Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

The SOR and AOR Methods with Stepwise Optimized Values of Parameters for the Iterative Solutions of Linear Systems

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Received: 3 July 2024; **Revised:** 3 September 2024; **Accepted:** 12 September 2024

Abstract: We modify the successive overrelaxation (SOR) method and accelerated overrelaxation (AOR) method for solving linear equations systems. The optimal value of the acceleration parameter is determined, using the maximal reduction method of the residual vector's length, or equivalently an orthogonality condition. Rather than the constant value, the MAOR method endows a step-by-step varying acceleration parameter to possess the property of absolute convergence and the orthogonality of consecutive residual vector. In SOR, the relaxation parameter is also optimized by using the orthogonality condition. Numerical examples ensure that the MSOR and MAOR iterative schemes converge faster than the original SOR and AOR iterative schemes. They are easily implemented with low computational cost, and without needing of a detailed spectral analysis to determine the optimal values of parameters has a great advantage.

Keywords: linear equations system, accelerated overrelaxation (AOR) method, modification of AOR (MAOR), step-by-step varying parameters, convergence

MSC: 65F10, 15A06

1. Introduction

The splitting iterative schemes involve the Jacobi method, the Gauss-Seidel method, the successive overrelaxation (SOR) method, and the accelerated overrelaxation (AOR) method as special cases.

In the paper, we improve the SOR and AOR methods for solving

$$\mathbf{A}\mathbf{x}=\mathbf{b},\ \mathbf{b},\ \mathbf{x}\in\mathbb{R}^n,\ \ \mathbf{A}\in\mathbb{R}^{n\times n}. \tag{1}$$

We suppose that the rank of $\mathbf{A}$ is full, and hence is invertible. The most popular splitting of $\mathbf{A}$ is

$$\mathbf{A}=\mathbf{D}-\mathbf{U}-\mathbf{L}, \tag{2}$$

DOI: https://doi.org/10.37256/cm.5320245246

where $\mathbf{D}$ is a nonsingular diagonal matrix, and $\mathbf{U}$ and $\mathbf{L}$ are strictly upper and lower matrices.

The successive overrelaxation (SOR) method developed in 1950 by Young and Frankel was modified from the Gauss-Seidel method and inherited the advantages of the Gaussian elimination method and the iterative method. The SOR method can be written as [1, 2]

$$(\mathbf{D} - w\mathbf{L})\mathbf{x}_{k+1} = w\mathbf{b} + [(1-w)\mathbf{D} + w\mathbf{U}]\mathbf{x}_k, \tag{3}$$

where w is known as the relaxation parameter.

The progress of SOR up to 2003 can be seen in [3]. Owing to its efficiency and simplicity in numerical implementation as shown in equation (3), the SOR method is becoming an important solver for equation (1) rapidly.

Among the many splitting methods which have more parameters engaged in the iteration formula, the AOR method is a two-parameter generalization of SOR [4]:

$$(\mathbf{D} - w\mathbf{L})\mathbf{x}_{k+1} = \eta\mathbf{b} + [(1-\eta)\mathbf{D} + \eta\mathbf{U} + (\eta - w)\mathbf{L}]\mathbf{x}_k, \tag{4}$$

where η is an acceleration parameter. The SOR iterative method is recovered from equation (4) by taking $\eta = w$. Given an initial guess $\mathbf{x}_0$ of $\mathbf{x}$ at the beginning, equation (4) can generate a sequence of iterative solution $\mathbf{x}_k$ of equation (1) at the kth step.

The generalizations of the AOR method can be referred to [5, 6]. The analysis of the convergence behavior for the AOR type methods have been derived in [7, 8]; the preconditioned AOR methods can be considered to improve the convergence speed [9].

More analysis of SOR, SSOR, AOR and SAOR methods can be seen in [10–12]. The different extensions of SOR and AOR for different kinds of linear systems were reported in [13–18].

The SOR and AOR methods have been reformulated in the frame of descent and residual vectors [19]. The re-accelerated version of AOR was discussed in [20]. Some generalizations to the three-parameter methods of AOR were made in [21].

In addition to the linear problem with a real coefficient matrix, the complex symmetric linear systems are also appeared in many applications [22, 23]. The generalizations of splitting iterative methods for different linear problems are given in [24–27].

The conventional spectral analysis is an efficient way to pick up the optimal values of parameters. However, the theoretical optimal values can be obtained only for certain linear systems whose coefficient matrices have special structures like as symmetry and positiveness.

In the iterative scheme based on the Krylov subspace method, it is very often to employ the principle of orthogonality to seek the optimal descent vector. In the fields of splitting iterative schemes the orthogonality principle is rarely considered as a main tool to seek the optimal descent vector.

