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Research Article

D-Continuity

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Abstract: We call a map $f: X \to Y$ D-continuous if its restriction to any set of points that do not possess compact neighborhoods is continuous. We investigate this weaker version of continuity and provide examples to compare D-continuity with other types of continuity. Let $f: X \to Y$ be a D-continuous bijective map such that $f(A)$ is locally finite, where A is the set of all points that do not possess compact neighborhoods in X. Then, we show that $f|_A$ is a homeomorphism. We also show that if X is a countably generated topological space, then any D-continuous $f: X \to Y$ is continuous. We discuss C-normality, illustrating the relationship between this property and D-continuity. Finally, we investigate the space of all real-valued D-continuous maps of an arbitrary topological space X and obtain some results.

***Keywords*:** topological spaces, real-valued functions, subspaces, compactness, local compactness, continuous maps

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1. Introduction

Topology is essential beyond its theoretical foundations. The study of topological spaces provides an understanding of practical problems. Recent works have demonstrated inventive ways that topology can effectively solve these issues. One of the applications is utilizing topological concepts to create new rough set models. Rough sets are a framework Pawlak introduced for tackling uncertainty and vagueness in data analysis [1]. By including topological ideas, researchers have developed new models to enhance rough sets' capability when dealing with complex data structures [2]. These models benefit from inherent attributes found in topological spaces, providing more advanced and accurate approximations of data. In the present paper, we focus on the theoretical component. Arhangelskii introduced the notion of τ-continuous maps in [3]. This notion represents a weaker version of continuity, examining continuity concerning subspaces rather than the entire space. Similarly, this paper introduces the concept of D-continuous maps. Given a topological space X, a set of points lacking compact neighborhoods is called a set of defects, as discussed in [4]. We define a map $f: X \to Y$ as D-continuous if its restriction to any set of defects is continuous. Initially, we establish a connection between τ-continuity and D-continuity, illustrating that one type of continuity does not necessarily imply the existence of the other. In this paper, we demonstrate that if a space X is countably generated and a map $f: X \to Y$ is D-continuous, then f is continuous, see [5] for more details about countably generated spaces. Arhangelskii introduced the notion of a C-normal topological property in 2012 during his visit to the Mathematics Department at King Abdulaziz University in Jeddah, Saudi Arabia. This notion of C-normality was discussed in detail in [6]. We establish that if X is a non-locally compact C-normal space,

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and the subspace consisting of all defects is T_2 and can be expressed as a union of pairwise disjoint compact subsets, then there exists a D-continuous map on X. Furthermore, we explore the space of all real-valued D-continuous maps and derive several results. A space X is considered T_4 if it is a T_1 normal space. A Tychonoff space is a T_1 completely regular space, denoted by $T_{3\frac{1}{2}}$. By a T_{1c} topological space, we mean a T_1 space such that every compact subset is closed. Following the notations in [7], $\aleph_0$ represents the cardinality of the set of natural numbers. In a topological space X, the set of cardinal numbers of the form $|\beta|$, where β is a base for X, possesses the smallest element known as the weight of X, denoted by $\omega(X)$. By a neighborhood of a point $x \in X$, we mean an open subset containing x. We denote $\overline{U}$ the closure of the subset $U \subseteq X$.

2. *D*-continuous maps

Definition 2.1 A map $f : X \to Y$ is declared to be τ-continuous if, for every subspace A of X with $|A| = \tau$, the restriction $f|_A$ is continuous [3].

It is obvious that every continuous map is τ-continuous.

Now, we introduce the notion of D-continuity. We will then show that, in some cases, if a map $f : X \to Y$ is D-continuous, then it is continuous on the entire topological space X.

Definition 2.2 Let X and Y be topological spaces. A map $f : X \to Y$ is called D-continuous if for every set $A \subseteq X$ of points that do not possess compact neighborhoods, the restriction $f|_A$ is continuous.

