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Research Article

Analyzing an $M/G/1$ Double Orbit Retrial Queue with the Implementation of a Working Vacation Policy

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Abstract: This paper introduces a double-orbit retrial queue model designed to address patient dissatisfaction by taking into account their willingness to pay for enhanced comfort and service. The system incorporates non-Markovian characteristics and employs a probability-generating function approach to solve its equations. Key performance metrics investigated include expected queue length, system length, and the precision of numerical results derived from empirical data. The model's application is specifically examined within a hospital management system (HMS), underscoring its relevance in healthcare operations. Overall, the paper offers valuable insights into optimizing design strategies for retrial queue systems featuring unreliable servers, with a focus on enhancing patient satisfaction through personalized service offerings.

Keywords: retrial queues, ordinary orbit, premium orbit, single server, working vacations

MSC: 65L05, 34K06, 34K28

1. Introduction

In queueing theory, Krishna Kumar and Arivudainambi [1] provide us an overview of the fundamentals of the retrial queueing system and analysis, an $M/G/1$ retrial queue on a single server with a Bernoulli service schedule, and typical retrial timings. Ke and Chang [2] introduced a retrial queueing system with server vacations limited to J, balking, Bernoulli feedback, and repeated attempts were implemented. Aissani [3] examines an $M/G/1$ retrial queue server vacations, analyzing its distribution in a stationary regime. It provides a heavy traffic approximation, decomposes the system size into two random variables, and solves optimization problems for vacation and retrial policies. Sherman and Kharoufeh [4] examine a standard queue and an $M/M/1$ retry queue with the orbit of infinite capacity, analyzing its conditions for stability and stochastic decomposability results. Jain and Mehta [5] presented imperfect repair and patient discontent by presenting performance modeling and the ideal configuration for an unstable server retrial queue with two retrial orbits. patients' desire to pay more for higher levels of comfort and service is taken into account by the system. Using ANFIS, the system's performance measures are investigated. Jain and Sanga [6] investigated a Markovian model that has been created in the study to analyze double orbit and other faulty single-server retrial queues. patient-discouraging behavior is addressed, and performance measurements are derived using probability-generating algorithms. The approach is applied

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to manufacturing, computer networks, and telecommunications. The lowest cost and ideal service rates are shown in the model. Jain et al. [7] examined finite double retrial orbit queues with service interruptions and priority patients into account. They explained how the suggested queueing paradigm was implemented in the cellular radio network. Kumar et al. [8] discussed the working vacation features, double retrial orbits, and in most practical instances of queueing, the server might not be dependable. Specific retrial queue models consider server unreliability as a potential solution in case of server failure. Sanga and Jain [9] presented A study that looks at the balking behavior of regular and premium class patients in a double-orbit retrial queueing system. It makes use of a fuzzy $FM/FM/1$ queueing system including balking, a profit function, and a steady-state analytical solution. Performance metrics are defined using the ranking index approach. Dimitriou [10] studied coupled orbit queues and retry queueing systems with two classes of patients. Dhibar and Jain [11] discussed a study of designing the optimal strategy for a Markovian double orbit retrial queue by involving the realistic features of imperfect service and vacation interruption. The approach is used for telecommunication systems, computer networks, call centers, etc. Servi and Finn [12] presented the idea of a working vacation, in which the server runs services slower while on vacation instead of discontinuing them altogether and working vacation used in communication, manufacturing, and service systems. Rajadurai [13] studied a retrial queueing system on a single server that included working vacations and vacation interruptions. Three patient types are taken into account: priority, ordinary, and harmful. The server goes into vacation mode and operates more slowly when the system is empty. Gupta and Kumar [14] discovered an $M/M/1$ retrial model for a server during working vacations, considering breakdowns and repair. PGF is used to determine steady-state solutions, and performance measures are developed for different server states. Li et al. [15] examined a queue for $M/G/1$ retrial with generic retrial durations, in which the Bernoulli schedule governs a single working vacation. Jain et al. [16] examined The Markovian retrial queue has various applications such as communication networks, manufacturing businesses' call centers, and cyber centers. It does take into consideration the idea of a working vacation, patients' balking behavior, and faulty service. Rajadurai et al. [17] investigates a queueing system with a single server feedback retrial that experiences multiple working vacations and interruptions. The system operates at a lower rate during vacations, and unsatisfied patients may rejoin to receive another service. Further, the steady-state PGF, Performance metrics, and numerical examples are presented. Zhang [18] investigated a single-server retrial queue with exponential service times and a Poisson arrival process. It talks about two different ways that patients retrial: independent, in which each one looks for services on their own, and constant retrial policy, in which the server charges a set price to users who sign up for the system. Varalakshmi et al. [19] studied the steady state behavior of an $M/G/1$ retrial queueing system including two service stages and rapid feedback while operating under a working vacation policy. The system is impacted by negative consumers arriving, with patients entering an orbit if the server is busy or fails. Boualem et al. [20] investigated an $M/G/1$ retrial queue with vacations and identified various stochastic comparison features, such as strong stochastic ordering and convex ordering. Boualem et al. [21] examined a mathematical method for obtaining performance index boundaries by comparing Markov chains, focusing on the monotonicity characteristics of a single server retrial queue. Boualem [22] discussed the monotonicity of a single server retrial queue, revealing insensitive constraints for the stationary distribution of the embedded Markov chain. Gao et al. [23] examined a queue for $M/G/1$ retrials that had both idle and busy breakdowns. It presents steady-state joint queue length distribution, reliability indices, performance metrics, and stable conditions using probability generating function (PGF) and embedded Markov chain approaches. Additionally, it examines a patient's sojourn time in a stable condition and provides numerical examples to highlight the implications of system characteristics. Boualem et al. [24] presented stochastic comparison techniques to establish performance bounds in a single-server retrial queue, demonstrating the monotonic transition operator under strong stochastic and increasing convex ordering. Boualem [25] examines the stochastic analysis of a single server's unreliable queue, focusing on its balking and general retrial time.

Our investigation into the double orbit retrial queue was prompted by the literature we have read thus far. Unreliable servers have been noted. The existing literature has not yet examined the $M/G/1$ double orbit retrial queue with the implementation of a working vacation policy in the queuing systems.

The remaining section of this article is organized using the following structure: Section 2 includes a system description and analysis. Section 3 provides an overview of the steady-state analysis, including determining the number of consumers and their orbit. And an actual application of the model we covered. Section 4 focuses on system measurements.

Section 5 investigates the model's key exceptional cases. Section 6 several numerical figures are provided to demonstrate the impact of various factors on system performance measures.