Our main contribution can unify the splitting iterative schemes including SOR and AOR from the frame of residual vector and descent vector. Hence, in that space the orthogonality principle replacing the conventional spectral analysis method is used to seek the optimal descent vector, and the optimal values of w and η used in the SOR and AOR methods. In general, the optimal values can be obtained easily for linear systems whose coefficient matrices are non-singular.

In the paper we sketch some new ideas to solve the linear system by using the AOR iterative method, which is accompanied by the maximization technique and the orthogonality of residual vector to determine w and η.

2. New form of AOR and optimal value of η

2.1 *A new form of AOR*

Upon defining a descent vector by

$$\mathbf{u} = \mathbf{x} - \mathbf{x}_k, \tag{5}$$

equation (1) is equivalent to

$$\mathbf{A}\mathbf{u} = \mathbf{r}_k, \tag{6}$$

where

$$\mathbf{r}_k = \mathbf{b} - \mathbf{A}\mathbf{x}_k \tag{7}$$

is the kth step residual vector.

Let $\mathbf{u}_k$ be the kth step descent vector, which is defined as $\mathbf{u}_k = \mathbf{x}_{k+1} - \mathbf{x}_k$. One goal of the iterative method is designing a good manner for computing the descent vector $\mathbf{u}_k$ at each iteration step, such that $\mathbf{x}_{k+1}$ is better than $\mathbf{x}_k$ to close the real solution $\mathbf{x}$ of equation (1).

Because SOR is a special case of AOR, the following analysis is also applicable to the SOR method.

Theorem 1 For the AOR method in equation (4), an equivalent iterative form is

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{u}_k,\ (\mathbf{D} - w\mathbf{L})\mathbf{u}_k = \eta\mathbf{r}_k, \tag{8}$$

which is expressed in terms of the kth step descent vector $\mathbf{u}_k$ and residual vector $\mathbf{r}_k$.

Proof. Subtracting both sides of equation (4) by $(\mathbf{D} - w\mathbf{L})\mathbf{x}_k$ yields

$$\begin{aligned}(\mathbf{D} - w\mathbf{L})\mathbf{x}_{k+1} - (\mathbf{D} - w\mathbf{L})\mathbf{x}_k &= \eta\mathbf{b} + [(1-\eta)\mathbf{D} + \eta\mathbf{U} + (\eta - w)\mathbf{L}]\mathbf{x}_k - (\mathbf{D} - w\mathbf{L})\mathbf{x}_k \\ &= \eta\mathbf{b} + [(1-\eta)\mathbf{D} + \eta\mathbf{U} + (\eta - w)\mathbf{L} - \mathbf{D} + w\mathbf{L}]\mathbf{x}_k,\end{aligned} \tag{9}$$

which can be simplified as

$$(\mathbf{D} - w\mathbf{L})\mathbf{x}_{k+1} - (\mathbf{D} - w\mathbf{L})\mathbf{x}_k = \eta\mathbf{b} - \eta(\mathbf{D} - \mathbf{U} - \mathbf{L})\mathbf{x}_k. \tag{10}$$

It follows from equations (2), (7) and (10) that

$$(\mathbf{D} - w\mathbf{L})\mathbf{x}_{k+1} - (\mathbf{D} - w\mathbf{L})\mathbf{x}_k = \eta(\mathbf{b} - \mathbf{A}\mathbf{x}_k) = \eta\mathbf{r}_k, \tag{11}$$

after inserting $\mathbf{x}_{k+1}-\mathbf{x}_k=\mathbf{u}_k$ which becomes

$$(\mathbf{D}-w\mathbf{L})\mathbf{u}_k=\eta\mathbf{r}_k. \tag{12}$$

The proof of equation (8) is complete, which has a neater form than equation (4). □

2.2 *Determining η in AOR*

Theorem 2 For the AOR method in equation (8), the optimal value of the accelerating parameter η is given by

$$\eta=\frac{\mathbf{r}_k^{\mathrm{T}}\mathbf{A}\mathbf{v}_k}{\|\mathbf{A}\mathbf{v}_k\|^2}, \tag{13}$$

where

$$(\mathbf{D}-w\mathbf{L})\mathbf{v}_k=\mathbf{r}_k. \tag{14}$$

Then equation (8) is modified to the following iterative method:

$$\mathbf{x}_{k+1}=\mathbf{x}_k+\eta_k\mathbf{u}_k,\ \ (\mathbf{D}-w\mathbf{L})\mathbf{u}_k=\mathbf{r}_k, \tag{15}$$

where

$$\eta_k=\frac{\mathbf{r}_k^{\mathrm{T}}\mathbf{A}\mathbf{u}_k}{\|\mathbf{A}\mathbf{u}_k\|^2}. \tag{16}$$

Proof. Upon letting

$$\mathbf{v}_k=\frac{\mathbf{u}_k}{\eta}, \tag{17}$$

equation (14) is the same to the second one in equation (8); meanwhile the first one in equation (8) changes to

$$\mathbf{x}_{k+1}=\mathbf{x}_k+\eta\mathbf{v}_k. \tag{18}$$

Equations (18) and (14) constitute an alternative form of AOR. Our objective is to derive the optimal value of η, rather than a constant value of η in AOR.

