Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

# Efficient Solutions for Fractional Order Kuramoto-Sivashinsky Equation: Aboodh Residual Power Series Method

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**Received:** 8 July 2024; **Revised:** 18 August 2024; **Accepted:** 29 August 2024

**Abstract:** In this research, the Aboodh residual power series method is applied to investigate the behavior of the fractional order of the Kuramoto-Sivashinsky (KS) equation. The KS equation is a nonlinear partial differential equation (PDE) that can model the chaotic behavior of different physical processes. The presence of fractional derivatives leads to memory effects and non-local interactions, making the analysis non-trivial and fundamental for understanding various processes. The Aboodh residual power series method (ARPSM) is a highly effective analytical approach for analyzing more complicated problems and deriving highly accurate approximations. Using a combination of analytical derivation and computational studies, we discuss the features of the fractional KS equation and explain how the fractional derivatives affect its behavior. Altogether, the results indicate that the ARPSM is beneficial in obtaining necessary information about the complexity of this equation for further analysis and contributions to nonlinear dynamics field and fractional calculus. This method does not require complex calculations and is characterized by ease and flexibility in analyzing more complicated problems. In addition, you do not need highly efficient computers; a personal computer with normal capabilities can be used to perform all calculations for the most complicated problems.

***Keywords***: Aboodh residual power series method, time fractional Kuramoto-Sivashinsky equation, fractional PDEs, Caputo operator, fractional shock waves

**MSC:** 26A33, 35R11

## 1. Introduction

Fractional calculus (FC) is a field that deals with derivatives and integrals of a fractional nature concerning some variable [1]. It helps find different complex objects' initial properties and aspects, including the memory effect. More and more FC applications have been used to convert classical derivatives and integrals into nonintegers to analyze the


DOI: https://doi.org/10.37256/cm.6520255269

dynamics of physical processes on a gigantic scale. A wide range of engineering and physical science fields employ FC in their applications, such as mathematical biology, signal processing, relaxation schemes, flow models, and viscoelasticity [2–4]. Applications in various domains, from research to practical work, involve analyzing nonlinear processes. Topics such as chemical kinetics, mathematical biology, Quantum science, fluid dynamics, solid-state physics, and nonlinear spectroscopics are some of the topics covered by this category [5–10]. They are defined as the nonlinear partial differential equations of a higher order, giving rise to the non-linearity concept. As a result of nonlinear principles, all physical systems still explain basic phenomena [11, 12]. Caputo, Hilfer, Riemann-Liouville, Caputo Fabrizio, Atangana Baleanu, and Grunwald Letnikov are new fractional derivatives. Out of all the various forms of the fractional derivative, Caputo's definition is the most widely used in fractional calculus and the analysis of FDEs, because virtually every fractional derivative can be derived from it by simply modifying its parameters. There are a number of important operators commonly used to mimic a variety of physical systems, with Caputo's being one of the most familiar, which utilizes the power-law kernel to a significant degree [13, 14].

Nonlinearity characterizes the vast majority of complex phenomena in nature. Nonlinear equations capture the fundamental characteristics of the most crucial processes in the world. The mathematical and practical implications of nonlinear FDE solutions are considerable. Some instances of dynamic systems with fractional calculus are as follows: fractional diffusion-reaction equation [15], Klein-Gordon equations of time-fractional [16], the biological population diffusion model [17], percolation in porous media [18], anomalous transport in disordered systems [19], fractional diabetes model [20], fractional-order sliding mode based extremum seeking control of a class of nonlinear systems [21], design of optimal lighting control strategy based on multi-variable technique of fractional order extremum searching [22], Buck Master's equation [23], space-fractional telegraph equation [24], fractional KdVBurger-Kuramoto equation [25], fractal vehicular traffic flow [26], fractional Drinfeld-Sokolov-Wilson equation [27], time fractional modified anomalous sub-diffusion equation [28], fractional model for the dynamics of Hepatitis B virus [29], model for tuberculosis [30] etc.

The Kuramoto-Sivashinsky (KS) equation has been widely employed as a powerful model for addressing many nonlinear processes in various fields of science and engineering. However, in practice, the classical KS equation could be better for capturing all the behaviors of systems with anomalous diffusion or memory effect. These limitations require fractional calculus extension, resulting in the fractional order KS equation. The fractional KS equation involves fractional derivatives necessary for adequately describing systems with long-range interactions and non-locality. These features make the fractional KS equation especially relevant in fluid mechanics, plasma physics, and material science. However, it is essential to note that additional mathematical complications come with the inclusion of these fractional derivatives. However, the fractional KS equation has some drawbacks as it stands in its current state of development. However, there is a lack of an appropriate understanding of the analytical and numerical methods for the fractional KS equation, the effect of the fractional orders on solution properties, how the fractional model can be used to solve physical problems, and the methods for increasing computing speed. This paper attempts to fill these voids by discussing new approaches, giving detailed analysis, and presenting real-world solutions to the fractional KS equation. In this way, it aims to expand the relevance and knowledge of this crucial mathematical model to new areas.

The KS equation is applicable in the simulation of many occurrences including plasma instabilities, chemical reaction diffusion, flames, viscous flow difficulties, and magnetized plasmas [31, 32]. Thus, in this investigation focuses on studying the following FKS equation [33, 34]

$$D_{\Theta}^{p}\varphi+\varphi\frac{\partial\varphi}{\partial\sigma}+\alpha\frac{\partial^{2}\varphi}{\partial\sigma^{2}}\varphi+\beta\frac{\partial^{3}\varphi}{\partial\sigma^{3}}+\gamma\frac{\partial^{4}\varphi}{\partial\sigma^{4}}=0, \tag{1}$$

with the initial condition (IC):

$$\varphi(\sigma,\,0)=\varphi_{0}(\sigma), \tag{2}$$

where $(\alpha,\ \beta,\ \gamma)$ are constant coefficients depending on the physical model under consideration, $\varphi \equiv \varphi(\sigma,\ \Theta)$, and $0 < p \leq 1$. It is clear that this equation can cover several wave equations used to model solitary and shock waves in different plasma models. For instance, for $\gamma = 0$ Eq. (1) reduces to the KdV-Burgers equation which this equation has been widely used to study shock waves in different plasma systems having viscosity inertial charges. Also, for $\beta = \gamma = 0$, the fractional Burgers equation is recovered, which this equation is also used for describing shock waves in different plasma systems having viscosity inertial charges. Furthermore, for $\alpha = \gamma = 0$, the fractional KdV is recovered, which this equation was widely used in modeling many nonlinear waves that arise and propagate in seas and oceans, water pools, and various plasma systems without the viscosity of moving charges.

Numerous authors employed a variety of methods to study the classical order KS equation, including the Lattice-Boltzmann method [35], cubic B-spline finite difference-collocation method [36], He's variational iteration method [37], Chebyshev spectral collocation methods [38], finite-difference discretization [39], and other schemes [40, 41]. These procedures demand a lengthy process and much processing power to solve the problems. Conversely, applying the suggested strategy to find and analyze the answer for nonlinear issues is relatively easy.

Residual power series method (RPSM) was established by Omar Abu Arqub in 2013 [42]. It is obtained from the multiplication between the Taylor series and the residual error function. The optimization problems can be solved using an infinite convergence series [43]. This is because current RPSM algorithms have been developed to produce accurate and efficient aproximation solutions for many differential equations (DEs) including Boussinesq DEs, fuzzy DEs, KdV Burgers equation and others [44, 45].

A new approach was introduced to the solving of FODEs using a two important methods. The categories that consist of the Sumudu transform in combination with the following are as follows: the Laplace transform with RPSM; the natural transform; the homotopy perturbation method; and the Shehu transformation combined with the Adomian decomposition technique [46–51].

To learn more about how the two approaches were integrated, see [52–54]. For this study's time-fractional partial differential equations (PDEs), we obtained both exact and approximate solutions using a novel combination approach called the Aboodh residual power series method (ARPSM). This new approach is noteworthy since it combines the RPSM and the Aboodh transform [55, 56]. Some researchers have widely used this technique to analyze many complicated linear and nonlinear fractional evolution equations, and its success has been proven in analyzing these equations and obtaining high-accuracy approximations compared to many other methods [57–61]. Thus, the primary goal of this study is to apply ARPSM to analyze some fractional nonlinear evolutionary equations, namely, fractional KS equations, which are widely used in interpreting many nonlinear phenomena in fluid mechanics and plasma physics. We also compare the derived approximations using the ARPSM with the exact solutions for the integer case to these equations.

## 2. Fractional calculus basic concepts

**Definition 1** [62] Let us consider that $\varphi \equiv \varphi(\sigma,\ \Theta)$ is a piecewise continuous function that is ordered exponentially. The Aboodh transform (AT) can be defined as follows, under the assumption that $\tau \geq 0$ for $\varphi$

$$A[\varphi] = \Psi(\sigma,\ \xi) = \frac{1}{\xi}\int_0^\infty \varphi e^{-\Theta\xi} d\Theta,\ \ r_1 \leq \xi \leq r_2.$$

The precise definition of the inverse AT (IAT) is as follows:

$$A^{-1}[\Psi(\sigma,\ \xi)] = \varphi = \frac{1}{2\pi i}\int_{u-i\infty}^{u+i\infty} \Psi(\sigma,\ \Theta)\xi e^{\Theta\xi} d\Theta,$$

where $\sigma = (\sigma_1,\ \sigma_2,\ \cdots,\ \sigma_p) \in \mathbb{R}$ and $p \in \mathbb{N}$.

**Lemma 2** [63, 64] $\varphi_1 \equiv \varphi_1(\sigma,\ \Theta)$ and $\varphi_2 \equiv \varphi_2(\sigma,\ \Theta)$ represent two distinct functions. They are regarded as piecewise continuous and exponentially ordered functions on the interval $[0,\ \infty]$. Consider $A[\varphi_1] = \Psi_1 \equiv \Psi_1(\sigma,\ \xi)$, $A[\varphi_2] = \Psi_2 \equiv \Psi_2(\sigma,\ \xi)$ and $(\chi_1,\ \chi_2)$ are constants. These characteristics are therefore true:

1. $A[\chi_1\varphi_1 + \chi_2\varphi_2] = \chi_1\Psi_1 + \chi_2\Psi_2$,
2. $A^{-1}[\chi_1\Psi_1 + \chi_2\Psi_2] = \chi_1\varphi_1(\sigma,\ \Theta) + \chi_2\varphi_2(\sigma,\ \Theta)$,
3. $A[J^p_\Theta\varphi(\sigma,\ \Theta)] = \dfrac{\Psi(\sigma,\ \xi)}{\xi^p}$,
4. $A[D^p_\Theta\varphi(\sigma,\ \Theta)] = \xi^p\Psi(\sigma,\ \xi) - \sum_{K=0}^{r-1}\dfrac{\varphi^K(\sigma,\ 0)}{\xi^{K-p+2}},\ r-1 < p \leq r,\ r \in \mathbb{N}$.

