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Research Article

Approximate and Exact Solutions of Some Nonlinear Differential Equations Using the Novel Coupling Approach in the Sense of Conformable Fractional Derivative

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Abstract: Several scientific fields utilize fractional nonlinear partial differential equations to model various phenomena. However, most of these equations lack exact solutions. Consequently, techniques for obtaining approximate solutions, which sometimes yield exact solutions, are essential. In this research, we develop a new approach by combining the homotopy perturbation method (HPM) and the conformable natural transform to solve the gas-dynamic equation (GDE), the Fokker-Planck equation (FPE), and the Swift-Hohenberg equation (SHE) in the context of conformable derivatives. The proposed approach is called the conformable natural homotopy perturbation method (CNHPM). This approach has the advantage of not requiring assumptions about significant or minor physical factors. Consequently, it eliminates some of the constraints associated with conventional perturbation methods and can solve both weak and highly nonlinear problems. We consider the absolute, relative, and residual errors numerically and graphically to assess the correctness of our approach. The results show that our approach serves as a suitable alternative to the approximate methods in the literature for solving fractional differential equations.

***Keywords*:** fokker-planck equation, gas-dynamic equation, swift-hohenberg equation, approximate solutions, exact solutions

MSC: 35R11, 80M20, 39A14

1. Introduction

Fractional calculus (Fr-Cal) is a branch of mathematical analysis that generalizes the concepts of differentiation and integration to non-integer (fractional) orders. While traditional calculus deals with integer-order operations (like the first derivative or second integral), Fr-Cal extends these operations to real or complex numbers, providing a more flexible and comprehensive framework.

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Fr-Cal, with its ability to extend the concepts of differentiation and integration to non-integer orders, offers several notable advantages over traditional calculus. These advantages make it particularly useful in modeling complex systems and phenomena across various scientific and engineering disciplines.

The advantages of Fr-Cal include its ability to model systems with memory and hereditary effects, which traditional calculus cannot capture effectively. This approach is particularly useful in fields such as viscoelasticity, where materials display both elastic and viscous characteristics, and in anomalous diffusion, where the movement deviates from classical Brownian motion. Additionally, Fr-Cal provides improved tools for analyzing systems with dynamics that span multiple scales, enhancing the accuracy and predictive capability in various scientific and engineering applications.

Different definitions of fractional order derivatives (FrOD) exist; however, not all of them are often applied. Conformable, Caputo, Riemann-Liouville (R-L), Atangana-Baleanu, Riesz, and Caputo-Fabrizio are the most prominent operators [1–7]. FrOD can simulate and analyze massive systems with complex non-linear processes and higher-order behaviors, making them often a better alternative for modeling than integer-order derivatives. The conformable fractional derivative (Con-FrD), a novel definition of the FrOD, is much easier to compute than earlier forms.

Compared to other types of fractional derivatives, such as the Riemann-Liouville or Caputo derivatives, the Con-FrD is simpler and more intuitive. It retains many of the properties of classical derivatives, such as the product rule and chain rule, making it easier to apply and understand.

For given a function $\Psi : [0, \infty) \to \mathrm{R}$ the Con-FrD of $\Psi(\vartheta)$ order ζ is defined as follows [8]:

$$\mathrm{T}_{\vartheta}^{\zeta}\Psi = \lim_{\in \to 0} \frac{\Psi\left(\vartheta + \in \vartheta^{1-\zeta}\right) - \Psi(\vartheta)}{\in},$$

for $\vartheta > 0$ and $\zeta \in (0, 1]$. If Ψ is ζ-differentiable in some $(0, \mathrm{E})$, $\mathrm{E} > 0$ and $\lim_{\vartheta \to 0^+} \left(\mathrm{T}_{\vartheta}^{\zeta}\Psi\right)(\vartheta)$ exist, then it is defined as

$$\left(\mathrm{T}_{\vartheta}^{\zeta}\Psi\right)(0) = \lim_{\vartheta \to 0^+} \left(\mathrm{T}_{\vartheta}^{\zeta}\Psi\right)(\vartheta).$$

The following are the key characteristics of Con-FrD [9–14]:

i. With Con-FrD, the analytical and numerical solutions to the fractional order differential equations (FrODEs) and systems can be done quickly and easily.

ii. In comparison with the other FrOD, the Con-FrD complies with all specifications for an ordinary derivative, including Rolle's theorem, chain rules, the product, the mean-value theorem, and the quotient.

iii. It enables the development and expansion of new definitions, such as class conformable FrOD, Katugampola FrOD, M-conformable FrOD, fuzzy generalized conformable FrOD, and deformable FrOD.

iv. It makes popular integral transformations, such as Laplace and Sumudu, easier to utilize to solve some FrODEs.

v. In a variety of applications, it generates new evaluations between the Con-FrD and the prior FrOD.

Differential equations (DEs) and integral equations can both be solved using integral transformations, one of the most useful mathematical methods. A suitable integral transform (IT) can convert DEs into the terms of a simple algebraic equation. The ordinary and partial DEs were first addressed using the Fourier and Laplace ITs, two well-known transformations. Then, these ITs were utilized with FODEs. Researchers have created several new ITs in recent years to tackle a variety of mathematical issues. FrODEs are solved using the fractional complex, Elzaki, Sumudu, Aboodh, the traveling wave, and ZZ ITs [15–20]. These transformations are combined with various numerical, analytical, or homotopy-based methods to handle FrODEs. Many mathematicians are becoming interested in a novel IT termed the "natural transform" (NT).

DEs, especially those that are non-homogeneous or have variable coefficients, can be difficult to solve directly. The NT simplifies these equations by converting them into algebraic equations in a transform space. These algebraic equations are easier to manipulate and solve. The NT can be seen as a generalization of other transforms, like the Laplace or Fourier transform. It combines the strengths of these transforms and can be applied to a broader class of functions and equations. For instance, where Laplace transforms might be limited by conditions like the existence of certain integrals, the NT can often be applied more broadly. After solving the equation in the transform space, the inverse NT is applied to revert to the original space, providing the solution to the original problem. This two-step process (transforming, solving, and then inverse transforming) is powerful because it breaks down a complex problem into more manageable parts.

The conformable natural transform (CNT) of $\Psi(\Theta,\ \vartheta)$ can then be defined as [21]:

$$\check{\mathrm{N}}_{\zeta}[\Psi(\Theta,\ \vartheta)] = \mathrm{M}_{\zeta}(\xi,\ \phi) = \frac{1}{\phi}\int_{0}^{\infty} \mathrm{e}^{-\frac{\xi\theta^{\zeta}}{\phi\zeta}}\Psi(\Theta,\ \vartheta)\mathrm{d}_{\zeta}\vartheta,\ \phi>0,\ \zeta>0$$

The fundamental advantages of the NT are as follows [22]:

i. The Fourier, Sumudu, Laplace, and Mellin transforms could each be constructed from the NT and are related to it.

ii. It has the potential to tackle a wide range of difficult concerns in fluid mechanics, dynamics, engineering, physics, and chemistry, including concerns with Maxwell's equations.

iii.Many nonlinear FrODEs with variable coefficients, most notably wavelike ones, can be solved using it.

iv. Problems are solved without the necessity for a new frequency domain.

Nonlinear DEs are crucial in various scientific, engineering, and mathematical contexts due to their ability to model complex systems and phenomena that cannot be captured by linear equations. There are various kinds of nonlinear DEs [23–26].

The nonlinear partial fractional differential equations (NPFDEs) provide a powerful tool for modeling and analyzing complex systems by incorporating memory effects, non-local interactions, and generalized dynamics. They offer improved accuracy, predictive capability, and applicability across a range of scientific and engineering disciplines [27, 28]. Applications regularly involve NPFDEs that are so complex that exact solutions are frequently impractical. For solving NPFDEs under the given initial conditions, methods that provide approximate solutions present a potent alternative tool.

The most commonly employed techniques to solve FrODEs are perturbation methods. Nearly all perturbation techniques are built on the premise that the equation has a small parameter. This so-called ”small parameter” assumption severely restricts the applicability of perturbation techniques. It is widely understood that extremely few nonlinear situations have even one small parameter. Second, discovering small parameters takes special understanding as well as methodologies. Results are perfect when a few parameters are chosen wisely. A bad choice of small factors, however, can have serious negative effects. Furthermore, approximation solutions established by perturbation techniques are often only appropriate for small parameter values. The small parameter assumption is responsible for each of these restrictions. He presented a new perturbation approach in conjunction with the homotopy approach [29]. The constraints of conventional perturbation approaches are easily overcome by the proposed approaches because this approach does not need small parameters.

The following class of fractional gas dynamics equation (FrGDE) is used [30]:

$$\mathrm{T}_{\vartheta}^{\zeta}\Psi(\Theta,\ \vartheta) = -\Psi(\Theta,\ \vartheta)D_{\Theta}\Psi(\Theta,\ \vartheta)+\mathrm{G}\Psi(\Theta,\ \vartheta)(1-\Psi(\Theta,\ \vartheta))+\mathrm{Z}(\Theta,\ \vartheta)$$

with the initial condition:

$$\Psi(\Theta,\ 0) = \mathrm{H}(\Theta)$$

where $\vartheta \geq 0,\ 0<\zeta\leq 1,\ G$ is an appropriate constant and $Z(\Theta,\ \vartheta)$ represent the basis term.

FrGDEs are used to model anomalous diffusion processes where gas particles spread in a non-classical manner. This is particularly useful for describing pollutant dispersion in the atmosphere or the spread of contaminants in various environments. These equations help describe gas flow through porous media, such as soil or industrial filters. They account for complex interactions between the gas and the porous structure, providing more accurate models for applications like groundwater remediation or gas extraction. In situations where gases exhibit non-Newtonian characteristics, such as varying viscosity with the rate of shear, FrGDEs offer a more precise description. This is relevant for high-speed aerodynamic flows and other contexts where classical models are insufficient. In engineering, these equations improve the simulation of processes involving gases, such as combustion in engines or reactions in chemical reactors. They help optimize designs and predict performance more accurately [31].

The fractional Fokker-Planck equation (FrFPE) is a versatile tool for modeling various complex phenomena that involve anomalous diffusion or non-local effects [32]. In finance, it is used to better understand asset price dynamics and market behavior, capturing complexities like volatility clustering and heavy tails that traditional models might miss. In biology, it helps describe the movement and diffusion of particles or organisms in complex environments, such as the spread of diseases or intracellular processes.

In environmental science, the equation improves predictions of pollutant dispersion in atmospheric and aquatic systems, particularly when the dispersion process is influenced by non-standard factors. It also finds applications in materials science for studying diffusion in heterogeneous or porous materials, offering insights into how particles or heat spread through complex media. Astrophysics benefits from this equation in modeling cosmic ray diffusion and the interaction of particles with interstellar mediums, enhancing our understanding of particle transport in space. In neuroscience, the fractional Fokker-Planck equation can model neural activity and signal propagation in neural networks, providing a more detailed picture of brain dynamics and potentially offering new insights into neurological disorders [33].

The FrFPE uses the general system listed below [34, 35]:

$$\mathrm{T}_{\vartheta}^{\zeta}\Psi(\Theta,\ \vartheta)=\left(-\mathrm{D}_{\Theta}\mathrm{R}(\Theta)+\mathrm{D}_{\Theta\Theta}\mathrm{S}(\Theta)\right)\Psi(\Theta,\ \vartheta),$$

with the initial condition:

$$\Psi(\Theta,\ 0)=\mathrm{L}(\Theta),\ \Theta\in\mathrm{R},$$

where $S(\Theta)>0$ is the diffusion factor and $\mathrm{R}(\Theta)>0$ is the drift factor.

The fractional Swift-Hohenberg equation (FrSHE) is a powerful tool for studying complex patterns and dynamics in various scientific fields. In material science, it is used to model pattern formation and phase transitions in systems undergoing structural changes. This includes the study of phenomena such as crystal growth, surface roughening, and the formation of spatial structures in complex materials [36]. In fluid dynamics, the FrSHE helps in understanding the development of turbulent patterns and instability in fluid flows. It provides insights into the formation of coherent structures and patterning in fluids, which can be crucial for predicting and controlling turbulence. In biological systems, the FrSHE is applied to study spatial patterns in biological organisms, such as the development of patterns in animal skin or the distribution of organisms in an ecosystem. It offers a framework to model how these patterns evolve and space.