Multiplying equation (18) by $\mathbf{A}$ and using equation (7) yields

$$\mathbf{r}_{k+1}=\mathbf{r}_k-\eta\mathbf{A}\mathbf{v}_k, \tag{19}$$

whose squared norm is given by

$$\|\mathbf{r}_{k+1}\|^2 = \|\mathbf{r}_k\|^2 - 2\eta \mathbf{r}_k^{\mathrm{T}} \mathbf{A}\mathbf{v}_k + \eta^2 \|\mathbf{A}\mathbf{v}_k\|^2. \tag{20}$$

Let

$$f = 2\eta \mathbf{r}_k^{\mathrm{T}} \mathbf{A}\mathbf{v}_k - \eta^2 \|\mathbf{A}\mathbf{v}_k\|^2 \tag{21}$$

be a merit function to measure the reduction quantity of residual vector's length; hence, we have

$$\|\mathbf{r}_{k+1}\|^2 = \|\mathbf{r}_k\|^2 - f. \tag{22}$$

To determine η, we encounter a maximization problem depicted by

$$\max_{\eta} \{f = 2\eta \mathbf{r}_k^{\mathrm{T}} \mathbf{A}\mathbf{v}_k - \eta^2 \|\mathbf{A}\mathbf{v}_k\|^2\}. \tag{23}$$

By using the maximality condition of equation (23), i.e., $df/d\eta = 0$, we can derive equation (13). Since f is a concave function of η, for any $\mathbf{v}_k \neq \mathbf{0}$, we have $d^2 f/d\eta^2 = -2\|\mathbf{A}\mathbf{v}_k\|^2 < 0$. Consequently, at η_k given in equation (13), f obtains its maximal value owing to $df/d\eta = 0$ and $d^2 f/d\eta^2 = -2\|\mathbf{A}\mathbf{v}_k\|^2 < 0$, and η_k is the unique maximal point.

Renaming $\mathbf{v}_k$ in equations (18), (14) and (13) to $\mathbf{u}_k$, we can derive equations (15) and (16); the MAOR method is a modification of the AOR method in equation (8). □

In MAOR, $\mathbf{v}_k$ is solved from equation (14) by a forward substitution method, which is not a zero vector, because $\mathbf{r}_k$ is not a zero vector on the right-hand side. The range space of $\mathbf{A}$ is not zero owing to the full rank assumption of the linear system. Hence, we can guarantee that $\|\mathbf{A}\mathbf{v}_k\| > 0$ and η_k in equation (13) is well-defined during the iteration process.

Equations (15) and (16) constitute a modification of the accelerated overrelaxation method, namely the MAOR method; it is different from the AOR in equation (4), where the parameter η is in general a given constant value, not that determined by equation (16). Usually, if the value of η is not properly given, AOR may be divergent very fast. To determine the theoretical value of η, the spectral analysis of the iteration matrix is necessary [4, 5]. However, the spectral analysis is not an easy task for the general linear system.

2.3 *Equivalent class*

Theorem 3 For any $\beta \neq 0 \in \mathbb{R}$, the following iterative method:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \frac{\mathbf{r}_k^{\mathrm{T}} \mathbf{A}\mathbf{v}_k}{\|\mathbf{A}\mathbf{v}_k\|^2} \mathbf{v}_k, \quad (\mathbf{D} - w\mathbf{L})\mathbf{v}_k = \beta \mathbf{r}_k \tag{24}$$

is equivalent to that in equations (15) and (16).

Proof. Let

$$\mathbf{v}_k = \beta \mathbf{u}_k;$$

$(\mathbf{D}-w\mathbf{L})\mathbf{v}_k = \beta\mathbf{r}_k$ in equation (24) changes to the second one $(\mathbf{D}-w\mathbf{L})\mathbf{u}_k = \mathbf{r}_k$ in equation (15). Inserting $\mathbf{v}_k = \beta\mathbf{u}_k$ into the first one in equation (24) yields

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \frac{\beta\mathbf{r}_k^{\mathrm{T}}\mathbf{A}\mathbf{u}_k}{\beta^2\|\mathbf{A}\mathbf{u}_k\|^2}\beta\mathbf{u}_k = \mathbf{x}_k + \frac{\mathbf{r}_k^{\mathrm{T}}\mathbf{A}\mathbf{u}_k}{\|\mathbf{A}\mathbf{u}_k\|^2}\mathbf{u}_k.$$

Hence, equation (24) is returned to the iterative method in equations (15) and (16). □

Theorem 3 indicates that SOR ($\beta = 1$) and the reaccelerated over relaxation (ROR) method ($\beta = 1 - w$) derived in [20] can be incorporated into the iterative method in equations (15) and (16). In the equivalent class, the iterative schemes have the same form.