**Definition 3** [65] The Caputo formula of the derivative of fractional order for $\varphi$ is expressed for order $p$ as:

$$D^p_\Theta\varphi = J^{m-p}_\Theta\varphi^{(m)},\ r \geq 0,\ m-1 < p \leq m,$$

where $\sigma = (\sigma_1,\ \sigma_2,\ \cdots,\ \sigma_p) \in \mathbb{R}^p$ and $m,\ p \in R,\ J^{m-p}_\Theta$ is the integral of R-L of $\varphi$.

**Definition 4** [66] The power series representation is formatted in the subsequent manner.

$$\sum_{r=0}^{\infty}\hbar_r(\sigma)(\Theta-\Theta_0)^{rp} = \hbar_0(\Theta-\Theta_0)^0 + \hbar_1(\Theta-\Theta_0)^p + \hbar_2(\Theta-\Theta_0)^{2p} + \cdots,$$

where $\sigma = (\sigma_1,\ \sigma_2,\ \cdots,\ \sigma_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$. Multiple fractional power series (MFPS) are series centred on $\Theta_0$, where the series coefficients are $\hbar_r(\sigma)$'s and $\Theta$ is a variable.

**Lemma 5** Assume that the exponential order function is denoted by $\varphi$. In this particular case, the AT is mathematically represented as $A[\varphi] = \Psi(\sigma,\ \xi)$. Hence, we get

$$A[D^{rp}_\Theta\varphi(\sigma,\ \Theta)] = \xi^{rp}\Psi(\sigma,\ \xi) - \sum_{j=0}^{r-1}\xi^{p(r-j)-2}D^{jp}_\Theta\varphi(\sigma,\ 0),\ \ 0 < p \leq 1, \tag{3}$$

where $\sigma = (\sigma_1,\ \sigma_2,\ \cdots,\ \sigma_p) \in R^p$ and $p \in N$ and $D^{rp}_\Theta = D^p_\Theta.D^p_\Theta.\ \cdots\ .D^p_\Theta(r-times)$.

**Proof.** Induction will be used to prove Eq. (3). The results for substituting $r = 1$ into Eq. (3) is given as:

$$A[D^{2p}_\Theta\varphi(\sigma,\ \Theta)] = \xi^{2p}\Psi(\sigma,\ \xi) - \xi^{2p-2}\varphi(\sigma,\ 0) - \xi^{p-2}D^p_\Theta\varphi(\sigma,\ 0).$$

Here, $r = 1$, according to Lemma 2 part four. Solving Eq. (3) with the value of $r = 2$ yields

$$A[D^{2p}_r\varphi] = \xi^{2p}\Psi(\sigma,\ \xi) - \xi^{2p-2}\varphi(\sigma,\ 0) - \xi^{p-2}D^p_\Theta\varphi(\sigma,\ 0). \tag{4}$$

The expression on the left-hand side (LHS) of Eq. (4) can be reformulated as follows:

$$L.H.S = A[D^{2p}_\Theta\varphi] = A[D^p_\Theta(D^p_\Theta\varphi)]. \tag{5}$$

Furthermore, Eq. (5) can be restated subsequently

$$L.H.S = A[D_{\Theta}^{p} z], \tag{6}$$

with

$$z \equiv z(\sigma,\ \Theta) = D_{\Theta}^{p} \varphi. \tag{7}$$

By utilizing the Caputo derivative on Eq. (6), we get

$$L.H.S = A[J^{1-p} z']. \tag{8}$$

The following can be deduced using the R-L integral for AT:

$$L.H.S = \frac{A[z']}{\xi^{1-p}}. \tag{9}$$

The present formulation of Eq. (9) is achieved through the utilization of the differential property of AT

$$L.H.S = \xi^{p} Z(\sigma,\ \xi) - \frac{z(\sigma,\ 0)}{\xi^{2-p}}, \tag{10}$$

which, we obtain

$$Z(\sigma,\ \xi) = \xi^{p} \Psi(\sigma,\ \xi) - \frac{\varphi(\sigma,\ 0)}{\xi^{2-p}},$$

where $A[z] = Z(\sigma,\ \xi)$.

Accordingly, Eq. (10) becomes

$$L.H.S = \xi^{2p} \Psi(\sigma,\ \xi) - \frac{\varphi(\sigma,\ 0)}{\xi^{2-2p}} - \frac{D_{\Theta}^{p} \varphi(\sigma,\ 0)}{\xi^{2-p}}. \tag{11}$$

Note that Eqs. (5) and (11) are compatible for $r = K$. With the assumption that Eq. (5) remains valid when $r = K$. Therefore, we substitute $r = K$ into Eq. (5):

$$A[D_{\Theta}^{Kp} \varphi] = \xi^{Kp} \Psi(\sigma,\ \xi) - \sum_{j=0}^{K-1} \xi^{p(K-j)-2} D_{\Theta}^{jp} D_{\Theta}^{jp} \varphi(\sigma,\ 0),\ \ 0 < p \leq 1. \tag{12}$$

The following step is prove that Eq. (5) is valid for $r = K+1$. Eq. (5) enables us to obtain:

$$A[D_{\Theta}^{(K+1)p}\varphi] = \xi^{(K+1)p}\Psi(\sigma,\ \xi) - \sum_{j=0}^{K}\xi^{p((K+1)-j)-2}D_{\Theta}^{jp}\varphi(\sigma,\ 0). \tag{13}$$

An analysis of the left hand side of Eq. (13) yields

$$L.H.S = A[D_{\Theta}^{Kp}(D_{\Theta}^{Kp})], \tag{14}$$

and by assuming

$$D_{\Theta}^{Kp} = g(\sigma,\ \Theta),$$

then Eq. (14) becomes

$$L.H.S = A[D_{\Theta}^{p}g(\sigma,\ \Theta)]. \tag{15}$$

The integral R-L and derivative of Caputo are applied to Eq. (15) to derive the subsequent results

$$L.H.S = \xi^{p}A[D_{\Theta}^{Kp}\varphi] - \frac{g(\sigma,\ 0)}{\xi^{2-p}}. \tag{16}$$

In order to obtain Eq. (16), we employ Eq. (12)

$$L.H.S = \xi^{rp}\Psi(\sigma,\ \xi) - \sum_{j=0}^{r-1}\xi^{p(r-j)-2}D_{\Theta}^{jp}\varphi(\sigma,\ 0). \tag{17}$$

In addition, the subsequent outcome is attained by employing Eq. (17):

$$L.H.S = A[D_{\Theta}^{rp}\varphi(\sigma,\ 0)].$$

Equation (5) remains valid when $r = K+1$. Thus, by means of mathematical induction it is proved that Eq. (5) holds for every positive integer.

The subsequent lemma offers a novel perspective on the subject of multiple fractional Taylor's series (MFTS). This formula is advantageous for the ARPSM, which will be elaborated upon in subsequent sections. □

**Lemma 6** Consider the exponential order function to be denoted as $\varphi$. The AT for $\varphi$ is given as when $A[\varphi(\sigma,\ \Theta)] = \Psi(\xi,\ \sigma)$:

$$\Psi(\sigma,\ \xi) = \sum_{r=0}^{\infty}\frac{r(\sigma)}{\xi^{rp+2}},\ \ \xi > 0, \tag{18}$$

where, $\sigma = (s_1,\ \sigma_2,\ \cdots,\ \sigma_p) \in R^p,\ p \in N$.

**Proof.** An analysis of the expression for Taylor's series is as follows:

$$\varphi(\sigma,\ \Theta) = \hbar_0(\sigma) + \hbar_1(\sigma)\frac{\Theta^p}{\Gamma[p+1]} + \hbar_2(\sigma)\frac{\Theta^{2p}}{\Gamma[2p+1]} + \cdots. \tag{19}$$

By applying AT to Eq. (19) we determine:

$$A\left[\varphi(\sigma,\ \Theta)\right] = A\left[\hbar_0(\sigma)\right] + A\left[\hbar_1(\sigma)\frac{\Theta^p}{\Gamma[p+1]}\right] + A\left[\hbar_1(\sigma)\frac{\Theta^{2p}}{\Gamma[2p+1]}\right] + \cdots.$$

Using the characteristic of AT we obtain:

$$A\left[\varphi(\sigma,\ \Theta)\right] = \hbar_0(\sigma)\frac{1}{\xi^2} + \hbar_1(\sigma)\frac{\Gamma[p+1]}{\Gamma[p+1]}\frac{1}{\xi^{p+2}} + \hbar_2(\sigma)\frac{\Gamma[2p+1]}{\Gamma[2p+1]}\frac{1}{\xi^{2p+2}}\cdots.$$

As a result, (18), which is an AT specific version of Taylor's series, is obtained. □

**Lemma 7** The MFPS can be mathematically expressed as $A[\varphi(\sigma,\ \Theta)] = \Psi(\sigma,\ \xi)$ by employing the revised form of Taylor's series (18).

$$\hbar_0(\sigma) = \lim_{\xi \to \infty} \xi^2 \Psi(\sigma,\ \xi) = \varphi(\sigma,\ 0). \tag{20}$$

**Proof.** By applying the recently modified form of Taylor's series, the following is obtain:

$$\hbar_0(\sigma) = \xi^2 \Psi(\sigma,\ \xi) - \frac{\hbar_1(\sigma)}{\xi^p} - \frac{\hbar_2(\sigma)}{\xi^{2p}} - \cdots. \tag{21}$$

The desired outcome Eq. (21) can be obtained by applying a rapid calculation and utilizing $\lim_{\xi \to \infty}$ to Eq. (20). □

**Theorem 8** In MFPS form, the function $A[\varphi(\sigma,\ \Theta)] = \Psi(\sigma,\ \xi)$ may given as:

$$\Psi(\sigma,\ \xi) = \sum_0^{\infty} \frac{\hbar_r(\sigma)}{\xi^{rp+2}},\ \ \xi > 0,$$

where $\sigma = (\sigma_1,\ \sigma_2,\ \cdots,\ \sigma_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$. Then we have

$$\hbar_r(\sigma) = D_r^{rp} \varphi(\sigma,\ 0),$$

where, $D_\Theta^{rp} = D_\Theta^p . D_\Theta^p . \cdots . D_\Theta^p (r - times)$.