In optics, the FrSHE is used to analyze the propagation of light waves in nonlinear media. It aids in understanding how spatial patterns and instabilities develop in optical systems, which is important for applications such as laser design and optical communication. In meteorology, the equation helps model atmospheric patterns and climate phenomena. It can be used to study the formation and evolution of weather patterns and climate variability, providing better predictions and understanding of complex atmospheric processes.

The generic form of FrSHE is provided by [37, 38]:

$$\mathrm{T}_{\vartheta}^{\zeta}\Psi(\Theta,\ \vartheta)=\mathrm{Q}\Psi(\Theta,\ \vartheta)-\left(1+\nabla^{2}\right)^{2}\Psi(\Theta,\ \vartheta)-\Psi^{3}(\Theta,\ \vartheta),$$

where $\Theta \in R$, $\vartheta > 0$, and Q is bifurcation parameter, $\Psi(\Theta, \vartheta)$ is a scaler function of Θ and ϑ defined on the line or the plane.

In the literature, the solutions of FrGDE, FrFPE, and FrSHE in terms of Caputo's fractional derivative are discussed [30–38]. We provided a new algorithm that combined CNT and HPM to offer approximate solutions (App-S) and exact solutions (Ex-S) for FrGDE, FrFPE, and FrSHE in terms of Con-FrD. Relative error (Rel-E), absolute error (Abs-E), and residual error (Res-E) measurements have been used to validate the accuracy of the App-S obtained by CNHPM for nonlinear problems. This approach has the advantage of not requiring any parameters in the equation. As a result, it can solve both weak and highly nonlinear FrODEs and avoids some of the drawbacks of conventional perturbation approaches. The results led us to the conclusion that our approach is precise and user-friendly. In the future, we plan to solve various nonlinear fractional models that develop in various fields of engineering and biological systems using the CNHPM.

The remaining sections are organized as follows: in Section 2, we deliberate the procedure of the CNHPM. In Section 3, applications of the algorithm are offered. Section 4 evaluates the proposed approach based on the graphical and numerical outcomes. The conclusion is then presented in Section 5.

2. The algorithm of the CNHPM

The primary goal of this section is to provide an algorithm for series solutions for the nonlinear model using CNHPM. The main algorithms of CNHPM are as follows: first, we apply the CNT to both sides of the problem to convert the given model into algebraic expressions, and then we use the inverse CNT to convert the obtained algebraic expression into the model's real domain. In the next step, we provided the series solutions of the model by using the He's polynomial on the algebraic expressions that were attained with the help of CNT and inverse CNT.

To achieve the solution in the original space, CNHPM must first determine the CNT of the target equations and then execute the inverse CNT. Therefore, the source functions for nonhomogeneous equations must be piecewise continuous and of exponential order, and the inverse CNT must exist after the calculations.

We need the CNT of various functions to obtain series solutions using CNHPM.

Suppose, $\mathbb{C}$, $\gamma \in \mathrm{R}$ and $0 < \zeta \leq 1$. Then we have:

I. $\check{\mathrm{N}}_{\zeta}[\mathbb{C}] = \dfrac{\mathbb{C}}{\xi}$.

II. $\tilde{\mathrm{N}}_{\zeta}\left[e^{\gamma \frac{\vartheta^{\zeta}}{\zeta}}\right] = \dfrac{1}{\xi - \gamma\phi}$, $\xi > \phi\gamma$.

III. $\check{\mathrm{N}}_{\zeta}\left[\sin\left(\gamma \dfrac{\vartheta^{\zeta}}{\zeta}\right)\right] = \dfrac{\gamma\phi}{\xi^2 + \gamma^2\Phi^2}$, $\xi > 0$.

IV. $\check{\mathrm{N}}_{\zeta}\left[\cos\left(\gamma \dfrac{\vartheta^{\zeta}}{\zeta}\right)\right] = \dfrac{\gamma\phi}{\xi^2 + \gamma^2\phi^2}$, $\xi > 0$.

V. $\check{\mathrm{N}}_{\zeta}\left[\sinh\left(\gamma \dfrac{\vartheta^{\zeta}}{\zeta}\right)\right] = \dfrac{\gamma\phi}{\xi^2 - \gamma^2\phi^2}$, $\xi > |\phi\gamma|$.

VI. $\check{\mathrm{N}}_{\zeta}\left[\cosh\left(\gamma \dfrac{\vartheta^{\zeta}}{\zeta}\right)\right] = \dfrac{\gamma\phi}{\xi^2 - \phi^2}$, $\xi > |\phi\gamma|$.

VII. $\check{\mathrm{N}}_{\zeta}\left[\dfrac{\vartheta^{k\zeta}}{\xi^k}\right] = \Gamma(k+1)\dfrac{\phi^k}{\xi^{k+1}}$, $\xi > 0$.

Below, using CNT and HPM is the general analytic solution of the NPFDEs in the standard operator structure.

$$\mathrm{T}_{\vartheta}^{\zeta}\Psi(\Theta, \vartheta) + \mathrm{A}(\Psi(\Theta, \vartheta)) + \mathrm{B}(\Psi(\Theta, \vartheta)) = \mathrm{C}(\Theta, \vartheta), \tag{1}$$

with the initial condition:

$$\Psi(\Theta,\ 0) = \Xi(\Theta), \tag{2}$$

where $\Theta > 0,\ \vartheta > 0,\ 0 < \zeta \leq 1,\ \ \mathrm{T}_{\vartheta}^{\zeta}\Psi(\Theta,\ \vartheta)$ is the Con-FrD of $\Psi(\Theta,\ \vartheta)$, $A(\Psi(\Theta,\ \vartheta))$ stands for a linear differential operator, $B(\Psi(\Theta,\ \vartheta))$ for one that is nonlinear, $\mathrm{C}(\Theta,\ \vartheta)$ for a specified term that is nonzero, and $\Xi(\Theta)$ for the function of Θ.

Apply CNT to Eq. (1).

$$\check{\mathrm{N}}_{\zeta}\left[\mathrm{T}_{\vartheta}^{\zeta}\Psi(\Theta,\ \vartheta) + \mathrm{A}(\Psi(\Theta,\ \vartheta)) + \mathrm{B}(\Psi(\Theta,\ \vartheta))\right] = \check{\mathrm{N}}_{\zeta}[\mathrm{C}(\Theta,\ \vartheta)]. \tag{3}$$

When we apply the linear property of CNT Eq. (3), we get

$$\check{\mathrm{N}}_{\zeta}\left[\mathrm{T}_{\vartheta}^{\zeta}\Psi(\Theta,\ \vartheta)\right] + \check{\mathrm{N}}_{\zeta}[\mathrm{A}(\Psi(\Theta,\ \vartheta))] + \check{\mathrm{N}}_{\zeta}[\mathrm{B}(\Psi(\Theta,\ \vartheta))] = \check{\mathrm{N}}_{\zeta}[\mathrm{C}(\Theta,\ \vartheta)]. \tag{4}$$

By using $\check{\mathrm{N}}_{\zeta}\left[\mathrm{T}_{\vartheta}^{\zeta}\Psi(\Theta,\ \vartheta)\right] = \frac{\xi}{\phi}\check{\mathrm{N}}_{\zeta}[\Psi(\Theta,\ \vartheta)] - \frac{1}{\phi}\Psi(\Theta,\ \vartheta)$, Eq. (4) can be rewritten as

$$\check{\mathrm{N}}_{\zeta}[\Psi(\Theta,\ \vartheta)] = \frac{1}{\xi}\Psi(\Theta,\ 0) + \frac{\phi}{\xi}\check{\mathrm{N}}_{\zeta}[\mathrm{C}(\Theta,\ \vartheta)] - \frac{\phi}{\xi}\check{\mathrm{N}}_{\zeta}\left[[\mathrm{A}(\Psi(\Theta,\ \vartheta))] - \frac{\phi}{\xi}\check{\mathrm{N}}_{\zeta}[\mathrm{B}(\Psi(\Theta,\ \vartheta))]\right.. \tag{5}$$

We achieved the following results by applying the inverse CNT to Eq. (5):

$$\begin{aligned}\Psi(\Theta,\ \vartheta) =& \check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{1}{\xi}\Psi(\Theta,\ 0) + \frac{\phi}{\xi}\check{\mathrm{N}}_{\zeta}[\mathrm{C}(\Theta,\ \vartheta)]\right] - \check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{\mathrm{N}}_{\zeta}[\mathrm{A}(\Psi(\Theta,\ \vartheta))] + \right. \\ &\left. \frac{\phi}{\xi}\check{\mathrm{N}}_{\zeta}[\mathrm{B}(\Psi(\Theta,\ \vartheta))]\right].\end{aligned} \tag{6}$$

According to HPM, the series solution to the problem can be obtained by following the expansion form:

$$\Psi(\Theta,\ \vartheta) = \sum_{v=0}^{\infty}\Xi^{v}\Psi_{v}(\Theta,\ \vartheta) \tag{7}$$

The non-linear term $\mathrm{B}(\Psi(\Theta,\ \vartheta))$ can be decomposed into the following parts using HPM:

$$\mathrm{B}(\Psi(\Theta,\ \vartheta)) = \sum_{v=0}^{\infty}\Xi^{v}\mathrm{X}_{v}(\Psi(\Theta,\ \vartheta)) \tag{8}$$

where $\mathrm{X}_{v}(\Psi(\Theta,\ \vartheta))$ is He's polynomial, which is defined as follows:

$$X_v\left(\Psi_0,\ \Psi_1,\ \ldots,\ \Psi_v\right)=\frac{1}{v!}\frac{\partial^v}{\partial\Xi^v}\left[B\left(\sum_{j=0}^{\infty}\Xi^j\Psi_j(\Theta,\ \vartheta)\right)\right]_{\Xi=0}, \tag{9}$$

where $v=0,\ 1,\ 2,\ \ldots$.

Eqs. (7) and (8) are substituted into Eq. (6), and the result is as follows:

$$\begin{aligned}\sum_{v=0}^{\infty}\Xi^v\Psi_v(\Theta,\ \vartheta)=&\check{\mathrm{N}}_\zeta^{-1}\left[\frac{1}{\xi}\Psi(\Theta,\ 0)+\frac{\phi}{\xi}\check{\mathrm{N}}_\zeta[C\Psi(\Theta,\ \vartheta)]\right]-\\ &\check{\mathrm{N}}_\zeta^{-1}\left[\frac{\phi}{\xi}\check{\mathrm{N}}_\zeta\left[A\left(\sum_{v=0}^{\infty}\Xi^v\Psi_v(\Theta,\ \vartheta)\right)\right]+\frac{\phi}{\xi}\check{N}_\zeta\left[\sum_{v=0}^{\infty}\Xi^vX_v(\Psi(\Theta,\ \vartheta))\right]\right].\end{aligned} \tag{10}$$

By matching Ξ^0 of Eq. (10), it is straightforward to extract the first term from the expansion solution to Eq. (1).

$$\Psi_0(\Theta,\ \vartheta)=\check{N}_\zeta^{-1}\left[\frac{\phi}{\xi}\Psi(\Theta,\ 0)+\frac{\phi}{\xi}\check{N}_\zeta[C(\Theta,\ \vartheta)]\right]$$

The second term is obtained by matching Ξ^1 at both ends of Eq. (10) is as follows:

$$\Psi_1(\Theta,\ \vartheta)=-\check{\mathrm{N}}_\zeta^{-1}\left[\frac{\phi}{\xi}\check{\mathrm{N}}_\zeta\left[\mathrm{A}\left(\Psi_0(\Theta,\ \vartheta)\right]+\frac{\phi}{\xi}\check{\mathrm{N}}_\zeta\left[\mathrm{X}_0(\Psi(\Theta,\ \vartheta))\right]\right]\right..$$

Similarly, to this, we established the third term of the expansion solution of Eq. (1) as follows:

$$\Psi_2(\Theta,\ \vartheta)=-\check{N}_\zeta^{-1}\left[\frac{\phi}{\xi}\check{N}_\zeta\left[\mathrm{A}\left(\Psi_1(\Theta,\ \vartheta)\right)\right]+\frac{\phi}{\xi}\check{\mathrm{N}}_\zeta\left[\mathrm{X}_1(\Psi(\Theta,\ \vartheta))\right]\right].$$

Repeating the process for the fourth term

$$\Psi_3(\Theta,\ \vartheta)=-\check{N}_\zeta^{-1}\left[\frac{\phi}{\xi}\check{N}_\zeta\left[\mathrm{A}\left(\Psi_2(\Theta,\ \vartheta)\right)\right]+\frac{\phi}{\xi}\check{N}_\zeta\left[\mathrm{X}_2(\Psi(\Theta,\ \vartheta))\right]\right].$$

By making the expansion solutions more general for $\Psi(\Theta,\ \vartheta)$, we get to the following, which includes a description:

$$\Psi_{v+1}(\Theta,\ \vartheta)=-\check{\mathrm{N}}_\zeta^{-1}\left[\frac{\phi}{\xi}\check{\mathrm{N}}_\zeta\left[\mathrm{A}\left(\Psi_v(\Theta,\ \vartheta)\right)\right]+\frac{\phi}{\xi}\check{\mathrm{N}}_\zeta\left[\mathrm{X}_v(\Psi(\Theta,\ \vartheta))\right]\right].$$

The following theorem establishes the conditions for the convergence of the series solution.