If η_k is replaced by w_k as that SOR is recovered from AOR by setting $\eta = w$, then we can compute the value of relaxation parameter at each step by

$$w_k = \frac{\mathbf{r}_k^{\mathrm{T}}\mathbf{A}\mathbf{u}_k}{\|\mathbf{A}\mathbf{u}_k\|^2}. \tag{25}$$

This will be named an MSOR method as a modification of successive overrelaxation (SOR) method, which replaces the constant value of w in SOR by a step-by-step varying parameter w_k in equation (25).

3. Convergence and orthogonality

Theorem 4 The MAOR method in equation (15) is absolute convergence, i.e.,

$$\|\mathbf{r}_{k+1}\| < \|\mathbf{r}_k\|, \ \forall\, k = 0,\ 1,\ \ldots. \tag{26}$$

Proof. Multiplying the first one in equation (15) by $\mathbf{A}$ and using equation (7), we have

$$\mathbf{r}_{k+1} = \mathbf{r}_k - \eta\mathbf{A}\mathbf{u}_k; \tag{27}$$

the squared norm is

$$\|\mathbf{r}_{k+1}\|^2 = \|\mathbf{r}_k\|^2 - 2\eta\mathbf{r}_k^{\mathrm{T}}\mathbf{A}\mathbf{u}_k + \eta^2\|\mathbf{A}\mathbf{u}_k\|^2. \tag{28}$$

Inserting equation (16) for the optimal value of η into equation (28) yields

$$\|\mathbf{r}_{k+1}\|^2 = \|\mathbf{r}_k\|^2 - \frac{(\mathbf{r}_k^{\mathrm{T}}\mathbf{A}\mathbf{u}_k)^2}{\|\mathbf{A}\mathbf{u}_k\|^2} < \|\mathbf{r}_k\|^2, \tag{29}$$

because of $(\mathbf{r}_k^{\mathrm{T}}\mathbf{A}\mathbf{u}_k)^2/\|\mathbf{A}\mathbf{u}_k\|^2 > 0$. Equation (26) is proven. □

Theorem 5 For MAOR in equations (15) and (16), the consecutive residual vector is perpendicular to $\mathbf{A}\mathbf{u}_k$, i.e.,

$$\mathbf{r}_{k+1}^{\mathrm{T}}\mathbf{A}\mathbf{u}_k = 0. \tag{30}$$

Proof. It follows from equations (15) and (16) that

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \frac{\mathbf{r}_k^{\mathrm{T}}\mathbf{A}\mathbf{u}_k}{\|\mathbf{A}\mathbf{u}_k\|^2}\mathbf{u}_k. \tag{31}$$

Multiplying equation (15) by $\mathbf{A}$ and using equation (16) yields

$$\mathbf{r}_{k+1} = \mathbf{r}_k - \frac{\mathbf{r}_k^{\mathrm{T}}\mathbf{A}\mathbf{u}_k}{\|\mathbf{A}\mathbf{u}_k\|^2}\mathbf{A}\mathbf{u}_k. \tag{32}$$

The inner product to $\mathbf{A}\mathbf{u}_k$ is

$$\mathbf{r}_{k+1}^{\mathrm{T}}\mathbf{A}\mathbf{u}_k = \mathbf{r}_k^{\mathrm{T}}\mathbf{A}\mathbf{u}_k - \frac{\mathbf{r}_k^{\mathrm{T}}\mathbf{A}\mathbf{u}_k}{\|\mathbf{A}\mathbf{u}_k\|^2}\|\mathbf{A}\mathbf{u}_k\|^2 = 0. \tag{33}$$

The orthogonality of consecutive residual vector is a very important property for the iterative method, which guarantees the stepwise convergence of the MAOR method.

Theorem 5 indicates that the optimal value of η can also be determined by the orthogonality condition.

The orthogonality condition (30) after multiplying by η implies

$$\mathbf{r}_{k+1}^{\mathrm{T}}(\eta\mathbf{A}\mathbf{u}_k) = 0. \tag{34}$$

In view of equation (27), we have

$$\mathbf{r}_k = \mathbf{r}_{k+1} + \eta\mathbf{A}\mathbf{u}_k. \tag{35}$$

Therefore $\mathbf{r}_k$, $\mathbf{r}_{k+1}$ and $\eta\mathbf{A}\mathbf{u}_k$ constitute the three sides of a perpendicular triangle. According to the Pythagorean theorem, we have

$$\|\mathbf{r}_k\|^2 = \|\mathbf{r}_{k+1}\|^2 + \|\eta\mathbf{A}\mathbf{u}_k\|^2; \tag{36}$$

it indicates that $\|\mathbf{r}_{k+1}\|^2 < \|\mathbf{r}_k\|^2$ during the iteration processes. In the context of iteration method, the orthogonality condition guarantees that the residual is strictly decreased step-by-step, which means that the iteration method is absolutely convergent.