Remark 2.3 It is evident that any continuous map is also D-continuous. Moreover, if a space X is not locally compact such that no points have compact neighborhoods, then any map that is D-continuous on X is also continuous.

Remark 2.4 If X is a non-locally compact space such that the set of defects A has cardinality no greater than τ, then it is evident that any τ-continuous map on X is also D-continuous. Particularly, in the case where X is a locally compact space, any τ-continuous map $f : X \to Y$ is D-continuous since the set of defects is empty.

Remark 2.5 The following example shows that if $f : X \to Y$ is τ-continuous, then it is not necessarily D-continuous.

Remark 2.6 Let $\mathbb{R}$ be the set of real numbers. We define a topology C on $\mathbb{R}$ by declaring open sets to include $\mathbb{R}$ and the empty set, along with all sets whose complements are countable, i.e., $\mathbb{R}$ is equipped with the countable complement topology. Note that the space $(\mathbb{R},\ C)$ is not locally compact, and it's evident that the set of defect points is the entire space, as the only compact sets are finite sets.

Now, consider the identity map $f : (\mathbb{R},\ C) \to (\mathbb{R},\ U)$, where U is the usual topology on $\mathbb{R}$. It's straightforward to observe that f is $\aleph_0$continuous. However, it is not D-continuous because the preimage of any interval $(a,\ b)$ is not open, where $a < b$. (Here, we consider the set of defect points to be the entire space.)

Lemma 2.7 (Lemma 2.4, [4]) Let X be a topological space, then the set of all points that do not possess compact neighborhoods, say A, is a closed subset of X.

Proof. We need to demonstrate that $X \backslash A$ is open. If $A = \emptyset$, then $X \backslash A = X$ is open. If $A = X$, then $X \backslash A = \emptyset$ is open. Now, suppose $A \neq X$ and $A \neq \emptyset$, and let x be any arbitrary point in $X \backslash A$. Suppose all neighborhoods of the point x intersect with the set A. Take a neighborhood U_x of x such that $\overline{U_x}$ is compact. Our assumption implies $U_x \cap A \neq \emptyset$. However, this implies that U_x is a compact neighborhood of some point $y \in A$. This is a contradiction, as all points in A do not possess compact neighborhoods.

Example 2.8 (Modified Arens-Fort Space (Example 2, [4])):

Let's consider the modification of the Arens-fort space. We define $(S,\ \rho)$ as the set of all ordered pairs of $\mathbb{N} \times \mathbb{N}$. In this space, all the singletons are open sets except for the points $<0,\ 0>,\ <1,\ 0>,\ ...,\ <m,\ 0>$ for some positive integer m. Open neighborhoods of each point $\{<0,\ 0>,\ <1,\ 0>,\ ...,\ <m,\ 0>\}$ are defined as any set V such that $\{<0,\ 0>,\ <1,\ 0>,\ ...,\ <m,\ 0>\} \subset V$, and all but a finite number of points of each set $K_d = \{<l,\ d>: l \text{ is fixed and } d \in \mathbb{N}\}$. It's important to note that this space is not locally compact because the points $<0,\ 0>,\ <1,\ 0>,\ ...,\ <m,\ 0>$ do not have compact neighborhoods.

Let's verify that the point $<0,\ 0>$ does not have a compact neighborhood, and all other points can be verified similarly. Consider an open neighborhood V of the point $<0,\ 0>$. Let $V = \{\{a_s\},\ U\}$, where each a_s is distinct from all the points $< 0,\ 0 >,\ < 1,\ 0 >,\ ...,\ < m,\ 0 >$, and U is an open neighborhood of $< 0,\ 0 >$ which is distinct from V. Specifically, if $L = \{< l,\ d >: l$ is fixed $\} \in V$, we require that $L \not\subset U$. Then, V is an open cover of the closure $\overline{V}$, which does not have a finite open subcover.