2. Model description and analysis

This model aims to estimate performance indices using a non-Markovian model of a double-orbit retrial queue with unstable servers, showing like Figure 1. Depending on their ability to pay and the needs of the facility in case the server is busy, patients using the retrial queueing system have the option to wait in any of the two orbits: premium or ordinary. FCFS is the patient service discipline. The retrial queueing system that has been developed has the following mathematical formulation based on the subsequent assumptions:

The arrival process: Patients enter the system using a Poisson process with rate λ.

The process of retrial and service: Presuming there isn't a waiting area, a new patient who finds the server available will take advantage of his services. Suppose the server is on vacation or busy, the patient has the option to leave the service area and join a group of patients who have been blocked, which is referred to as "orbit." When a new patient arrives. Upon arrival, patients can join any one of the retrial orbits, either the premium or ordinary orbit. The patients receive service from the server based on a general distribution, with patients from the premium and ordinary orbits receiving service first. If the ordinary orbit patient finds the server is accessible, with the rate λ, they can receive service. If the premium orbit patient finds the server is accessible, with the rate δ, they can receive service. The ordinary orbit patient joins with the probability κ, whereas the premium orbit patient joins with the probability $\bar{\kappa} = (1-\kappa)$. Inter-ordinary retrial time of the patient has an arbitrary distribution $R_1(\varepsilon)$ and "Laplace-Stieltijes transform" (LST) $R_1^*(\nu)$ and inter-premium retrial time of the patient have an arbitrary distribution $R_2(\varepsilon)$ and (LST) $R_2^*(\nu)$. The service times are assumed to be generally distributed and shown as the random variable S with a distribution function $S(\varepsilon)$ and (LST) $S^*(\nu)$ respectively, $E(S)$ and $E(S)^2$ represent the first and second moments.

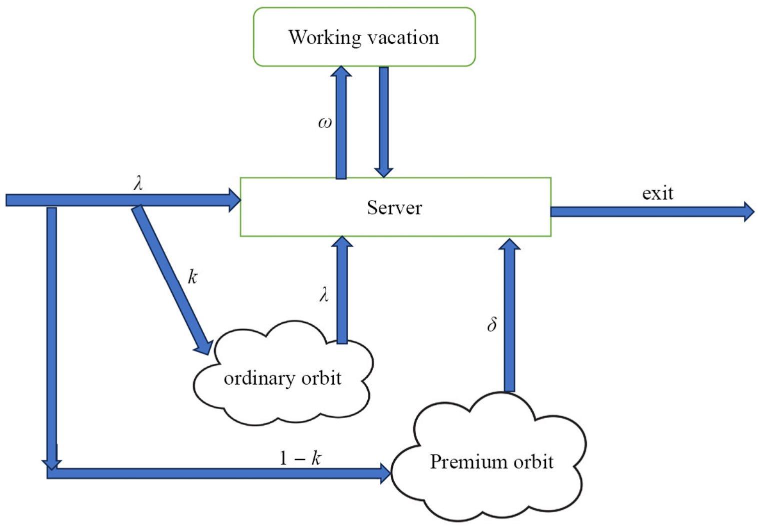

Figure 1. Structure of the queuing model

Working vacation procedures: Whenever the orbit is empty, the server takes a vacation, with an amount of time determined by an exponential distribution with parameter V. The server continues operating at a reduced speed service rate if a patient arrives during a vacation, commonly called a "working vacation." In the event of a working vacation, processes move more slowly. Over the working vacation period, the service time is a generic random variable ψ_ω, using $E(V)$ and $E(V)^2$ as the first and second moments, with distribution functions $\psi_\omega(\varepsilon)$ and LST $\psi_\omega^*(\nu)$.

2.1 *Practical application of the model*

In a hospital environment, a system with a single server consists of the same pool of resources, such as physicians, nurses, and facilities, that serve both ordinary and premium patients. The hospital's structure just like Figure 2. The words "ordinary" and "premium" patients can refer to distinct types of patients depending on many circumstances, including the amount of treatment they get, their insurance coverage, and their willingness to pay for additional amenities or services.

Ordinary patients who receive regular or critical healthcare treatments in the hospital. They might have government-sponsored insurance, minimal health insurance policies, or no insurance. Ordinary patients frequently receive care based on regular processes and norms, with no added benefits or facilities. They may have less access to doctors, treatment alternatives, and facilities than premium patients. Because of the high volume of patients and limited resources, they may have to wait longer for appointments or operations.

Premium patients choose higher-level treatments or facilities at the hospital, sometimes by paying more fees or having premium insurance. They may be eligible for expedited appointments, priority scheduling for operations or treatments, and reduced wait times. Premium patients may have access to a greater number of experts, including top-tier doctors and specialists. During their stay, they may be offered additional facilities such as private rooms, individualized care plans, concierge services, or increased comfort alternatives. Premium patients may have access to unique hospital amenities such as VIP lounges or dedicated patient service personnel. They may receive more customized treatment from medical professionals, such as faster appointment periods and more complete follow-up care. Premium patients may have access to premium treatment choices.

The only time the doctor's assistant has the opportunity to provide care is when the chief doctor is away on vacation, and even then, the doctor's assistant frequently offers services at a slower rate than the chief doctor. Within the context of our queueing paradigm, the hospital administration system serves as a concrete illustration.

