$-\frac{35}{8}e^2x^{10}c_1 + \frac{187{,}133x^{11}c_1}{20{,}160} - \frac{11{,}407ex^{11}c_1}{1{,}680} + \frac{149}{120}e^2x^{11}c_1$$

$$-\frac{2{,}633x^{12}c_1}{1{,}260} + \frac{7{,}699ex^{12}c_1}{5{,}040} - \frac{67}{240}e^2x^{12}c_1 + \frac{163x^{13}c_1}{420}$$

$$-\frac{1{,}429ex^{13}c_1}{5{,}040} + \frac{29}{560}e^2x^{13}c_1 - \frac{1{,}199x^{14}c_1}{20{,}160} + \frac{73ex^{14}c_1}{1{,}680}$$

$$-\frac{1}{126}e^2x^{14}c_1 + \frac{121x^{15}c_1}{20{,}160} - \frac{11ex^{15}c_1}{2{,}520} + \frac{e^2x^{15}c_1}{1{,}260} - 2xc_2$$

$$-3x^2c_2 + \frac{93x^3c_2}{2} - 18ex^3c_2 + \frac{121x^4c_2}{6} - 8ex^4c_2$$

$$-\frac{15,143x^5c_2}{24}+452ex^5c_2-81e^2x^5c_2+\frac{126,721x^6c_2}{120}-780ex^6c_2$$

$$+144e^2x^6c_2-\frac{396,499x^7c_2}{720}+401e^7c_2-73e^2x^7c_2+\frac{785,471x^8c_2}{5,040}$$

$$-\frac{1,373}{12}ex^8c_2+21e^2x^8c_2-\frac{170,721x^9c_2}{1,120}+\frac{559}{5}ex^9c_2$$

$$-\frac{41}{2}e^2x^9c_2+\frac{172,751x^{10}c_2}{2,016}-\frac{3,767}{60}ex^{10}c_2+\frac{23}{2}e^2x^{10}c_2$$

$$-\frac{658,403x^{11}c_2}{20,160}+\frac{12,049}{504}ex^{11}c_2-\frac{35}{8}e^2x^{11}c_2+\frac{187,133x^{12}c_2}{20,160}$$

$$-\frac{11,407ex^{12}c_2}{1,680}+\frac{149}{120}e^2x^{12}c_2-\frac{2,633x^{13}c_2}{1,260}+\frac{7,699ex^{13}c_2}{5,040}$$

$$-\frac{67}{240}e^2x^{13}c_2+\frac{163x^{14}c_2}{420}-\frac{1,429ex^{14}c_2}{5,040}+\frac{29}{560}e^2x^{14}c_2$$

$$-\frac{1,199x^{15}c_2}{20,160}+\frac{73ex^{15}c_2}{1,680}-\frac{1}{126}e^2x^{15}c_2+\frac{121x^{16}c_2}{20,160}-\frac{11ex^{16}c_2}{2,520}$$

$$+\frac{e^2x^{16}c_2}{1,260}-2x^2c_3-3x^3c_3+\frac{93x^4c_3}{2}-18ex^4c_3+\frac{121x^5c_3}{6}$$

$$-8ex^5c_3-\frac{15,143x^6c_3}{24}+452ex^6c_3-81e^2x^6c_3$$

$$+\frac{126,721x^7c_3}{120}-780ex^7c_3+144e^2x^7c_3-\frac{396,499x^8c_3}{720}$$

$$+401e^8c_3-73e^2x^8c_3+\frac{785,471x^9c_3}{5,040}-\frac{1,373}{12}ex^9c_3$$

$$+21e^2x^9c_3-\frac{170,721x^{10}c_3}{1,120}+\frac{559}{5}ex^{10}c_3-\frac{41}{2}e^2x^{10}c_3$$

$$+\frac{172,751x^{11}c_3}{2,016}-\frac{3,767}{60}ex^{11}c_3+\frac{23}{2}e^2x^{11}c_3-\frac{658,403x^{12}c_3}{20,160}$$

