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Research Article

Certain Fixed Point Results for (α-F)-Contraction in New Controlled S-Metric Type Spaces

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Abstract: In this paper, we introduce the concept of a new Controlled S-Metric type Space (CSMS) by using a function $\Xi : \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B} \to [1, +\infty)$ in the right hand side of the triangle inequality of S-metric space. The triangle inequality in the new CSMS will have the form $S(\rho^o, \iota^o, z) \leq \Xi(\rho^o, \rho^o, w)S(\rho^o, \rho^o, w) + \Xi(\iota^o, \iota^o, w)S(\iota^o, \iota^o, w) + \Xi(z, z, w)S(z, z, w)$. Further, we provide some non-trivial examples, investigate (α, F)-contraction and prove several fixed point results in the setting of new CSMS. Furthermore, we present some of its consequences to illustrate the significance of our results. At the end, we provide an application to support the main result.

***Keywords*:** fixed point, S-metric-type space, controlled function, integral equations

MSC: 47H10, 54H25, 47H99

1. Introduction

Fixed-point theory has a lot of attention because of its numerous applications in diverse branches of mathematics, such as differential equations, nonlinear analysis, and graph theory, see [1–3]. Several researchers concentrated on establishing new metric spaces and different contractions, see [4–10]. There are multiple findings that contribute to the advancement of this theory. A b-metric space is a well-known generalization of a metric space presented by Bakhtin [11] in 1989 and provided later by Czerwik [12] by introducing a new fixed-point theorem in this space.

The notion of the b-metric space has encouraged numerous scholars to examine fixed-point results under various hypotheses, see [13–16]. Also, authors in [17–22] presented a number of generalizations of b-metric spaces. In 2017, Kamran et al. [23] generalized the concept of the b-metric space by replacing the constant $s \geq 1$ with a function $\Xi(\rho^o, \iota^o)$ in the triangle inequality and obtained an extended b-metric space. Mlaiki et al. [24] provided one of the generalizations of b-metric space. They introduced the concept of a controlled metric-type space by applying the control function $\Xi : \mathfrak{B} \times \mathfrak{B} \to [1, +\infty)$ to the triangle inequality as:

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$$d(\rho^o, \iota^o) \leq \Xi(\rho^o, z)d(\rho^o, z) + \Xi(z, \iota^o)d(z, \iota^o).$$

Various generalizations of controlled metric-type spaces have also been defined [15–18]. Notice that a controlled metric space is not an extended b-metric space; see [24]. Sedghi et al. [25] introduced the notion of an S-metric space. Recently, Yazici et al. [26] presented the notion of controlled S-metric-type space by utilizing the S-metric [25] and control function. An extension of S-metric spaces was given in [27], where S_b-metric spaces was introduced with a symmetry condition. The notion of extended S-metric space of type (α, β) is introduced in [28]. This extension is a generalization of S-metric space, defined by employing two functions instead of considering a constant in the second condition of the S-metric space definition. The concept of a new extended b-metric space was defined in [29], by extending the triangle inequality to a functional triangular inequality.

In this study, we introduce a new CSMS, where the controlled function Ξ is replaced by three variables. We establish several fixed point results by using (α, F)-contractions in new CSMS. Moreover, we provide some consequences to illustrate the significance of our results. At the end, we provide an application to show the validity of a main result.

2. Preliminaries

In this section, we provide some basic definitions from the existing literature for better understanding of readers.

Definition 1 [25] Let $\mathfrak{B}$ be a nonempty set. A function $S : \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B} \to [0, +\infty)$ is called S-metric, if for each $\rho^o, \iota^o, z, w \in \mathfrak{B}$, it satisfies:

(1) $S(\rho^o, \iota^o, z) \geq 0$;

(2) $S(\rho^o, \iota^o, z) = 0$ if and only if $\rho^o = \iota^o = z$;

(3) $S(\rho^o, \iota^o, z) \leq S(\rho^o, \rho^o, w) + S(\iota^o, \iota^o, w) + S(z, z, w)$.

The pair $(\mathfrak{B}, S)$ is called S-metric space.

Example 1 [25] Let $\mathfrak{B} = [-1, 1]$ and $\|.\|$ be a norm on $\mathfrak{B}$, then $S : \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B} \to [0, +\infty)$ defined by

$$S(\rho^o, \iota^o, z) = \|\iota^o + z - 2\rho^o\| + \|\iota^o - z\|$$

is S-metric on $\mathfrak{B}$.

Definition 2 [24] Let $\mathfrak{B}$ be a nonempty set and $\Xi : \mathfrak{B} \times \mathfrak{B} \to [1, +\infty)$ be a function. A function $d : \mathfrak{B} \times \mathfrak{B} \to [0, +\infty)$ is called controlled metric on $\mathfrak{B}$ if for each $\rho^o, \iota^o, z \in \mathfrak{B}$, it satisfies:

(1) $d(\rho^o, \iota^o) = 0$ if and only if $\rho^o = \iota^o$;

(2) $d(\rho^o, \iota^o) = d(\iota^o, \rho^o)$;

(3) $d(\rho^o, \iota^o) \leq \Xi(\rho^o, z)d(\rho^o, z) + \Xi(z, \iota^o)d(z, \iota^o)$.

The pair $(\mathfrak{B}, d)$ is called controlled metric-type space.

Example 2 [24] Let $\mathfrak{B} = \{0, 1, 2\}$ and a function $d : \mathfrak{B} \times \mathfrak{B} \to [0, +\infty)$ be defined as

$$d(\rho^o, \iota^o) = \begin{cases} 0, & \text{if } \rho^o = \iota^o; \\ 1, & \text{if } \rho^o = 1, \iota^o = 0 \text{ or } \rho^o = 0, \iota^o = 1; \\ \frac{1}{2}, & \text{if } \rho^o = 0, \iota^o = 2 \text{ or } \rho^o = 2, \iota^o = 0; \\ \frac{2}{5}, & \text{if } \rho^o = 1, \iota^o = 2 \text{ or } \rho^o = 2, \iota^o = 1. \end{cases}$$

Define a mapping $\Xi : \mathfrak{B} \times \mathfrak{B} \to [1, +\infty)$ by

$$\Xi(0, 0) = \Xi(1, 1) = \Xi(2, 2) = \Xi(0, 2) = 1, \ \Xi(1, 2) = \frac{5}{4}, \ \Xi(0, 1) = \frac{11}{10}.$$

Then, $(\mathfrak{B}, d)$ is controlled metric type space. In the below Figure 1, we added a heatmap of an Example 2.

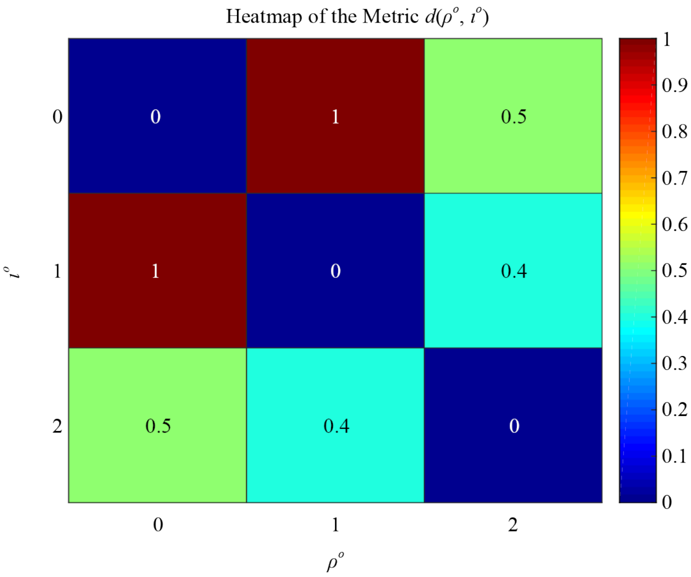

Figure 1. Heatmap of the above example

Definition 3 [26] Let $\mathfrak{B}$ be a nonempty set and $\Xi : \mathfrak{B} \times \mathfrak{B} \to [1, +\infty)$ be a function. A function $S : \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B} \to [0, +\infty)$ is called controlled S-metric, if for each $\rho^o, \iota^o, z, w \in \mathfrak{B}$, it satisfies:

(1) $S(\rho^o, \iota^o, z) \geq 0$;

(2) $S(\rho^o, \iota^o, z) = 0$ if and only if $\rho^o = \iota^o = z$;

(3) $S(\rho^o, \iota^o, z) \leq \Xi(\rho^o, w)S(\rho^o, \rho^o, w) + \Xi(\iota^o, w)S(\iota^o, \iota^o, w) + \Xi(z, w)S(z, z, w)$.