Theorem 1 Suppose that $\mathscr{M}$ is Banach space with suitable norm over partial sums of $\sum_{v=0}^{\infty} \Psi_v(\Theta,\ \vartheta)$. Further, let $\Psi_0(\Theta)$ is inside the ball $\mathscr{B}_r(\Psi)$ of the solution $\Psi(\Theta,\ \vartheta)$. Then, $\sum_{v=0}^{\infty} \Psi_v(\Theta,\ \vartheta)$ converges if $\exists \in > 0$ as $\|\Psi_{v+1}(\Theta,\ \vartheta)\| \leq \in \|\Psi_v(\Theta,\ \vartheta)\|$.

Proof. A series of partial sums is defined as follows:

$$\mho_0 = \Psi_0(\Theta)$$

$$\mho_1 = \Psi_0(\Theta) + \Psi_1(\Theta,\ \vartheta)$$

$$\mho_2 = \Psi_0(\Theta) + \Psi_1(\Theta,\ \vartheta) + \Psi_2(\Theta,\ \vartheta)$$

$$\mho_3 = \Psi_0(\Theta) + \Psi_1(\Theta,\ \vartheta) + \Psi_2(\Theta,\ \vartheta) + \Psi_3(\Theta,\ \vartheta)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\mho_v = \Psi_0(\Theta) + \Psi_1(\Theta,\ \vartheta) + \Psi_2(\Theta,\ \vartheta) + \Psi_3(\Theta,\ \vartheta) + \ldots + \Psi_v(\Theta,\ \vartheta).$$

Now, we will prove that $\{\Omega_v\}_{v=0}^{\infty}$ is a Cauchy sequence in $\mathscr{M}$. For this, we have as follows:

$$\|\mho_{v+1} - \mho_v\| = \|\Psi_{v+1}(\Theta,\ \vartheta)\| \leq \in \|\Psi_v(\Theta,\ \vartheta)\| \leq \in^2 \|\Psi_{v-1}(\Theta,\ \vartheta)\| \leq \varepsilon^3 \|\Psi_{v-2}(\Theta,\ \vartheta)\|$$

$$\leq \in^4 \|\Psi_{v-3}(\Theta,\ \vartheta)\| \leq \ \ldots \ \leq \in^{v+1} \|\Psi_0(\Theta)\|.$$

Where $v = 0,\ 1,\ 2,\ \ldots$.

For every $v,\ w \in N,\ v \geq w$, we get

$$\|\mho_v - \mho_{\mathrm{w}}\| = \|(\mho_v - \mho_{v-1}) + (\mho_{v-1} - \mho_{v-2}) + (\mho_{v-2} - \mho_{v-3}) + (\mho_{v-3} - \mho_{v-4})$$

$$+ \cdots + (\mho_{\mathrm{w}+1} - \mho_{\mathrm{w}})\|.$$

By triangle inequality, we get

$$\|(\mho_v - \mho_{v-1}) + (\mho_{v-1} - \mho_{v-2}) + (\mho_{v-2} - \mho_{v-3}) + (\mho_{v-3} - \mho_{v-4})$$

$$+ \cdots + (\mho_{\mathrm{w}+1} - \mho_{\mathrm{w}})\|$$

$$\leq \|(\mho_v - \mho_{v-1})\| + \|(\mho_{v-1} - \mho_{v-2})\| + \|(\mho_{v-2} - \mho_{v-3})\| + \|(\mho_{v-3} - \mho_{v-4})\|$$

$$+ \cdots + \|(\mho_{\mathrm{w}+1} - \mho_{\mathrm{w}})\|.$$

Further, we have

$$\|(\mho_v - \mho_{v-1})\| + \|(\mho_{v-1} - \mho_{v-2})\| + \|(\mho_{v-2} - \mho_{v-3})\| + \|(\mho_{v-3} - \mho_{v-4})\|$$

$$+ \cdots + \|(\mho_{\mathrm{w}+1} - \mho_{\mathrm{w}})\|$$

$$\leq \in^v \|\Psi_0(\Theta)\| + \in^{v-1} \|\Psi_0(\Theta)\| + \in^{v-2} \|\Psi_0(\Theta)\| + \cdots + \in^{w+1} \|\Psi_0(\Theta)\|$$

$$= \|\Psi_0(\Theta)\| \left(\in^v + \in^{v-1} + \in^{v-2} + \cdots + \in^{w+1}\right),$$

and,

$$\|\Psi_0(\Theta)\| \left(\in^v + \in^{v-1} + \in^{v-2} + \cdots + \in^{w+1}\right) = \frac{1- \in^{v-w}}{1- \in} \in^{w+1} \|\Psi_0(\Theta)\|.$$

Consequently, the following outcome follows from above.

$$\|(\mho_v - \mho_{\mathrm{w}})\| \leq \frac{1- \in^{v-\mathrm{w}}}{1- \in} \in^{w+1} \|\Psi_0(\Theta)\|.$$

Demonstrating the boundedness of the sequence and obtaining the following result for $0 < \in < 1$:

$$\lim_{v,\ w \to \infty} \|(\mho_v - \mho_w)\| = 0$$

This confirms that the partial sum produced by CNHPM is Cauchy, and thus convergent. We now consider the maximum truncation error as follows: for $\|(\mho_v - \mho_w)\|$ with for $v \geq w$,

$$\|(\mho_v - \mho_w)\| \leq \frac{1- \in^{v-w}}{1- \in} \in^{w+1} \|\Psi_0(\Theta)\|$$

For $0 < \in < 1$; for $v \geq w,\ 1- \in^{v-w} < 1$,

$$\|(\mho_v - \mho_{\mathrm{w}})\| \leq \frac{\in^{w+1}}{1- \in} \|\Psi_0(\Theta)\|$$

demonstrating the bounds of $\{\mho_v\}_{v=0}^{\infty}$.

Let $E_w = \dfrac{\in^{w+1}}{1- \in}$,

$$\|(\mho_v - \mho_w)\| = E_w \|\Psi_0(\Theta)\|.$$

3. Applications of the CNHPM

Five nonlinear problems are solved in this section to assess the effectiveness of the CNHPM.

Problem 1 Consider the non-linear and time-fractional GDE that follows [31]:

$$\mathrm{T}_{\vartheta}^{\zeta}\Psi(\Theta,\ \vartheta)+\Psi(\Theta,\ \vartheta)\mathrm{D}_{\Theta}\Psi(\Theta,\ \vartheta)-\Psi(\Theta,\ \vartheta)(1-\Psi(\Theta,\ \vartheta))=0 \tag{11}$$

where $\vartheta\geq 0,\ 0\leq\Theta\leq 1,\ 0<\zeta\leq 1$, with the initial condition:

$$\Psi(\Theta,\ 0)=\mathrm{e}^{-\Theta}. \tag{12}$$

The result of applying $\check{N}_{\zeta}$ to both sides of Eq. (11) is as follows:

$$\check{\mathrm{N}}_{\zeta}\left[\mathrm{T}_{\vartheta}^{\zeta}\Psi(\Theta,\ \vartheta)\right]+\check{\mathrm{N}}_{\zeta}\left[\Psi(\Theta,\ \vartheta)\mathrm{D}_{\Theta}\Psi(\Theta,\ \vartheta)-\Psi(\Theta,\ \vartheta)(1-\Psi(\Theta,\ \vartheta))\right]=0. \tag{13}$$

Using the process outlined in Section 2, we get as follows:

$$\begin{aligned}\check{\mathrm{N}}_{\zeta}[\Psi(\Theta,\ \vartheta)]=&\frac{1}{\xi}\Psi(\Theta,\ 0)-\frac{\phi}{\xi}\check{\mathrm{N}}_{\zeta}\left[\Psi(\Theta,\ \vartheta)\mathrm{D}_{\Theta}\Psi(\Theta,\ \vartheta)+\Psi^{2}(\Theta,\ \vartheta)\right]+\\&\frac{\phi}{\xi}\check{\mathrm{N}}_{\zeta}[\Psi(\Theta,\ \vartheta)].\end{aligned} \tag{14}$$

As a result of applying the inverse CNT to Eq. (14), we obtain:

$$\begin{aligned}\Psi(\Theta,\ \vartheta)=&\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{1}{\xi}\mathrm{e}^{-\Theta}\right]-\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{\mathrm{N}}_{\zeta}\left[\Psi(\Theta,\ \vartheta)\mathrm{D}_{\Theta}\Psi(\Theta,\ \vartheta)+\Psi^{2}(\Theta,\ \vartheta)\right]\right]+\\&\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{\mathrm{N}}_{\zeta}[\Psi(\Theta,\ \vartheta]]\right].\end{aligned} \tag{15}$$

Using HPM, the non-linear terms $B(\Psi(\Theta,\ \vartheta)$ split into

$$\Psi(\Theta,\ \vartheta)\mathrm{D}_{\Theta}\Psi(\Theta,\ \vartheta)+\Psi(\Theta,\ \vartheta)\Psi(\Theta,\ \vartheta)=\sum_{v=0}^{\infty}\Xi^{v}\mathrm{X}_{v}(\Psi(\Theta,\ \vartheta)) \tag{16}$$

where $\mathrm{X}_{v}(\Psi(\Theta,\ \vartheta))$ represents He's polynomial, which denotes the nonlinear term $\Psi(\Theta,\ \vartheta)\mathrm{D}_{\Theta}\Psi(\Theta,\ \vartheta)+\Psi(\Theta,\ \vartheta)\Psi(\Theta,\ \vartheta)$.