**Proof.** Taylor's most recent series consists of the following form:

$$\hbar_1(\sigma) = \xi^{p+2}\Psi(\sigma,\ \xi) - \xi^p \hbar_0(\sigma) - \frac{\hbar_2(\sigma)}{\xi^p} - \frac{\hbar_3(\sigma)}{\xi^{2p}} - \cdots . \tag{22}$$

By taking $\lim_{\xi\to\infty}$ to Eq. (22), we get

$$\hbar_1(\sigma) = \lim_{\xi\to\infty}(\xi^{p+2}\Psi(\sigma,\ \xi) - \xi^p \hbar_0(\sigma)) - \lim_{\xi\to\infty}\frac{\hbar_2(\sigma)}{\xi^p} - \lim_{\xi\to\infty}\frac{\hbar_3(\sigma)}{\xi^{2p}} - \cdots .$$

Applying the limit yields the subsequent equivalence:

$$\hbar_1(\sigma) = \lim_{\xi\to\infty}(\xi^{p+2}\Psi(\sigma,\ \xi) - \xi^p \hbar_0(\sigma)). \tag{23}$$

The following result is derived after implementing Lemma 5 on Eq. (23):

$$\hbar_1(\sigma) = \lim_{\xi\to\infty}(\xi^2 A[D_\Theta^p \varphi(\sigma,\ \Theta)](\xi)). \tag{24}$$

Moreover, the Eq. (24) is transformed using Lemma 6:

$$\hbar_1(\sigma) = D_\Theta^p \varphi(\sigma,\ 0).$$

When the limit $\xi \to \infty$ is taken and the new form of Taylor's series is utilized once more, the result is

$$\hbar_2(\sigma) = \xi^{2p+2}\Psi(\sigma,\ \xi) - \xi^{2p}\hbar_0(\sigma) - \xi^p \hbar_1(\sigma) - \frac{\hbar_3(\sigma)}{\xi^p} - \cdots .$$

The following is the result of Lemma 6:

$$\hbar_2(\sigma) = \lim_{\xi\to\infty}\xi^2(\xi^{2p}\Psi(\sigma,\ \xi) - \xi^{2p-2}\hbar_0(\sigma) - \xi_1^{p-2}(\sigma)). \tag{25}$$

Equation (25) is converted due to the application of Lemmas 5 and 7

$$\hbar_2(\sigma) = D_\Theta^{2p}\varphi(\sigma,\ 0).$$

The following results are acquired when the new Taylor's series is subjected to the same procedure:

$$\hbar_3(\sigma) = \lim_{\xi\to\infty}\xi^2(A[D_\Theta^{2p}\varphi(\sigma,\ p)](\xi)).$$

The final equation is derived by employing Lemma 7

$$\hbar_3(\sigma) = D_\Theta^{3p}\varphi(\sigma,\ 0).$$

In general

$$\hbar_r(\sigma) = D_\Theta^{rp}\varphi(\sigma,\ 0).$$

Hence proved. □

The subsequent theorem explains and ensures the conditions in which Taylor's series, in its modified form, will converge.

**Theorem 9** Lemma 6 presents the mathematical expression that represents multiple fractional Taylor's formula as: $A[\varphi(\sigma,\ \Theta)] = \Psi(\sigma,\ \xi)$. When $|\xi^a A[D_\Theta^{(K+1)p}\varphi(\sigma,\ \Theta)]| \leq T$, for all $0 < \xi \leq s$ and $0 < p \leq 1$, the residual $R_K(\sigma,\ \xi)$ of the newly developed MFTS satisfies the subsequent inequality:

$$|R_K(\sigma,\ \xi)| \leq \frac{T}{\xi^{(K=1)p+2}},\ \ 0 < \xi \leq s.$$

**Proof.** Assume $A[D_\Theta^{rp}\varphi(\sigma,\ \Theta)](\xi)$ is defined on $0 < \xi \leq s$ for $r = 0,\ 1,\ 2,\ \cdots,\ K+1$. Let consider $|\xi^2 A[D_{\Theta^{K+1}}$ $\varphi(\sigma,\ \tau)]| \leq T$, on $0 < \xi \leq s$. Use the new Taylor's series to establish the following relationship:

$$R_K(\sigma,\ \xi) = \Psi(\sigma,\ \xi) - \sum_{r=0}^{K} \frac{\hbar_r(\sigma)}{\xi^{rp+2}}. \tag{26}$$

By employing the Theorem 8, Eq. (26) is transformed.

$$R_K(\sigma,\ \xi) = \Psi(\sigma,\ \xi) - \sum_{r=0}^{K} \frac{D_\Theta^{rp}\varphi(\sigma,\ 0)}{\xi^{rp+2}}. \tag{27}$$

To obtain the solution for Eq. (27), multiply $\xi(K+1)a+2$ on both sides.

$$\xi^{(K+1)p+2}R_K(\sigma,\ \xi) = \xi^2\left(\xi^{(K+1)p}\Psi(\sigma,\ \xi) - \sum_{r=0}^{K}\xi^{(K+1-r)p-2}D_\Theta^{rp}\varphi(\sigma,\ 0)\right). \tag{28}$$

Implementing Lemma 5 on Eq. (28) we obtain:

$$\xi^{(K+1)p+2}R_K(\sigma,\ \xi) = \xi^2 A[D_\Theta^{(K+1)p}\varphi(\sigma,\ \Theta)]. \tag{29}$$

Using the absolute of Eq. (29), we derive

$$|\xi^{(K+1)p+2} R_K(\sigma,\ \xi)| = |\xi^2 A[D_\Theta^{(K+1)p} \varphi(\sigma,\ \Theta)]|. \tag{30}$$

We can achieve the subsequent result by implementing the conditions specified in Eq. (30).

$$\frac{-T}{\xi^{(K+1)p+2}} \le R_K(\sigma,\ \xi) \le \frac{T}{\xi^{(K+1)p+2}}. \tag{31}$$

To achieve the desired result, Eq. (31) is utilized

$$|R_K(\sigma,\ \xi)| \le \frac{T}{\xi^{(K+1)p+2}}.$$

Thus, new criteria have been developed with regard to series convergence. □

# 3. Methodology

## 3.1 *General implementation of ARPSM*

The ARPSM of principles steps, upon which our overall model solution was built, is described in this section.

**Step 1** Consider the following PDE of general form:

$$D_\Theta^{qp} \varphi(\sigma,\ \Theta) + \vartheta(\sigma) N(\varphi) - \zeta(\sigma,\ \varphi) = 0. \tag{32}$$

**Step 2** Both sides of Eq. (32) are subject to the AT in which we get:

$$A[D_\Theta^{qp} \varphi(\sigma,\ \Theta) + \vartheta(\sigma) N(\varphi) - \zeta(\sigma,\ \varphi)] = 0. \tag{33}$$

Now, by applying Lemma 5 on Eq. (33), we get

$$\Psi(\sigma,\ s) = \sum_{j=0}^{q-1} \frac{D_\Theta^j \varphi(\sigma,\ 0)}{s^{qp+2}} - \frac{\vartheta(\sigma) Y(s)}{s^{qp}} + \frac{F(\sigma,\ s)}{s^{qp}}, \tag{34}$$

where, $A[\zeta(\sigma,\ \varphi)] = F(\sigma,\ s)$ and $A[N(\varphi)] = Y(s)$.

**Step 3** The expression representing the result to Eq. (34) is defined as:

$$\Psi(\sigma,\ s) = \sum_{r=0}^{\infty} \frac{\hbar_r(\sigma)}{s^{rp+2}},\ \ s > 0.$$

**Step 4** Following procedure is consider

$$\hbar_0(\sigma) = \lim_{s\to\infty} s^2\Psi(\sigma,\ s) = \varphi(\sigma,\ 0).$$

Also, the following outcome can be achieved by applying Theorem 9

$$\hbar_1(\sigma) = D_{\Theta}^{p}\varphi(\sigma,\ 0),$$

$$\hbar_2(\sigma) = D_{\Theta}^{2p}\varphi(\sigma,\ 0),$$

$$\vdots$$

$$\hbar_w(\sigma) = D_{\Theta}^{wp}\varphi(\sigma,\ 0).$$

**Step 5** After the $K^{th}$ truncation, the $\Psi(\sigma,\ s)$ series can be obtained using the subsequent formula:

$$\Psi_K(\sigma,\ s) = \sum_{r=0}^{K} \frac{\hbar_r(\sigma)}{s^{rp+2}},\ s > 0,$$

$$= \frac{\hbar_0(\sigma)}{s^2} + \frac{\hbar_1(\sigma)}{s^{p+2}} + \cdots + \frac{\hbar_w(\sigma)}{s^{wp+2}} + \sum_{r=w+1}^{K} \frac{\hbar_r(\sigma)}{s^{rp+2}}.$$

**Step 6** For these latter results, it is required to analyze the Aboodh residual function (ARF) mentioned in Eq. (34); the $K^{th}$-truncated ARF has to be discussed independently.

$$ARes(\sigma,\ s) = \Psi(\sigma,\ s) - \sum_{j=0}^{q-1} \frac{D_{\Theta}^{j}\varphi(\sigma,\ 0)}{s^{jp+2}} + \frac{\vartheta(\sigma)Y(s)}{s^{jp}} - \frac{F(\sigma,\ s)}{s^{jp}},$$

and

$$ARes_K(\sigma,\ s) = \Psi_K(\sigma,\ s) - \sum_{j=0}^{q-1} \frac{D_{\Theta}^{j}\varphi(\sigma,\ 0)}{s^{jp+2}} + \frac{\vartheta(\sigma)Y(s)}{s^{jp}} - \frac{F(\sigma,\ s)}{s^{jp}}. \tag{35}$$

**Step 7** In Eq. (35), $\Psi_K(\sigma,\ s)$ should be substituted for its expansion form

$$ARes_K(\sigma,\ s) = \left[\frac{\hbar_0(\sigma)}{s^2} + \frac{\hbar_1(\sigma)}{s^{p+2}} + \cdots + \frac{\hbar_w(\sigma)}{s^{wp+2}} + \sum_{r=w+1}^{K} \frac{\hbar_r(\sigma)}{s^{rp+2}}\right]$$
$$- \sum_{j=0}^{q-1} \frac{D_\Theta^j \varphi(\sigma,\ 0)}{s^{jp+2}} + \frac{\vartheta(\sigma)Y(s)}{s^{jp}} - \frac{F(\sigma,\ s)}{s^{jp}}. \tag{36}$$

**Step 8** the following result is obtained, by multiplying each side of Eq. (36) by $s^{Kp+2}$:

$$s^{Kp+2}ARes_K(\sigma,\ s) = s^{Kp+2}\left[\frac{\hbar_0(\sigma)}{s^2} + \frac{\hbar_1(\sigma)}{s^{p+2}} + \cdots + \frac{\hbar_w(\sigma)}{s^{wp+2}} + \sum_{r=w+1}^{K} \frac{\hbar_r(\sigma)}{s^{rp+2}}\right.$$
$$\left. - \sum_{j=0}^{q-1} \frac{D_\Theta^j \varphi(\sigma,\ 0)}{s^{jp+2}} + \frac{\vartheta(\sigma)Y(s)}{s^{jp}} - \frac{F(\sigma,\ s)}{s^{jp}}\right]. \tag{37}$$

**Step 9** Taking $\lim_s \to \infty$, for Eq. (37) yields

$$\lim_{s\to\infty} s^{Kp+2}ARes_K(\sigma,\ s) = \lim_{s\to\infty} s^{Kp+2}\left(\frac{\hbar_0(\sigma)}{s^2} + \frac{\hbar_1(\sigma)}{s^{p+2}} + \cdots + \frac{w(\sigma)}{s^{wp+2}} + \sum_{r=w+1}^{K} \frac{\hbar_r(\sigma)}{s^{rp+2}}\right.$$
$$\left. - \sum_{j=0}^{q-1} \frac{D_\Theta^j \varphi(\sigma,\ 0)}{s^{jp+2}} + \frac{\vartheta(\sigma)Y(s)}{s^{jp}} - \frac{F(\sigma,\ s)}{s^{jp}}\right).$$

**Step 10** In order find $\hbar_K(\sigma)$, we solve the given equation

$$\lim_{s\to\infty} \left[s^{Kp+2}ARes_K(\sigma,\ s)\right] = 0,$$

where $K = w+1,\ w+2,\ \cdots$.