The nonstationary Richardson method, known as the semi-iterative method, is described as follows [28]:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{P}^{-1}\mathbf{r}_k,\ k \geq 0. \tag{37}$$

Comparing to equation (15), the MAOR is a special case of the nonstationary Richardson method if we take $\alpha_k = \eta_k$ and $\mathbf{P} = \mathbf{D} - \omega\mathbf{L}$. In this paper we further enhanced the convergence speed by specifying η_k with equation (16).

4. Results and discussions

4.1 *Algorithms*

According to Theorem 2 we have the following iterative algorithms. The factor α is a parameter to accelerate the speed of convergence.

Algorithms MAOR and MSOR

1: Given $\mathbf{x}_0$, $\alpha \geq 1$, and ε

2: Do $k = 0,\ 1,\ \ldots$, until $\|\mathbf{r}_k\| < \varepsilon$

3: $\mathbf{r}_k = \mathbf{b} - \mathbf{A}\mathbf{x}_k$

4: Solve $(\mathbf{D} - w\mathbf{L})\mathbf{u}_k = \mathbf{r}_k$ (MAOR), $(\mathbf{D} - w_k\mathbf{L})\mathbf{u}_k = \mathbf{r}_k$ (MSOR)

5: $\eta_k = \dfrac{\mathbf{r}_k^{\mathrm{T}}\mathbf{A}\mathbf{u}_k}{\|\mathbf{A}\mathbf{u}_k\|^2}$ (MAOR), $w_k = \dfrac{\mathbf{r}_k^{\mathrm{T}}\mathbf{A}\mathbf{u}_k}{\|\mathbf{A}\mathbf{u}_k\|^2}$ (MSOR)

6: $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha\eta_k\mathbf{u}_k$ (MAOR), $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha w_k\mathbf{u}_k$ (MSOR)

4.2 *Computational cost of MSOR and MAOR*

In MSOR, we first give an initial guess of w_k to compute $\mathbf{u}_k$ in step 4, and then w_k is corrected in step 5. In doing so, steps 4 and 5 are uncoupled, such that the inner iterations can be avoided. The computational cost of MSOR compared to SOR is slightly increased by needing one matrix-vector product and two inner products of two vectors to compute w_k.

Compared to AOR, of which the value η is computed one time, in MAOR η_k is computed at every iteration. The computational cost of MAOR than AOR is slightly increased by needing one matrix-vector product and two inner products of two vectors to compute η_k.

The steepest-descent method (SDM):

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \frac{\|\mathbf{r}_k\|^2}{\mathbf{r}_k^{\mathrm{T}}\mathbf{A}\mathbf{r}_k}\mathbf{r}_k \tag{38}$$

is in general applied to solve the linear system with $\mathbf{A}$ being a symmetric and positive definite matrix. Compared to SDM, the computational cost of MAOR, besides the computation of $\mathbf{u}_k$, is the same for needing one matrix-vector product and two inner products of two vectors.

In step 6 of MAOR and MSOR there appears a factor α to accelerate the convergence speed, which can be explained from the extrapolation technique as follows. For any splitting of $\mathbf{A}$ given by

$$\mathbf{A} = \mathbf{M} - \mathbf{N}, \tag{39}$$

with $\mathbf{M}$ nonsingular, an iterative scheme for equation (1) is

$$\mathbf{M}\mathbf{x}_{k+1} = \mathbf{N}\mathbf{x}_k + \mathbf{b}. \tag{40}$$

Let

$$\rho = \rho(\mathbf{G}) < 1 \tag{41}$$

be the spectral radius of $\mathbf{G}$ and is smaller than one for the convergence of equation (40), where $\mathbf{G} = \mathbf{M}^{-1}\mathbf{N}$ is the iteration matrix.

The splitting iterative scheme (40) can be recast to

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{u}_k, \ \ \mathbf{M}\mathbf{u}_k = \mathbf{r}_k. \tag{42}$$

We can prove the following result.