Using the process outlined in Section 2, we get as follows:

$$\sum_{v=0}^{\infty}\Xi^{v}\Psi_{v}(\Theta,\ \vartheta)=\check{N}_{\zeta}^{-1}\left[\frac{1}{\xi}e^{-\Theta}\right]-\check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}\left[\sum_{v=0}^{\infty}\Xi^{v}X_{v}(\Psi(\Theta,\ \vartheta))\right]\right]+$$
$$\check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}\left[\sum_{v=0}^{\infty}\Xi^{v}\Psi_{v}(\Theta,\ \vartheta)\right]\right]. \tag{17}$$

For the first three terms, the term nonlinear can be expressed in the form of:

$$X_0(\Psi(\Theta,\ \vartheta))=\Psi_0^2(\Theta,\ \vartheta)+\Psi_0(\Theta,\ \vartheta)D_\Theta\Psi_0(\Theta,\ \vartheta),$$

$$X_1(\Psi(\Theta,\ \vartheta))=2\Psi_0(\Theta,\ \vartheta)\Psi_1(\Theta,\ \vartheta)+\Psi_0(\Theta,\ \vartheta)D_\Theta\Psi_1(\Theta,\ \vartheta)+\Psi_1(\Theta,\vartheta)D_\Theta\Psi_0(\Theta,\ \vartheta),$$

$$X_2(\Xi(\Theta,\ \vartheta))=2\Psi_0(\Theta,\ \vartheta)\Psi_2(\Theta,\ \vartheta)+\Psi_0(\Theta,\ \vartheta)D_\Theta\Psi_2(\Theta,\ \vartheta)+\Psi_1(\Theta,\ \vartheta)D_\Theta\Psi_1(\Theta,\ \vartheta)+$$
$$\Psi_2(\Theta,\ \vartheta)D_\Theta\Psi_0(\Theta,\ \vartheta)+\Psi_1^2(\Theta,\ \vartheta).$$

By matching the coefficients of Ξ^0 of Eq. (17), we can find the first term of the expansion solution to Eq. (11).

$$\Psi_0(\Theta,\ \vartheta)=e^{-\Theta}.$$

We obtain the second term of the expansion solution to Eq. (11) as follows:

$$\Psi_1(\Theta,\ \vartheta)=-\check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}\left[X_0(\Psi(\Theta,\ \vartheta))\right]\right]+\check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}\left[\Psi_0(\Theta,\ \vartheta)\right]\right].$$

$$\Psi_1(\Theta,\ \vartheta)=-\check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}\left[\Psi_0^2(\Theta,\ \vartheta)+\Psi_0(\Theta,\ \vartheta)D_\Theta\Psi_0(\Theta,\ \vartheta)\right]\right]+\check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}\left[\Psi_0(\Theta,\ \vartheta)\right]\right].$$

$$\Psi_1(\Theta,\ \vartheta)=e^{-\Theta}\frac{\vartheta^{\zeta}}{\zeta}.$$

We can find the third term of the expansion solution for Eq. (11).

$$\Psi_2(\Theta,\ \vartheta)=-\check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}\left[X_1(\Psi(\Theta,\ \vartheta))\right]\right]+\check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}\left[\Psi_1(\Theta,\ \vartheta)\right]\right],$$

$$\Psi_2(\Theta,\ \vartheta)=-\check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}\left[\Psi_0(\Theta,\ \vartheta)\Psi_1(\Theta,\ \vartheta)+\Psi_0(\Theta,\ \vartheta)D_\Theta\Psi_1(\Theta,\ \vartheta)+\right.\right.$$

$$\Psi_1(\Theta,\ \vartheta)\mathrm{D}_\Theta\Psi_0(\Theta,\ \vartheta)]] + \check{\mathrm{N}}_\zeta^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_\zeta\,[\Psi_1(\Theta,\ \vartheta)]\right],$$

$$\Psi_2(\Theta,\ \vartheta) = \mathrm{e}^{-\Theta}\frac{1}{2!}\left(\frac{\vartheta^\zeta}{\zeta}\right)^2.$$

The fourth term of the expansion solution to Eq. (11).

$$\Psi_3(\Theta,\ \vartheta) = -\check{\mathrm{N}}_\zeta^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_\zeta\,[\mathrm{X}_2(\Psi(\Theta,\ \vartheta))]\right] + \check{\mathrm{N}}_\zeta^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_\zeta\,[\Psi_2(\Theta,\ \vartheta)]\right],$$

$$\Psi_3(\Theta,\ \vartheta) = -\check{\mathrm{N}}_\zeta^{-1}\left[\frac{\phi}{\xi}\check{\mathrm{N}}_\zeta\,[2\Psi_0(\Theta,\ \vartheta)\Psi_2(\Theta,\ \vartheta) + \Psi_0(\Theta,\ \vartheta)\mathrm{D}_\Theta\Psi_2(\Theta,\ \vartheta)\right.$$

$$\left. + \Psi_1(\Theta,\ \vartheta)\mathrm{D}_\Theta\Psi_1(\Theta,\ \vartheta) + \Psi_2(\Theta,\ \vartheta)\mathrm{D}_\Theta\Psi_0(\Theta,\ \vartheta) + \Psi_1^2(\Theta,\ \vartheta)]\right]$$

$$+\check{\mathrm{N}}_\zeta^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_\zeta\,[\Psi_2(\Theta,\ \vartheta)]\right],$$

$$\Psi_3(\Theta,\ \vartheta) = \mathrm{e}^{-\Theta}\frac{1}{3!}\left(\frac{\vartheta^\zeta}{\zeta}\right)^3.$$

Compare the coefficients of Ξ^4 and Ξ^5 of Eq. (17) to find the fifth and sixth terms.

$$\Psi_4(\Theta,\ \vartheta) = \mathrm{e}^{-\Theta}\frac{1}{4!}\left(\frac{\vartheta^\zeta}{\zeta}\right)^4.$$

$$\Psi_5(\Theta,\ \vartheta) = \mathrm{e}^{-\Theta}\frac{1}{5!}\left(\frac{\vartheta^\zeta}{\zeta}\right)^5.$$

In this way, the expansion-form solution of Eq. (11) is provided as

$$\Psi(\Theta,\ \vartheta) = \mathrm{e}^{-\vartheta}\left(1 + \frac{\vartheta^\zeta}{\zeta} + \frac{1}{2!}\left(\frac{\vartheta^\zeta}{\zeta}\right)^2 + \frac{1}{3!}\left(\frac{\vartheta^\zeta}{\zeta}\right)^3 + \frac{1}{4!}\left(\frac{\vartheta^\zeta}{\zeta}\right)^4 + \frac{1}{5!}\left(\frac{\vartheta^\zeta}{\zeta}\right)^5 + \cdots\right)$$

$$\Psi(\Theta,\ \vartheta) = \mathrm{e}^{-\Theta}\mathrm{e}^{\frac{\vartheta^\zeta}{\zeta}}$$

Problem 2 Consider the non-linear and time-fractional GDE that follows [31]:

$$\mathrm{T}_{\vartheta}^{\zeta}\Psi(\Theta,\ \vartheta)+\Psi(\Theta,\ \vartheta)\mathrm{D}_{\Theta}\Psi(\Theta,\ \vartheta)-\Psi(\Theta,\ \vartheta)(1-\Psi(\Theta,\ \vartheta))\log\beta=0, \tag{18}$$

where $\vartheta\geq 0,\ \beta>0,\ 0\leq\Theta\leq 1$, and $0<\zeta\leq 1$, with the initial condition:

$$\Psi(\Theta,\ 0)=\beta^{-\theta}. \tag{19}$$

Using the process outlined in Section 2, we get as follows:

$$\begin{aligned}\check{\mathrm{N}}_{\zeta}[\Psi(\Theta,\ \vartheta)]=&\frac{1}{\xi}\Psi(\Theta,\ 0)-\frac{\phi}{\xi}\check{\mathrm{N}}_{\zeta}\log\beta\left[\Psi(\Theta,\ \vartheta)\mathrm{D}_{\Theta}\Psi(\Theta,\ \vartheta)+\Psi^{2}(\Theta,\ \vartheta)\right]+\\&\frac{\phi}{\xi}\check{\mathrm{N}}_{\zeta}[\Psi(\Theta,\ \vartheta)].\end{aligned} \tag{20}$$

As a result of applying the inverse CNT to Eq. (20), we obtain:

$$\begin{aligned}\Psi(\Theta,\ \vartheta)=&\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{1}{\xi}\beta^{-\Theta}\right]-\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{\mathrm{N}}_{\zeta}\log\beta\left[\Psi(\Theta,\ \vartheta)\mathrm{D}_{\Theta}\Psi(\Theta,\ \vartheta)+\Psi^{2}(\Theta,\ \vartheta)\right]\right]+\\&\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{\mathrm{N}}_{\zeta}[\Psi(\Theta,\ \vartheta)]\right].\end{aligned} \tag{21}$$

We acquire the following by employing the iteration process described in Section 3:

$$\begin{aligned}\sum_{v=0}^{\infty}\Xi^{v}\Psi_{v}(\Theta,\ \vartheta)=&\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{1}{\xi}\beta^{-\Theta}\right]-\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{\mathrm{N}}_{\zeta}\log\Theta\left[\sum_{v=0}^{\infty}\Xi^{v}\mathrm{X}_{v}(\Psi(\Theta,\ \vartheta))\right]\right]+\\&\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{\mathrm{N}}_{\zeta}\left[\sum_{v=0}^{\infty}\Xi^{v}\Psi_{v}(\Theta,\ \vartheta)\right]\right].\end{aligned} \tag{22}$$

The first six terms of the expansion solution of Eq. (18) are obtained by comparing the coefficients of Ξ^{v} on the two sides of Eq. (22). By matching the coefficients of Ξ^{0} on both sides of Eg. (22), we established the first term of the expansion solution to Eg. (18).

$$\Psi_{0}(\Theta,\ \vartheta)=\beta^{-\theta}.$$

By matching the coefficients of Ξ^{1} of Eq. (22), we obtained the second term of the expansion solution to Eq. (18).

$$\Psi_1(\Theta,\ \vartheta) = -\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_{\zeta}\log\beta\,[\mathrm{X}_0(\Psi(\Theta,\ \vartheta))]\right] + \check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_{\zeta}\,[\Psi_0(\Theta,\ \vartheta)]\right].$$

$$\Psi_1(\Theta,\ \vartheta) = -\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_{\zeta}\log\beta\,\left[\Psi_0^2(\Theta,\ \vartheta) + \Psi_0(\Theta,\ \vartheta)D_{\Theta}\Psi_0(\Theta,\ \vartheta)\right]\right] + \check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_{\zeta}\,[\Psi_0(\Theta,\ \vartheta)]\right].$$

$$\Psi_1(\Theta,\ \vartheta) = \beta^{-\theta}\log\beta\frac{\mathrm{t}^{\zeta}}{\zeta}.$$

The third term of the series solution to Eq. (18) is established as follows:

$$\Psi_2(\Theta,\ \vartheta) = -\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_{\zeta}\log\beta\,[\mathrm{X}_1(\Psi(\rho,\ \mu))]\right] + \check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_{\zeta}\,[\Psi_1(\Theta,\ \vartheta)]\right].$$

$$\Psi_2(\Theta,\ \vartheta) = -\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{\mathrm{N}}_{\zeta}\log\beta\,[\Psi_0(\Theta,\ \vartheta)\Psi_1(\Theta,\ \vartheta) + \Psi_0(\Theta,\ \vartheta)\mathrm{D}_{\Theta}\Psi_1(\Theta,\ \vartheta)\right.$$

$$\left. + \Psi_1(\Theta,\ \vartheta)\mathrm{D}_{\Theta}\Psi_0(\Theta,\ \vartheta)]\right] + \check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_{\zeta}\,[\Psi_1(\Theta,\ \vartheta)]\right].$$

$$\Psi_2(\Theta,\ \vartheta) = \beta^{-\theta}\frac{1}{2!}(\log\beta)^2\left(\frac{\theta^{\zeta}}{\zeta}\right)^2.$$

To find the fourth term, compare the coefficients of Ξ^3 on the two sides of Eq. (22).

$$\Psi_3(\Theta,\ \vartheta) = -\check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}\log\beta\,[\mathrm{X}_2(\Psi(\Theta,\ \vartheta)]\right] + \check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}\,[\Psi_2(\Theta,\ \vartheta)]\right]$$

$$\Psi_3(\Theta,\ \vartheta) = -\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\log\beta\check{\mathrm{N}}_{\zeta}\,[2\Psi_0(\Theta,\ \vartheta)\Psi_2(\Theta,\ \vartheta) + \Psi_0(\Theta,\ \vartheta)D_{\Theta}\Psi_2(\Theta,\ \vartheta) + \right.$$

$$\left.\Psi_1(\Theta,\ \vartheta)D_{\Theta}\Psi_1(\Theta,\ \vartheta) + \Psi_2(\Theta,\ \vartheta)D_{\Theta}\Psi_0(\Theta,\ \vartheta) + \Psi_1^2(\Theta,\ \vartheta)]\right] + \check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}\,[\Psi_2(\Theta,\ \vartheta)]\right]$$

$$\Psi_3(\Theta,\ \vartheta) = \beta^{-\Theta}\frac{1}{3!}(\log\beta)^3\left(\frac{\vartheta^{\zeta}}{\zeta}\right)^3$$