$$+\frac{12{,}049}{504}ex^{12}c_3-\frac{35}{8}e^2x^{12}c_3+\frac{187{,}133x^{13}c_3}{20{,}160}-\frac{11{,}407ex^{13}c_3}{1{,}680}$$

$$+\frac{149}{120}e^2x^{13}c_3-\frac{2{,}633x^{14}c_3}{1{,}260}+\frac{7{,}699ex^{14}c_3}{5{,}040}-\frac{67}{240}e^2x^{14}c_3$$

$$+\frac{163x^{15}c_3}{420}-\frac{1{,}429ex^{15}c_3}{5{,}040}+\frac{29}{560}e^2x^{15}c_3-\frac{1{,}199x^{16}c_3}{20{,}160}$$

$$+\frac{73ex^{16}c_3}{1{,}680}-\frac{1}{126}e^2x^{16}c_3+\frac{121x^{17}c_3}{20{,}160}-\frac{11ex^{17}c_3}{2{,}520}+\frac{e^2x^{17}c_3}{1{,}260}. \tag{43}$$

Therefore, the OHAM approximate solutions of first order is given by using Eqs. (41) and the solution of Eq. (43) into Eq. (44),

$$\tilde{u}\left(x, c_1, c_2, c_3\right)=u_0(x)+u_1(x). \tag{44}$$

Following the methodology outlined in Section 2, we conducted our analysis within the specified domain of $a = 0$ and $b = 1$. During this process,

$$R=\tilde{u}^{(5)}\left(x, c_1, c_3\right)-e^{-x}\tilde{u}'^2\left(x, c_1, c_2, c_3\right), \tag{45}$$

then, we use the least-square method

$$J\left(c_1, c_2, c_3\right)=\int_0^1 R^2 dx, \tag{46}$$

which resulted the following system of algebraic equations

$$\frac{dJ}{dc_1}=\frac{dJ}{dc_2}=\frac{dJ}{dc_3}, \tag{47}$$

then by solving it, we obtain the following optimal values of $c_i's$

$$c_1=-0.502126212667799,\ c_2=-0.502126212667799,\ c_3=-0.03116245509891681.$$

By considering these values into Eq. (43), our approximate solution of one order becomes

$$\tilde{u}(x)=1+x+\frac{x^2}{2}-7.98818462806945x^3+3ex^3+5.478227860867282x^4-2ex^4$$

$$+0.008368770211129984x^5+0.0013561817289601823x^6$$

$$+0.0001933073024596107x^7+0.000036164175867814996x^8$$

$$+0.0000041750370105375 35x^9 - 0.000001680099691118745x^{10}$$

$$-6.058939511878777\times 10^{-8}x^{11} - 1.059618121976791\times 10^{-7}x^{12}$$

$$+1.81235643089278\times 10^{-8}x^{13} - 4.222893579921716\times 10^{-9}x^{14}$$

$$+2.09478324142723\times 10^{-9}x^{15} - 7.580136119035265\times 10^{-10}x^{16}$$

$$+1.936013020441608\times 10^{-10}x^{17} - 3.929151635204326\times 10^{-11}x^{18}$$

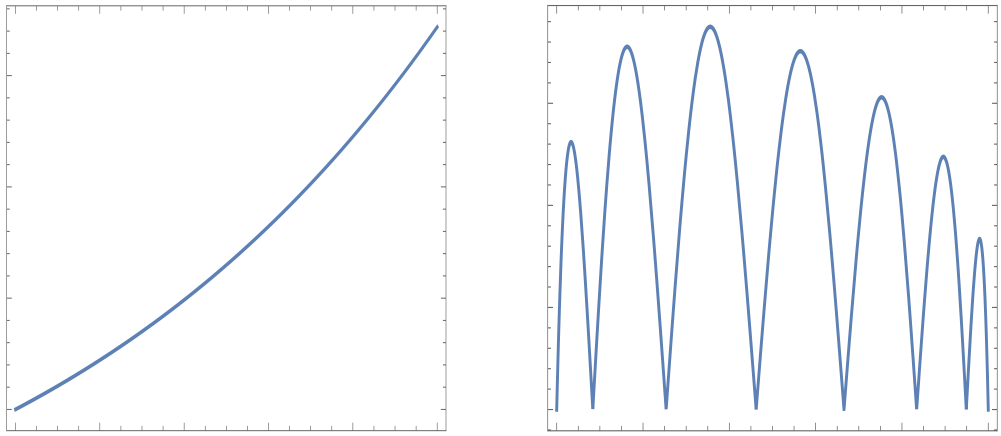
$$+6.640133188641945\times 10^{-12}x^{19} - 8.515214065753627\times 10^{-13}x^{20}$$