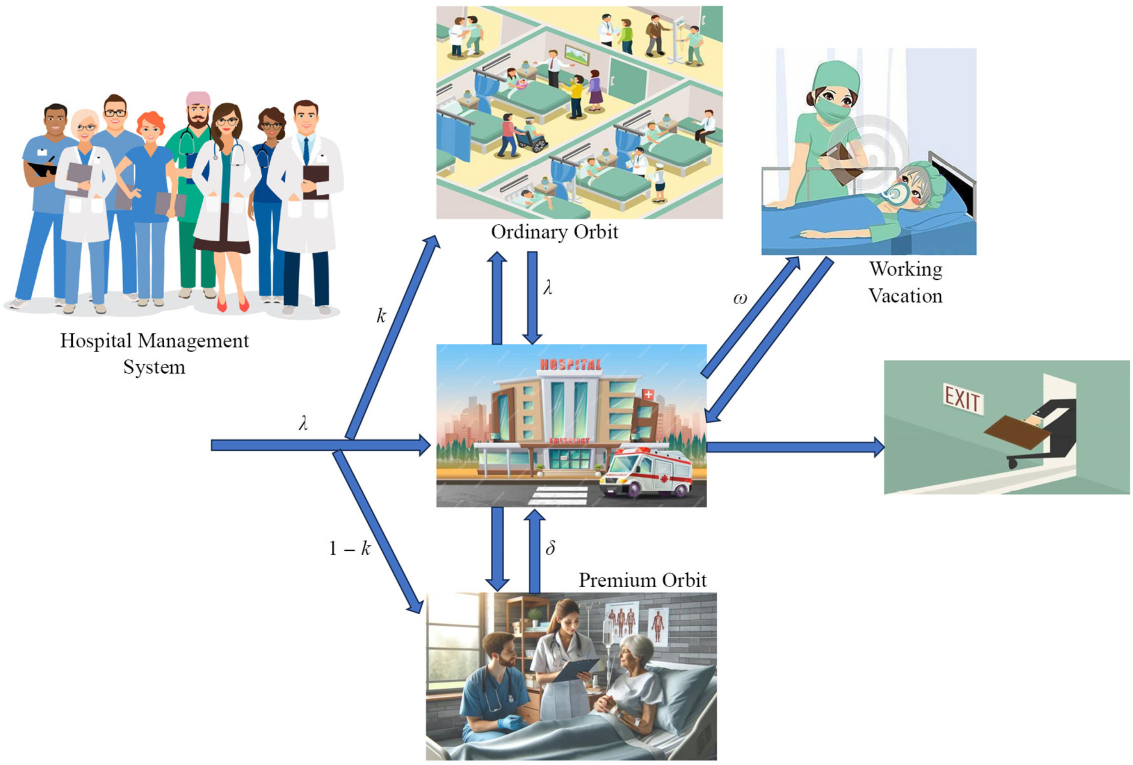

Figure 2. Structure of the application model

3. Probability analysis for steady states

This division develops the retrial system's steady state equations by treating the elapsed times of normal service, elapsed premium retrial time, elapsed ordinary retrial time, and lower rate service times that occurred via supplementary variables. Next, we calculate the orbit size generating functions (GFs) for various server states and the PGF of the total number of patients in the orbit and system.

3.1 *The steady state equations*

For steady state techniques, we consider $R_1(0) = 0$, $R_1(\infty) = 1$, $R_2(0) = 0$, $R_2(\infty) = 1$, $S(0) = 0$, $S(\infty) = 1$ and $V(0) = 0$, $V(\infty) = 1$ are continuous at $\varepsilon = 0$. The hazard rates for ordinary retrial, premium retrial, normal service, and lower rate service are signified by the functions $a_1(\varepsilon)$, $a_2(\varepsilon)$, $\mu(\varepsilon)$, and $\psi_\omega(\varepsilon)$, respectively.

$$a_1(\varepsilon)d\varepsilon = \frac{dR_1(\varepsilon)}{1-R_1(\varepsilon)}$$

$$a_2(\varepsilon)d\varepsilon = \frac{dR_2(\varepsilon)}{1-R_2(\varepsilon)}$$

$$\mu(\varepsilon)d\varepsilon = \frac{dS(\varepsilon)}{1-S(\varepsilon)}$$

$$\psi_\omega(\varepsilon)d\varepsilon = \frac{dV(\varepsilon)}{1-V(\varepsilon)}$$

Apart from it, let $R_1^0(\iota)$, $R_2^0(\iota)$, $S^0(\iota)$, and $V^0(\iota)$ represent the elapsed retrial times of ordinary orbit, premium orbit, normal service, and the lower rate service times, accordingly, at time ι. Additionally providing the random variables,

$$\Phi(\iota) = \begin{cases} 0, & \text{The server seems to be in working vacation period and the server is free} \\ 1, & \text{The server seems to be ordinary orbit patient and the server is free} \\ 2, & \text{The server seems to be premium orbit patient and the server is free} \\ 3, & \text{The server seems to be normal service at time } \iota \text{ and the server is busy} \\ 4, & \text{The server seems to be in lower service pace at time } \iota \text{ and the server is busy} \end{cases}$$

Thus the SV $R_1^0(\iota)$, $R_2^0(\iota)$, $S^0(\iota)$, and $V^0(\iota)$ are motivated to design a bivariate Markov process$\{\Phi(\iota),\ \mathscr{X}(\iota);\ \iota \geq 0\}$ where $\Phi(\iota)$ represents the state of the server as $(0,\ 1,\ 2,\ 3,\ 4)$ depending on whether the server is free or busy on both normal service and working vacation periods. $\mathscr{X}(\iota)$ Identifies the available patient number in the orbit.

We provide four additional variables in order to work with a Markov process when $\Phi(\iota) > 0$: If $\Phi(\iota) = 1$ and $\mathscr{X}(\iota) > 0$, then $R_1^0(\iota)$ indicates the elapsed ordinary retrial time. If $\Phi(\iota) = 2$ and $\mathscr{X}(\iota) > 0$, then $R_2^0(\iota)$ represents the elapsed premium retrial time time. If $\Phi(\iota) = 3$ and $\mathscr{X}(\iota) \geq 0$, then $S^0(\iota)$ represents the elapsed time of the patient served in a normal busy period. If $\Phi(\iota) = 4$ and $\mathscr{X}(\iota) \geq 0$, then $V^0(\iota)$ indicates the elapsed time of the patient being served in a lower rate service period.

Theorem 1 The embedded Markov chain $\{V_n;\ n\varepsilon N\}$ is ergodic if $\rho < 1$ for our system to be stable, where $\rho = \kappa\{1-R_1^*(\lambda)-(\lambda+\delta)E(s)+1\}-(1-\kappa)\{1-R_2^*(\delta)-(\lambda+\delta)E(s)+1\}$.

Proof. Pakes [26] Foster's criteria can be used to show that the chain $\{V_n;\, n \in N\}$ appears to be both an aperiodic Markov chain and irreducible, which is a necessary requirement for ergodicity. If $e(l)$ exists as non-negative function with $l \in N$ and $\varepsilon > 0$, the Markov chain is ergodic. The mean drift is $\eta_l = E[e(v_{n+1}) - e(v_n)/v_n = l]$. Excluding only a finite number of l's, all $l \in N$ have $\eta_l \leq -\in$. We take the function $e(l) = l$ for the given instance. Then we obtain

$$\eta_l = \begin{cases} (1-\kappa)\{1 - R_2^*(\delta) - E(s)(\lambda+\delta) + 1, & \text{if } l = 0 \\ \kappa\{1 - R_1^*(\lambda) - E(s)(\lambda+\delta) + 1\} - (1-\kappa)\{1 - R_2^*(\delta) - E(s)(\lambda+\delta) + 1, & \text{if } l = 1, 2, ... \end{cases}$$

Here $\kappa\{1 - R_1^*(\lambda) - (\lambda+\delta)E(s) + 1\} - (1-\kappa)\{1 - R_2^*(\delta) - (\lambda+\delta)E(s) + 1 < 1$ is without a doubt a necessary requirement for ergodicity.