Similar results include the following:

$$\Psi_4(\Theta,\ \vartheta) = \beta^{-\theta}\frac{1}{4!}(\log\beta)^4\left(\frac{\vartheta\zeta}{\zeta}\right)^4$$

$$\Psi_5(\Theta,\ \vartheta) = \beta^{-\theta}\frac{1}{5!}(\log\beta)^5\left(\frac{\vartheta\zeta}{\zeta}\right)^5.$$

In this way, the expansion-form solution of Eq. (18) is provided as

$$\begin{aligned}\Psi(\Theta,\ \vartheta) =&\beta^{-\theta}\left(1+(\log\beta)\left(\frac{\vartheta\zeta}{\zeta}\right)+\frac{1}{2!}(\log\beta)^2\left(\frac{\vartheta\zeta}{\zeta}\right)^2+\frac{1}{3!}(\log\beta)^3\left(\frac{\vartheta\zeta}{\zeta}\right)^3+\frac{1}{4!}(\log\beta)^4\left(\frac{\vartheta\zeta}{\zeta}\right)^4\right.\\ &\left.+\frac{1}{5!}(\log\beta)^5\left(\left(\frac{\vartheta\zeta}{\zeta}\right)^5\right)\right).\end{aligned}$$

$$\Psi(\Theta,\ \vartheta) =\beta^{-\theta}e^{(\log\beta)\frac{\vartheta\zeta}{\zeta}}.$$

Problem 3 Consider taking into account the following non-linear, time-fractional FPE [35]:

$$\mathrm{T}_{\vartheta}^{\zeta}\Psi(\Theta,\ \vartheta)+\mathrm{D}_{\Theta}\left(\frac{4}{\Theta}\Psi^2(\Theta,\ \vartheta)-\frac{\theta}{3}\Psi(\Theta,\ \vartheta)\right)-\mathrm{D}_{\Theta\Theta}\Psi^2(\Theta,\ \vartheta)=0 \tag{23}$$

where $\vartheta \geq 0,\ 0 \leq \Theta \leq 1$, and $0 \leq \zeta \leq 1$, with the initial condition:

$$\Psi(\Theta,\ \vartheta) = \Theta^2. \tag{24}$$

We acquire the following using the iteration procedure described in Section 2:

$$\check{N}_{\zeta}\left[\mathrm{T}_{\vartheta}^{\zeta}\Psi(\Theta,\ \vartheta)\right]+\check{N}_{\zeta}\left[\mathrm{D}_{\Theta}\left(\frac{4}{\Theta}\Psi^2(\Theta,\ \vartheta)-\frac{\theta}{3}\Psi(\Theta,\ \vartheta)\right)-\mathrm{D}_{\Theta\Theta}\Psi^2(\Theta,\ \vartheta)\right]=0. \tag{25}$$

By taking the inverse CNT on Eq. (25), we get the following:

$$\Psi(\Theta,\ \vartheta) = \check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{1}{\xi}\Theta^2\right]-\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_{\zeta}\left[\mathrm{D}_{\Theta}\left(\frac{4}{\Theta}\Psi^2(\Theta,\ \vartheta)-\frac{\theta}{3}\Psi(\Theta,\ \vartheta)\right)-\mathrm{D}_{\Theta\Theta}\Psi^2(\Theta,\ \vartheta)\right]\right]. \tag{26}$$

Using the iteration process outlined in Section 2, we get as follows:

$$\Psi(\Theta,\ \vartheta)=\Theta^2,$$

$$\Psi(\Theta,\ \vartheta)=\Theta^2\frac{\vartheta^{\zeta}}{1!\zeta},$$

$$\Psi(\Theta,\ \vartheta)=\Theta^2\frac{\vartheta^{2\zeta}}{2!\zeta^2},$$

$$\Psi(\Theta,\ \vartheta)=\Theta^2\frac{\vartheta^{3\zeta}}{3!\zeta^3},$$

$$\Psi(\Theta,\ \vartheta)=\Theta^2\frac{\vartheta^{4\zeta}}{4!\zeta^4},$$

$$\Psi(\Theta,\ \vartheta)=\Theta^2\frac{\vartheta^{5\zeta}}{5!\zeta^5}.$$

The following is the expansion solution to Eq. (23):

$$\Psi(\Theta,\ \vartheta)=\Theta^2\left(1+\frac{\vartheta^{1\zeta}}{1!\zeta^1}+\frac{\vartheta^{2\zeta}}{2!\zeta^2}+\frac{\vartheta^{3\zeta}}{3!\zeta^3}+\frac{\vartheta^{4\zeta}}{4!\zeta^4}+\frac{\vartheta^{5\zeta}}{5!\zeta^5}+\ldots\right).$$

$$\Psi(\Theta,\ \vartheta)=\Theta^2\mathrm{e}^{\frac{\vartheta^{\zeta}}{\zeta}}.$$

Problem 4 The following non-linear, time-fractional SHE is considered as follows [37]:

$$\begin{aligned}&\mathrm{T}_{\vartheta}^{\zeta}\Psi(\Theta,\ \vartheta)+(1-\delta)\Psi(\Theta,\ \vartheta)+2\mathrm{D}_{\Theta\Theta}\Psi(\Theta,\ \vartheta)+\mathrm{D}_{\Theta\Theta\Theta\Theta}\Psi(\Theta,\ \vartheta)-\beta\mathrm{D}_{\Theta\Theta\Theta}\Psi(\Theta,\vartheta)-\\&\Psi^2(\Theta,\ \vartheta)+(\mathrm{D}_{\Theta}\Psi(\Theta,\ \vartheta))^2=0,\ 0<\zeta\leq 1,\end{aligned}\tag{27}$$

the initial condition:

$$\Psi(\Theta,\ 0)=\mathrm{e}^{\Theta}.\tag{28}$$

We obtain the following using the iteration procedure described in Section 2.

$$\Psi(\Theta,\ \vartheta) = \check{N}_{\zeta}^{-1}\left[\frac{1}{\xi}e^{\theta}\right] - \check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}[(1-\delta)\Psi(\Theta,\ \vartheta)]\right] - \check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}[2D_{\Theta\Theta}\Psi(\Theta,\ \vartheta)]\right] -$$

$$\check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}[D_{\Theta\Theta\Theta\Theta}\Psi(\Theta,\ \vartheta)]\right] + \check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}[\beta D_{\Theta\Theta\Theta}\Psi(\Theta,\ \vartheta)]\right] + \tag{29}$$

$$\check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}\left[\Psi^{2}(\Theta,\ \vartheta)\right]\right] - \check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}[]\left(D_{\mathbf{x}}\Psi(\Theta,\ \vartheta)\right)^{2}\right].$$

By following the steps indicated in Section 2, we obtain the following outcome:

$$A(\Psi(\Theta,\ \vartheta)) = -\check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}[(1-\delta)\Psi(\Theta,\ \vartheta)]\right] - \check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}[2D_{\Theta\Theta}\Psi(\Theta,\ \vartheta)]\right] -$$

$$\check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}[D_{\Theta\Theta\Theta\Theta}\Psi(\Theta,\ \vartheta)]\right] + \check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}[\beta D_{\Theta\Theta\Theta}\Psi(\Theta,\ \vartheta)]\right].$$

$$B(\Psi(\Theta,\ \vartheta)) = \check{N}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}[\Psi(\Theta,\ \vartheta)]\right] - N_{\zeta}^{-1}\left[\frac{\phi}{\xi}\check{N}_{\zeta}\left[\left(D_{\Theta}\Psi(\Theta,\ \vartheta)\right)^{2}\right]\right].$$

Using the iteration process outlined in Section 2, we get as follows:

$$\Psi_0(\Theta,\ \vartheta) = e^{\Theta}.$$

$$\Psi_1(\Theta,\ \vartheta) = e^{\Theta}(\delta - 4 + \beta)^{1}\frac{\vartheta^{1\zeta}}{1!\zeta^{1}}.$$

$$\Psi_2(\Theta,\ \vartheta) = e^{\Theta}(\delta - 4 + \beta)^{2}\frac{\vartheta^{2\zeta}}{2!\zeta^{2}}.$$

$$\Psi_3(\Theta,\ \vartheta) = e^{\Theta}(\delta - 4 + \beta)^{3}\frac{\vartheta^{3\zeta}}{3!\zeta^{3}}.$$

$$\Psi_4(\Theta,\ \vartheta) = e^{\Theta}(\delta - 4 + \beta)^{4}\frac{\vartheta^{4\zeta}}{4!\zeta^{4}}.$$

$$\Psi_5(\Theta,\ \vartheta) = e^{\Theta}(\delta - 4 + \beta)^{5}\frac{\vartheta^{5\zeta}}{5!\zeta^{5}}$$

The series solution to Eq. (27) is defined below:

$$\Psi(\Theta,\ \vartheta) = \mathrm{e}^{\Theta} + \mathrm{e}^{\Theta}(\delta-4+\beta)^{1}\frac{\vartheta^{1\zeta}}{1!\zeta^{1}} + \mathrm{e}^{\Theta}(\delta-4+\beta)^{2}\frac{\vartheta^{2\zeta}}{2!\zeta^{2}} + \mathrm{e}^{\Theta}(\delta-4+\beta)^{3}\frac{\vartheta^{3\zeta}}{3!\zeta^{3}} +$$

$$\mathrm{e}^{\Theta}(\delta-4+\beta)^{4}\frac{\theta^{4\zeta}}{4!\zeta^{4}} + e^{\Theta}(\delta-4+\beta)^{5}\frac{\theta^{5\zeta}}{5!\zeta^{5}} + \ldots$$

$$\Psi(\Theta,\ \vartheta) = e^{\Theta+(\beta+\delta-4)\frac{\theta^{\zeta}}{\zeta}}.$$

Problem 5 The following non-linear, time-fractional SHE is considered as follows [37]:

$$\begin{aligned} &\mathrm{T}_{t}^{\zeta}\Psi(\Theta,\ \vartheta) + (1-\delta)\Psi(\Theta,\ \vartheta) + 2\mathrm{D}_{\Theta\Theta}\Psi(\Theta,\ \vartheta) + \mathrm{D}_{\Theta\Theta\Theta\Theta}\Psi(\Theta,\ \vartheta) - \\ &\Psi^{2}(\Theta,\ \vartheta) + (\mathrm{D}_{\Theta}\Psi(\Theta,\ \vartheta))^{2} = 0,\ 0<\zeta\leq 1 \end{aligned} \tag{30}$$

the initial condition

$$\Psi(\Theta,\ 0) = e^{\Theta}. \tag{31}$$

By applying the procedures described in Section 2 to Eq. (30), we obtain the conclusion shown below.

$$\begin{aligned} \Psi(\Theta,\ \vartheta) = &\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{1}{\xi}\mathrm{e}^{\Theta}\right] - \check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_{\zeta}[(1-\delta)\Psi(\Theta,\ \vartheta)]\right] - \check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_{\zeta}\left[2\mathrm{D}_{\Theta\Theta}\Psi(\Theta,\ \vartheta)\right]\right] - \\ &\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_{\zeta}\left[\mathrm{D}_{\Theta\Theta\Theta\Theta}\Psi(\Theta,\ \vartheta)\right]\right] + \check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_{\zeta}\left[\Psi^{2}(\Theta,\ \vartheta)\right]\right] - \\ &\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_{\zeta}\left[(\mathrm{D}_{\Theta}\Psi(\Theta,\ \vartheta))^{2}\right]\right]. \end{aligned} \tag{32}$$