$$+1.082201577472571\times 10^{-13}x^{21} - 7.873708466870311\times 10^{-15}x^{22}. \quad (48)$$

The reliability and precision of the improved OHAM procedure have been evaluated through various experiments involving nonlinear differential equations. The findings demonstrate that the improved OHAM procedure is highly effective in solving equations with significant nonlinearities in a precise manner. For example, numerical results are discussed in Tables 1-3 and also graphically represented in Figures 1-3. Our observations suggest that the precision of the solutions is contingent upon the order of the approximation terms in the standard OHAM, and that convergence to the exact solutions occurs as the number of terms approaches infinity. Thus, attaining greater accuracy requires additional computational work and effort. Consequently, to obtain accurate results, modifications should be made to improve the OHAM procedure by using one order of approximations which provides accurate solutions without the need to increase the number of approximation terms in the standard OHAM, and this was done in this work. Although OHAM has numerous advantages, it is not exempt from limitations. The method's effectiveness hinges on the selection of the initial guess and the construction of the homotopy, both of which demand a certain level of expertise and understanding of the problem at hand. In certain situations, choosing an unsuitable initial guess or homotopy can result in slower convergence or less precise solutions. Additionally, the optimization process, while generally straightforward, can become complex when dealing with highly nonlinear or multi-dimensional problems. To overcome these constraints, future studies should focus on developing systematic procedures for selecting the initial guess and constructing the homotopy. Additionally, OHAM could be made more powerful and versatile for various applications by integrating advanced optimization techniques such as machine learning algorithms. These algorithms could automate and improve the optimization process.

Table 1. Numerical results for example 1

x	Exact solution	OHAM solution	Absolute error
0.00	1.00000000	1.00000000	0.00
0.20	1.221402758	1.221402760	1.40×10^{-9}
0.40	1.491824698	1.491824696	1.48×10^{-9}
0.60	1.822118800	1.822118802	1.50×10^{-9}
0.80	2.225540928	2.225540928	9.52×10^{-10}
1.00	2.718281828	2.718281828	3.43×10^{-13}

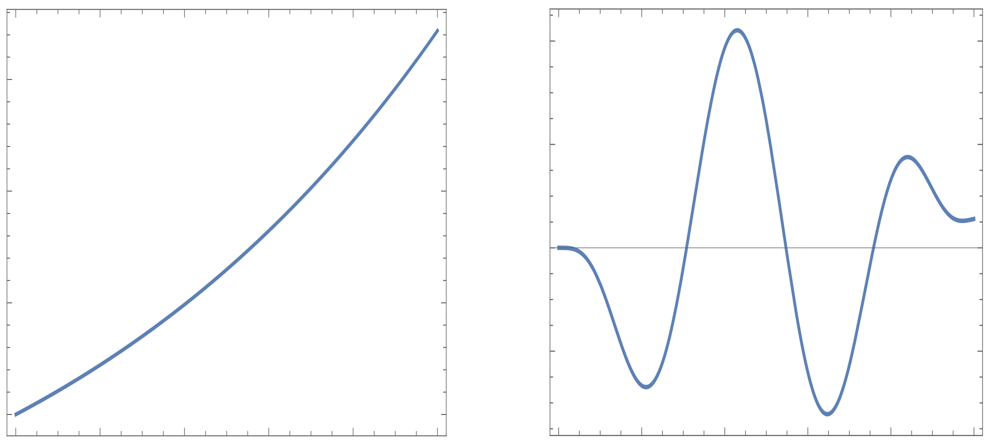


(a) (b)