As stated Sennott et al. [27], whenever the Markov chain $\{W_n;\, n\varepsilon N\}$ sees Kaplan's status, notably $\eta_l < \infty$ for all $l \geq 0$ and $\exists\ l_0 \in N$ s.t $\eta_l \geq 0$ for $l \geq l_0$, the necessary prerequisite is satisfied. $M = (m_{kl})$ is relates to one-step transition matrix of $\{V_n;\, n \in N\}$ for $l < k - i$ and $k > 0$, in which $M = (m_{kl})$ is relates to one-step transition matrix of $\{V_n;\, n \in N\}$. If $\rho \geq 1$ indicates that the Markov chain is non-ergodic.

Let's assume the epoch sequence $\{\iota_n;\, n = 1, 2, ...\}$, where a shorter service period or the completion of the service period occurs. The random vector sequence $V_n = \{\Delta(\iota_n+),\, S(\iota_n+)\}$. Incorporated in the RQ system, forming a Markov chain. Based on Theorem (1), $V_n;\, n \in N$ is ergodic iff $\rho < 1$, where $\rho = \kappa\{1 - R_1^*(\lambda) - (\lambda+\delta)E(s) + 1\} - (1 - \kappa)\{1 - R_2^*(\delta) - (\lambda+\delta)E(s) + 1 < 1$. This is necessary for our system to stay stable. The probabilities for the approach $\{\mathscr{X}(\iota),\, \iota \geq 0\}$ are given as follows: $R_0(\iota) = P\{\Phi(\iota) = 0,\, \mathscr{X}(\iota) = 0\}$; the prob. densities are □

$$R_{1,n}(\varepsilon,\, \iota)d\varepsilon = P\{\Phi(\iota) = 1,\, \mathscr{X}(\iota) = n,\, \varepsilon \leq R_1^0(\iota) < \varepsilon + d\varepsilon\}, \text{ for } \iota \geq 0,\, \varepsilon \geq 0 \text{ and } n \geq 1.$$

$$R_{2,n}(\varepsilon,\, \iota)d\varepsilon = P\{\Phi(\iota) = 2,\, \mathscr{X}(\iota) = n,\, \varepsilon \leq R_2^0(\iota) < \varepsilon + d\varepsilon\}, \text{ for } \iota \geq 0,\, \varepsilon \geq 0 \text{ and } n \geq 1.$$

$$S_n(\varepsilon,\, \iota)d\varepsilon = P\{\Phi(\iota) = 3,\, \mathscr{X}(\iota) = n,\, \varepsilon \leq S^0(\iota) < \varepsilon + d\varepsilon\}, \text{ for } \iota \geq 0,\, \varepsilon \geq 0 \text{ and } n \geq 0.$$

$$V_n(\varepsilon,\, \iota)d\varepsilon = P\{\Phi(\iota) = 4,\, \mathscr{X}(\iota) = n,\, \varepsilon \leq V^0(\iota) < \varepsilon + d\varepsilon\}, \text{ for } \iota \geq 0,\, \varepsilon \geq 0 \text{ and } n \geq 0.$$

The limiting densities are $R_0 = lim_{\iota \to \infty} R_0(\iota)$, if the sequel satisfies the stability conditions.

$$R_{1,n}(\varepsilon) = lim_{\iota \to \infty} R_{1,n}(\varepsilon,\, \iota);$$

$$R_{2,n}(\varepsilon) = lim_{\iota \to \infty} R_{2,n}(\varepsilon,\, \iota);$$

$$S_n(\varepsilon) = lim_{\iota \to \infty} S_n(\varepsilon,\, \iota);$$

$$V_n(\varepsilon,\, y) = lim_{\iota \to \infty} V_n(\varepsilon,\, y,\, \iota).$$

Applying the supplemental variable method, we build the subsequent system of equations.

$$(\lambda+\omega)R_0(\iota)=\int_0^\infty S_0(\varepsilon)\mu(\varepsilon)d\varepsilon+\int_0^\infty V_0(\varepsilon)\psi(\varepsilon)d\varepsilon \tag{1}$$

$$\frac{d}{d\varepsilon}R_{1,\,n}(\varepsilon)+(\lambda+a_1(\varepsilon))R_{1,\,n}(\varepsilon)=0,\, n\geq 1 \tag{2}$$

$$\frac{d}{d\varepsilon}R_{2,\,n}(\varepsilon)+(\delta+a_2(\varepsilon))R_{2,\,n}(\varepsilon)=0,\, n\geq 1 \tag{3}$$

$$\frac{d}{d\varepsilon}S_n(\varepsilon)+(\lambda+\delta+\mu(\varepsilon))S_n(\varepsilon)=(\lambda+\delta)S_{n-1}(\varepsilon),\, n\geq 1 \tag{4}$$

$$\frac{d}{d\varepsilon}V_n(\varepsilon)+(\lambda+\delta+\omega+\psi(\varepsilon))V_n(\varepsilon)=(\lambda+\delta)V_{n-1}(\varepsilon),\, n\geq 1 \tag{5}$$

For $\varepsilon=0$ and $y=0$, the following are the steady-state boundary conditions:

$$R_{1,\,n}(0)=\kappa\int_0^\infty S_{n-1}(\varepsilon)\mu(\varepsilon)d\varepsilon+\int_0^\infty V_n(\varepsilon)\mu(\varepsilon)d\varepsilon \tag{6}$$

$$R_{2,\,n}(0)=(1-\kappa)\int_0^\infty S_{n-1}(\varepsilon)\mu(\varepsilon)d\varepsilon \tag{7}$$

$$S_n(0)=\int_0^\infty R_{1,\,n+1}(\varepsilon)a_1(\varepsilon)d\varepsilon+\lambda\int_0^\infty R_{1,\,n}(\varepsilon)d\varepsilon+\int_0^\infty R_{2,\,n+1}(\varepsilon)a_2(\varepsilon)d\varepsilon$$

$$+\delta\int_0^\infty R_{2,\,n}(\varepsilon)d\varepsilon+\omega\int_0^\infty V_n(\varepsilon)d\varepsilon n\geq 1 \tag{8}$$

$$V_n(0)=\begin{cases}(\lambda+\delta)R_0,\, n=0\\ \\ 0,\, n\geq 1\end{cases} \tag{9}$$