Theorem 6 Suppose that a new splitting iterative scheme for equation (1) is given by

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha\mathbf{u}_k, \ \ \mathbf{M}\mathbf{u}_k = \mathbf{r}_k, \tag{43}$$

which is modified from equation (42) by inserting an accelerated parameter α preceding $\mathbf{u}_k$. If the parameter α satisfies

$$1 < \alpha < \frac{1}{1-\rho(\mathbf{G})}, \tag{44}$$

then the spectral radius for equation (43), denoted by ρ_α satisfies

$$\rho_\alpha < \rho(\mathbf{G}). \tag{45}$$

Proof. Applying $\mathbf{M}$ to the first one in equation (43), and using equations (7) and (39), and the second one in equation (43), we can derive

$$\begin{aligned} \mathbf{M}\mathbf{x}_{k+1} &= \mathbf{M}\mathbf{x}_k + \alpha\mathbf{M}\mathbf{u}_k = \mathbf{M}\mathbf{x}_k + \alpha\mathbf{r}_k \\ &= \mathbf{M}\mathbf{x}_k + \alpha(\mathbf{b} - \mathbf{A}\mathbf{x}_k) = \mathbf{M}\mathbf{x}_k + \alpha\mathbf{b} - \alpha(\mathbf{M}-\mathbf{N})\mathbf{x}_k \\ &= (1-\alpha)\mathbf{M}\mathbf{x}_k + \alpha\mathbf{N}\mathbf{x}_k + \alpha\mathbf{b}. \end{aligned} \tag{46}$$

The corresponding iteration matrix is

$$\mathbf{G}_\alpha = (1-\alpha)\mathbf{I}_n + \alpha\mathbf{G}. \tag{47}$$

Consequently, we can derive

$$\rho_\alpha = 1 - \alpha + \alpha\rho(\mathbf{G}). \tag{48}$$

If $\alpha > 1$, then by $0 < \rho(\mathbf{G}) < 1$, we can verify that

$$(\alpha - 1)\rho(\mathbf{G}) < \alpha - 1 \Rightarrow \rho_\alpha = 1 - \alpha + \alpha\rho(\mathbf{G}) < \rho(\mathbf{G}).$$

To satisfy

$$\rho_\alpha = 1 - \alpha + \alpha\rho(\mathbf{G}) > 0,$$

we can derive $\alpha < 1/[1 - \rho(\mathbf{G})]$. □

According to Theorem 6, we can chose the value of α in the range $\alpha \geq 1$ by some trials. The value of w can also be obtained by some trials for MAOR.

As that done in [19], the system of nonlinear equations can be linearized by using the splitting-linearizing method, which then takes advantage of the MAOR for its simple formulation and with low computational cost, the optimal combination between the splitting-linearizing method and MAOR may be developed to effectively solve the nonlinear equations.

5. Testing examples

5.1 *Example 1*

In the computation by an iteration method giving a suitable convergence criterion is required. When we compare the computed result with other iterative method we take the same convergence criterion. In general we take a moderate value of ε, for example $\varepsilon = 10^{-15}$ to $\varepsilon = 10^{-4}$. For each example the value of ε is specified.

Consider an example of equation (1) with $\mathbf{A} = [a_{ij}]$, $i,\ j = 1,\ \ldots,\ n$; $a_{ij} = 1/(10j) - 1/20$, $i > j$, $a_{ij} = 1/(10(i - j)) - 1/20$, $i < j$, and $a_{ij} = 1$, $i = j$ [6]. Suppose that the exact solutions are $x_i = 1$, $i = 1,\ \ldots,\ n$, and the initial values are $x_i^0 = 0$, $i = 1,\ \ldots,\ n$. We fix $n = 20$.

In Table 1, the number of iterations (NS) obtained by AOR and MAOR under the convergence criterion $\varepsilon = 10^{-6}$, are compared to that obtained in [6] by using the algorithms of quasi-AOR (QAOR) and quasi-SOR (QSOR). In QAOR, $\mathbf{M} = [(1 + \eta)\mathbf{D} - w\mathbf{L}]/\eta$, and $\mathbf{N} = [\mathbf{D} + (\eta - w)\mathbf{L} + \eta\mathbf{U}]/\eta$. $w = 0.15$ and $\alpha = 1$ are used in MAOR; $w = 0.3$ and $\eta = 0.9$ are used in AOR and QAOR. In MSOR, we take $\alpha = 1.5$. MSOR and MAOR converge faster than other iterative methods.

Table 1. Comparing NS obtained by QAOR, QSOR, AOR, and MAOR

	QAOR	QSOR	AOR	MSOR	MAOR
NS	155	141	81	33	35

5.2 *Example 2*

In equation (1), we take [20]

$$\mathbf{A}=\begin{bmatrix} 4 & 0 & 0 & 0 & -1 & -1 \\ 0 & 4 & 0 & 0 & -1 & -1 \\ 0 & 0 & 4 & -1 & -1 & 0 \\ 0 & 0 & -1 & 4 & 0 & 0 \\ -1 & -1 & -1 & 0 & 4 & 0 \\ -1 & -1 & 0 & 0 & 0 & 4 \end{bmatrix}, \quad \mathbf{x}=\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{49}$$

The initial guess of $\mathbf{x}$ is zero. Under the convergence criterion $\varepsilon = 10^{-15}$, we take $w = 1.08743$ and $\eta = 1.3$ in AOR. Optimal value $\eta = 1.118736$ is obtained from equation (13) with $k = 0$ at the first step. NSs obtained by some methods are compared in Table 2. It is interesting that in MSOR we do not need to specify any value of parameter, and it is still competitive to MAOR.