The pair $(\mathfrak{B}, S)$ is called controlled S-metric-type space.

Example 3 [26] Let $\mathfrak{B}$ be a non-empty set and $(\mathfrak{B}, d)$ be a controlled metric-type space with a symmetric function $\Xi : \mathfrak{B} \times \mathfrak{B} \to [1, +\infty)$. Define $S : \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B} \to [0, +\infty)$ by

$$S(\rho^o, \iota^o, z) = d(\rho^o, \iota^o) + d(\rho^o, z) + d(\iota^o, z).$$

Then, S is controlled S-metric.

3. Main results

In this section, we introduce the concept of new CSMS and prove a Banach version of fixed point.

Definition 4 Let $\mathfrak{B}$ be a nonempty set and $\Xi : \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B} \to [1, +\infty)$ be a function. A function $S : \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B} \to [0, +\infty)$ is called a new controlled S-metric, if for each $\rho^o, \iota^o, z, w \in \mathfrak{B}$, it satisfies:

$(S1)$ $S(\rho^o, \iota^o, z) \geq 0$;

$(S2)$ $S(\rho^o, \iota^o, z) = 0$ if and only if $\rho^o = \iota^o = z$;

$(S3)$ $S(\rho^o, \iota^o, z) \leq \Xi(\rho^o, \rho^o, w)S(\rho^o, \rho^o, w) + \Xi(\iota^o, \iota^o, w)S(\iota^o, \iota^o, w) + \Xi(z, z, w)S(z, z, w)$.

The pair $(\mathfrak{B}, S)$ is called a new controlled S-metric-type space.

Remark 1 If we take $\Xi(\rho^o, \rho^o, w) = \Xi(\rho^o, w)$, $\Xi(\iota^o, \iota^o, w) = \Xi(\iota^o, w)$ and $\Xi(z, z, w) = \Xi(z, w)$, then new controlled S-metric-type space becomes controlled S-metric-type space in [24].

Example 4 Let $\mathfrak{B} = \mathbb{N}$ and a function $S : \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B} \to [0, +\infty)$ be defined as

$$S(\rho^o, \iota^o, z) = \begin{cases} 0, & \text{if } \rho^o = \iota^o = z; \\ \dfrac{1}{\rho^o}, & \text{if } \rho^o = \iota^o \text{ is even and } z \text{ is odd}; \\ \dfrac{1}{z}, & \text{if } \rho^o = \iota^o \text{ is odd and } z \text{ is even}; \\ 1, & \text{otherwise.} \end{cases}$$

Define a mapping $\Xi : \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B} \to [1, +\infty)$ by

$$\Xi(\rho^o, \iota^o, z) = \begin{cases} 1, & \text{if } \rho^o = \iota^o = z\,; \\ \dfrac{2+\rho^o}{1+\rho^o}, & \text{if } \rho^o = \iota^o \text{ is even and } z \text{ is odd}; \\ \dfrac{2+z}{1+z}, & \text{if } \rho^o = \iota^o \text{ is odd and } z \text{ is even}; \\ \dfrac{3}{2}, & \text{otherwise.} \end{cases}$$

Conditions $(S1)$ and $(S2)$ are clearly satisfied. For condition $(S3)$, consider the cases:

Case 1. $\rho^o = \iota^o = z$;

Case 2. ρ^o, ι^o, z are all even;

Case 3. ρ^o and ι^o are even and z is odd;

Case 4. ρ^o and ι^o are even with $\rho^o = \iota^o$ and z is odd;

Case 5. ρ^o and z are even and ι^o is odd;
Case 6. ρ^o is even, ι^o and z are odd;
Case 7. ρ^o is odd, ι^o and z are even;
Case 8. ρ^o and z are odd and ι^o is even;
Case 9. ρ^o and ι^o are odd and z is even;
Case 10. ρ^o and ι^o are odd with $\rho^o = \iota^o$ and z is even;
Case 11. ρ^o, ι^o and z are all odd.
Then, (ρ^o, S) is a new controlled S-metric-type space. We plot the graph of this example in Figure 2.
Lemma 1 In a new CSMS, $S(\rho^o, \rho^o, \iota^o) = S(\iota^o, \iota^o, \rho^o)$.

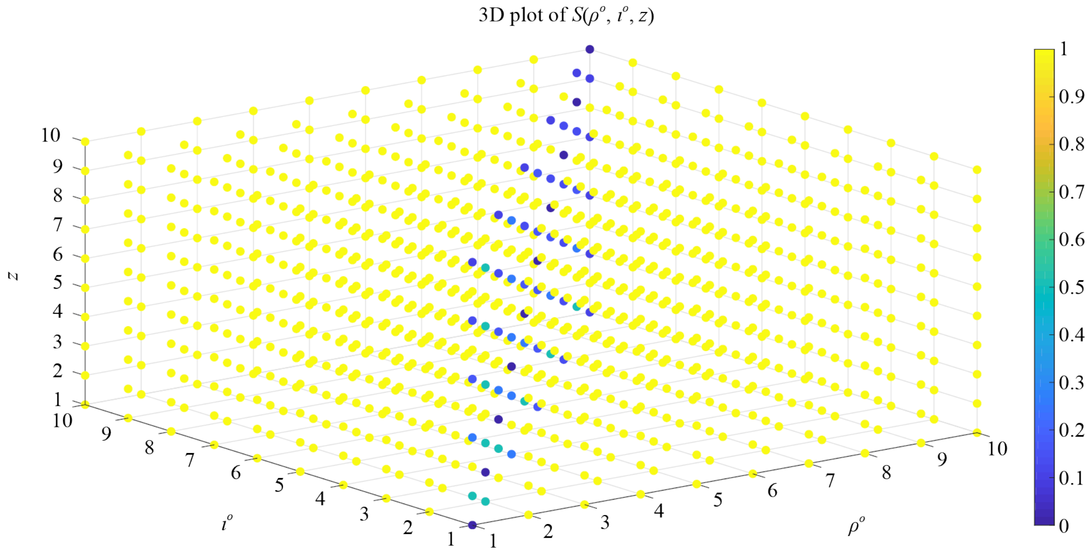

Figure 2. Depict the graphical behaviour of new CSMS in Example 4

Proof. By axiom ($S3$) of Definition 4, we have

$$S(\rho^o, \rho^o, \iota^o) \leq \Xi(\rho^o, \rho^o, \rho^o)S(\rho^o, \rho^o, \rho^o) + \Xi(\rho^o, \rho^o, \rho^o)S(\rho^o, \rho^o, \rho^o) + \Xi(\iota^o, \iota^o, \rho^o)S(\iota^o, \iota^o, \rho^o). \tag{1}$$

Similarly,

$$S(\iota^o, \iota^o, \rho^o) \leq \Xi(\iota^o, \iota^o, \iota^o)S(\iota^o, \iota^o, \iota^o) + \Xi(\iota^o, \iota^o, \iota^o)S(\iota^o, \iota^o, \iota^o) + \Xi(\rho^o, \rho^o, \iota^o)S(\rho^o, \rho^o, \iota^o). \tag{2}$$

From equation (1) and (2), we get

$$S(\rho^o, \rho^o, \iota^o) \leq \Xi(\iota^o, \iota^o, \rho^o)S(\iota^o, \iota^o, \rho^o), \tag{3}$$

$$S(\iota^o, \iota^o, \rho^o) \leq \Xi(\rho^o, \rho^o, \iota^o)S(\rho^o, \rho^o, \iota^o). \tag{4}$$