The state of normalization is

$$R_0+\sum_{n=1}^{\infty}\int_0^\infty\left(R_{1,\,n}(\varepsilon)d\varepsilon+R_{2,\,n}(\varepsilon)d\varepsilon\right)+\sum_{n=0}^{\infty}\int_0^\infty\left(S_n(\varepsilon)d\varepsilon+V_n(\varepsilon)d\varepsilon\right)=1 \tag{10}$$

3.2 *The steady-state solution*

The steady-state solution of the RQ model is obtained by the generating function approach. To solve the aforementioned equations, the GFs for $|\varphi|<1$ are defined as follows:

$$R_1(\varepsilon, \varphi) = \sum_{n=0}^{\infty} R_n(\varepsilon)\varphi^n; R_1(0, \varphi) = \sum_{n=0}^{\infty} R_n(0)\varphi^n;$$

$$R_2(\varepsilon, \varphi) = \sum_{n=0}^{\infty} R_n(\varepsilon)\varphi^n; R_2(0, \varphi) = \sum_{n=0}^{\infty} R_n(0)\varphi^n;$$

$$S(\varepsilon, \varphi) = \sum_{n=0}^{\infty} S_n(\varepsilon)\varphi^n; S(0, \varphi) = \sum_{n=0}^{\infty} S_n(0)\varphi^n;$$

$$V(\varepsilon, \varphi) = \sum_{n=0}^{\infty} V_n(\varepsilon)\varphi^n; V(0, \varphi) = \sum_{n=0}^{\infty} V_n(0)\varphi^n;$$

From (2) to (9), multiply the steady-state equation and steady-state boundary conditions by φ^n, then sum over n ($n = 0, 1, 2, ...$).

$$\frac{\partial}{\partial \varepsilon} R_1(\varepsilon, \varphi) + (\lambda + a_1(\varepsilon))R_1(\varepsilon, \varphi) = 0 \tag{11}$$

$$\frac{\partial}{\partial \varepsilon} R_2(\varepsilon, \varphi) + (\delta + a_2(\varepsilon))R_2(\varepsilon, \varphi) = 0 \tag{12}$$

$$\frac{\partial}{\partial \varepsilon} S(\varepsilon, \varphi) + ((\lambda + \delta)(1 - \varphi) + \mu(\varepsilon))S(\varepsilon, \varphi) = 0 \tag{13}$$

$$\frac{\partial}{\partial \varepsilon} V(\varepsilon, \varphi) + (\omega + (1 - \varphi)(\lambda + \delta) + \psi(\varepsilon))V(\varepsilon, \varphi) = 0 \tag{14}$$

$$R_1(0, \varphi) = \kappa\varphi \int_0^{\infty} S(\varepsilon, \varphi)\mu(\varepsilon)d\varepsilon + \int_0^{\infty} V(\varepsilon, \varphi)\mu(\varepsilon)d\varepsilon - \lambda R_0 \tag{15}$$

$$R_2(0, \varphi) = (1 - \kappa)\varphi \int_0^{\infty} S(\varepsilon, \varphi)\mu(\varepsilon)d\varepsilon \tag{16}$$

$$S(0, \varphi) = \frac{1}{\varphi} \int_0^{\infty} R_1(\varepsilon, \varphi)a_1(\varepsilon)d\varepsilon + \lambda \int_0^{\infty} R_1(\varepsilon, \varphi)d\varepsilon + \frac{1}{\varphi} \int_0^{\infty} R_2(\varepsilon, \varphi)a_2(\varepsilon)d\varepsilon$$
$$+ \delta \int_0^{\infty} R_2(\varepsilon, \varphi)d\varepsilon + \omega \int_0^{\infty} V(\varepsilon, \varphi)d\varepsilon \tag{17}$$

$$V(0, \varphi) = (\lambda + \delta)R_0 \tag{18}$$

Solving the partial differential eqns. (11) to (14), we obtain

$$R_1(\varepsilon,\,\varphi) = R_1(0,\,\varphi)e^{-\lambda(\varepsilon)}[1-R_1(\varepsilon)] \tag{19}$$

$$R_2(\varepsilon,\,\varphi) = R_2(0,\,\varphi)e^{-\delta(\varepsilon)}[1-R_2(\varepsilon)] \tag{20}$$

$$S(\varepsilon,\,\varphi) = S(0,\,\varphi)e^{-C_s(\varphi)\varepsilon}[1-S(\varepsilon)] \tag{21}$$

$$V(\varepsilon,\,\varphi) = V(0,\,\varphi)e^{-A_v(\varphi)\varepsilon}[1-V(\varepsilon)] \tag{22}$$

where $C_s(\varphi) = (\lambda+\delta)(1-\varphi)$, $A_v(\varphi) = \omega + (\lambda+\delta)(1-\varphi)$.

Inserting the eqns. (21) to (22) and (15) and (16) after doing some calculations, the result became,

$$R_1(0,\,\varphi) = \kappa\varphi S(0,\,\varphi)S^*(C_s(\varphi)) + V(0,\,\varphi)V^*(A_v(\varphi)) - \lambda R_0 \tag{23}$$

$$R_2(0,\,\varphi) = (1-\kappa)\varphi S(0,\,\varphi)S^*(C_s(\varphi)) - \delta R_0 \tag{24}$$

combining the equation (19) to (22) in (17), we obtain

$$S(0,\,\varphi) = R_0\left[\frac{N(\varphi)}{D(\varphi)}\right] \tag{25}$$

$$N(\varphi) = [R_1^*(\lambda) + \varphi(1-R_1^*(\lambda))]\{(\lambda+\delta)V^*(A_v(\varphi)) - \lambda\} - [R_2^*(\delta) + \delta\varphi(1-R_2^*(\delta))] + (\lambda+\delta)W(\varphi)$$

$$D(\varphi) = \varphi - \kappa\varphi[R_1^*(\lambda) + \varphi(1-R_1^*(\lambda))]S^*(C_s(\varphi)) - \varphi(1-\kappa)[R_2^*(\delta) + \varphi(1-R_2^*(\delta))]S^*(C_s(\varphi))$$

Where $W(\varphi) = \dfrac{\omega[1-V^*(A_v(\varphi))]}{A_v(\varphi)}$.