Table 2. Comparing NS obtained by different methods

	AOR	AOR (equation (13) with $k = 0$)	MSOR	ROR	MAOR
NS	31	19	21	18	18

5.3 *Example 3*

In equation (1), we take a cyclic matrix of $\mathbf{A}$ with its first row being $(1, \ldots, 5)$. $\mathbf{x} = (1, \ldots, 1)$ is the exact solution. We find that the spectral radius of $\mathbf{I}_n - \eta(\mathbf{D} - w\mathbf{L})^{-1}\mathbf{A}$ is greater than one for the AOR method, which means that the AOR method for this problem is divergent.

The initial guess of $\mathbf{x}$ is zero. Under the convergence criterion $\varepsilon = 10^{-5}$, in Table 3, we demonstrate the usefulness of MSOR method for different values of α.

Table 3. Comparing NS obtained by MSOR for different values of α

α	1.4	1.5	1.6	1.7	1.8
NS	130	103	124	153	177

In Table 4, we demonstrate the usefulness of MAOR method for different values of w. The best value is $w = 0.7$ for MAOR.

Table 4. Comparing NS obtained by MAOR for different values of w; $\alpha = 1.6$

w	0	0.1	0.4	0.6	0.7	0.8	1
NS	141	133	91	95	77	79	156

5.4 *Example 4*

We apply MAOR and SDM to solve the following boundary value problem:

$$u''(x) = f(x),\ u(0) = 0,\ \ u(1) = 0. \tag{50}$$

The exact solution is supposed to be

$$u(x) = \sin \pi x. \tag{51}$$

The finite difference discretization of equation (50) is

$$\begin{aligned} &\frac{1}{(\Delta x)^2}(u_{i+1} - 2u_i + u_{i-1}) - f(x_i) = 0,\ i = 1,\ \ldots,\ n, \\ &u_0 = 0,\ \ u_{n+1} = 0, \end{aligned} \tag{52}$$

where $\Delta x = 1/(n+1)$ and $x_i = i\Delta x = i/(n+1),\ i = 1,\ \ldots,\ n$.

Table 5 compares NS obtained by the methods of MAOR and SDM for different values of n, where $\varepsilon = 10^{-4}$ and the initial values are $x_i^0 = 0$. Both MAOR and SDM are convergent very fast, but MAOR is slightly more accurate than SDM.

Table 5. Comparing (NS, ME) obtained by MAOR and SDM for different values of n for Example 4

n	200	300	400	450	500
MAOR	$(2, 1.11 \times 10^{-5})$	$(2, 6.22 \times 10^{-6})$	$(2, 4.38 \times 10^{-6})$	$(2, 3.79 \times 10^{-6})$	$(2, 3.39 \times 10^{-6})$
SDM	$(2, 2.04 \times 10^{-5})$	$(2, 9.08 \times 10^{-6})$	$(2, 5.11 \times 10^{-6})$	$(2, 4.04 \times 10^{-6})$	$(2, 3.28 \times 10^{-6})$

5.5 *Example 5*

Consider a complex Helmholtz equation:

$$-\Delta u(x,\ y) + \sigma u(x,\ y) = p(x,\ y),\ (x,\ y) \in \Omega, \tag{53}$$

where $\Omega := \{(x,\ y),\ 0 < x < 1,\ \ 0 < y < 1\}$; $\sigma = \sigma_1 + i\sigma_2$, with $i^2 = -1$ and $\sigma_2 \geq 0$, is a complex-valued wave number; $u(x,\ y) = w(x,\ y) + iv(x,\ y)$ is a complex function.

After a five-point finite difference discretization of equation (53), it becomes a complex linear system:

$$(\mathbf{K} + \sigma_1 \mathbf{I}_n + i\sigma_2 \mathbf{I}_n)(\mathbf{w} + i\mathbf{v}) = \mathbf{f} + i\mathbf{g}. \tag{54}$$

$\mathbf{K} = \mathbf{I}_{n_0} \otimes \mathbf{S} + \mathbf{S} \otimes \mathbf{I}_{n_0}$ is the centered difference matrix approximation of the negative Laplacian operator in equation (53), where $\otimes$ is the Kronecker tensor product and $n = n_0^2$; the meshzsize is $h = 1/(n_0 + 1)$, and $\mathbf{S} = \text{tridiag}(-1,\ 2,\ -1)/h^2$; $\mathbf{w}$ consists of nodal values of the variable $w(x,\ y)$, and is a vectorization of $w(x_i,\ y_j)$ at all inner nodal points; $\mathbf{v}$ consists of nodal values of the variable $v(x,\ y)$, and is a vectorization of $v(x_i,\ y_j)$ at all inner nodal points.