Example 2.9 (Modified Dieudonne Plank (Example 1.10, [6]):

Let

$$X = ([0,\ \omega_2] \times [0,\ \omega_1]) \backslash < \omega_2,\ \omega_1 >$$

where ω_1 is the first uncountable ordinal and the ordinal ω_2 is the successor ordinal of ω_1. The topology τ generated by declaring open each point of each point of $([0,\ \omega_2) \times [0,\ \omega_1)$, together with the sets $U_\alpha(\kappa) = \{< \kappa,\ \gamma >: \alpha < \gamma \leq \omega_1)\}$ and $V_\alpha(\kappa) = \{< \gamma,\ \kappa >: \alpha < \gamma \leq \omega_2)\}$. This is a non-locally compact Tychonoff space. To see that this space is not locally compact, observe that the point $< \omega_2,\ 0 >$ does not have a compact neighborhood.

We will now construct a counterexample to illustrate that a D-continuous function may not always be continuous.

Example 2.10 Let's denote the topological spaces defined in Example 2.8 and Example 2.9 by X and Y, respectively. We define a function $f : X \to Y$ as follows: the set of points $A = \{< 0,\ 0 >,\ < 1,\ 0 >,\ ...,\ < m,\ 0 >\}$ is mapped onto the set $\{< \omega_2,\ 0 >\}$, and $X \backslash A$ is mapped onto $\{< 0,\ 0 >\}$. Clearly, the set A is the set of all points that do not possess compact neighborhoods and is closed. Since the restriction $f|_A : A \to f(A)$ is continuous, the restriction of f to any $B \subset A$ is also continuous. Hence, f is D-continuous. Since Y is a T_1 space, the set $\{< 0,\ 0 >\}$ is closed. If f were continuous, then the preimage of $\{< 0,\ 0 >\}$ would be closed, i.e., $X \backslash A$ would be a closed set. However, this implies that $X \backslash (X \backslash A) = A$ is open. Consequently, A would be a compact neighborhood of any point in A. This is a contradiction because all points in A do not possess compact neighborhoods.

Proposition 2.11 Let $f : X \to Y$ be a D-continuous bijective map. If $f(A)$ is locally finite where A is the set of all defects in X, then the restriction $f|_A : A \to f(A)$ is a homeomorphism.

Proof. Since f is a D-continuous map, the restriction $f|_A : A \to f(A)$ is continuous. To demonstrate that this restriction is a homeomorphism, it suffices to prove that $f|_A$ is a closed map, meaning the image of any closed subset is closed. Let $F \subseteq A$ be closed. We observe that

$$f|_A(F) = \bigcup_{i \in I} \{f(a_i)\}.$$

As the set $f(A)$ is locally finite, then $\bigcup_{i \in I} \{f(a_i)\}$ is also locally finite. Consequently,

$$\bigcup_{i \in I} \{f(a_i)\} = \bigcup_{i \in I} \overline{\{f(a_i)\}} = \overline{\bigcup_{i \in I} \{f(a_i)\}}.$$

Hence, the restriction $f|_A$ is a homeomorphism.

Corollary 2.12 Let $f : X \to Y$ be a D-continuous bijective map, and let A be the set of all defects in X. If $|A| = n$ for some $n \in \mathbb{N}$, then the restriction of f to A is a homeomorphism.

Definition 2.13 A topological space X is called countably generated if a set $A \subset X$ is closed if for each countable subset $D \subset X$, we have that $A \cap D$ is closed in the relative topology on D [5].

Theorem 2.14 Let X be a countably generated topological space. If a map $f : X \to Y$ is D-continuous, then f is continuous.

Proof. Let A be the set of all defects:

Case 1 if $A = \phi$, then it is clear that f is continuous.

Case 2 If $A \neq \emptyset$ and $|A| \leq \aleph_0$, consider any closed subset $F \subseteq Y$. It is straightforward to see that $F \cap f(A)$ is closed in the space $f(A)$. Now, observe:

$$f^{-1}(F \cap f(A)) = f|_A^{-1}(F \cap f(A))$$

$$= f|_A^{-1}(F) \cap A$$

$$= f^{-1}(F) \cap A.$$

Thus, $f^{-1}(F) \cap A$ is closed in A, implying $f^{-1}(F)$ is closed in X.