**Step 11** In order to derive the $K$-approximated result to Eq. (34), we replace $\hbar_K(\sigma)$ with a truncated $\Psi(\sigma,\ s)$ series.

**Step 12** To obtain the final solution apply the IAT, solve $\Psi_K(\sigma,\ s)$ in order to acquire the necessary function $\varphi_K(\sigma,\ \Theta)$.

## 4. Test problems

To demonstrate the efficiency and accuracy of the suggested method, we will present some examples of fractional evolution equations related to the family KS equation, as follows.

### 4.1 *Example-I*

Let us consider the following time fractional-order KS equation:

$$D_{\Theta}^{p}\varphi+\varphi\frac{\partial\varphi}{\partial\sigma}-\frac{\partial^{2}\varphi}{\partial\sigma^{2}}+\frac{\partial^{4}\varphi}{\partial\sigma^{4}}=0, \tag{38}$$

where $\varphi\equiv\varphi(\sigma,\ \Theta)$ and $0<p\leq 1$.

Using the wave transformation: $\varphi\equiv\psi\left[l\left(\sigma-\lambda\Theta\right)\right]$ and apply the tanh method, the following exact solution for $p=1$, is obtained

$$\varphi=\frac{1}{361}\left(361\lambda-45\sqrt{19}\tanh\left[l(\sigma-\lambda\Theta)\right]+15\sqrt{19}\tanh^{3}\left[l\left(\sigma-\lambda\Theta\right)\right]\right), \tag{39}$$

where $l=1/\left(2\sqrt{19}\right)$. According to solution (39), the IC to problem (38) reads

$$\varphi_{0}\equiv\varphi(\sigma,\ 0)=\frac{1}{361}\left(361\lambda-45\sqrt{19}\tanh\left[l\sigma\right]+15\sqrt{19}\tanh^{3}\left[l\sigma\right]\right) \tag{40}$$

After applying AT to Eq. (38) and using Eq. (40), we get

$$\begin{aligned}&\varphi(\sigma,\ s)-\frac{\varphi_{0}}{s^{2}}+\frac{1}{s^{p}}A_{\Theta}\left[A_{\Theta}^{-1}\varphi(\sigma,\ s)\times\frac{\partial}{\partial\sigma}A_{\Theta}^{-1}\varphi(\sigma,\ s)\right]\\&-\frac{1}{s^{p}}\left[\frac{\partial^{2}}{\partial\sigma^{2}}\varphi(\sigma,\ s)\right]+\frac{1}{s^{p}}\left[\frac{\partial^{4}}{\partial\sigma^{4}}\varphi(\sigma,\ s)\right]=0.\end{aligned} \tag{41}$$

Consequently, the $k^{th}$-truncated term series are as follows:

$$\varphi(\sigma,\ s)=\frac{\varphi_{0}}{s^{2}}+\sum_{r=1}^{k}\frac{f_{r}(\sigma,\ s)}{s^{rp+1}}, \tag{42}$$

where $r=1,\ 2,\ 3,\ 4\ \cdots$.

The ARF yields

$$\begin{aligned}A_{\Theta}Res(\sigma,\ s)=\varphi(\sigma,\ s)-\frac{\varphi_{0}}{s^{2}}+\frac{1}{s^{p}}A_{\Theta}\left[A_{\Theta}^{-1}\varphi(\sigma,\ s)\times\frac{\partial}{\partial\sigma}A_{\Theta}^{-1}\varphi(\sigma,\ s)\right]\\-\frac{1}{s^{p}}\left[\frac{\partial^{2}}{\partial\sigma^{2}}\varphi(\sigma,\ s)\right]+\frac{1}{s^{p}}\left[\frac{\partial^{4}}{\partial\sigma^{4}}\varphi(\sigma,\ s)\right]=0,\end{aligned} \tag{43}$$

and the $k^{th}$-LRFs as:

$$A_\Theta Res_k(\sigma,\ s) = \varphi_k(\sigma,\ s) - \frac{\varphi_0}{s^2} + \frac{1}{s^p} A_\Theta \left[ A_\Theta^{-1} \varphi_k(\sigma,\ s) \times \frac{\partial}{\partial \sigma} A_\Theta^{-1} \varphi_k(\sigma,\ s) \right]$$
$$- \frac{1}{s^p} \left[ \frac{\partial^2}{\partial \sigma^2} \varphi_k(\sigma,\ s) \right] + \frac{1}{s^p} \left[ \frac{\partial^4}{\partial \sigma^4} \varphi_k(\sigma,\ s) \right] = 0. \tag{44}$$

Determining $f_r(\sigma,\ s)$ for $r = 1,\ 2,\ 3,\ ...$ need some computation. Here are the instructions, the $r^{th}$-ARF equation is Releasing Eq. (44) into the second term on the right of Eq. (3) and replacing it with the $r^{th}$-truncated series Eq. (42), and we find limit $s^{rp+1}$ by post-multiplying it with $s^{rp+1}$. $A_\Theta Res_{\varphi,\ r}(\sigma,\ s) = 0$, and $r = 1,\ 2,\ 3,\ \ldots$. Few terms are obtained through the proposed method as follow:

$$f_1(\sigma,\ s) = \frac{45\lambda}{722} \operatorname{sech}^4(l\sigma)$$

$$f_2(\sigma,\ s) = \frac{45\lambda^2}{361\sqrt{19}} \operatorname{sech}^4(l\sigma) \tanh(l\sigma),$$

and so on.

Put $f_r(\sigma,\ s)$, for $r = 1,\ 2,\ 3,\ \cdots$, in Eq. (42):

$$\varphi(\sigma,\ s) = \frac{\varphi_0}{s^2} + \frac{f_1(\sigma,\ s)}{s^p + 1} + \frac{f_2(\sigma,\ s)}{s^{2p+1}}$$
$$= \frac{\varphi_0}{s^2} + \frac{\frac{45\lambda}{722} \operatorname{sech}^4(l\sigma)}{s^p + 1} + \frac{\frac{45\lambda^2}{361\sqrt{19}} \operatorname{sech}^4(l\sigma) \tanh(l\sigma)}{s^{2p+1}} \tag{45}$$

Applying IAT on Eq. (45), we finally get

$$\varphi(\sigma,\ \Theta) = \frac{1}{361} \left( 361\lambda - 45\sqrt{19} \tanh[l\sigma] + 15\sqrt{19} \tanh^3[l\sigma] \right)$$
$$+ \frac{\frac{45\lambda}{722} \Theta^p \operatorname{sech}^4(l\sigma)}{\Gamma(1+p)} + \frac{\frac{45\lambda^2}{361\sqrt{19}} \Theta^{2p} \operatorname{sech}^4(l\sigma) \tanh(l\sigma)}{\Gamma(1+2p)}. \tag{46}$$

The approximation (46) is analyzed graphically, as presented in Figure 1, to understand the effect of the fractional parameter on the dynamics of the propagation of shock waves, which is described by one of the solutions to the problem (38). Furthermore, to verify the suggested method's efficiency and the derived approximations' accuracy, we compared the approximation (46) at $p = 1$ with the exact solution (39), as shown in Figure 2. In addition, we calculated the absolute error to the approximation (46) at $p = 1$ compared to the exact solution (39), as shown in Figure 3.