As $\Xi(\iota^o, \iota^o, \rho^o) \geq 1$ and $\Xi(\rho^o, \rho^o, \iota^o) \geq 1$, so by equation (3) and (4), we conclude that

$$S(\rho^o, \rho^o, \iota^o) = S(\iota^o, \iota^o, \rho^o).$$

□

Theorem 1 Let $(\mathfrak{B}, S)$ be a complete new controlled S-metric-type space and $\Delta : \mathfrak{B} \to \mathfrak{B}$ satisfies:

$$S(\Delta\rho^o, \Delta\rho^o, \Delta\iota^o) \leq kS(\rho^o, \rho^o, \iota^o)$$

for all $\rho^o, \iota^o \in \mathfrak{B}$. For $\rho_0^o \in \mathfrak{B}$, take $\rho_n^o = \Delta^n \rho_0^o$. Suppose that

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\Xi(\rho_{i+1}^o, \rho_{i+1}^o, \rho_{i+2}^o)}{\Xi(\rho_i^o, \rho_i^o, \rho_{i+1}^o)} \Xi(\rho_{i+1}^o, \rho_m^o, \rho_m^o) < \frac{1}{k}$$

for $0 < k < 1$. Moreover, assume that for each $\rho^o \in \mathfrak{B}$, we have

$$\lim_{n\to+\infty} \Xi(\rho^o, \rho_n^o, \rho_{n+1}^o) \text{ and } \lim_{n\to+\infty} \Xi(\rho^o, \rho_n^o, \Delta\rho^o)$$

exist and are finite. Then, Δ has a unique fixed point.

Proof. For $\rho_0^o \in \mathfrak{B}$, consider the sequence $\rho_n^o = T^n \rho_0^o$. For all $n \geq 0$, we have

$$S(\rho_n^o, \rho_n^o, \rho_{n+1}^o) = S(\Delta^n \rho_0^o, \Delta^n \rho_0^o, \Delta^n \rho_1^o) \leq k^n S(\rho_0^o, \rho_0^o, \rho_1^o). \tag{5}$$

For $n < m$, we have

$$\begin{aligned}
&S(\rho_n^o, \rho_n^o, \rho_m^o)\\
&\leq \Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)S(\rho_n^o, \rho_n^o, \rho_{n+1}^o) + \Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)S(\rho_n^o, \rho_n^o, \rho_{n+1}^o) + \Xi(\rho_m^o, \rho_m^o, \rho_{n+1}^o)S(\rho_m^o, \rho_m^o, \rho_{n+1}^o).\\
&= 2\Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)S(\rho_n^o, \rho_n^o, \rho_{n+1}^o) + \Xi(\rho_m^o, \rho_m^o, \rho_{n+1}^o)S(\rho_{n+1}^o, \rho_{n+1}^o, \rho_m^o)\\
&\leq 2\Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)S(\rho_n^o, \rho_n^o, \rho_{n+1}^o) + \Xi(\rho_m^o, \rho_m^o, \rho_{n+1}^o)[\Xi(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o)S(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o)\\
&\quad + \Xi(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o)S(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o) + \Xi(\rho_{n+2}^o, \rho_{n+2}^o, \rho_m^o)S(\rho_{n+2}^o, \rho_{n+2}^o, \rho_m^o)].\\
&= 2\Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)S(\rho_n^o, \rho_n^o, \rho_{n+1}^o) + 2\Xi(\rho_m^o, \rho_m^o, \rho_{n+1}^o)\Xi(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o)S(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o)\\
&\quad + \Xi(\rho_m^o, \rho_m^o, \rho_{n+1}^o)\Xi(\rho_{n+2}^o, \rho_{n+2}^o, \rho_m^o)S(\rho_{n+2}^o, \rho_{n+2}^o, \rho_m^o)
\end{aligned}$$

$$
\begin{aligned}
&\le 2\Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)S(\rho_n^o, \rho_n^o, \rho_{n+1}^o) + 2\Xi(\rho_m^o, \rho_m^o, \rho_{n+1}^o)\Xi(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o)S(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o) \\
&\quad + \Xi(\rho_m^o, \rho_m^o, \rho_{n+1}^o)\Xi(\rho_{n+2}^o, \rho_{n+2}^o, \rho_m^o)[2\Xi(\rho_{n+2}^o, \rho_{n+2}^o, \rho_{n+3}^o)S(\rho_{n+2}^o, \rho_{n+2}^o, \rho_{n+3}^o) \\
&\quad + \Xi(\rho_{n+3}^o, \rho_{n+3}^o, \rho_m^o)S(\rho_{n+3}^o, \rho_{n+3}^o, \rho_m^o)] \\
&\le 2\Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)S(\rho_n^o, \rho_n^o, \rho_{n+1}^o) + 2\Xi(\rho_m^o, \rho_m^o, \rho_{n+1}^o)\Xi(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o)S(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o) \\
&\quad + 2\Xi(\rho_m^o, \rho_m^o, \rho_{n+1}^o)\Xi(\rho_{n+2}^o, \rho_{n+2}^o, \rho_m^o)\Xi(\rho_{n+2}^o, \rho_{n+2}^o, \rho_{n+3}^o)S(\rho_{n+2}^o, \rho_{n+2}^o, \rho_{n+3}^o) \\
&\quad + \Xi(\rho_m^o, \rho_m^o, \rho_{n+1}^o)\Xi(\rho_{n+2}^o, \rho_{n+2}^o, \rho_m^o)\Xi(\rho_{n+3}^o, \rho_{n+3}^o, \rho_m^o)S(\rho_{n+3}^o, \rho_{n+3}^o, \rho_m^o)] \\
&\quad \vdots \\
&\le 2\Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)S(\rho_n^o, \rho_n^o, \rho_{n+1}^o) + 2\sum_{i=n+1}^{m-2}\left\{\prod_{j=n+1}^{i}\Xi(\rho_m^o, \rho_m^o, \rho_j^o)\right\}\Xi(\rho_i^o, \rho_i^o, \rho_{i+1}^o)S(\rho_i^o, \rho_i^o, \rho_{i+1}^o) \\
&\quad + 2\left\{\prod_{i=n+1}^{m-1}\Xi(\rho_i^o, \rho_i^o, \rho_m^o)\right\}S(\rho_{m-1}^o, \rho_{m-1}^o, \rho_m^o) \\
&\le 2\Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)k^n S(\rho_0^o, \rho_0^o, \rho_1^o) + 2\sum_{i=n+1}^{m-2}\left\{\prod_{j=n+1}^{i}\Xi(\rho_m^o, \rho_m^o, \rho_j^o)\right\}\Xi(\rho_i^o, \rho_i^o, \rho_{i+1}^o)k^i S(\rho_0^o, \rho_0^o, \rho_1^o) \\
&\quad + 2\left\{\prod_{i=n+1}^{m-1}\Xi(\rho_i^o, \rho_i^o, \rho_m^o)\right\}k^{m-1}S(\rho_0^o, \rho_0^o, \rho_1^o) \\
&\le 2\Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)k^n S(\rho_0^o, \rho_0^o, \rho_1^o) + 2\sum_{i=n+1}^{m-2}\left\{\prod_{j=n+1}^{i}\Xi(\rho_m^o, \rho_m^o, \rho_j^o)\right\}\Xi(\rho_i^o, \rho_i^o, \rho_{i+1}^o)k^i S(\rho_0^o, \rho_0^o, \rho_1^o) \\
&\quad + 2\left\{\prod_{i=n+1}^{m-1}\Xi(\rho_i^o, \rho_i^o, \rho_m^o)\right\}\Xi(\rho_{m-1}^o, \rho_{m-1}^o, \rho_m^o)k^{m-1}S(\rho_0^o, \rho_0^o, \rho_1^o) \\
&\le 2\Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)k^n S(\rho_0^o, \rho_0^o, \rho_1^o) + 2\sum_{i=n+1}^{m-1}\left\{\prod_{j=n+1}^{i}\Xi(\rho_m^o, \rho_m^o, \rho_j^o)\right\}\Xi(\rho_i^o, \rho_i^o, \rho_{i+1}^o)k^i S(\rho_0^o, \rho_0^o, \rho_1^o)
\end{aligned}
$$

$$\leq 2\Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)k^n S(\rho_0^o, \rho_0^o, \rho_1^o) + 2\sum_{i=n+1}^{m-1}\left\{\prod_{j=0}^{i}\Xi(\rho_m^o, \rho_m^o, \rho_j^o)\right\}\Xi(\rho_i^o, \rho_i^o, \rho_{i+1}^o)k^i S(\rho_0^o, \rho_0^o, \rho_1^o).$$