Using the iteration process outlined in Section 2, we get as follows:

$$\begin{aligned} \mathrm{A}(\Psi(\Theta,\ \vartheta)) = &-\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_{\zeta}[(1-\delta)\Psi(\Theta,\ \vartheta)]\right] - \check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_{\zeta}\left[2\mathrm{D}_{\Theta\Theta}\Psi(\Theta,\ \vartheta)\right]\right] - \\ &\check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_{\zeta}\left[\mathrm{D}_{\Theta\Theta\Theta\Theta}\Psi(\Theta,\ \vartheta)\right]\right]. \end{aligned} \tag{33}$$

$$\mathrm{B}(\Psi(\Theta,\ \vartheta)) = \check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_{\zeta}\left[\Psi^{2}(\Theta,\ \vartheta)\right]\right] - \check{\mathrm{N}}_{\zeta}^{-1}\left[\frac{\phi}{\xi}\,\check{\mathrm{N}}_{\zeta}\left[(\mathrm{D}_{\Theta}\Psi(\Theta,\ \vartheta))^{2}\right]\right].$$

The following outcomes are attained by employing the iteration procedure described in Section 2:

$$\Psi_0(\Theta,\ \vartheta) = \mathrm{e}^{\Theta}.$$

$$\Psi_1(\Theta,\ \vartheta) = \mathrm{e}^{\Theta}(\delta-4)^1\frac{\vartheta^{\zeta}}{1!\zeta^1}.$$

$$\Psi_2(\Theta,\ \vartheta) = \mathrm{e}^{\Theta}(\delta-4)^2\frac{\vartheta^{2\zeta}}{2!\zeta^2}.$$

$$\Psi_3(\Theta,\ \vartheta) = \mathrm{e}^{\Theta}(\delta-4)^3\frac{\vartheta^{3\zeta}}{3!\zeta^3}.$$

$$\Psi_4(\Theta,\ \vartheta) = \mathrm{e}^{\Theta}(\delta-4)^4\frac{\vartheta^{4\zeta}}{4!\zeta^4}.$$

$$\Psi_5(\Theta,\ \vartheta) = \mathrm{e}^{\Theta}(\delta-4)^5\frac{\vartheta^{5\zeta}}{5!\zeta^5}.$$

The following is the expansion solution to Eq. (30):

$$\Psi(\Theta,\ \vartheta) = \mathrm{e}^{\Theta} + \mathrm{e}^{\Theta}(\delta-4)^1\frac{\vartheta^{\zeta}}{1!\zeta^1} + \mathrm{e}^{\Theta}(\delta-4)^2\frac{\vartheta^{2\zeta}}{2!\zeta^2} + \mathrm{e}^{\Theta}(\delta-4)^3\frac{\vartheta^{3\zeta}}{3!\zeta^3} +$$

$$\mathrm{e}^{\Theta}(\delta-4)^4\frac{\theta^{4\zeta}}{4!\zeta^4} + \mathrm{e}^{\Theta}(\delta-4)^5\frac{\theta^{5\zeta}}{5!\zeta^5} + \dots$$

$$\Psi(\Theta,\ \vartheta) = e^{\theta+(\delta-4)\frac{\theta^{\zeta}}{\zeta}}.$$

Based on their graphical and numerical outcomes, the approximations established by the CNHPM for the GDE, FPE, and SHE are evaluated in the next section.

4. Discussion and results

In this section, we evaluate the graphic and numerical results of the App-Ss for the five nonlinear problems solved in Section 3. To assess the precision and capabilities of the approximation method, error functions can be utilized. It is required to indicate the errors of the approximate solution since CNHPM offers an approximate analytical solution in terms of an infinite fractional power series. To illustrate the precision and capability of the CNHPM, we used the Abs-E, Rel-E, and Res-E functions.

The solutions to FrGDEs provide a more comprehensive and realistic description of gas behavior by incorporating memory, non-local effects, and anomalous diffusion. Similarly, the solutions to FrFPEs offer a deeper and more nuanced understanding of systems characterized by anomalous diffusion, memory effects, and non-local interactions, delivering a more accurate and flexible depiction of stochastic processes in complex scenarios. Additionally, the solutions to FrSHEs enhance our understanding of pattern formation, non-local interactions, and memory effects in spatially extended systems. By extending classical models, they capture more intricate and realistic behaviors in various physical and natural systems.

In this section, the graphical and numerical results of the App-S and Ex-S for the models described in Problems 1-5 are analyzed and assessed. The two-dimensional graphs of the App-S acquired from five iterations and the Ex-S derived by CNHPM for $\zeta = 0.6,\ 0.7,\ 0.8,\ 0.9$ and 1.0 are shown in Figures 1-3 of Problems 1-5.

These graphs show how, when $\zeta \to 1.0$ occurs the App-S converges to the Ex-S. The App-S interaction with the Ex-S when $\zeta = 1.0$ occurs demonstrating the accuracy and efficacy of the proposed approach. For Problems 1-5 in Figures 4-6, the 2D curve is used to compare the App-S and Ex-S in the sense of Abs-E. The comparison analysis indicates that the Ex-S and the fifth-step App-S are quite comparable. The Abs-E is displayed on graphs to show the excellent precision of CNHPM.

For Problems 1-5 in Figures 7-11, the App-S and Ex-S are compared using the 3D curve, respectively. The comparison analysis indicates that the Ex-S and the fifth-step App-S are quite similar. The 3D graphs show the excellent precision of CNHPM.

The Abs-E and Rel-E for appropriate specified locations between the Ex-S and fifth-order App S derived by CNHPM in Problems 1-5 at $\zeta = 1.0$ are shown in Tables 1-5. The tables demonstrate that the App-S and Ex-S are almost in agreement, which attests to the CNHPM's efficacy. Tables 6-10 show the Res-E for the 5th App-Ss in the interval $\vartheta \in [0,\ 0.5]$ as obtained by the CNHPM for Problems 1-5 at $\zeta = 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0$. From these tables, we observed that the Res-E for all the problems in the 5th step App-Ss is very small.

The findings in this section, illustrated in graphs and tables, show that the CNHPM is a useful and effective approach to solving nonlinear FrODEs, requiring fewer calculations and iterations.

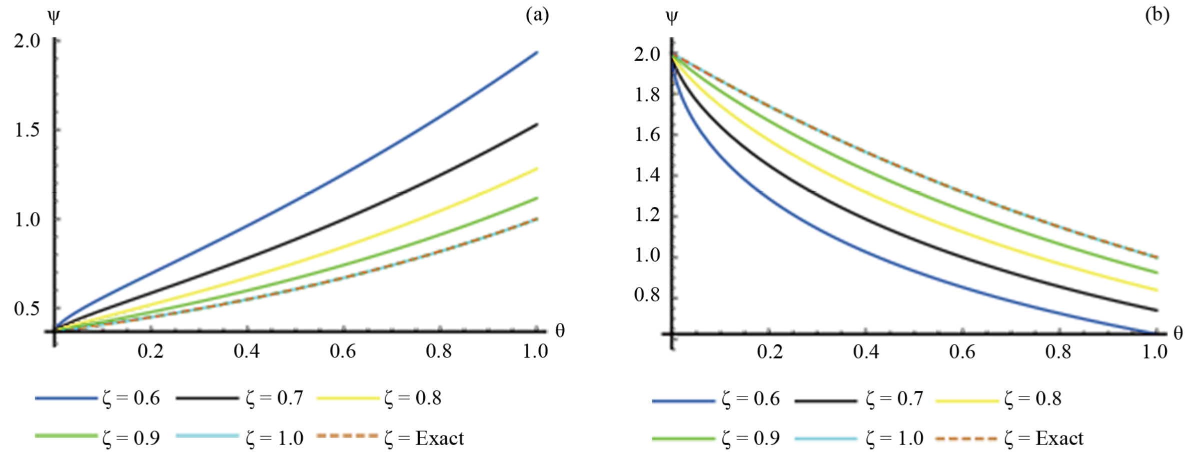

Figure 1. The 2D diagrams of App-S and Ex-S for various levels of ζ in the range $\theta \in [0,\ 1.0]$ at $\Theta = 1.0$ for Problem 1 and $\Theta = 1.0$ with $\beta = 0.5$ for Problem 2, respectively

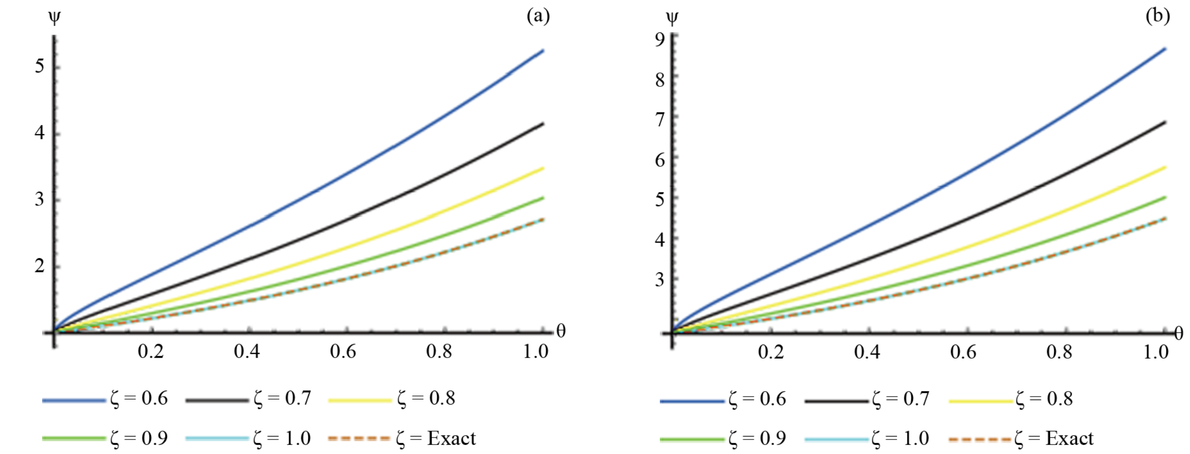

Figure 2. The 2D diagrams of App-S and Ex-S for various levels of ζ in the range $\vartheta \in [0,\ 1.0]$ at $\Theta = 1.0$ for Problem 3 and $\Theta = 0.5$ at $\delta = 3.0$ and $\beta = 2.0$ for Problems 4

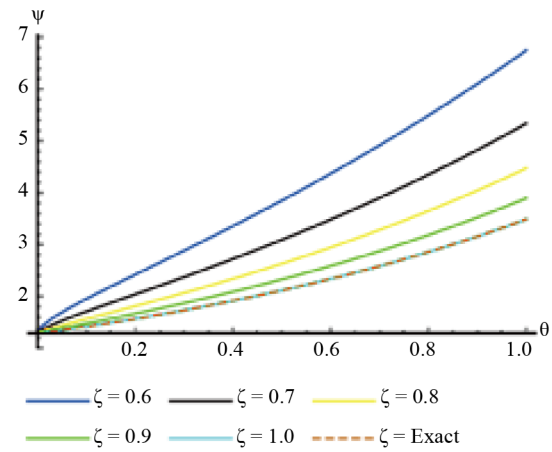

Figure 3. The 2D diagrams of App-S and Ex-S for various levels of ζ in the range $\vartheta \in [0,\ 1.0]$ at $\Theta = 0.25$ when $\delta = 5.0$ for Problem 5

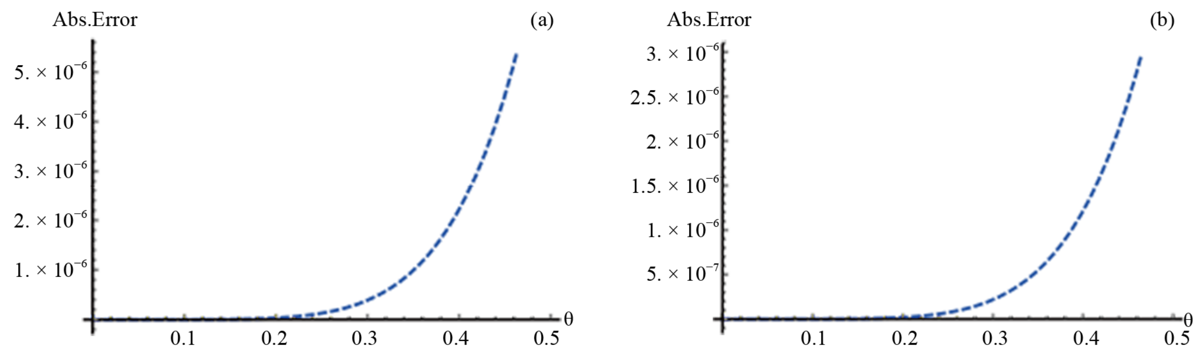

Figure 4. Two-dimensional Abs-E graphs in the range of $\vartheta \in [0,\ 0.5]$ between the fifth step App-S and Ex-S for $\zeta = 1.0$ when $\Theta = 1.0$ for Problems 1 and when $\Theta = 1.0$ with $\beta = 0.5$ for Problem 2