Case 3 Suppose $|A| > \aleph_0$. Take any countable subset $A' \subset A$. Since f is D-continuous, $f|_{A'} : A' \to f(A')$ is continuous. Now, for any closed set $F \subseteq Y$, $F \cap f(A')$ is closed in Y. Therefore:

$$f^{-1}(F \cap f(A')) = f|_{A'}^{-1}(F \cap f(A'))$$

$$= f|_{A'}^{-1}(F) \cap A'$$

$$= f^{-1}(F) \cap A'$$

implying that $f^{-1}(F) \cap A'$ is closed in A'. Hence, $f^{-1}(F)$ is closed in X.

Remark 2.15 Assuming $f : X \to Y$ is a D-continuous map, with Y being a countably generated space, continuity of f cannot be ensured. This is demonstrated when we consider the following example.

Example 2.16 Denote γ as the graph of the function $g(r) = \sin(1/r)$, where $0 < r \leq 1$, as a subset of the Euclidean space $\mathbb{R}^2$ with the relative topology. The set $\gamma^\star = \{(0, 0)\} \cup \gamma$ is not locally compact since the point $(0, 0)$ has no compact neighborhood. To see this, let S be any disc centered at $(0, 0)$ of radius ε. Let U be any open neighborhood of $(0, 0)$ such that $S \cap \gamma^\star \subseteq U$. Any horizontal line passing through intersects $S \cap \gamma^\star$ in a sequence of points with no accumulation point in U, proving that U is not compact. Therefore, the topological space $\gamma^\star$ is a locally compact space with defect of type local compactness (Page 137, [8]).

Take Y as any discrete space such that $|Y| \geq 2$. Clearly, the space Y is countably generated. Now, let's define a map $f : \gamma^\star \to Y$ as follows: $f((0, 0)) = y_1 \in Y$, and $f(x) = y_2$ such that $x \in \gamma$ and $y_1 \neq y_2$. One easily sees that the map f is D-continuous but it is not continuous.

Remark 2.17 One of the properties of continuity states that if we have a continuous map $f : Y_1 \to Y_2$, then the image $f(A)$ is connected for any connected subset $A \subseteq X$. This is equivalent to saying if $f(A)$ is disconnected, then A is disconnected. This property does not always hold in the case of D-continuity. To see this, consider the Example 2.16. Y is a discrete space, so it is disconnected, and any subspace of Y must also be disconnected. Therefore, $f(\gamma^*)$ is disconnected. However, the space γ^* is connected, see (Page 138, [8]).

Remark 2.18 It was proven in (Theorem 2.15, [4]) that if X is a T_{1c} topological space, where A is the set of all points that do not possess compact neighborhoods, and for each point $x \in A$, there exists an open neighborhood U such that its closure $\overline{U}$ can be expressed as $\bigcup_{s \in S} F_s$, a union of compact subsets. Moreover, if the family $\{F_s\}_{s \in S}$ is pairwise disjoint and locally finite except at a finite number of points, then X is $T_{3\frac{1}{2}}$.

Proposition 2.19 Let X be a non-locally compact space such that $\omega(X) \leq m$. If the set of all defects A satisfies the assumption in Remark 2.18, then a D-continuous map exists on X.

Proof. This follows from the fact that the Tychonoff cube I^m is universal for all Tychonoff spaces of weight m, as seen in (Page 83, [7]). Therefore, a continuous map $f : X \to Y \subseteq I^m exists$. Clearly, f is a D-continuous map.

Example 2.20 Consider the Modified Arens-Fort Space in Example 2.8. It was proved in [4] that this space is a Tychonoff space. Therefore, there exists a D-continuous map on this space.