Let

$$\rho_p^o = \sum_{i=0}^{p}\left\{\prod_{j=0}^{i}\Xi(\rho_m^o, \rho_m^o, \rho_j^o)\right\}\Xi(\rho_i^o, \rho_i^o, \rho_{i+1}^o)k^i.$$

Then

$$S(\rho_n^o, \rho_n^o, \rho_m^o) \leq [2\Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)S(\rho_n^o, \rho_n^o, \rho_{n+1}^o)k^n + 2(\rho_{m-1}^o - \rho_n^o)]$$

$$= 2S(\rho_0^o, \rho_0^o, \rho_1^o)[k^n\Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o) + (\rho_{m-1}^o - \rho_n^o)]$$

Using the given condition and by ratio test $\lim_{n\to+\infty}\rho_n^o$ exists. Hence, $\{\rho_n^o\}$ is a Cauchy sequence. Since, $\mathfrak{B}$ is complete so there exists $\rho^o \in \mathfrak{B}$ such that $\{\rho_n^o\} \to \rho^o$.

Now to prove ρ^o is fixed point of T, we have

$$S(\rho_{n+1}^o, \rho_{n+1}^o, \rho_n^o) \leq 2\Xi(\rho_{n+1}^o, \rho_{n+1}^o, \rho_n^o)S(\rho_{n+1}^o, \rho_{n+1}^o, \rho_n^o) + \Xi(\rho_n^o, \rho_n^o, \rho^o)S(\rho_n^o, \rho_n^o, \rho^o).$$

Applying $\lim n \to +\infty$, we get

$$\lim_{n\to+\infty} S(\rho_{n+1}^o, \rho_{n+1}^o, \rho_n^o) = 0.$$

Now,

$$S(\Delta\rho^o, \Delta\rho^o, \rho^o) = S(\rho^o, \rho^o, \Delta\rho^o)$$

$$\leq 2\Xi(\rho^o, \rho^o, \rho_{n+1}^o)S(\rho^o, \rho^o, \rho_{n+1}^o) + \Xi(\rho_{n+1}^o, \rho_{n+1}^o, \Delta\rho^o)S(\rho_{n+1}^o, \rho_{n+1}^o, \Delta\rho^o)$$

$$\leq 2\Xi(\rho^o, \rho^o, \rho_{n+1}^o)S(\rho^o, \rho^o, \rho_{n+1}^o) + \Xi(\rho_{n+1}^o, \rho_{n+1}^o, \Delta\rho^o)kS(\rho_n^o, \rho_n^o, \rho^o)$$

by taking $\lim n \to +\infty$, we have $S(\Delta\rho^o, \Delta\rho^o, \rho^o) = 0$. Which implies $\Delta\rho^o = \rho^o$. Now to show uniqueness, let us consider another fixed point ι^o such that $\Delta\iota^o = \iota^o$ and $\rho^o \neq \iota^o$.

Now,

$$S(\rho^o, \rho^o, \iota^o) = S(\Delta\rho^o, \Delta\rho^o, \Delta\iota^o) \leq kS(\rho^o, \rho^o, \iota^o),$$

which is a contradiction. Thus, we have $S(\rho^o, \rho^o, \iota^o) = 0$ that is $\rho^o = \iota^o$. □

Example 5 Let $\mathfrak{B} = [0, 1]$ and define $S : \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B} \to [0, +\infty)$ by

$$S(\rho^o, \iota^o, z) = d(\rho^o, z) + d(\iota^o, z),$$

where $d(\rho^o, \iota^o) = |\rho^o - \iota^o|^2$. Define $\Xi : \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B} \to [1, +\infty)$ by $\Xi(\rho^o, \iota^o, z) = 1 + \rho^o + \iota^{o2} + z^3$. Then, $(\mathfrak{B}, S)$ is a complete CSMS. Define a mapping $\Delta : \mathfrak{B} \to \mathfrak{B}$ by

$$\Delta(\rho^o) = \frac{\rho^o}{2}.$$

Then, all the conditions of Theorem 1 are satisfied and 0 is a unique fixed point of Δ. Below Figure 3 shows that the left hand side of the contraction is less than the right hand side of the contraction. Figure 4 shows that 0 is a unique fixed point of Δ.

Next, we will introduce a family of functions to investigate some fixed point results on a new controlled S-metric-type space.

Let $\mathfrak{M}$ be a family of all continuous functions $\mathscr{M}$ such that $\mathscr{M} : \mathbb{R}_+^5 \to \mathbb{R}_+$. For some $k \in [0, 1)$, we consider the following conditions:

(1) For all $\rho^o, \iota^o, z \in \mathbb{R}_+$, if $\iota^o \leq \mathscr{M}(\rho^o, \rho^o, 0, z, \iota^o)$ with $z \leq 2\rho^o + \iota^o$, then $\iota^o \leq k\rho^o$.

(2) For all $\iota^o \in \mathbb{R}_+$, if $\iota^o \leq \mathscr{M}(\iota^o, 0, \iota^o, \iota^o, 0)$, then $\iota^o = 0$.

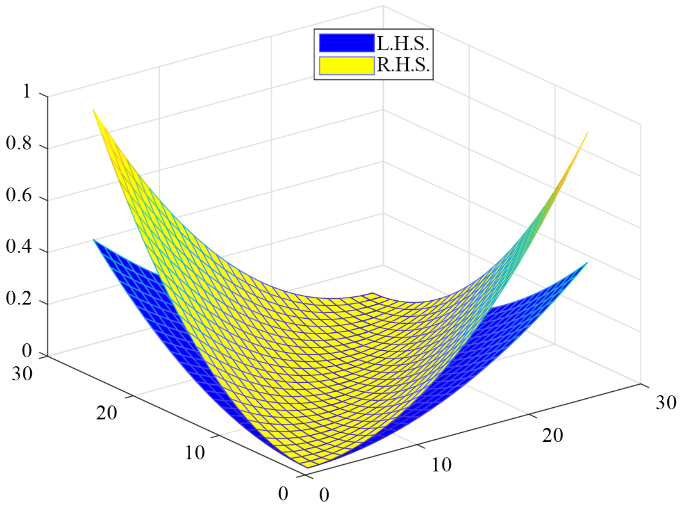

Figure 3. Depicts the graphical behaviour of an Example 5, which shows that contraction mapping in Theorem 1 is satisfied

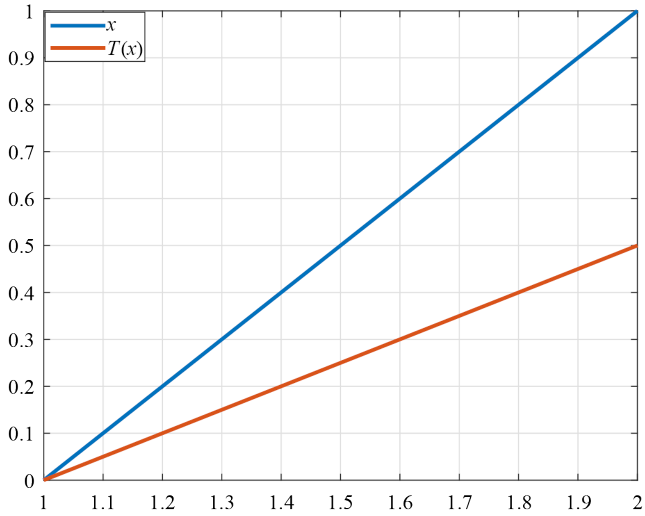

Figure 4. Depicts that 0 is a unique fixed point of T in Example 5

Theorem 2 Let $(\mathfrak{B}, S)$ be a new controlled S-metric-type space and $\Delta : \mathfrak{B} \to \mathfrak{B}$ satisfies