Theorem 2 The stationary dist. of the no. of consumers in the ordinary and premium orbit, while the server is free, regular busy, reduced speed service, and the prob. That the server is free is provided by $\rho < 1$ under the stability condition

$$R_1(\varphi) = R_0\left[\frac{Ne(R_1(\varphi))}{D(\varphi)}\right] \tag{26}$$

$$Ne(R_1(\varphi)) = \frac{(1-R_1^*(\lambda))}{\lambda}\Big\{\kappa\varphi S^*(C_s(\varphi))(\lambda+\delta)W(\varphi) + [R_2^*(\delta) + \varphi(1-R_2^*(\delta))]\{\lambda\varphi(1-\kappa)S^*(C_s(\varphi))$$

$$-\kappa\varphi S^*(C_s(\varphi))(\delta) - \varphi(1-\kappa)S^*(C_s(\varphi))(\lambda+\delta)V^*(A_v(\varphi))\} + \varphi(\lambda+\delta)V^*(A_v(\varphi)) - \lambda\varphi\Big\}$$

$$R_2(\varphi) = R_0\left[\frac{Ne(R_2(\varphi))}{D(\varphi)}\right] \tag{27}$$

$$Ne(R_2(\varphi)) = \frac{(1-R_2^*(\delta))}{\delta}\Big\{[R_1^*(\lambda)+\varphi(1-R_1^*(\lambda))]\{(1-\kappa)(\lambda+\delta)\varphi S^*(C_s(\varphi))V^*(A_v(\varphi))-\lambda(1-\kappa)$$

$$\varphi S^*(C_s(\varphi))+\delta\kappa\varphi S^*(C_s(\varphi))\}+(\lambda+\delta)(1-\kappa)\varphi S^*(C_s(\varphi))W(\varphi)-\delta\varphi\Big\}$$

$$S(\varphi) = R_0\left[\frac{Ne(S(\varphi))}{D(\varphi)}\right] \tag{28}$$

$$Ne(S(\varphi)) = \frac{(1-S^*(C_s(\varphi))}{C_s(\varphi)}\Big([R_1^*(\lambda)+\varphi(1-R_1^*(\lambda))]\{(\lambda+\delta)V^*(A_v(\varphi))-\lambda\}-[R_2^*(\delta)+\delta\varphi(1-R_2^*(\delta))]$$

$$+(\lambda+\delta)W(\varphi)\Big)$$

$$V(\varphi) = R_0\left[\frac{(\lambda+\delta)[1-V^*(A_v(\varphi))]}{A_v(\varphi)}\right] \tag{29}$$

where

$$R_0 = \left[\frac{Ne(R_0)}{De(R_0)}\right] \tag{30}$$

$$Ne(R_0) = (1-\kappa\{1-R_1^*(\lambda)-E(S)(\lambda+\delta)+1\}-(1-\kappa)\{1-R_2^*(\delta)-E(S)(\lambda+\delta)+1\})$$

$$De(R_0) = (1-\kappa\{1-R_1^*(\lambda)-E(S)(\lambda+\delta)+1\}-(1-\kappa)\{1-R_2^*(\delta)-E(S)(\lambda+\delta)+1\})$$

$$(1+\left(\frac{(\lambda+\delta)[1-V^*(\omega)}{\omega}\right))+\left(\frac{1-R_1^*(\lambda)}{\lambda}\right)\Bigg[\left(\frac{1}{\omega}\right)\Big(\omega\kappa(\lambda+\delta)[1-V^*(\omega)]-\omega\kappa(\lambda+\delta)^2E(S)$$

$$[1-V^*(\omega)]+\kappa(\lambda+\delta)^2[E(V)+(1-V^*(\omega))]+\omega(\lambda+\delta)V^*(\omega)-\omega(\lambda+\delta)^2E(V)-\omega\lambda$$

$$+\omega\Big[1-R_2^*(\delta)\Big]\Big\{\lambda(1-\kappa)-\lambda(\lambda+\delta)(1-\kappa)E(S)-\kappa\delta+\kappa\delta(\lambda+\delta)E(S)-(1-\kappa)V^*(\omega)$$

$$(\lambda+\delta)+(1-\kappa)(\lambda+\delta)^2E(S)V^*(\omega)+(1-\kappa)(\lambda+\delta)^2E(V)\Big\}\Big)\Bigg]+\left(\frac{1-R_2^*(\delta)}{\delta}\right)$$

$$\left[\left(\frac{1}{\omega}\right)\left(\omega\left(1-R_1^*(\lambda)\right)\left\{\lambda+\delta)(1-\kappa)V^*(\omega)-(\lambda+\delta)^2(1-\kappa)E(S)V^*(\omega)-(\lambda+\delta)^2(1-\kappa)\right.\right.\right.$$

$$\left.E(V)-\lambda(1-\kappa)+\lambda(\lambda+\delta)(1-\kappa)E(S)+\kappa\delta-\kappa\delta(\lambda+\delta)E(S)\right\}+\omega(\lambda+\delta)(1-\kappa)$$

$$\left.\left.[1-V^*(\omega)]-\omega(\lambda+\delta)^2(1-\kappa)E(S)[1-V^*(\omega)]+(\lambda+\delta)^2(1-\kappa)[E(V)+[1-V^*(\omega)]]-\omega\delta\right)\right]$$

$$-E(S)\left[\left(\frac{1}{\omega}\right)\left((\lambda+\delta)^2[E(V)+[1-V^*(\omega)]-E(S)[\omega[1-R_1^*(\lambda)][-(\lambda+\delta)^2E(V)-\lambda]\right.\right.$$

$$\left.\left.-\omega[1-R_2^*(\delta)]\delta\right)\right]$$

Proof. Taking the eqns. (19)-(22) then computing their integration with respect to ε to ascertain the partial PG $R_1(\varphi)=\int_0^\infty R_1(\varepsilon,\varphi)d\varepsilon$, $R_2(\varphi)=\int_0^\infty R_2(\varepsilon,\varphi)d\varepsilon$, $S(\varphi)=\int_0^\infty S(\varepsilon,\varphi)d\varepsilon$, $V(\varphi)=\int_0^\infty V(\varepsilon,\varphi)d\varepsilon$, We estimate the server's available probability. Using the normalization condition (R_0) by establishing functions as, when there is no patient in the orbit $\varphi=1$ in (26)-(29) additionally, anytime the l'Hospital Rule is necessary, we get $R_0+R_1(1)+R_2(1)+S(1)+V(1)=1$. □

Theorem 3 Regarding the stability restriction $\rho<1$, We calculate the PGF of the number of patients in the system and the distribution of orbit sizes at a stationary moment in time.