Definition 2.21 Suppose $\{C_i\}_{i\in I}$ is a cover of the space X. Let's consider a family of continuous maps $\{h_i\}_{i\in I}$, where $h_i : C_i \to Y$. The maps h_i are said to be compatible if for every $i_1, i_2 \in I$ we have $h_{i_1}|_{C_{i_1}\cap C_{i_2}} = h_{i_2}|_{C_{i_1}\cap C_{i_2}}$. The combined map is defined as $h = \bigcup_{i\in I} h_i : X \to Y$. (Page 71, [7]).

Remark 2.22 It was proven in (Page 71, [7]) that if $\{S_i\}_{i\in I}$ is a locally finite closed cover of X and $\{h_i\}_{i\in I}$ is a family of compatible maps, where $h_i : S_i \to Y$. Then, the combination is continuous.

Definition 2.23 A topological space X is called C-normal if there exists a normal space Y and a bijective function $f : X \to Y$ such that the restriction $f|_C : C \to f(C)$ is a homeomorphism for each compact subspace $C \subseteq X$ [6].

Theorem 2.24 Let X be a non-locally compact C-normal space such that the subspace consisting of all defects is T_2 and can be written as a union of pairwise disjoint compact subsets. Then, there exists a D-continuous map on X.

Proof. Let A be the set of all defects, defined as $A = \bigcup_{i\in I} F_i$, where $\{F_i\}_{i\in I}$ is a family of pairwise disjoint compact subsets. Based on our assumption, there exists a normal space Y and a bijective map $f : X \to Y$ such that the restriction to any compact subset is a homeomorphism. Since each F_i is compact, the restrictions $f|_{F_i} : F_i \to f(F_i)$ are homeomorphisms for all $i \in I$. Furthermore, from Lemma 2.7, it's established that the set A is closed. Given that the subspace A is T_2, all compact subsets of A are also closed. The family $\{F_i\}_{i\in I}$ forms a locally finite closed cover of A. As the map f is bijective, we conclude that the family of maps $\{f_{F_i}\}_{i\in I}$ is a family of compatible maps. In other words, the combination

$$\bigcup_{i\in I} f_i : F_i \to f(F_i)$$

is continuous. Hence, $f|_A : A \to f(A)$ is a continuous map.

Let's recall that a topological space (X, ρ) is called submetrizable if there exists a metric d on X such that the topology ρ_d generated by d is coarser than ρ, i.e., $\rho_d \subseteq \rho$ (see [9]). During his visit in 2012, Arhangelskii introduced the notion of epinormal spaces. A topological space (X, ρ) is said to be epinormal if there exists a coarser topology ρ' on X such that (X, ρ') is T_4.

Remark 2.25 It has been proven in (Theorem 1.2, [6]) that every submetrizable space is C-normal. As an immediate consequence, every epinormal space is also C-normal. From these results, we derive the following two results.

Theorem 2.26 Suppose X is a non-locally compact submetrizable space, with the subspace consisting of all defects being T_2 and can be expressed as a union of pairwise disjoint compact subsets. Then, there exists a D-continuous map on X.

Corollary 2.27 Let X be a non-locally compact epinormal space such that the subspace consisting of all defects is T_2 and can be written as a union of pairwise disjoint compact subsets. Then, there exists a D-continuous map on X.

Proposition 2.28 Suppose $\{X_i\}_{i\in I}$ is a family of pairwise disjoint topological spaces. If there exists a D-continuous map on each X_i, then there also exists a D-continuous map on $\oplus_{i\in I} X_i$.

Proof. Each X_i is closed in the space $\oplus_{i\in I} X_i$. Note that the family $\{X_i\}_{i\in I}$ forms a locally finite closed cover of $\oplus_{i\in I} X_i$. Since there exists a D-continuous map $f_i : X_i \to Y_i$ for each X_i, and the family of maps $\{f_i\}_{i\in I}$ is compatible, we define the map $f = \bigcup_{i\in I} f_i : X_i \to Y_i$. It is clear that f is D-continuous.