$$S(\Delta\rho^o, \Delta\rho^o, \Delta\iota^o) \leq \mathscr{M}(S(\rho^o, \rho^o, \iota^o), S(\Delta\rho^o, \Delta\rho^o, \rho^o), S(\Delta\rho^o, \Delta\rho^o, \iota^o), S(\Delta\iota^o, \Delta\iota^o, \rho^o), S(\Delta\iota^o, \Delta\iota^o, \iota^o)), \quad (6)$$

for all $\rho^o, \iota^o, z \in S$ and $\mathscr{M} \in \mathfrak{M}$. For $\rho_0^o \in \mathfrak{B}$, take $\rho_n^o = \Delta^n \rho_0^o$. Suppose that

$$\sup_{m \geq 1} \lim_{i \to +\infty} \frac{\Xi(\rho_{i+1}^o, \rho_{i+1}^o, \rho_{i+2}^o)}{\Xi(\rho_i^o, \rho_i^o, \rho_{i+1}^o)} \Xi(\rho_{i+1}^o, \rho_m^o, \rho_m^o) < \frac{1}{k},$$

for $0 < k < 1$. Moreover, assume that for each $\rho^o \in \mathfrak{B}$ we have

$$\lim_{n \to +\infty} \Xi(\rho^o, \rho_n^o, \rho_{n+1}^o) \text{ and } \lim_{n \to +\infty} \Xi(\rho^o, \rho_n^o, \Delta\rho^o)$$

exist and are finite. Then, we have the following:

(i) If $\mathscr{M}$ satisfies condition (1), then Δ has a fixed point.

(ii) If $\mathscr{M}$ satisfies condition (2) and Δ has a fixed point. Then, this fixed point is unique.

Proof. For $\rho_0^o \in \mathfrak{B}$, consider the sequence $\rho_n^o = \Delta^n \rho_0^o$. By using equation (6) and Lemma 1, we have

$$S(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o)$$

$$= S(\Delta\rho_n^o, \Delta\rho_n^o, \Delta\rho_{n+1}^o)$$

$$\leq \mathscr{M}(S(\rho_n^o, \rho_n^o, \rho_{n+1}^o), S(\Delta\rho_n^o, \Delta\rho_n^o, \rho_n^o), S(\Delta\rho_n^o, \Delta\rho_n^o, \rho_{n+1}^o),$$

$$S(\Delta\rho^o_{n+1}, \Delta\rho^o_{n+1}, \rho^o_n), S(\Delta\rho^o_{n+1}, \Delta\rho^o_{n+1}, \rho^o_{n+1}))$$

$$= \mathscr{M}(S(\rho^o_n, \rho^o_n, \rho^o_{n+1}), S(\rho^o_{n+1}, \rho^o_{n+1}, \rho^o_n), S(\rho^o_{n+1}, \rho^o_{n+1}, \rho^o_{n+1}),$$

$$S(\rho^o_{n+2}, \rho^o_{n+2}, \rho^o_n), S(\rho^o_{n+2}, \rho^o_{n+2}, \rho^o_{n+1}))$$

$$= \mathscr{M}(S(\rho^o_n, \rho^o_n, \rho^o_{n+1}), S(\rho^o_{n+1}, \rho^o_{n+1}, \rho^o_n), 0,$$

$$S(\rho^o_n, \rho^o_n, \rho^o_{n+2}), S(\rho^o_{n+1}, \rho^o_{n+1}, \rho^o_{n+2})).$$

From the Definition of new controlled S-type-metric space, we have

$$S(\rho^o_n, \rho^o_n, \rho^o_{n+2}) \leq 2\Xi(\rho^o_n, \rho^o_n, \rho^o_{n+1})S(\rho^o_n, \rho^o_n, \rho^o_{n+1}) + \Xi(\rho^o_{n+2}, \rho^o_{n+2}, \rho^o_{n+1})S(\rho^o_{n+2}, \rho^o_{n+2}, \rho^o_{n+1}).$$

As $\mathscr{M}$ satisfies condition (1), we get

$$S(\rho^o_{n+1}, \rho^o_{n+1}, \rho^o_{n+2}) \leq kS(\rho^o_n, \rho^o_n, \rho^o_{n+1}) \leq k^{n+1}S(\rho^o_0, \rho^o_0, \rho^o_1), \tag{7}$$

for $k \in (0, 1)$. By using the same procedure as Theorem 1, we get a Cauchy sequence $\{\rho^o_n\}$. Since $\mathfrak{B}$ is complete so there exists $\mathfrak{B} \in \mathfrak{B}$ such that $\{x_n\} \to x$.

(i) Now we prove that ρ^o is a fixed point of Δ.

$$S(\rho^o_{n+1}, \rho^o_{n+1}, \Delta\rho^o) = S(\Delta\rho^o_n, \Delta\rho^o_n, \Delta\rho^o)$$

$$\leq \mathscr{M}(S(\rho^o_n, \rho^o_n, \rho^o), S(\Delta\rho^o_n, \Delta\rho^o_n, \rho^o_n), S(\Delta\rho^o_n, \Delta\rho^o_n, \rho^o), S(\Delta\rho^o, \Delta\rho^o, \rho^o_n),$$

$$S(\Delta\rho^o, \Delta\rho^o, \rho^o))$$

$$= \mathscr{M}(S(\rho^o_n, \rho^o_n, \rho^o), S(\rho^o_{n+1}, \rho^o_{n+1}, \rho^o_n), S(\rho^o_{n+1}, \rho^o_{n+1}, \rho^o), S(\Delta\rho^o, \Delta\rho^o, \rho^o_n),$$

$$S(\Delta\rho^o, \Delta\rho^o, \rho^o)).$$

Since $\mathscr{M} \in \mathfrak{M}$. Using Lemma 1 and by taking limit as $n \to +\infty$, we get

$$S(\rho^o, \rho^o, \Delta\rho^o) \leq \mathscr{M}(0, 0, 0, S(\rho^o, \rho^o, \Delta\rho^o), S(\rho^o, \rho^o, \Delta\rho^o)).$$

Since $\mathscr{M}$ satisfies (1), $S(\rho^o, \rho^o, \Delta\rho^o) \leq 0$, that is $\Delta\rho^o = \rho^o$.

(ii) For uniqueness, let ι^o be another fixed point of Δ. By using Lemma 1 and equation (6), we have

$$S(\rho^o, \rho^o, \iota^o) = S(\Delta\rho^o, \Delta\rho^o, \Delta\iota^o)$$

$$\leq \mathscr{M}(S(\rho^o, \rho^o, \iota^o), S(\Delta\rho^o, \Delta\rho^o, \rho^o), S(\Delta\rho^o, \Delta\rho^o, \iota^o), S(\Delta\iota^o, \Delta\iota^o, \rho^o), S(\Delta\iota^o, \Delta\iota^o, \iota^o))$$

$$= \mathscr{M}(S(\rho^o, \rho^o, \iota^o), S(\rho^o, \rho^o, \rho^o), S(\rho^o, \rho^o, \iota^o), S(\iota^o, \iota^o, \rho^o), S(\iota^o, \iota^o, \iota^o))$$

$$= \mathscr{M}(S(\rho^o, \rho^o, \iota^o), 0, S(\rho^o, \rho^o, \iota^o), S(\rho^o, \rho^o, \iota^o), 0).$$

As $\mathscr{M}$ satisfies condition (2), we have $\iota^o = S(\rho^o, \rho^o, \iota^o) = 0$, that is $\rho^o = \iota^o$. □

Corollary 1 In Theorem 2, if we take $\mathscr{M}(\rho^o, \iota^o, z, s, t) = L\rho^o$, then we obtain Theorem 1.

4. (α, F)-contractions

In the setting of complete metric spaces, Samet et al. [25] established the concept of α-admissible mappings and demonstrated several associated fixed point results. In the context of complete metric spaces, Wardowski [11] recently developed a novel kind of contractions known as F-contractions and established several new related fixed point theorems.

Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a function fulfilling:

$(F1)$ F is strictly increasing, that is, for all $\rho_1^o, \rho_2^o \in \mathbb{R}^+$ such that $\rho_1^o < \rho_2^o$ implies that $F(\rho_1^o) < F(\rho_2^o)$.

$(F2)$ For every sequence ρ_n^o of positive real numbers, $\lim_{n\to+\infty} \rho_n^o = 0$ and $\lim_{n\to+\infty} F(\rho_n^o) = -\infty$ are equivalent.

$(F3)$ There is $h \in (0, 1)$ so that $\lim_{\rho^o\to 0^+} \rho^{oh} F(\rho^o) = 0$.

Let $\mathbb{F}$ be the set of above functions F satisfying $(F1)$-$(F3)$.

A self mapping $\Delta : \mathfrak{B} \to \mathfrak{B}$ on a metric $\mathfrak{B}$ is said to be an F-contraction if there is a function F satisfying $(F1)$-$(F3)$ and a constant $\tau > 0$ so that for all $\rho^o, \iota^o \in \mathfrak{B}$

$$d(\Delta\rho^o, \Delta\iota^o) > 0 \Longrightarrow \tau + F(d(\Delta\rho^o, \Delta\tilde{\iota^o})) \leq F(d(\rho^o, \iota^o)).$$

Definition 5 [28] Let $\mathfrak{B}$ be a nonempty set and $\alpha : \mathfrak{B} \times \mathfrak{B} \to [0, +\infty)$ be a given function. A self mapping Δ on $\mathfrak{B}$ is called α-admissible if for all $\rho^o, \iota^o \in \rho^o$

$$\alpha(\rho^o, \iota^o) > 1 \Longrightarrow \alpha(\Delta\rho^o, \Delta\iota^o) > 1.$$

Definition 6 A self mapping $\Delta : \mathfrak{B} \to \mathfrak{B}$ on a controlled metric $\mathfrak{B}$ is said to be an (α, F)-contraction if there is a map $\alpha : \mathfrak{B} \times \mathfrak{B} \to [0, +\infty)$, $F \in \mathbb{F}$ and a constant $\tau > 0$ so that

$$\tau + \alpha(\rho^o, \iota^o) F(d(\Delta\rho^o, \Delta\iota^o)) \leq F(d(\rho^o, \iota^o))$$

for all $\rho^o, \iota^o \in \mathfrak{B}$ with $d(\Delta\rho^o, \Delta\iota^o) > 0$.

Now, we first define (α, F)-contraction for new controlled S-metric-type space.

Definition 7 Let $(\mathfrak{B}, S)$ be a new controlled S-metric-type space. A self mapping $\Delta : \mathfrak{B} \to \mathfrak{B}$ on $\mathfrak{B}$ is said to be an (α, F)-contraction if there is a map $\alpha : \mathfrak{B} \times \mathfrak{B} \to [0, +\infty)$, $F \in \mathbb{F}$ and a constant $\tau > 0$ so that

$$\tau + \alpha(\rho^o, \iota^o)F(S(\Delta\rho^o, \Delta\iota^o, \Delta z)) \leq F(S(\rho^o, \iota^o, z)) \tag{8}$$

for all $\rho^o, \iota^o, z \in \mathfrak{B}$ with $S(\Delta\rho^o, \Delta\iota^o, \Delta z) > 0$.

Theorem 3 Let $(\mathfrak{B}, S)$ be a complete new controlled S-metric-type space. Let $\Delta : \mathfrak{B} \to \mathfrak{B}$ be an (α, F)-contraction so that

(1) Δ is α-admissible,

(2) there exists an $\rho_0^o \in \mathfrak{B}$ such that $\alpha(\rho_0^o, \Delta\rho_0^o) \geq 1$,

(3) for $\rho_0^o \in \mathfrak{B}$, define a Picard sequence $\{\rho_n^o = \Delta^n \rho_0^o\}$ such that

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\Xi(\rho_{i+1}^o, \rho_{i+1}^o, \rho_{i+2}^o)}{\Xi(\rho_i^o, \rho_i^o, \rho_{i+1}^o)} \Xi(\rho_{i+1}^o, \rho_m^o, \rho_m^o) < 1.$$

Moreover, assume that for each $\rho^o \in \mathfrak{B}$, we have

$$\lim_{n\to+\infty} \Xi(\rho^o, \rho_n^o, \rho_{n+1}^o) \text{ and } \lim_{n\to+\infty} \Xi(\rho^o, \rho_n^o, \Delta\rho^o)$$

exist and are finite. Then, Δ has a unique fixed point.

Proof. Let $\rho_0^o \in \mathfrak{B}$ be such that $\alpha(\rho_0^o, \Delta\rho_0^o) \geq 1$. We define a sequence $\{\rho_n^o\} \in \mathfrak{B}$ by $\rho_{n+1}^o = \Delta\rho_n^o$ for all $n \in \mathbb{N}$. Clearly, if there is n_0 so that $\rho_{n_0+1}^o = \rho_{n_0}^o$, then there is nothing to prove. So, assume that $\rho_{n+1}^o = \rho_n^o$ for each $n \in \mathbb{N}$. By using (1) and (2), we have

$$\alpha(\rho_0^o, \Delta\rho_0^o) \geq 1. \tag{9}$$

Now, by using the definition of (α, F)-contraction

$$\begin{aligned}\tau + F(S(\rho_n^o, \rho_n^o, \rho_{n+1}^o)) &= \tau + F(S(\Delta\rho_{n-1}^o, \Delta\rho_{n-1}^o, \Delta\rho_n^o))\\ &\leq \tau + \alpha(\rho_n^o, \rho_{n+1}^o)F(S(\Delta\rho_{n-1}^o, \Delta\rho_{n-1}^o, \Delta\rho_n^o))\\ &\leq F(S(\rho_{n-1}^o, \rho_{n-1}^o, \rho_n^o))\end{aligned}$$

which implies

$$\begin{aligned}&F(S(\rho_n^o, \rho_n^o, \rho_{n+1}^o))\\ &\leq F(S(\rho_{n-1}^o, \rho_{n-1}^o, \rho_n^o)) - \tau\end{aligned}$$

$$\leq F(S(\rho_{n-2}^o, \rho_{n-2}^o, \rho_{n-1}^o)) - 2\tau$$

$$\leq F(S(\rho_{n-3}^o, \rho_{n-3}^o, \rho_{n-2}^o)) - 3\tau$$

$$\ldots$$

$$\leq F(S(\rho_0^o, \rho_0^o, \rho_1^o)) - n\tau.$$

Taking $n \to +\infty$, we have,

$$\lim_{n \to +\infty} F(S(\rho_n^o, \rho_n^o, \rho_{n+1}^o)) = -\infty.$$

By $(F2)$, we get

$$\lim_{n \to +\infty} S(\rho_n^o, \rho_n^o, \rho_{n+1}^o) = 0.$$

Now, by $(F3)$, there is $h \in (0, 1)$ so that

$$\lim_{n \to +\infty} [S(\rho_n^o, \rho_n^o, \rho_{n+1}^o)]^h F(S(\rho_n^o, \rho_n^o, \rho_{n+1}^o)) = 0.$$

Using equation (8), we obtain

$$[S(\rho_n^o, \rho_n^o, \rho_{n+1}^o)]^h F(S(\rho_n^o, \rho_n^o, \rho_{n+1}^o)) - [S(\rho_n^o, \rho_n^o, \rho_{n+1}^o)]^h F(S(\rho_0^o, \rho_0^o, \rho_{n+1}^o))$$

$$\leq -n\tau [S(\rho_n^o, \rho_n^o, \rho_{n+1}^o)]^h \leq 0.$$

Taking $n \to +\infty$, we have

$$\lim_{n \to \infty} n(S(\rho_n^o, \rho_n^o, \rho_{n+1}^o))^h = 0.$$

Equivalently,

$$\lim_{n \to \infty} n^{\frac{1}{h}} (S(\rho_n^o, \rho_n^o, \rho_{n+1}^o)) = 0$$

there exists $n_1 \in \mathbb{N}$ such that

$$n^{\frac{1}{\hbar}}(S(\rho_n^o, \rho_n^o, \rho_{n+1}^o)) \leq 1 \ \forall\, n \geq n_1.$$