Let

$$\mathbf{W} := \mathbf{K} + \sigma_1 \mathbf{I}_n,\ \mathbf{T} := \sigma_2 \mathbf{I}_n, \tag{55}$$

where $\mathbf{W},\ \mathbf{T} \in \mathbb{R}^{n\times n}$ are symmetric positive and positive semi-definite; equation (54) is re-written as

$$(\mathbf{W} + i\mathbf{T})(\mathbf{w} + i\mathbf{v}) = \mathbf{f} + i\mathbf{g}. \tag{56}$$

Upon letting

$$\mathbf{A} = \begin{bmatrix} \mathbf{W} & -\mathbf{T} \\ \mathbf{T} & \mathbf{W} \end{bmatrix},\ \mathbf{y} = \begin{bmatrix} \mathbf{w} \\ \mathbf{v} \end{bmatrix},\ \mathbf{h} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}, \tag{57}$$

equation (56) can be written as

$$\mathbf{A}\mathbf{y} = \mathbf{h},\ \mathbf{h},\ \mathbf{y} \in \mathbb{R}^N,\ \ \mathbf{A} \in \mathbb{R}^{N\times N}, \tag{58}$$

where $N = 2n$.

In [29], a complex linear system (56) was considered with

$$\mathbf{W} = \mathbf{K} + \frac{3-\sqrt{3}}{h}\mathbf{I}_n,\ \mathbf{T} = \mathbf{K} + \frac{3+\sqrt{3}}{h}\mathbf{I}_n, \tag{59}$$

$$\mathbf{f} = \frac{j}{h(j+1)^2},\ \mathbf{g} = -\frac{j}{h(j+1)^2},\ j = 1,\ \ldots,\ n. \tag{60}$$

With $\|\mathbf{r}\|/\|\mathbf{b}\| \le \varepsilon = 10^{-6}$ and $n_0 = 32$, the MSOR method with $\alpha = 1.2$ is convergence with 226 steps. In contrast, the MAOR method with $\alpha = 1$ and $w = 1.6$ is convergence with 108 steps. For this linear system with a large dimension $N = 2n_0^2 = 2{,}048$, MAOR is convergent faster than MSOR.

By using the data reported in [29], we compare NS obtained by different methods in Table 6. HSS was developed in [30], and MHSS was developed in [31]. The GSOR was chosen according to Table 1 in [32]. Upon comparing the existing iterative methods for solving the complex linear system the convergence speed of MSOR and MAOR is still slower. However, taking advantage of the MAOR for its easy formulation and low computational cost, the computational efficiency of MAOR is also very good, which without needing of the complicated spectral analysis to determine the values of parameters is a competitive method compared to other iterative methods.

Table 6. Example 5: NS obtained by different methods

Method	HSS	MHSS	SBTS	GSOR	MAOR
NS	65	54	31	22	108

6. Conclusions

The AOR iterative scheme was reformulated from the framework of the descent vector and residual vector. Rather than a constant value of the acceleration parameter η used in AOR, we examined the modified AOR (MAOR) from two aspects of preserving the orthogonality and maximizing the decreasing length of the residual vector. The step-by-step varying parameters were introduced in MSOR and MAOR, which can enhance the speed of convergence. The property of absolute convergence and the orthogonality of consecutive residual vector were proven, which are very important for the iterative schemes. Testing examples confirmed the significant improvement of the convergence speed by using the MSOR and MAOR methods. Even if the original SOR and AOR are unstable for one testing case, MSOR and MAOR are still available.

Among many splitting iterative schemes, SOR and AOR are the most simple ones. To test a complex linear system resulting from the complex Helmholtz equation, the efficiency of MSOR and MAOR needed to be improved further. Upon comparing to the existing iterative methods for solving the complex linear system the convergence speed of MSOR and MAOR is still slower. However some advantages over other iterative methods are that the MAOR method is free of complicated spectral analysis and has a low computational cost.

In the near future we may consider more complex splitting iterative schemes and extend the new idea by adding η to these iterative schemes to accelerate the convergence speed.

Funding statement

NSTC 113-2221-E-019-043-MY3 granted by the National Science and Technology Council partially supporting this study is gratefully acknowledged.

Data availability

The data presented in this study are available on request from the corresponding authors.

Acknowledgments

The authors would like to express their thanks to the reviewers, who supplied valuable opinions to improve the quality of this paper.

Conflict of interest

The authors declare there is no conflict of interest at any point with reference to research findings.

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