2.1 *Spaces of D-continuous maps*

Consider a non-locally compact space X with a non-empty set of defects A. Let $Y = \mathrm{Hom}(X, \mathbb{R})$ denote the set of all D-continuous real-valued functions on X. It is evident that the set of all continuous real-valued functions on X is a subset of Y. We define the closure $\overline{F}$ such that $g \in \overline{F}$ whenever g is the limit of a sequence $\{g_1, g_2, \ldots\}$ in F. We then establish a topology on Y using this closure operator, i.e., $O = \{Y \backslash \overline{F}\}$ forms the topology on Y, known as the topology of uniform convergence on Y. It's worth noting that for any $g \in Y$, the family $\{V_i(g)\}_{i=1}^{\infty}$ serves as a base at the point g for the space Y, where

$$V_i(g) = \{f \in Y : \exists\ an\ c < \frac{1}{i}\}$$

such that

$$|g(x) - f(x)| < c, \ \ \forall\, x \in A.$$

Theorem 2.29 Let $\{f_i\}$ be a sequence of D-continuous map from a topological space X to $\mathbb{R}$. If $\{f_i\}$ is uniformly convergent to a map $f : X \to \mathbb{R}$, then f is a D-continuous map.

Proof. Consider the set A representing all defects in X. We need to show that for every $x_0 \in A$ and $\varepsilon > 0$, there exists a neighborhood V_{x_0} of x_0. such that

$$|f(x_0) - f(x_1)|, \text{ for every } x_1 \in V_{x_0}.$$

Pick an integer l such that $|f(x) - f_i(x)| < \dfrac{\varepsilon}{3}$ for every $x \in X$ and $i \geq l$. Then, it is clear that $|f(x) - f_i(x)| < \dfrac{\varepsilon}{3}$ for every $x \in A$ and $i \geq l$. Since f_l is D-continuous, then there exists a neighborhood V_{x_0} of x_0 such that $|f_i(x_0) - f_i(x_1)| < \dfrac{\varepsilon}{3}$ for every $x_1 \in V_{x_0}$. Pick an arbitrary point $x_1 \in V_{x_0}$, then

$$|f(x_0) - f(x_1)| \leq |f(x_0) - f_l(x_0)| + |f_l(x_0) - f_l(x_1)| + |f_l(x_1) - f(x_1)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence, f is a D-continuous map.

Remark 2.30 According to Theorem 2.14, if the topological space X is countably generated, then any D-continuous $f : X \to \mathbb{R}$ is continuous. The following corollary is an immediate consequence.

Corollary 2.31 Consider $Y = \mathrm{Hom}(X, \ \mathbb{R})$ equipped with the uniform convergence topology. If X is countably generated, then Y represents the space of all continuous real-valued functions on X.

Conflict of interest

The author declares no competing financial interest.

References

[1] Pawlak Z. Rough sets. *International Journal of Computer Information Sciences.* 1982; 11: 341-356.

[2] Yao YY. Relational interpretations of neighborhood operators and rough set approximation operators. *Information Sciences.* 1998; 111(1-4): 239-259.

[3] Alexander A. Functional tightness, Q-spaces and τ-embeddings. *Commentationes Mathematicae Universitatis Carolinae.* 1983; 24(1): 105-120.
[4] Zailai M. Locally compact spaces with defects. *European Journal of Pure and Applied Mathematics.* 2023; 16(2): 1228-1235.
[5] Herrlich H. Topologische Epireflexionen und (mono-) coreflexionen. *Lecture Notes in Mathematics.* 1968; 78: 114-152.
[6] AlZahrani S, Kalantan L. C-normal topological property. *Filomat.* 2017; 31(2): 407-411.
[7] Engelking R. *General Topology.* Berlin: Heldermann Verlag; 1989.
[8] Steen LA, Seebach JA. *Counterexamples in Topology.* New York: Dover Publ; 1995.
[9] Gruenhage G. *Generalized Metric Spaces.* Handbook of Set-Theoretic Topology; 1984.