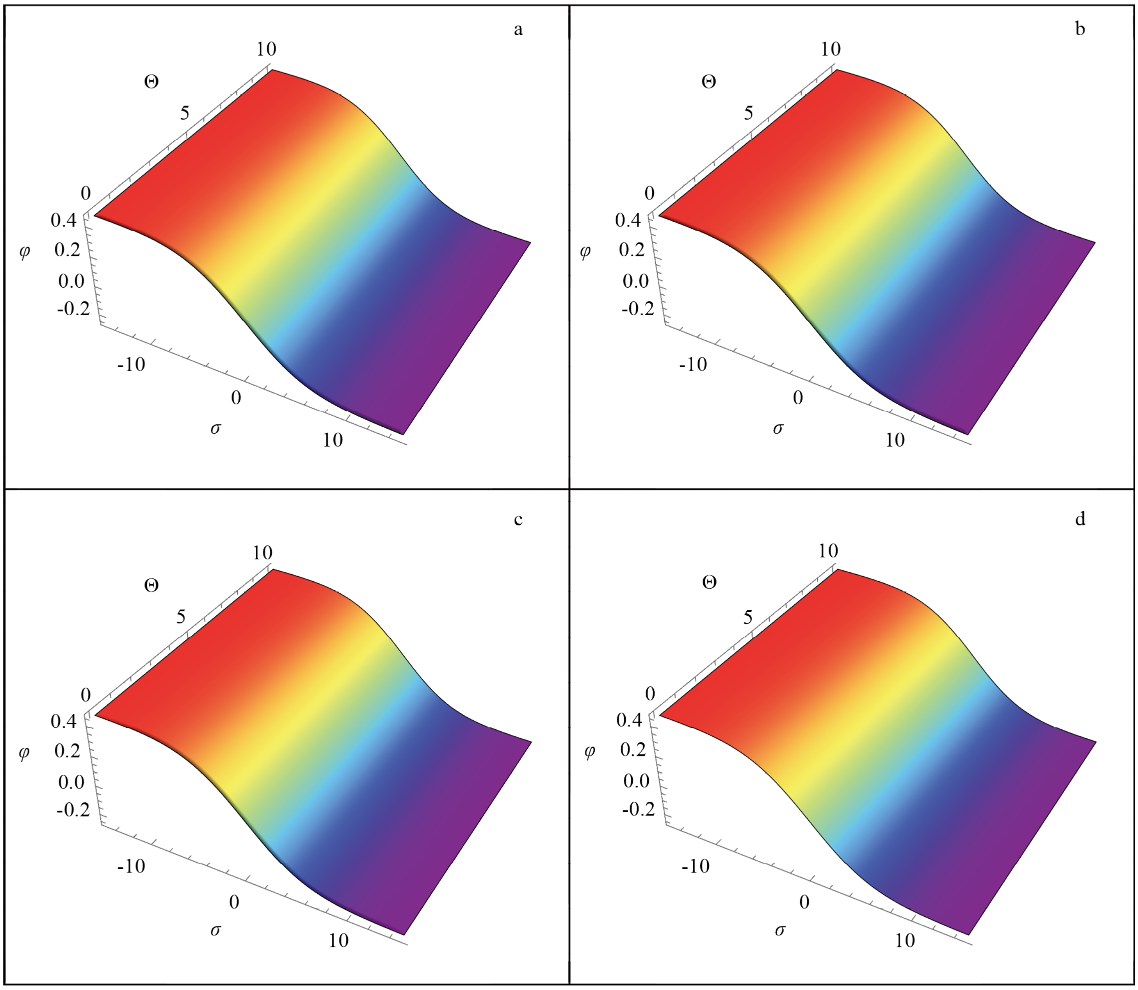


**Figure 1.** The approximation (46) to problem (38) is plotted against $p$: (a) 3D graphic for $\varphi(\sigma, \Theta)$ at $p = 0.1$, (b) 3D graphic for $\varphi(\sigma, \Theta)$ at $p = 0.3$, (c) 3D graphic for $\varphi(\sigma, \Theta)$ at $p = 0.5$, (d) 3D graphic for $\varphi(\sigma, \Theta)$ at $p = 1$

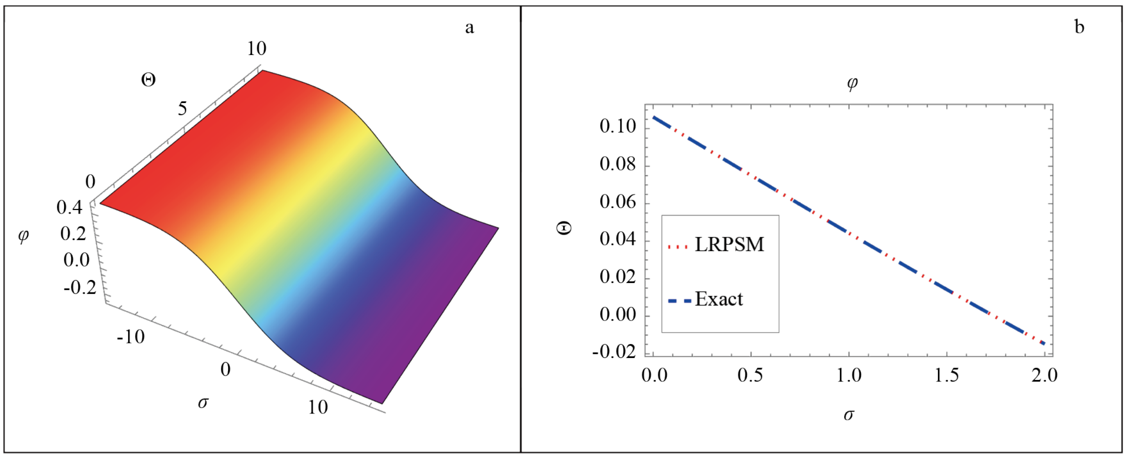


**Figure 2.** The approximation (46) to problem (38) at $p = 1$ against the exact solution (39) (a) 3D graphic in the $(\sigma, \Theta)$ plane and (b) 2D graphic at $\Theta = 1$

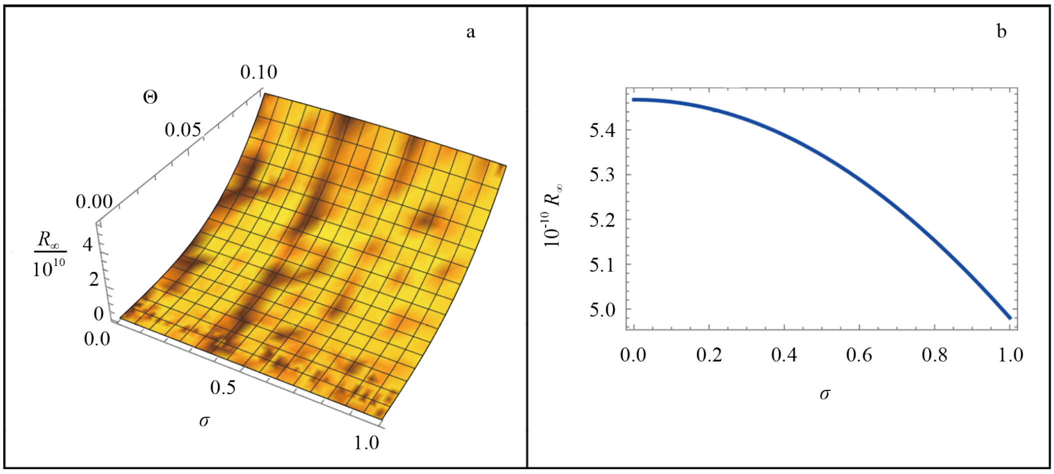


**Figure 3.** The absolute error $R_\infty$ for the approximation (46): (a) $R_\infty$ is plotted in the $(\sigma, \Theta)$ plane and (b) $R_\infty$ is plotted at $\Theta = 0.1$

## 4.2 *Problem 2*

Let's examine the time fractional KS equation:

$$D_{\Theta}^{p}\varphi+\varphi\frac{\partial\varphi}{\partial\sigma}+\frac{\partial^{2}\varphi}{\partial\sigma^{2}}+\frac{\partial^{4}\varphi}{\partial\sigma^{4}}=0. \tag{47}$$

Using the wave transformation: $\varphi \equiv \psi[l(\sigma-\lambda\Theta)]$ and apply the tanh method, the following exact solution for $p = 1$, is obtained

$$\varphi=\frac{1}{361}\left(361\lambda-135\sqrt{209}\tanh\left[l(\sigma-\lambda\Theta)\right]+1,655\sqrt{209}\tanh^{3}\left[l(\sigma-\lambda\Theta)\right]\right), \tag{48}$$

where $l = \frac{1}{2}\sqrt{11/19}$. According to solution (48), the IC to problem (47) reads

$$\varphi_0\equiv\varphi(\sigma,\,0)=\frac{1}{361}\left(361\lambda-135\sqrt{209}\tanh\left[l(\sigma)\right]+1,655\sqrt{209}\tanh^{3}\left[l(\sigma)\right]\right) \tag{49}$$

Applying AT to Eq. (47) and using Eq. (49), we get

$$\begin{aligned}&\varphi(\sigma,\,s)-\frac{\varphi_0}{s^2}+\frac{1}{s^p}A_{\Theta}\left[A_{\Theta}^{-1}\varphi(\sigma,\,s)\times\frac{\partial}{\partial\sigma}A_{\Theta}^{-1}\varphi(\sigma,\,s)\right]\\&+\frac{1}{s^p}\left[\frac{\partial^2}{\partial\sigma^2}\varphi(\sigma,\,s)\right]+\frac{1}{s^p}\left[\frac{\partial^4}{\partial\sigma^4}\varphi(\sigma,\,s)\right]=0.\end{aligned} \tag{50}$$

Consequently, the series which is $k^{th}$-truncated are:

$$\varphi(\sigma,\ s) = \frac{\varphi_0}{s^2} + \sum_{r=1}^{k} \frac{f_r(\sigma,\ s)}{s^{rp+1}},\ \ r = 1,\ 2,\ 3,\ 4\cdots \tag{51}$$

ARF is given by

$$\begin{aligned} A_\Theta Res(\sigma,\ s) &= \varphi(\sigma,\ s) - \frac{\varphi_0}{s^2} + \frac{1}{s^p} A_\Theta \left[ A_\Theta^{-1} \varphi(\sigma,\ s) \times \frac{\partial}{\partial \sigma} A_\Theta^{-1} \varphi(\sigma,\ s) \right] \\ &+ \frac{1}{s^p} \left[ \frac{\partial^2}{\partial \sigma^2} \varphi(\sigma,\ s) \right] + \frac{1}{s^p} \left[ \frac{\partial^4}{\partial \sigma^4} \varphi(\sigma,\ s) \right] = 0, \end{aligned} \tag{52}$$

and the $k^{th}$-LRFs as:

$$\begin{aligned} A_\Theta Res_k(\sigma,\ s) &= \varphi_k(\sigma,\ s) - \frac{\varphi_0}{s^2} + \frac{1}{s^p} A_\Theta \left[ A_\Theta^{-1} \varphi_k(\sigma,\ s) \times \frac{\partial}{\partial \sigma} A_\Theta^{-1} \varphi_k(\sigma,\ s) \right] \\ &+ \frac{1}{s^p} \left[ \frac{\partial^2}{\partial \sigma^2} \varphi_k(\sigma,\ s) \right] + \frac{1}{s^p} \left[ \frac{\partial^4}{\partial \sigma^4} \varphi_k(\sigma,\ s) \right] = 0, \end{aligned} \tag{53}$$

Determining $f_r(\sigma,\ s)$ for $r = 1,\ 2,\ 3,\ ...$ need some computation. Please see the following steps, when taking the $r^{th}$-ARF: On this basis, we substitute Eq. (53) for the $r^{th}$-truncated series. Taking limit on the equations of (51) and solving $\lim_{s\to\infty}(s^{rp+1})$ with the help of multiplication of final equation with $s^{rp+1}$. $A_\Theta Res_{\varphi,\ r}(\sigma,\ s) = 0$, and $r = 1,\ 2,\ 3,\ \cdots$. Some term are achieved through the suggested method as follow:

$$f_1(\sigma,\ s) = -\frac{495}{722}\lambda \left[ -7 + 4\cosh\left( \sqrt{\frac{11}{19}}\sigma \right) \right] \operatorname{sech}^4 \left( \frac{1}{2}\sqrt{\frac{11}{19}}\sigma \right),$$

$$f_2(\sigma,\ s) = -\frac{495}{361}\sqrt{\frac{11}{19}}\lambda^2 \operatorname{sech}^5 \left( \frac{1}{2}\sqrt{\frac{11}{19}}\sigma \right) \left[ -10\sinh\left( \frac{1}{2}\sqrt{\frac{11}{19}}\sigma \right) + \sinh\left( \frac{3}{2}\sqrt{\frac{11}{19}}\sigma \right) \right],$$

and so on.