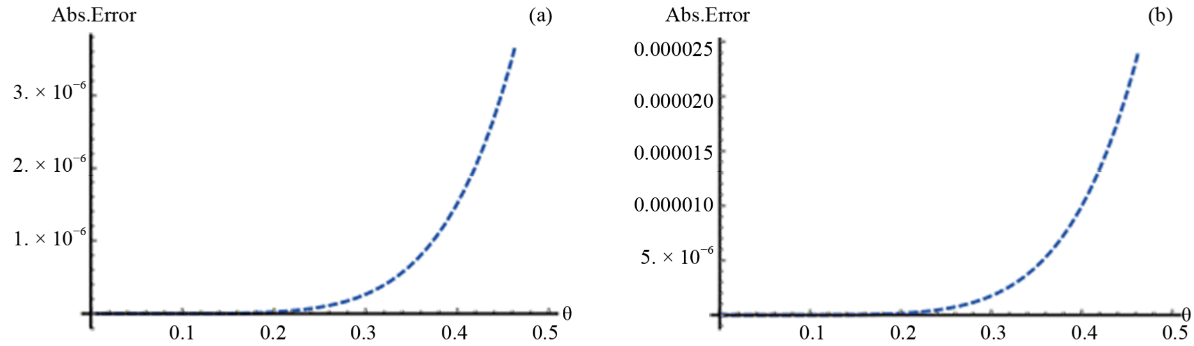

Figure 5. Two-dimensional Abs-E graphs in the range of $\vartheta \in [0,\ 0.5]$ between the fifth step App-S and Ex-S for $\zeta = 1.0$ when $\Theta = 0.5$ for Problems 3 and when $\Theta = 0.5$ with $\delta = 3.0$ and $\beta = 2.0$ for Problem 4

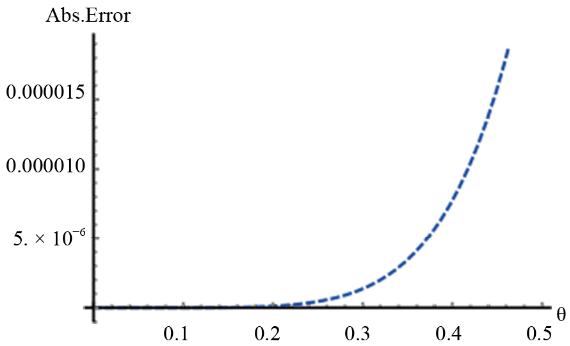

Figure 6. The 2D diagrams of Abs-E in the range of $\vartheta \in [0,\ 0.5]$ between the fifth step App-S and E x-S for $\zeta = 1.0$ when $\Theta = 0.25$ at $\delta = 5.0$ for Problems 5

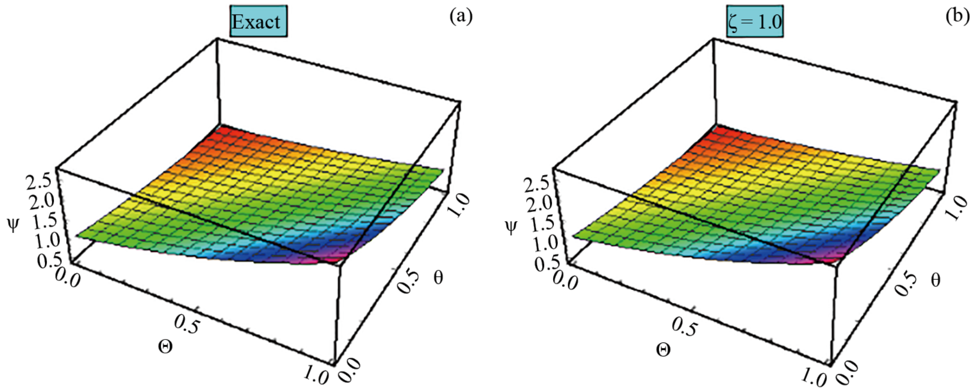

Figure 7. The 3D diagrams of the App-S and Ex-S of $\Psi(\Theta,\ \vartheta)$ in the range $\vartheta \in [0,\ 1.0]$ and $\Theta \in [0,\ 1.0]$ for Problem 1

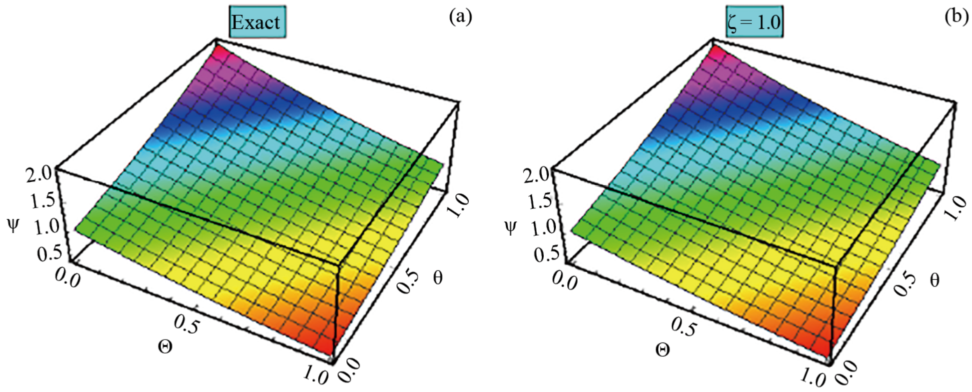

Figure 8. The 3D diagrams of the App-S and Ex-S of $\Psi(\Theta,\ \vartheta)$ in the range $\vartheta \in [0,\ 1.0]$ and $\Theta \in [0,\ 1.0]$ for Problem 2 when $\beta = 0.5$

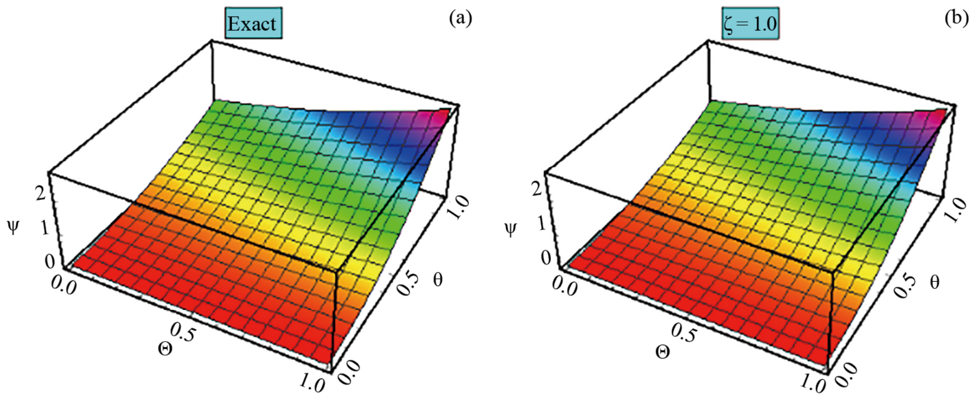

Figure 9. The 3D diagrams of the App-S and Ex-S of $\Psi(\Theta,\ \vartheta)$ in the range $\vartheta \in [0,\ 1.0]$ and $\Theta \in [0,\ 1.0]$ for Problem 3

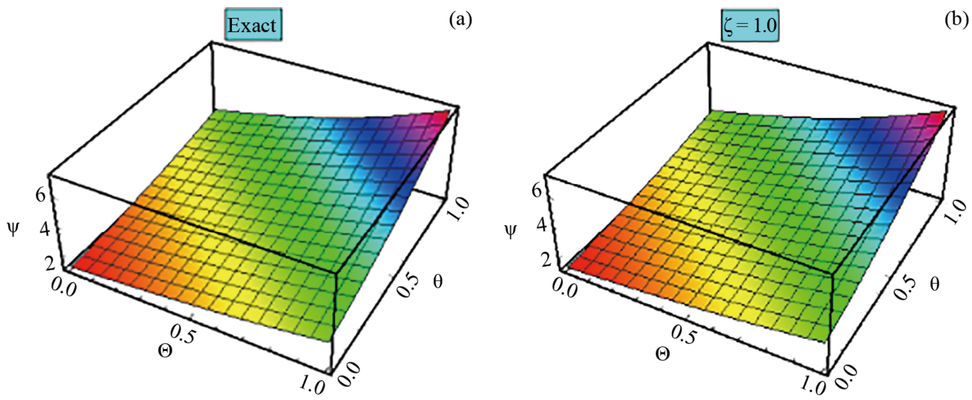

Figure 10. The 3D diagrams of the App-S and Ex-S of $\Psi(\Theta,\ \vartheta)$ in the ranges $\vartheta \in [0,\ 1.0]$ and $\Theta \in [0,\ 1.0]$ with $\delta = 3.0$ and $\beta = 2.0$ for Problem 4

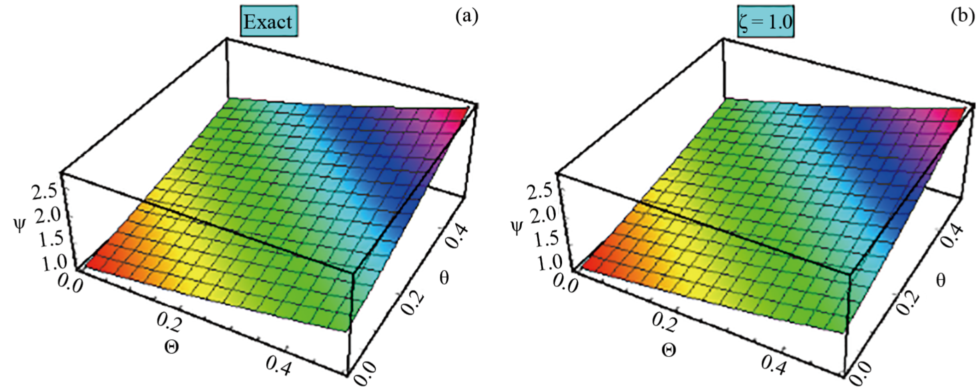

Figure 11. The 3D diagrams of the App-S and Ex-S of $\Psi(\Theta,\ \vartheta)$ in the ranges $\vartheta \in [0,\ 0.5]$ and $\Theta \in [0,\ 0.5]$ with $\delta = 3.0$ for Problem 5

Table 1. The Abs-E and Rel-E at varying values of ϑ when $\Theta = 1.0$ for Problem 1

ϑ	Ψ	$\Psi^{(5)}$	Rel. Error $= \frac{\lvert\Psi - \Psi(5)\rvert}{\lvert\Psi\rvert}$	Abs. Error $= \left\lvert\Psi - \Psi^{(5)}\right\rvert$
0.1	0.4065696598023920	0.40656965922226396	$1.274898753901053 \times 10^{-9}$	$5.1833515257726500 \times 10^{-10}$
0.2	0.44932896417792534	0.44932893045864280	$7.490854461965522 \times 10^{-8}$	$3.3658578757478350 \times 10^{-8}$
0.3	0.49658530385011157	0.49658491473094050	$7.834715728350320 \times 10^{-7}$	$3.8906046900821780 \times 10^{-7}$
0.4	0.5488116361493420	0.54880941742245950	0.00000404268317416354	0.000002218671567022490
0.5	0.60653065976384690	0.60652206824385460	0.0000146493732224820	0.000008591468778851308
0.6	0.67032004608091900	0.67029400002754420	0.0000385607815120771	0.000026046008095081950
0.7	0.74081822071924920	0.74075152752212270	0.0000902634888462165	0.000066693159595176970
0.8	0.81873075310563430	0.81857982618677230	0.00018434252120401530	0.000150926891209657300
0.9	0.90483741805123990	0.90452661098567340	0.00034349491310917050	0.000310807050286188160
1.0	1.00000000000000000	0.99940581535119140	0.00059418464880856000	0.000594184648808560000