Figure 1. (a) Plot of exact and approximate solutions, (b) Plot of the absolute error for example 1

Table 2. Numerical results for example 2

x	Exact solution	OHAM solution	Absolute error
0.00	1.00000000	1.00000000	0.00
0.20	0.221402758	0.221402758	2.66×10^{-10}
0.40	0.491824698	0.491824697	3.87×10^{-10}
0.60	0.822118800	0.822118800	2.41×10^{-10}
0.80	1.225540928	1.225540928	1.32×10^{-10}
1.00	1.718281828	1.718281828	5.59×10^{-10}

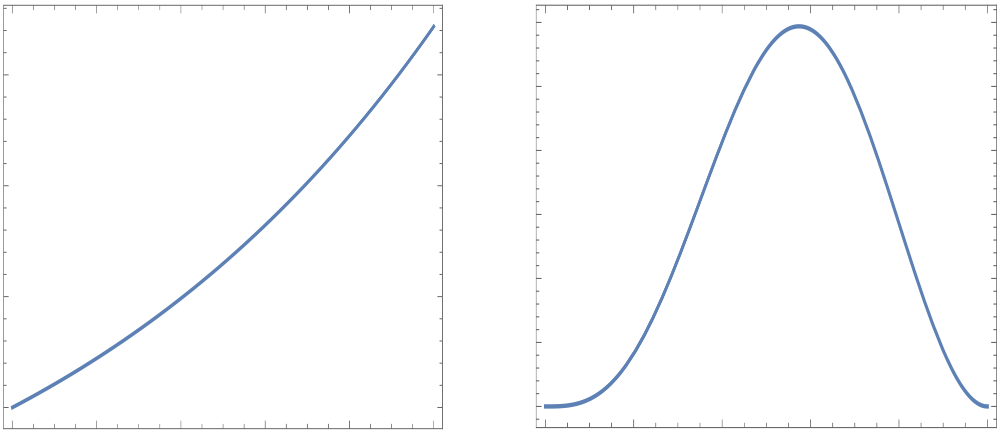


(a) (b)

Figure 2. (a) Plot of exact and approximate solutions, (b) Plot of the absolute error for example 2

Table 3. Numerical results for example 3

x	Exact solution	OHAM solution	Absolute error
0.00	1.00000000	1.00000000	0.00
0.20	1.221402758	1.221402716	4.12×10^{-8}
0.40	1.491824698	1.491824491	2.07×10^{-7}
0.60	1.822118800	1.822118506	2.95×10^{-7}
0.80	2.225540928	2.225540786	1.43×10^{-7}
1.00	2.718281828	2.718281828	0.00



(a) (b)

Figure 3. (a) Plot of exact $u(x) = e^x$ and approximate solutions, (b) Plot of the absolute error for example 3

4. Conclusion

The accuracy and resilience of the improved solutions of the Optimal Homotopy Asymptotic Method (OHAM) were evaluated by applying it to various nonlinear differential equations. The results show that OHAM is highly effective in generating precise solutions, particularly for equations with significant nonlinearities. Our suggested method has the ability to solve ordinary differential equations of any order. The same method can be applied to systems of differential equations, integro-differential equations, and partial differential equations. We found that the excellent accuracy of the obtained results and the decreased computational effort enable more rapid and dependable complex system analysis in future work. OHAM is especially effective for tackling nonlinear problems and situations where analytical or semi-analytical solutions are desired. Its notable efficiency stems from its capacity to manage and enhance convergence. However, it should be noted that OHAM may require more computational resources compared to purely numerical methods such as Adomian decopostion method, Finite Difference and Runge-Kutta Methods or Runge-Kutta.

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Conflict of interest

There is no conflict of interest.

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