Inserting the values of $f_r(\sigma,\ s)$, for $r = 1,\ 2,\ 3,\ \cdots$, into Eq. (51) yields

$$
\begin{aligned}
\varphi(\sigma,\, s) &= \frac{\varphi_0}{s^2} + \frac{f_1(\sigma,\, s)}{s^{p+1}} + \frac{f_2(\sigma,\, s)}{s^{2p+1}} \\
&= \frac{\varphi_0}{s^2} - \frac{495}{722} \frac{\lambda \left[-7+4\cosh\left(\sqrt{\frac{11}{19}}\sigma\right)\right] \operatorname{sech}^4\left(\frac{1}{2}\sqrt{\frac{11}{19}}\sigma\right)}{s^{p+1}} \\
&\quad - \frac{495}{361}\sqrt{\frac{11}{19}} \frac{\lambda^2 \operatorname{sech}^5\left(\frac{1}{2}\sqrt{\frac{11}{19}}\sigma\right)\left[-10\sinh\left(\frac{1}{2}\sqrt{\frac{11}{19}}\sigma\right) + \sinh\left(\frac{3}{2}\sqrt{\frac{11}{19}}\sigma\right)\right]}{s^{2p+1}}.
\end{aligned} \tag{54}
$$

Applying IAT on Eq. (54) implies

$$
\begin{aligned}
\varphi(\sigma,\, \Theta) &= \lambda - \frac{135}{19}\sqrt{\frac{11}{19}} \tanh[l\sigma] + \frac{165}{19}\sqrt{\frac{11}{19}} \tanh^3[l\sigma] \\
&\quad + \frac{\lambda \Theta^p}{\Gamma[1+p]} \left\{ \frac{3,465}{722} \operatorname{sech}^4[l\sigma] - \frac{990}{361} \cosh[2l\sigma] \operatorname{sech}^4[l\sigma] \right\} \\
&\quad + \frac{\lambda^2 \Theta^{2p}}{\Gamma[1+2p]} \left\{ -\frac{495}{361}\sqrt{\frac{11}{19}} \operatorname{sech}^5[l\sigma] \sinh[3l\sigma] + \frac{4,950}{361}\sqrt{\frac{11}{19}} \operatorname{sech}^4[l\sigma] \tanh[l\sigma] \right\}.
\end{aligned} \tag{55}
$$

The graphical analysis of the approximation (55) is depicted in Figure 4 to comprehend the impact of the fractional parameter on the wave dynamics described by this approximation. Moreover, to validate the proposed technique's effectiveness and the resulting estimations' precision, we compared the approximation (55) at $p = 1$ and the exact solution (48), as depicted in Figure 5. Furthermore, we computed the absolute error for the approximation (55) at $p = 1$ compared to the exact solution (48), as illustrated in Figure 6.

## 5. Conclusion

In summary, our study of the fractional KS equation using the ARPSM has yielded valuable insights into the behavior of this intricate nonlinear system. By applying ARPSM, we have effectively examined the influence of fractional derivatives on the equation's dynamics, revealing its memory effects and non-local interactions. The results from thorough analysis and computational simulations confirm the effectiveness and dependability of the proposed approach in offering precise solutions and understanding the complex dynamics of the KS equation. Our discoveries contribute to advancing knowledge in nonlinear dynamics and fractional calculus, providing valuable insights for researchers across various scientific and engineering disciplines. Future research opportunities involve exploring further applications of the Aboodh method and expanding its relevance to other nonlinear systems, thereby enriching our understanding and analytical capabilities in fractional partial differential equations. It should also be noted that this method does not require a high-efficiency computer; all calculations are performed on a personal device with normal capabilities.

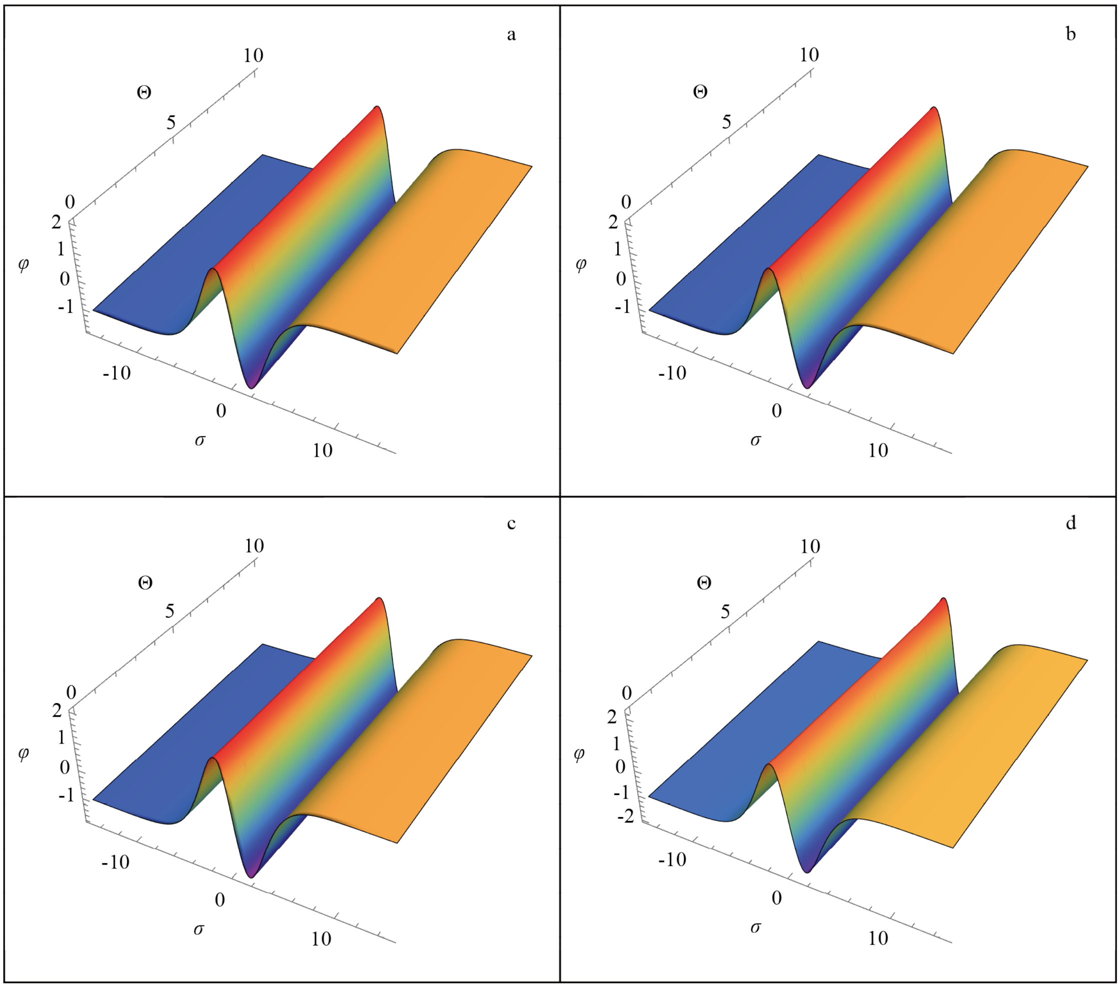


**Figure 4.** The approximation (55) to problem (47) is plotted against $p$: (a) 3D graphic for $\varphi(\sigma, \Theta)$ at $p = 0.1$, (b) 3D graphic for $\varphi(\sigma, \Theta)$ at $p = 0.3$, (c) 3D graphic for $\varphi(\sigma, \Theta)$ at $p = 0.5$, (d) 3D graphic for $\varphi(\sigma, \Theta)$ at $p = 1$

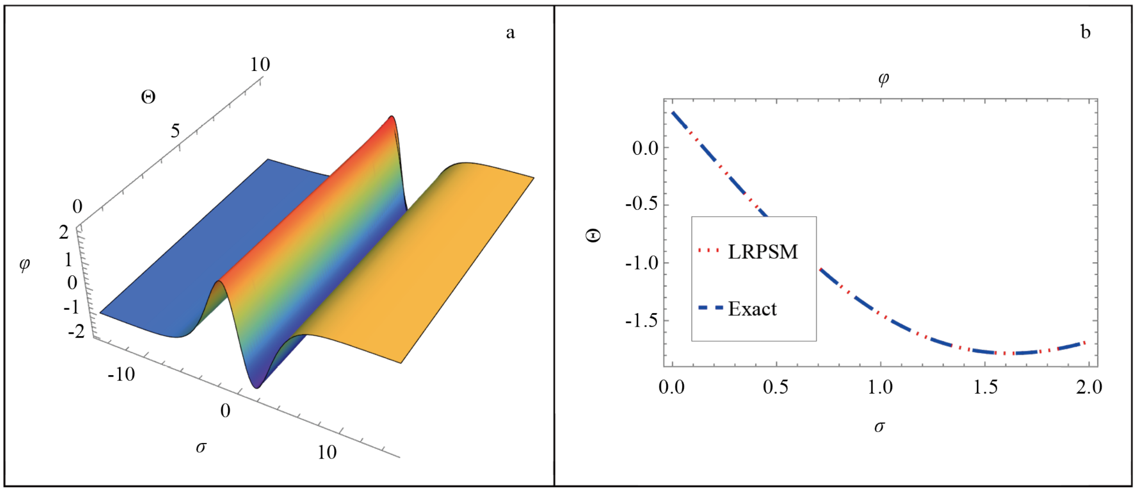


**Figure 5.** The approximation (55) to problem (47) at $p = 1$ against the exact solution (39) (a) 3D graphic in the $(\sigma, \Theta)$ plane and (b) 2D graphic at $\Theta = 1$

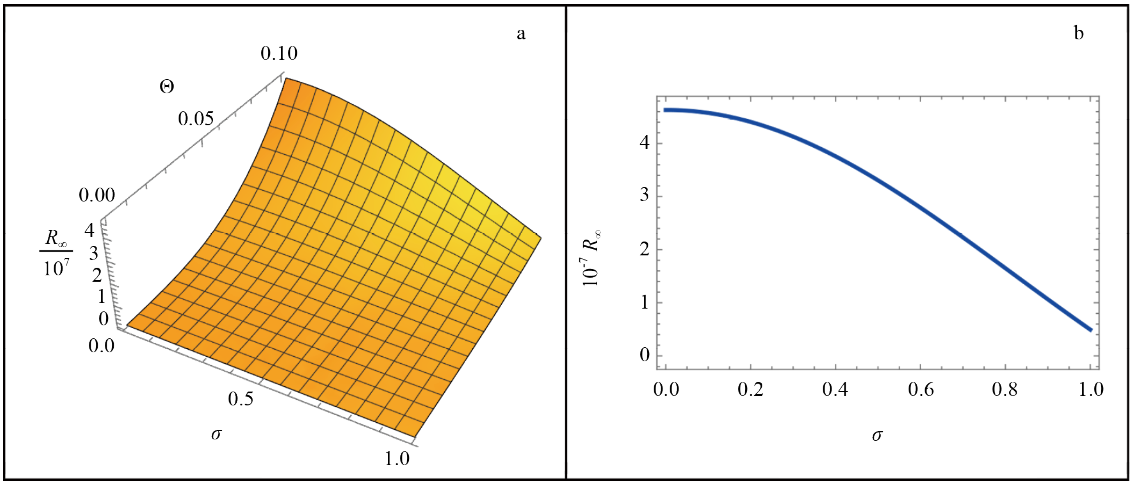


**Figure 6.** The absolute error $R_\infty$ for the approximation (55): (a) $R_\infty$ is plotted in the $(\sigma, \Theta)$ plane and (b) $R_\infty$ is plotted at $\Theta = 0.1$

## 6. Future work

The obtained results showcased the effectiveness and reliability of the used method, as well as the exceptional precision of the developed approximations. Consequently, several scholars can utilize this approach to analyze many complicated and strong nonlinearity engineering and physics problems and fractional evolutionary equations used in modeling and simulating many nonlinear structures that arise in different plasma systems. For instance, we can apply this approach to studying the family of KdV-type equations [67–73], Kawahara-type equations [74–78], and nonlinear Schrödinger-type equations [79–86] in their time fractional forms. This will enable us to comprehend the dynamics of solitary, shock, periodic, and rogue waves, considering the impact of the fractional parameter. Consequently, this will help us understand the mystery behaviors behind different nonlinear structures that arise in various plasma models.