$$K_s(\varphi)=R_0\frac{Ne_s(\varphi)}{(1-\varphi)D(\varphi)} \tag{31}$$

$$Ne_s(\varphi)=(1-\varphi)\left\{\frac{(1-R_1^*(\lambda))}{\lambda}\left(\kappa\varphi S^*(C_s(\varphi))(\lambda+\delta)W(\varphi)+[R_2^*(\delta)+\varphi(1-R_2^*(\delta))]\{\lambda\varphi(1-\kappa)S^*(C_s(\varphi))\right.\right.$$

$$\left.-\kappa S^*(C_s(\varphi))(\delta)-\varphi(1-\kappa)S^*(C_s(\varphi))(\lambda+\delta)V^*(A_v(\varphi))\}+\varphi(\lambda+\delta)V^*(A_v(\varphi))-\lambda\varphi\right)$$

$$+\frac{(1-R_2^*(\delta))}{\delta}\left([R_1^*(\lambda)+\varphi(1-R_1^*(\lambda))]\{(\lambda+\delta)(1-\kappa)\varphi S^*(C_s(\varphi))V^*(A_v(\varphi))-\lambda(1-\kappa)\varphi\right.$$

$$\left.\left.S^*(C_s(\varphi))+\delta\kappa\varphi S^*(C_s(\varphi))\}+(\lambda+\delta)(1-\kappa)\varphi s^*(C_s(\varphi))W(\varphi)-\delta\varphi\right)\right\}+\varphi(1-\varphi)D(\varphi)(\lambda+\delta)$$

$$\frac{(1-V^*(A_v(\varphi)))}{A_v(\varphi)}+\varphi\frac{(1-S^*(C_s(\varphi)))}{C_s(\varphi)}\left[[R_1^*(\lambda)+\varphi(1-R_1^*(\lambda))]\{(\lambda+\delta)V^*(A_v(\varphi))-\lambda\}\right.$$

$$-[R_2^*(\delta)+\varphi(1-R_2^*(\delta))]\delta+(\lambda+\delta)W(\varphi)\Big]$$

$$H_o(\varphi)=R_0\frac{Ne_o(\varphi)}{(1-\varphi)D(\varphi)} \tag{32}$$

$$Ne_o(\varphi)=(1-\varphi)\Bigg\{\frac{(1-R_1^*(\lambda))}{\lambda}\Bigg(\kappa\varphi S^*(C_s(\varphi))(\lambda+\delta)W(\varphi)+[R_2^*(\delta)+\varphi(1-R_2^*(\delta))]\{\lambda\varphi(1-\kappa)$$

$$S^*(C_s(\varphi))-\kappa\varphi S^*(C_s(\varphi))(\delta)-\varphi(1-\kappa)S^*(C_s(\varphi))(\lambda+\delta)V^*(A_v(\varphi))\}+\varphi(\lambda+\delta)$$

$$V^*(A_v(\varphi))-\lambda\varphi\Bigg)+\frac{(1-R_2^*(\delta))}{\delta}\Bigg([R_1^*(\lambda)+\varphi(1-R_1^*(\lambda))]\{(1-\kappa)(\lambda+\delta)$$

$$\varphi S^*(C_s(\varphi))V^*(A_v(\varphi))-\lambda(1-\kappa)\varphi S^*(C_s(\varphi))+\delta\kappa\varphi S^*(C_s(\varphi))\}+(\lambda+\delta)(1-\kappa)\varphi s^*(C_s(\varphi))$$

$$W(\varphi)-\delta\varphi\Bigg)\Bigg\}+(1-\varphi)D(\varphi)(\lambda+\delta)\frac{(1-V^*(A_v(\varphi)))}{A_v(\varphi)}+\frac{(1-S^*(C_s(\varphi)))}{C_s(\varphi)}\Big[[R_1^*(\lambda)+\varphi(1-R_1^*(\lambda))]$$

$$\{(\lambda+\delta)V^*(A_v(\varphi))-\lambda\}-[R_2^*(\delta)+\delta\varphi(1-R_2^*(\delta))]+(\lambda+\delta)W(\varphi)\Big]$$

where R_0 is denoted by eqn. (30).

Proof. Finally, by applying (26)-(29) and a little mathematical arithmetic, $K_s(\varphi)=R_0+R_1(\varphi)+R_2(\varphi)+\varphi S(\varphi)+\varphi W(\varphi)$ PGF of available patients number in the system is derived (31). Use the formula $H_0(\varphi)=R_0+R_1(\varphi)+R_2(\varphi)+S(\varphi)+W(\varphi)$ to find the PGF of available patients number in the orbit $(H_0(\varphi))$. It is possible to compute (32). □

4. System performance measures

The mean busy time and mean busy cycle algorithms were discovered, several pertinent system probabilities, and system efficiency measures even if the system is in various states.

4.1 *System state probabilities*

Using equations (26)-(29) putting $\varphi\to 1$ and using l'Hospital's rule until it's feasible, we reach at the subsequent conclusions.

(i) For the period of the Ordinary retrial, let R_1 represent the steady state probability of the server being available,

$$R_1=R_1(1)=R_0\left\{\frac{Ne_1(1)}{De(1)}\right\}$$

$$Ne_1(1) = \left(\frac{1-R_1^*(\lambda)}{\lambda}\right)\left\{\left(\frac{1}{\omega}\right)\left(\omega\kappa(\lambda+\delta)[1-V^*(\omega)] - \omega\kappa(\lambda+\delta)^2 E(s)[1-V^*(\omega)] + \kappa(\lambda+\delta)^2\right.\right.$$

$$[E(v)+(1-V^*(\omega))] + \omega(\lambda+\delta)V^*(\omega) - \omega(\lambda+\delta)^2 E(v) - \omega\lambda + \omega[1-R_2^*(\delta)]\{\lambda(1-\kappa)$$

$$-\lambda(\lambda+\delta)(1-\kappa)E(s) - \kappa\delta + \kappa\delta(\lambda+\delta)E(s) - (1-\kappa)V^*(\omega)(\lambda+\delta) + (1-\kappa)(\lambda+\delta)^2 E(s)$$

$$\left.\left. V^*(\omega) + (\lambda+\delta)^2 E(v)(1-\kappa)\}\right)\right\}$$

$$De(1) = 1 - \kappa\{1-R_1^*(\lambda) - (\lambda+\delta)E(s)+1\} - (1-\kappa)\{1-R_2^*(\delta) - (\lambda+\delta)E(s)+1\}$$