Table 2. The Abs-E and Rel-E at varying values of ϑ when $\Theta = 1.0$ at $\beta = 0.5$ for Problem 2

ϑ	Ψ	$\Psi^{(5)}$	Rel. Error $= \frac{\lvert\Psi - \Psi(5)\rvert}{\lvert\Psi\rvert}$	Abs. Error $= \left\lvert\Psi - \Psi^{(5)}\right\rvert$
0.1	1.8660659827685684	1.8660659830736148	$1.634703356286456 \times 10^{-10}$	$3.050464325582425 \times 10^{-10}$
0.2	1.741101107259536	1.7411011265922482	$1.110372741975478 \times 10^{-8}$	$1.933271231990829 \times 10^{-8}$
0.3	1.624504574631055	1.624504792712471	$1.342448585040598 \times 10^{-7}$	$2.180814160368527 \times 10^{-7}$
0.4	1.5157153529530853	1.5157165665103982	$8.006492372491392 \times 10^{-7}$	0.000001213557312862434
0.5	1.414208977153438	1.4142135623730951	0.000003242239912745257	0.000004585219657071704
0.6	1.3194943489638191	1.3195079107728942	0.000010277929343474892	0.000013561809075079978
0.7	1.1884431012121797	1.2311444133449163	0.034684243107370935000	0.042701312132736646000
0.8	1.148623576037244	1.148698354997035	0.000065098865568772420	0.000074778959791022000
0.9	1.0716232682975848	1.0717734625362931	0.000140136179853620200	0.000150194238708323270
1.0	0.9997199820546389	1.000000000000000	0.000280017945361144000	0.000280017945361144000

Table 3. The Abs-E and Rel-E at varying values of ϑ when $\Theta = 1.0$ for Problem 3

ϑ	Ψ	$\Psi^{(5)}$	Rel. Error $= \frac{\lvert\Psi - \Psi(5)\rvert}{\lvert\Psi\rvert}$	Abs. Error $= \left\lvert\Psi - \Psi^{(5)}\right\rvert$
0.1	0.27629272951891193	0.27629272916666664	$1.274898869421280 \times 10^{-9}$	$3.522452884929805 \times 10^{-10}$
0.2	0.30535068954004246	0.30535066666666666	$7.490854479152386 \times 10^{-8}$	$2.287337580453297 \times 10^{-8}$
0.3	0.33746470189400080	0.33746443750000000	$7.834715729382329 \times 10^{-7}$	$2.643940008040246 \times 10^{-7}$
0.4	0.37295617441031760	0.37295466666666666	0.000004042683174006208	0.000001507743650930315
0.5	0.41218031767503205	0.41217447916666666	0.000014164937322380167	0.000005838508365385575
0.6	0.45552970009762720	0.45551200000000003	0.000038856078151216220	0.000017700097627193490
0.7	0.50343817686761920	0.50339285416666660	0.000090026348884752570	0.000045322700952588060
0.8	0.55638523212311700	0.55628266666666670	0.000184342521204030960	0.000102565456450265380
0.9	0.61490077778923750	0.61468956250000000	0.000343494913109226050	0.000211215289237509650
1.0	0.67916666672122200	0.67916666666666670	0.000393494913109226050	0.000291666666666667000

Table 4. The Abs-E and Rel-E at varying values of ϑ and Θ with $\delta = 3.0$ and $\beta = 2.0$ for Problem 4

$(\Theta,\ \vartheta)$	$\Psi^{(5)}$	Ψ	Rel. Error $= \frac{\left\lvert\Psi - \Psi\Psi^{(5)}\right\rvert}{\lvert\Psi\rvert}$	Abs. Error $= \left\lvert\Psi - \Psi^{(5)}\right\rvert$
(0.03, 0.03)	1.06183654654432	1.06183654654535	$9.865985960312 \times 10^{-13}$	$1.0476064460362 \times 10^{-12}$
(0.13, 0.13)	1.29693007888701	1.29693008666577	$5.9977915677191 \times 10^{-9}$	$7.7787163377252 \times 10^{-9}$
(0.23, 0.23)	1.58407371746601	1.58407398499448	$1.6888630985918 \times 10^{-7}$	$2.6752840986965 \times 10^{-7}$
(0.33, 0.33)	1.93478971675811	1.93479233440203	0.000001352932710503	0.0000026176438372431
(0.43, 0.43)	2.36314632122551	2.36316069370579	0.000006081888669505	0.0000143724802472712
(0.53, 0.53)	2.88631445113781	2.88637098926795	0.000019587963696605	0.0000565381301527101
(0.63, 0.63)	3.52524252279690	3.52542148736538	0.000050764020450205	0.0001789645684802111
(0.73, 0.73)	4.30547337653070	4.30595952834520	0.000112902086352814	0.0004861518145009562
(0.83, 0.83)	5.25813191201622	5.25931084444689	0.000224161009976689	0.0011789324306725742
(0.93, 0.93)	6.42111750626145	6.42373677142913	0.000407747898283598	0.0026192651676772981

Table 5. The Abs-E and Rel-E at varying values of ϑ and Θ with $\delta = 3.0$ for Problem 5

$(\Theta,\ \vartheta)$	$\Psi^{(5)}$	Ψ	Rel. Error $= \frac{\left\lvert\Psi - \Psi\Psi^{(5)}\right\rvert}{\lvert\Psi\rvert}$	Abs. Error $= \left\lvert\Psi - \Psi^{(5)}\right\rvert$
(0.05, 0.05)	1.1051709153149	1.1051709180756	$2.497951209498 \times 10^{-9}$	$2.7606630315091 \times 10^{-9}$
(0.15, 0.15)	1.3498580535708	1.3498588075760	$5.585807849316 \times 10^{-7}$	$7.5400519228274 \times 10^{-7}$
(0.25, 0.25)	1.6487103698322	1.6487212707001	0.00000661171056119	0.000010900867837948
(0.35, 0.35)	2.0136867849099	2.0137527074704	0.00003273617475320	0.000065922560541498
(0.45, 0.45)	2.4593426274569	2.4596031111569	0.00010590476928346	0.000260483700015967
(0.55, 0.55)	3.0033668369534	3.0041660239464	0.00026602624043272	0.000799186992986200
(0.65, 0.65)	3.6672236017846	3.6692966676192	0.00056497634897884	0.002073065834591769
(0.75, 0.75)	4.4769175156003	4.4816890703380	0.00106467777278576	0.004771554737725836
(0.85, 0.85)	5.4639046334569	5.4739473917272	0.00183464647201958	0.010042758270253138
(0.95, 0.95)	6.6661756508830	6.6858944422792	0.00294931240186632	0.019718791396183377

Table 6. The Res-E for Problem 1 at various values of ζ with specific values of ϑ in the interval of $[0, 0.5]$, when $\beta = 1.0$

ϑ	$\zeta = 0.6$	$\zeta = 0.7$	$\zeta = 0.8$	$\zeta = 0.9$	$\zeta = 1.0$
0.1	0.0862136	0.0400950	0.0174227	0.0058774	3.0656620×10^{-8}
0.2	0.1555370	0.0758171	0.0346282	0.0123075	9.8101184×10^{-7}
0.3	0.2284715	0.1146656	0.0539696	0.0197834	7.4495586×10^{-6}
0.4	0.3071385	0.1577112	0.0759680	0.0285120	3.1392371×10^{-5}
0.5	0.3924850	0.2055702	0.1010150	0.0386948	9.5801931×10^{-5}

Table 7. The Res-E for Problem 2 at various values of ζ with specific values of ϑ in the interval of $[0, 0.5]$, when $\Theta = 1.0$ and $\beta = 0.5$

ϑ	$\zeta = 0.6$	$\zeta = 0.7$	$\zeta = 0.8$	$\zeta = 0.9$	$\zeta = 1.0$
0.1	0.1840312	0.1001817	0.0485237	0.0176771	3.2303145×10^{-6}
0.2	0.2471696	0.1463037	0.0769433	0.0303745	5.1258358×10^{-5}
0.3	0.2854440	0.1770869	0.0975785	0.0402774	2.5733540×10^{-4}
0.4	0.3105766	0.1986929	0.1128573	0.0478413	8.0648017×10^{-4}
0.5	0.3273033	0.2138209	0.1238968	0.0532125	1.9522787×10^{-4}

Table 8. The Res-E for Problem 3 at various values of ζ with specific values of ϑ in the interval of $[0, 0.5]$, when $\Theta = 1.0$

ϑ	$\zeta = 0.6$	$\zeta = 0.7$	$\zeta = 0.8$	$\zeta = 0.9$	$\zeta = 1.0$
0.1	0.2343530	0.1089895	0.0473598	0.0159764	8.3333331×10^{-8}
0.2	0.4227935	0.2060929	0.0941293	0.0334553	2.6666666×10^{-8}
0.3	0.6210501	0.3116934	0.1467045	0.0537771	0.000020249991
0.4	0.8348892	0.4287034	0.2065025	0.0775037	0.000085333331
0.5	0.9934889	0.5587979	0.2745873	0.1051835	0.000260416661

Table 9. The Res-E for Problem 4 at various values of ζ with specific values of ϑ in the interval of $[0, 0.5]$, when $\Theta = 1.0,\ \delta = 3.0$ and $\beta = 2.0$

ϑ	$\zeta = 0.6$	$\zeta = 0.7$	$\zeta = 0.8$	$\zeta = 0.9$	$\zeta = 1.0$
0.1	0.3103751	0.2962643	0.1287374	0.0434284	2.2652348×10^{-7}
0.2	0.5727935	0.5602172	0.2558702	0.0909410	7.2487515×10^{-6}
0.3	0.6210501	0.3116934	0.3987844	0.1461813	5.5045207×10^{-5}
0.4	0.6348892	0.4287034	0.5613320	0.2106770	2.3196004×10^{-4}
0.5	0.6934889	0.5587979	0.7464059	0.2859184	7.0788589×10^{-4}

Table 10. The Res-E for Problem 5 at various values of ζ with specific values of ϑ in the interval of $[0, 0.5]$ when $\Theta = 1.0$, $\delta = 5.0$

0.1	0.3103751	0.2962643	0.1287374	0.0434284	2.2652348×10^{-7}
0.2	0.5727935	0.5602172	0.2558702	0.0909410	7.2487515×10^{-6}
0.3	0.6210501	0.3116934	0.3987844	0.1461813	5.5045207×10^{-5}
0.4	0.6348892	0.4287034	0.5613320	0.2106770	2.3196004×10^{-4}
0.5	0.6934889	0.5587979	0.7464059	0.2859184	7.0788589×10^{-4}

5. Conclusion

To obtain both App-S and Ex-S for the FrGDE, FrFPE, and FrSHE in the sense of Con-FrD, a new coupling algorithm has been developed in this study. An analysis in the form of error functions, both graphical and numerical, has been done to illustrate the correctness of this approach. Results in graphs and tables demonstrated the validity of the proposed approach.

The CNHPM distinguishes itself from other approximate analytical methods with the following features: The advantage of this method is that it does not call for any presumptions regarding significant or minor physical factors that are unique to the problem at hand. Because of this, it circumvents some of the drawbacks of conventional perturbation techniques and may be applied to both weakly and severely nonlinear problems; therefore, the computations needed to solve nonlinear FrODEs are minimal. In contrast to earlier analytic approximation methods, the CNHPM may generate expansion solutions for FrODEs without the need for perturbation, linearization, or discretization. In light of the outcomes, we concluded that CNHPM is straightforward to use, precise, and effective.

We intend to use the CNHPM in the future to solve various nonlinear fractional models that arise in engineering and biological systems.

Conflict of interest

The authors declare there is no conflict of interest at any point with reference to research findings

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