## Conflict of interest

The authors claim that there is no conflict of interest.

## References


[1] Lizzy RM, Balachandran K, Trujillo JJ. Controllability of nonlinear stochastic fractional neutral systems with multiple time varying delays in control. *Chaos, Solitons and Fractals*. 2017; 102: 162-167.

[2] Rahman F, Ali A, Saifullah S. Analysis of time-fractional 4-equation with singular and non-singular kernels. *International Journal of Applied and Computational Mathematics*. 2021; 7(5): 192.

[3] Ray SS, Atangana A, Noutchie SC, Kurulay M, Bildik N, Kilicman A. Fractional calculus and its applications in applied mathematics and other sciences. *Mathematical Problems in Engineering*. 2014; 2014: 849395.

[4] Abd-Elhameed WM, Youssri YH, Amin AK, Atta AG. Eighth-kind Chebyshev polynomials collocation algorithm for the nonlinear time-fractional generalized Kawahara equation. *Fractal and Fractional*. 2023; 7(9): 652.

[5] Alhejaili W, Az-Zo'bi E, Shah R, El-Tantawy SA. On the analytical soliton approximations to fractional forced Korteweg-de Vries equation arising in fluids and plasmas using two novel techniques. *Communications in Theoretical Physics*. 2024; 76: 085001.

[6] Noor S, Alyousef HA, Shafee A, Shah R, El-Tantawy SA. A novel analytical technique for analyzing the (3+1)-dimensional fractional calogero-bogoyavlenskii-schiff equation: investigating solitary/shock waves and many others physical phenomena. *Physica Scripta*. 2024; 99(6): 065257.

[7] Alqudah M, Mukhtar S, Alyousef HA, Ismaeel SME, El-Tantawy SA, Ghani F. Probing the diversity of soliton phenomena within conformable Estevez-Mansfield-Clarkson equation in shallow water. *AIMS Mathematics*. 2024; 9(8): 21212-21238.

[8] Yasmin H, Alyousef HA, Asad S, Khan I, Matoog RT, El-Tantawy SA. The Riccati-Bernoulli sub-optimal differential equation method for analyzing the fractional Dullin-Gottwald-Holm equation and modeling nonlinear waves in fluid mediums. *AIMS Mathematics*. 2024; 9(6): 16146-16167.

[9] Al-Sawalha MM, Mukhtar S, Alrowaily AW, Alshammari S, Ismaeel SME, El-Tantawy SA. Analytical solutions to time-space fractional Kuramoto-Sivashinsky Model using the integrated Backlund transformation and Riccati-Bernoulli sub-ODE method. *AIMS Mathematics*. 2024; 9(5): 12357-12374.

[10] Mukhtar S, Hammad MA, Shah R, Alrowaily AW, Ismaeel SME, El-Tantawy SA. On the localized and periodic solutions to the time-fractional Klein-Gordan equations: Optimal additive function method and new iterative method. *Open Physics*. 2023; 21(1): 20230116.

[11] Youssri YH, Abd-Elhameed WM, Atta AG. Spectral Galerkin treatment of linear one-dimensional telegraph type problem via the generalized Lucas polynomials. *Arabian Journal of Mathematics*. 2022; 11(3): 601-615.

[12] Moustafa M, Youssri YH, Atta AG. Explicit Chebyshev-Galerkin scheme for the time-fractional diffusion equation. *International Journal of Modern Physics C: Computational Physics & Physical Computation*. 2024; 35(1): 2450002.

[13] Caputo M, Fabrizio M. A new definition of fractional derivative without singular kernel. *Progress in Fractional Differentiation and Applications*. 2015; 1(2): 73-85.

[14] Forte F, Mudambi R, Navarra PM. *A Handbook of Alternative Theories of Public Economics*. Cheltenham: Edward Elgar Publishing; 2001.

[15] Hosseini K, Mayeli P, Bekir A, Guner O. Density-dependent conformable space-time fractional diffusion-reaction equation and its exact solutions. *Communications in Theoretical Physics*. 2018; 69(1): 1.

[16] Hosseini K, Bekir A, Kaplan M, Guner O. On a new technique for solving the nonlinear conformable time-fractional differential equations. *Optical and Quantum Electronics*. 2017; 49: 1-12.

[17] Singh J, Kumar D, Kilicman A. Numerical solutions of nonlinear fractional partial differential equations arising in spatial diffusion of biological populations. *Abstract and Applied Analysis*. 2014; 2014: 535793.

[18] Stevik TK, Aa K, Ausland G, Hanssen JF. Retention and removal of pathogenic bacteria in wastewater percolating through porous media: a review. *Water Research*. 2004; 38(6): 1355-1367.

[19] Bisquert J, Garcia-Belmonte G, Fabregat-Santiago F, Compte A. Anomalous transport effects in the impedance of porous film electrodes. *Electrochemistry Communications*. 1999; 1(9): 429-435.

[20] Singh J, Kumar D, Baleanu D. On the analysis of fractional diabetes model with exponential law. *Advances in Difference Equations*. 2018; 2018(1): 1-15.

[21] Yin C, Chen Y, Zhong SM. Fractional-order sliding mode based extremum seeking control of a class of nonlinear systems. *Automatica*. 2014; 50(12): 3173-3181.

[22] Yin C, Huang X, Dadras S, Cheng Y, Cao J, Malek H, et al. Design of optimal lighting control strategy based on multi-variable fractional-order extremum seeking method. *Information Sciences*. 2018; 465: 38-60.

[23] Jain S. Numerical analysis for the fractional diffusion and fractional Buckmaster equation by the two-step Laplace Adam-Bashforth method. *The European Physical Journal Plus*. 2018; 133(1): 19.

[24] Prakash A. Analytical method for space-fractional telegraph equation by homotopy perturbation transform method. *Nonlinear Engineering*. 2016; 5(2): 123-128.

[25] Safari M, Ganji DD, Moslemi M. Application of He's variational iteration method and Adomian's decomposition method to the fractional KdV-Burgers-Kuramoto equation. *Computers and Mathematics with Applications*. 2009; 58(11-12): 2091-2097.

[26] Kumar D, Tchier F, Singh J, Baleanu D. An efficient computational technique for fractal vehicular traffic flow. *Entropy*. 2018; 20(4): 259.

[27] Singh J, Kumar D, Baleanu D, Rathore S. An efficient numerical algorithm for the fractional Drinfeld-Sokolov-Wilson equation. *Applied Mathematics and Computation*. 2018; 335: 12-24.

[28] Shivanian E, Jafarabadi A. Time fractional modified anomalous sub-diffusion equation with a nonlinear source term through locally applied meshless radial point interpolation. *Modern Physics Letters B*. 2018; 32(22): 1850251.

[29] Ullah S, Altaf Khan M, Farooq M. A new fractional model for the dynamics of the hepatitis B virus using the Caputo-Fabrizio derivative. *The European Physical Journal Plus*. 2018; 133: 1-14.