Thus, we have

$$(S(\rho_n^o, \rho_n^o, \rho_{n+1}^o)) \leq \frac{1}{n^{\frac{1}{\hbar}}} \ \forall\, n \geq n_1.$$

For $n < m$, we have

$$\begin{aligned}
& S(\rho_n^o, \rho_n^o, \rho_m^o) \\
& \leq \Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)S(\rho_n^o, \rho_n^o, \rho_{n+1}^o) + \Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)S(\rho_n^o, \rho_n^o, \rho_{n+1}^o) \\
& \quad + \Xi(\rho_m^o, \rho_m^o, \rho_{n+1}^o)S(\rho_m^o, \rho_m^o, \rho_{n+1}^o) \\
& = 2\Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)S(\rho_n^o, \rho_n^o, \rho_{n+1}^o) + \Xi(\rho_m^o, \rho_m^o, \rho_{n+1}^o)S(\rho_{n+1}^o, \rho_{n+1}^o, \rho_m^o) \\
& \leq 2\Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)S(\rho_n^o, \rho_n^o, \rho_{n+1}^o) + \Xi(\rho_m^o, \rho_m^o, \rho_{n+1}^o)[\Xi(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o) \\
& \quad S(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o) + \Xi(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o)S(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o) \\
& \quad + \Xi(\rho_{n+2}^o, \rho_{n+2}^o, \rho_m^o)S(\rho_{n+2}^o, \rho_{n+2}^o, \rho_m^o)] \\
& = 2\Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)S(\rho_n^o, \rho_n^o, \rho_{n+1}^o) + 2\Xi(\rho_m^o, \rho_m^o, \rho_{n+1}^o)\Xi(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o) \\
& \quad S(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o) + \Xi(\rho_m^o, \rho_m^o, \rho_{n+1}^o)\Xi(\rho_{n+2}^o, \rho_{n+2}^o, \rho_m^o) \\
& \quad S(\rho_{n+2}^o, \rho_{n+2}^o, \rho_m^o) \\
& \leq 2\Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)S(\rho_n^o, \rho_n^o, \rho_{n+1}^o) + 2\Xi(\rho_m^o, \rho_m^o, \rho_{n+1}^o)\Xi(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o) \\
& \quad S(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o) + \Xi(\rho_m^o, \rho_m^o, \rho_{n+1}^o)\Xi(\rho_{n+2}^o, \rho_{n+2}^o, \rho_m^o)[2\Xi(\rho_{n+2}^o, \rho_{n+2}^o, \rho_{n+3}^o) \\
& \quad S(\rho_{n+2}^o, \rho_{n+2}^o, \rho_{n+3}^o) + \Xi(\rho_{n+3}^o, \rho_{n+3}^o, \rho_m^o)S(\rho_{n+3}^o, \rho_{n+3}^o, \rho_m^o)] \\
& \leq 2\Xi(\rho_n^o, \rho_n^o, \rho_{n+1}^o)S(\rho_n^o, \rho_n^o, \rho_{n+1}^o) + 2\Xi(\rho_m^o, \rho_m^o, \rho_{n+1}^o)\Xi(\rho_{n+1}^o, \rho_{n+1}^o, \rho_{n+2}^o)
\end{aligned}$$

$$S(\rho^o_{n+1}, \rho^o_{n+1}, \rho^o_{n+2}) + 2\Xi(\rho^o_m, \rho^o_m, \rho^o_{n+1})\Xi(\rho^o_{n+2}, \rho^o_{n+2}, \rho^o_m)\Xi(\rho^o_{n+2}, \rho^o_{n+2}, \rho^o_{n+3})$$

$$S(\rho^o_{n+2}, \rho^o_{n+2}, \rho^o_{n+3}) + \Xi(\rho^o_m, \rho^o_m, \rho^o_{n+1})\Xi(\rho^o_{n+2}, \rho^o_{n+2}, \rho^o_m)\Xi(\rho^o_{n+3}, \rho^o_{n+3}, \rho^o_m)$$

$$S(\rho^o_{n+3}, \rho^o_{n+3}, \rho^o_m)$$

$$\vdots$$

$$\leq 2\Xi(\rho^o_n, \rho^o_n, \rho^o_{n+1})S(\rho^o_n, \rho^o_n, \rho^o_{n+1}) + 2\sum_{i=n+1}^{m-2}\left\{\prod_{j=n+1}^{i}\Xi(\rho^o_m, \rho^o_m, \rho^o_j)\right\}\Xi(\rho^o_i, \rho^o_i, \rho^o_{i+1})$$

$$S(\rho^o_i, \rho^o_i, \rho^o_{i+1}) + 2\left\{\prod_{i=n+1}^{m-1}\Xi(\rho^o_i, \rho^o_i, \rho^o_m)\right\}S(\rho^o_{m-1}, \rho^o_{m-1}, \rho^o_m)$$

$$\leq 2\Xi(\rho^o_n, \rho^o_n, \rho^o_{n+1})S(\rho^o_n, \rho^o_n, \rho^o_{n+1}) + 2\sum_{i=n+1}^{m-2}\left\{\prod_{j=n+1}^{i}\Xi(\rho^o_m, \rho^o_m, \rho^o_j)\right\}\Xi(\rho^o_i, \rho^o_i, \rho^o_{i+1})$$

$$S(\rho^o_i, \rho^o_i, \rho^o_{i+1})$$

$$\leq 2\Xi(\rho^o_n, \rho^o_n, \rho^o_{n+1})S(\rho^o_n, \rho^o_n, \rho^o_{n+1}) + 2\sum_{i=n+1}^{m-1}\left\{\prod_{j=n+1}^{i}\Xi(\rho^o_m, \rho^o_m, \rho^o_j)\right\}\Xi(\rho^o_i, \rho^o_i, \rho^o_{i+1})\frac{1}{i^{\frac{1}{h}}}.$$

Let

$$\rho^o_p = \sum_{i=0}^{p}\left\{\prod_{j=0}^{i}\Xi(\rho^o_m, \rho^o_m, \rho^o_j)\right\}\Xi(\rho^o_i, \rho^o_i, \rho^o_{i+1})\frac{1}{i^{\frac{1}{h}}}$$

then

$$S(\rho^o_n, \rho^o_n, \rho^o_m) \leq [2\Xi(\rho^o_n, \rho^o_n, \rho^o_{n+1})S(\rho^o_n, \rho^o_n, \rho^o_{n+1}) + 2(\rho^o_{m-1} - \rho^o_n)].$$

Using the given condition and by ratio test $\lim_{n\to+\infty}\rho^o_n$ exists. Hence $\{\rho^o_n\}$ is a Cauchy sequence. Since $\mathfrak{B}$ is complete so there exists $\rho^o \in \mathfrak{B}$ such that $\{\rho^o_n\} \to \rho^o$. Thus, $\lim_{n\to+\infty} S(\rho^o_n, \rho^o_n, \rho^o_m) = 0$. We claim that $\Delta\rho^o = \rho^o$. Since $\{\rho^o_n\} \to \rho^o$ as $n \to +\infty$ and T is continuous, we have $\Delta\rho^o_n \to \Delta\rho^o$ as $n \to +\infty$. Thus, we have

$$S(\rho^o, \rho^o, \Delta\rho^o) = \lim_{n\to+\infty} S(\rho^o_{n+1}, \rho^o_{n+1}, \Delta\rho^o)$$

$$= \lim_{n\to+\infty} S(\Delta\rho^o_n, \Delta\rho^o_n, \Delta\rho^o)$$

$$= 0.$$

Hence, $\Delta\rho^o = \rho^o$. Thus ρ^o is a fixed point of Δ.