[30] Khan MA, Ullah S, Farooq M. A new fractional model for tuberculosis with relapse via Atangana-Baleanu derivative. *Chaos, Solitons and Fractals*. 2018; 116: 227-238.
[31] Kuramoto Y, Tsuzuki T. Persistent propagation of concentration waves in dissipative media far from thermal equilibrium. *Progress of Theoretical Physics*. 1976; 55(2): 356-369.
[32] Sivashinsky GI. Instabilities, pattern formation, and turbulence in flames. *Annual Review of Fluid Mechanics*. 1983; 15(1): 179-199.
[33] Xu Y, Shu CW. Local discontinuous Galerkin methods for the Kuramoto-Sivashinsky equations and the Ito-type coupled KdV equations. *Computer Methods in Applied Mechanics and Engineering*. 2006; 195(25-28): 3430-3447.
[34] Shah R, Khan H, Baleanu D. Fractional Whitham-Broer-Kaup equations within modified analytical approaches. *Axioms*. 2019; 8(4): 125.
[35] Ye L, Yan G, Li T. Numerical method based on the lattice Boltzmann model for the Kuramoto-Sivashinsky equation. *Journal of Scientific Computing*. 2011; 49(2): 195-210.
[36] Lakestani M, Dehghan M. Numerical solutions of the generalized Kuramoto-Sivashinsky equation using B-spline functions. *Applied Mathematical Modelling*. 2012; 36(2): 605-617.
[37] Porshokouhi MG, Ghanbari B. Application of He's variational iteration method for solution of the family of Kuramoto-Sivashinsky equations. *Journal of King Saud University-Science*. 2011; 23(4): 407-411.
[38] Khater AH, Temsah R. Numerical solutions of the generalized Kuramoto-Sivashinsky equation by Chebyshev spectral collocation methods. *Computers and Mathematics with Applications*. 2008; 56(6): 1465-1472.
[39] Akrivis GD. Finite difference discretization of the Kuramoto-Sivashinsky equation. *Numerische Mathematik*. 1992; 63: 1-11.
[40] Singh BK, Arora G, Kumar P. A note on solving the fourth-order Kuramoto-Sivashinsky equation by the compact finite difference scheme. *Ain Shams Engineering Journal*. 2018; 9(4): 1581-1589.
[41] Sahoo S, Ray SS. New approach to find exact solutions of time-fractional Kuramoto-Sivashinsky equation. *Physica A: Statistical Mechanics and Its Applications*. 2015; 434: 240-245.
[42] Arqub OA. Series solution of fuzzy differential equations under strongly generalized differentiability. *Journal of Advanced Research in Applied Mathematics*. 2013; 5(1): 31-52.
[43] Abu Arqub O, Abo-Hammour Z, Al-Badarneh R, Momani S. A reliable analytical method for solving higher-order initial value problems. *Discrete Dynamics in Nature and Society*. 2013; 2013(1): 673829.
[44] Arqub OA, El-Ajou A, Zhour ZA, Momani S. Multiple solutions of nonlinear boundary value problems of fractional order: a new analytic iterative technique. *Entropy*. 2014; 16(1): 471-493.
[45] El-Ajou A, Arqub OA, Momani S. Approximate analytical solution of the nonlinear fractional KdV-Burgers equation: a new iterative algorithm. *Journal of Computational Physics*. 2015; 293: 81-95.
[46] Alquran M, Ali M, Alsukhour M, Jaradat I. Promoted residual power series technique with Laplace transform to solve some time-fractional problems arising in physics. *Results in Physics*. 2022; 19: 103667.
[47] Eriqat T, El-Ajou A, Moa'ath NO, Al-Zhour Z, Momani S. A new attractive analytic approach for solutions of linear and nonlinear neutral fractional pantograph equations. *Chaos, Solitons and Fractals*. 2020; 138: 109957.
[48] Alquran M, Alsukhour M, Ali M, Jaradat I. Combination of Laplace transform and residual power series techniques to solve autonomous n-dimensional fractional nonlinear systems. *Nonlinear Engineering*. 2021; 10(1): 282-292.
[49] Khan A, Junaid M, Khan I, Ali F, Shah K, Khan D. Application of homotopy analysis natural transform method to the solution of nonlinear partial differential equations. *Science International (Lahore)*. 2017; 29(1): 297-303.
[50] Zhang MF, Liu YQ, Zhou XS. Efficient homotopy perturbation method for fractional non-linear equations using Sumudu transform. *Thermal Science*. 2015; 19(4): 1167-1171.
[51] Liaqat MI, Khan A, Alam MA, Pandit MK, Etemad S, Rezapour S. Approximate and closed-form solutions of Newell-Whitehead-Segel equations via modified conformable Shehu transform decomposition method. *Mathematical Problems in Engineering*. 2022; 2022(2): 1-14.
[52] Al-Deiakeh R, Ali M, Alquran M, Sulaiman TA, Momani S, Al-Smadi M. On finding closed-form solutions to some nonlinear fractional systems via the combination of multi-Laplace transform and the Adomian decomposition method. *Romanian Reports in Physics*. 2022; 74(111): 1-17.
[53] Eltayeb H, Kilicman A. A note on double Laplace transform and telegraphic equations. *Romanian Reports in Physics*. 2013; 2013: 932578.
[54] Alquran M, Al-Khaled K, Ali M, Ta'any A. The combined Laplace transform-differential transform method for solving linear non-homogeneous PDEs. *Journal of Mathematical and Computational Science*. 2012; 2(3): 690-701.

[55] Liaqat MI, Etemad S, Rezapour S, Park C. A novel analytical Aboodh residual power series method for solving linear and nonlinear time-fractional partial differential equations with variable coefficients. *AIMS Mathematics*. 2022; 7(9): 16917-16948.
[56] Liaqat MI, Akgul A, Abu-Zinadah H. Analytical investigation of some time-fractional Black-Scholes models by the Aboodh residual power series method. *Mathematics*. 2023; 11(2): 276.
[57] Ganie AH, Noor S, Al Huwayz M, Shafee A, El-Tantawy SA. Numerical simulations for fractional Hirota-Satsuma coupled Korteweg-de Vries systems. *Open Physics*. 2024; 22(1): 20240008.
[58] Aboodh KS. The new integral transform Aboodh transform. *Global Journal of Pure and Applied Mathematics*. 2013; 9(1): 35-43.
[59] Aggarwal S, Chauhan RA. A comparative study of Mohand and Aboodh transforms. *International Journal of Research in Advent Technology*. 2019; 7(1): 520-529.
[60] Benattia ME, Belghaba K. Application of the Aboodh transform for solving fractional delay differential equations. *Universal Journal of Mathematics and Applications*. 2020; 3(3): 93-101.
[61] Noor S, Hammad MMA, Alrowaily AW, El-Tantawy SA. Numerical investigation of fractional-order Fornberg-Whitham equations in the framework of Aboodh transformation. *Symmetry*. 2023; 15(7): 1353.
[62] Aboodh KS. The new integral transform "Aboodh Transform". *Global Journal of Pure and Applied Mathematics*. 2013; 9(1): 35-43.
[63] Aggarwal S, Chauhan R. A comparative study of Mohand and Aboodh transforms. *International Journal of Research in Advent Technology*. 2019; 7(1): 520-529.
[64] Benattia ME, Belghaba K. Application of the Aboodh transform for solving fractional delay differential equations. *Universal Journal of Mathematics and Applications*. 2020; 3(3): 93-101.
[65] Delgado BB, Macias-Diaz JE. On the general solutions of some non-homogeneous Div-curl systems with Riemann-Liouville and Caputo fractional derivatives. *Fractal and Fractional*. 2021; 5(3): 117.
[66] Alshammari S, Al-Smadi M, Hashim I, Alias MA. Residual power series technique for simulating fractional Bagley-Torvik Problems emerging in applied physics. *Applied Sciences*. 2019; 9(23): 5029.
[67] Almutlak SA, Parveen S, Mahmood S, Qamar A, Alotaibi BM, El-Tantawy SA. On the propagation of cnoidal wave and overtaking collision of slow shear Alfvén solitons in low $\beta$-magnetized plasmas. *Physics of Fluids*. 2023; 35: 075130.
[68] Albalawi W, El-Tantawy SA, Salas AH. On the rogue wave solution in the framework of a Korteweg-de Vries equation. *Results in Physics*. 2021; 30: 104847.
[69] Hashmi T, Jahangir R, Masood W, Alotaibi BM, Ismaeel SME, El-Tantawy SA. Head-on collision of ion-acoustic (modified) Korteweg-de Vries solitons in Saturn's magnetosphere plasmas with two temperature superthermal electrons. *Physics of Fluids*. 2023; 35: 103104.
[70] El-Tantawy SA. Nonlinear dynamics of soliton collisions in electronegative plasmas: The phase shifts of the planar KdV-and mkdV-soliton collisions. *Chaos Solitons Fractals*. 2016; 93: 162.
[71] Tariq MS, Masood W, Siddiq M, Asghar S, Alotaibi BM, Ismaeel SME, et al. Bäcklund transformation for analyzing a cylindrical Korteweg-de Vries equation and investigating multiple soliton solutions in a plasma. *Physics of Fluids*. 2023; 35: 103105.
[72] Ali I, Masood W, Rizvi H, Alrowaily AW, Ismaeel SME, El-Tantawy SA. Archipelagos, islands, necklaces, and other exotic structures in external force-driven chaotic dusty plasmas. *Chaos, Solitons & Fractals*. 2023; 175: 113931.
[73] Wazwaz AM, Alhejaili W, El-Tantawy SA. Study on extensions of (modified) Korteweg-de Vries equations: Painlevé integrability and multiple soliton solutions in fluid mediums. *Physics of Fluids*. 2023; 35: 093110.
[74] Alkhateeb SA, Hussain S, Albalawi W, El-Tantawy SA, El-Awady EI. Dissipative Kawahara ion-acoustic solitary and cnoidal waves in a degenerate magnetorotating plasma. *Journal of Taibah University for Science*. 2023; 17(1): 2187606.
[75] Alharbey RA, Alrefae WR, Malaikah H, Tag-Eldin E, El-Tantawy SA. Novel approximate analytical solutions to the nonplanar modified Kawahara equation and modeling nonlinear structures in electronegative plasmas. *Symmetry*. 2023; 15(1): 97.
[76] El-Tantawy SA, Salas AH, Alyouse HA, Alharthi MR. Novel exact and approximate solutions to the family of the forced damped Kawahara equation and modeling strong nonlinear waves in a plasma. *Chinese Journal of Physics*. 2022; 77: 2454.

[77] Alyousef HA, Salas AH, Alharthi MR, El-tantawy SA. New periodic and localized traveling wave solutions to a Kawahara-Type Equation: Applications to plasma physics. *Chinese Journal of Physics*. 2022; 2022: 9942267.

[78] Alharthi MR, Alharbey RA, El-Tantawy SA. Novel analytical approximations to the nonplanar Kawahara equation and its plasma applications. *The European Physical Journal Plus*. 2022; 137: 1172.

[79] Irshad M, Rahman A, Khalid M, Khan S, Alotaibi BM, El-Sherif LS, et al. Effect of Īṛ-deformed Kaniadakis distribution on the modulational instability of electron-acoustic waves in a non-Maxwellian plasma. *Physics of Fluids*. 2023; 35: 105116.

[80] El-Tantawy SA, Wazwaz AM, Rahman A. Three-dimensional modulational instability of the electrostatic waves in $e-p-i$ magnetoplasmas having superthermal particles. *Physics of Plasmas*. 2017; 24: 022126.

[81] El-Tantawy SA, Aboelenen T. Simulation study of planar and nonplanar super rogue waves in an electronegative plasma: Local discontinuous Galerkin method. *Physics of Plasmas*. 2017; 24: 052118.

[82] El-Tantawy SA, El-Awady EI. Cylindrical and spherical Akhmediev breather and freak waves in ultracold neutral plasmas. *Physics of Plasmas*. 2018; 25: 012121.

[83] Aljahdaly NH, El-Tantawy SA, Wazwaz AM, Ashi HA. Adomian decomposition method for modelling the dissipative higher-order rogue waves in a superthermal collisional plasma. *Journal of Taibah University for Science*. 2021; 15: 971.

[84] El-Tantawy SA, Salas AH, Alharthi MR. On the analytical and numerical solutions of the linear damped NLSE for modeling dissipative freak waves and breathers in nonlinear and dispersive mediums: an application to a pair-ion plasma. *Frontiers in Physics*. 2021; 9: 580224.

[85] El-Tantawy SA, Alharbey RA, Salas AH. Novel approximate analytical and numerical cylindrical rogue wave and breathers solutions: An application to electronegative plasma. *Chaos, Solitons & Fractals*. 2022; 155: 111776.

[86] El-Tantawy SA, Salas AH, Alyousef HA, Alharthi MR. Novel approximations to a nonplanar nonlinear Schrödinger equation and modeling nonplanar rogue waves/breathers in a complex plasma. *Chaos, Solitons & Fractals*. 2022; 1635: 112612.