For uniqueness, let us consider another fixed point ι^o such that $\Delta\iota^o = \iota^o$ and $\rho^o \neq \iota^o$.

$$\tau + F(S(\rho^o, \rho^o, \iota^o) = \tau + F(S(\Delta\rho^o, \Delta\rho^o, T\iota^o))$$

$$\leq \tau + \alpha(\rho^o, \iota^o)F(S(\Delta\rho^o, \Delta\rho^o, \Delta\iota^o))$$

$$\leq F(S(\rho^o, \rho^o, \iota^o)),$$

which is a contradiction. Thus ρ^o is the unique fixed point of Δ. □

Corollary 2 Let $(\mathfrak{B}, S)$ be a new controlled S-metric-type space. A self mapping $\Delta : \mathfrak{B} \to \mathfrak{B}$ be continuous so that

$$\tau + \alpha(\rho^o, \iota^o)F(S(\Delta\rho^o, \Delta\iota^o, \Delta z)) \leq F(S(\rho^o, \iota^o, z))$$

for all $\rho^o, \iota^o, z \in \mathfrak{B}$. For $\rho^o_0 \in \mathfrak{B}$, define a picard sequence $\{\rho^o_n = \Delta^n\rho^o_0\}$ such that

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\Xi(\rho^o_{i+1}, \rho^o_{i+1}, \rho^o_{i+2})}{\Xi(\rho^o_i, \rho^o_i, \rho^o_{i+1})} \Xi(\rho^o_{i+1}, \rho^o_m, \rho^o_m) < 1.$$

Moreover, assume that for each $\rho^o \in \mathfrak{B}$, we have

$$\lim_{n\to+\infty} \Xi(\rho^o, \rho^o_n, \rho^o_{n+1}) \text{ and } \lim_{n\to+\infty} \Xi(\rho^o, \rho^o_n, \Delta\rho^o)$$

exist and are finite. Then, Δ has a unique fixed point.

Proof. Taking $\alpha : \mathfrak{B} \times \mathfrak{B} \to [0, +\infty)$ by $\alpha(\rho^o, \iota^o) = 1$ for all $\rho^o, \iota^o \in \mathfrak{B}$ in Theorem 3. □

Corollary 3 Let $(\mathfrak{B}, d)$ be a complete metric space and $\Delta : \mathfrak{B} \to \mathfrak{B}$ be a continuous, α-admissible and (α, F)-contraction so that there is $\rho^o_0 \in \mathfrak{B}$ in order that $\alpha(\rho^o_0, \Delta\rho^o_0) \geq 1$. Then, Δ has a unique fixed point.

Proof. Taking $\Xi : \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B} \to [0, +\infty)$ by $\Xi(\rho^o, \iota^o, z) = 1$, for all $\rho^o, \iota^o \in \mathfrak{B}$ in Theorem 3. □

5. An application to Riemann-Liouville fractional integrals

In this section, we examine the existence and uniqueness of a solution of Riemann-Liouville fractional integral equation by using our main result. Let X be the set of all continuous functions from $[0, 1]$ onto $\mathbb{R}$. We can write the Riemann-Liouville fractional integral equation as follows:

$$ {}^{\mathbb{RL}}_{\alpha}E^{v\,o}_{\rho} = \frac{1}{\Gamma(v)} \int^{\rho}{}^{o}_{\alpha} A(t)(\rho^{o} - t)^{v-1} dt; \quad \Gamma(v) > 0, \tag{10} $$

where $v \in \mathbb{R}$, $A(\rho^{o}) \in \mathfrak{B}$ and $x, t \in [0, 1]$. We define a $\Xi : \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B} \to [1, +\infty)$ by $\Xi(\rho^{o}, \iota^{o}, z) = 1 + \rho^{o} + \iota^{o} + z$ and $S : \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B} \to [0, +\infty)$ by

$$ S(A, B, C) = |A(\rho^{o}) - C(\rho^{o})|^{2} + |B(\rho^{o}) - C(\rho^{o})|^{2} $$

for all $A(\rho^{o}), B(\rho^{o}), C(\rho^{o}) \in \mathfrak{B}$ and $\rho^{o} \in [0, 1]$. Then, it is clear that S is a new CSMS. Consider

$$ \frac{1}{\Gamma^{2}(\rho^{o} + 1)} \frac{(\rho^{o} - t)^{v-1}(\rho^{o} - \alpha)^{2v}}{|(\rho^{o} - v)^{v-1}|} \leq L, \text{ where } L \in (0, 1) \text{ and } \rho^{o} \neq t. $$

Now, by using the above condition, we examine the existence and uniqueness of a solution of equation (10). An operator $T : \mathfrak{B} \to \mathfrak{B}$ defined by

$$ \Delta A(\rho^{o}) = \frac{1}{\Gamma(v)} \int^{\rho}{}^{o}_{\alpha} A(t)(\rho^{o} - t)^{v-1} dt. \tag{11} $$

Observe that the unique solution of the fractional integral (10) is same as the integral operator (11) has a unique fixed point. Assume that

$$ \begin{aligned} & S(\Delta A, \Delta A, \Delta B) \\ & = |\Delta A - \Delta B|^{2} + |\Delta A - \Delta B|^{2} \\ & = 2|\Delta A - \Delta B|^{2} \\ & = 2\left| \frac{1}{\Gamma(v)} \int^{\rho}{}^{o}_{\alpha} A(t)(\rho^{o} - t)^{v-1} dt - \frac{1}{\Gamma(v)} \int^{\rho}{}^{o}_{\alpha} B(t)(\rho^{o} - t)^{v-1} dt \right| \\ & \leq 2\left(\frac{1}{\Gamma(v)} \int^{\rho}{}^{o}_{\alpha} (\rho^{o} - t)^{v-1} dt \right)^{2} |A(t) - B(t)|^{2} \end{aligned} $$

$$\leq 2\frac{1}{\Gamma^2(v)}\left(\int^{\rho}{}_{\alpha}^{o}|(\rho^o-t)^{v-1}|dt\right)^2|A(t)-B(t)|^2$$

$$=2\frac{1}{\Gamma^2(v)}\frac{(\rho^o-t)^{v-1})}{|(\rho^o-t)^{v-1}|}\left(\int^{\rho}{}_{\alpha}^{o}|(\rho^o-t)^{v-1}|dt\right)^2|A(t)-B(t)|^2$$

$$=-2\frac{1}{\Gamma^2(v)}\frac{(\rho^o-t)^{v-1})}{|(\rho^o-t)^{v-1}|}\left(\frac{(\rho^o-t)^{v}}{v}|^{\rho_o}_{a}\right)^2|A(t)-B(t)|^2$$

$$=2\frac{1}{\Gamma^2(v)}\frac{(\rho^o-t)^{v-1})}{|(\rho^o-t)^{v-1}|}\left(\frac{(\rho^o-a)^{v}}{v}\right)^2|A(t)-B(t)|^2$$

$$=2\frac{1}{\Gamma(v+1)}\frac{(\rho^o-t)^{v-1})(\rho^o-a)^{2v}}{|(\rho^o-t)^{v-1}|}|A(t)-B(t)|^2$$

$$\leq 2L|A(t)-B(t)|^2$$

$$=LS(A, A, B).$$

Observe that all conditions of Theorem 1 are satisfied and equation (10) has a unique solution.

6. Conclusions and future works

In this study, we presented the notion of a new CSMS by using a function $\Xi:\mathfrak{B}\times\mathfrak{B}\times\mathfrak{B}\to[1,+\infty)$ in the right-hand side of the triangle inequality of S-metric space. Further, we proved the Banach contraction principle and fixed point result by using a family $\mathfrak{M}$ of all continuous functions $\mathscr{M}$ such that $\mathscr{M}:\mathbb{R}_+^5\to\mathbb{R}_+$ in the setting of new CSMS. Furthermore, we proved several fixed point results for (α, F)-contraction in the setting of new CSMS. At the end, we provided an application to Riemann-Liouville fractional integrals to show the validity of a main result. As future work, researchers should have a new approach toward this space. Furthermore, the fixed point result used in this paper for Riemann-Liouville Fractional Integrals can be applied to other differential and integral equations.

Conflict of interest

The authors declare no competing financial interest.

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