(ii) Let R_2 represent the steady state probability of the server will remain available during the Premium retrial,

$$R_2 = R_2(1) = R_0\left\{\frac{Ne_2(1)}{De(1)}\right\}$$

$$Ne_2(1) = \left(\frac{1-R_2^*(\delta)}{\delta}\right)\left\{\left(\frac{1}{\omega}\right)\left(\omega[1-R_1^*(\lambda)]\{(\lambda+\delta)(1-\kappa)V^*(\omega) - (\lambda+\delta)^2(1-\kappa)E(s)V^*(\omega)\right.\right.$$

$$-(\lambda+\delta)^2(1-\kappa)E(v) - \lambda(1-\kappa) + \lambda(\lambda+\delta)(1-\kappa)E(s) + \kappa\delta - \kappa\delta(\lambda+\delta)E(s)\} + \omega(\lambda+\delta)$$

$$(1-\kappa)[1-V^*(\omega)] - \omega(\lambda+\delta)^2(1-\kappa)E(s)[1-V^*(\omega)] + (\lambda+\delta)^2(1-\kappa)[E(v)+[1-V^*(\omega)]]$$

$$\left.\left. -\omega\delta\right)\right\}$$

(iii) Let S represent the steady state probability of the server being occupied.

$$S = S(1) = R_0\left\{\frac{Ne_s(1)}{De(1)}\right\}$$

$$Ne_s(1) = -E(S)\left\{\left(\frac{1}{\omega}\right)\left((\lambda+\delta)^2[E(v)+[1-V^*(\omega)] - E(s)[\omega[1-R_1^*(\lambda)][-(\lambda+\delta)^2 E(v) - \lambda]\right.\right.$$

$$\left.\left. -\omega[1-R_2^*(\delta)]\delta\right)\right\}$$

(iv) Let V represent the steady state probability of the server's working vacation,

$$V = V(1) = R_0\left[\frac{(\lambda+\delta)[1-V^*(\omega)}{\omega}\right]$$

4.2 *The size of an orbit and the average size of a system*

Once a steady state has been reached by the system,
(i) Given φ, (32), and $\varphi = 1$, the expected number of patients in the orbit (L_q) is obtained.

$$L_q = H_0^{'}(1) = \lim_{\varphi\to 1}\frac{d}{d\varphi}H_0(\varphi)$$

$$= R_0\left[\frac{Ne_q^{'''}(1)De_q^{''}(1) - De_q^{'''}(1)Ne_q^{''}(1)}{3(De_q^{''}(1))^2}\right] \tag{33}$$

$$Ne_q^{''}(1) = -2\left\{\left(\frac{(1-R_1^*(\lambda))}{\lambda}\right)\left[\left(\frac{1}{\omega}\right)\omega\kappa(\lambda+\delta)[1-V^*(\omega)] - \omega\kappa(\lambda+\delta)^2E(S)[1-V^*(\omega)]\right.\right.$$

$$+\kappa(\lambda+\delta)^2[E(V)+(1-V^*(\omega))] + \omega(\lambda+\delta)V^*(\omega) - \omega(\lambda+\delta)^2E(V) - \omega\lambda + \omega[1-R_2^*(\delta)]$$

$$\{\lambda(1-\kappa) - \lambda(\lambda+\delta)E(S)(1-\kappa) - \kappa\delta + \kappa\delta(\lambda+\delta)E(S) - (1-\kappa)V^*(\omega)(\lambda+\delta)$$

$$\left.\left. + (1-\kappa)V^*(\omega)(\lambda+\delta)^2E(S) + (1-\kappa)(\lambda+\delta)^2E(V)\}\right]\right\} - 2\left\{\left(\frac{(1-R_2^*(\delta))}{\delta}\right)\right.$$

$$\left[\left(\frac{1}{\omega}\right)\omega[1-R_1^*(\lambda)]\{(\lambda+\delta)(1-\kappa)V^*(\omega) - (\lambda+\delta)^2(1-\kappa)E(S)V^*(\omega) - (\lambda+\delta)^2\right.$$

$$(1-\kappa)E(V) - \lambda(1-\kappa) + \lambda(\lambda+\delta)(1-\kappa)E(S) + \kappa\delta - \kappa\delta(\lambda+\delta)E(S)\} + \omega(\lambda+\delta)(1-\kappa)$$

$$\left.\left.[1-V^*(\omega)] - \omega(\lambda+\delta)^2(1-\kappa)E(S)[1-V^*(\omega)] + (\lambda+\delta)^2(1-\kappa)[E(V)+[1-V^*(\omega)]] - \omega\delta\right]\right\}$$

$$+2\Big((\lambda+\delta)E(S)\Big)\left\{(-E(S))\left(\frac{1}{\omega}\right)(\lambda+\delta)^2[E(V)+[1-V^*(\omega)] - E(S)[\omega[1-R_1^*(\lambda)]\right.$$

$$\left.[-(\lambda+\delta)^2E(V) - \lambda] - \delta\omega[1-R_2^*(\delta)]\right\} - 2(1-\kappa\{1-R_1^*(\lambda) - (\lambda+\delta)E(S)+1\}$$

$$-(1-\kappa)\{1-R_2^*(\delta) - (\lambda+\delta)E(S)+1\})$$

$$De_q^{''}(1) = -2(1-\kappa\{1-R_1^*(\lambda)-(\lambda+\delta)E(S)+1\}-(1-\kappa)\{1-R_2^*(\delta)-(\lambda+\delta)E(S)+1\})$$

$$Ne_q^{'''}(1) = -3\Bigg\{ \left(\frac{1-R_1^*(\lambda)}{\lambda}\right)\left(\left(\frac{1}{\omega^3}\right)2\omega^2\kappa(\lambda+\delta)^2[E(V)+[1-V^*(\omega)]]-2\omega^3\kappa(\lambda+\delta)^2E(S)\right.$$

$$[1-V^*(\omega)]+\omega^3\kappa(\lambda+\delta)^3E(S)^2[1-V^*(\omega)]-2\omega^2\kappa(\lambda+\delta)^3E(S)[E(V)+[1-V^*(\omega)]]$$

$$+\kappa(\lambda+\delta)\Big[\omega(\lambda+\delta)^2E(V)[3\omega-\omega E(V)-1]+2[1-V^*(\omega)](\lambda+\delta)^2\Big]-2\omega^3(\lambda+\delta)^2E(V)$$

$$+\omega^3(\lambda+\delta)^3E(V)^2+\omega^3[1-R_2^*(\delta)]\Big[-2\lambda(\lambda+\delta)E(S)(1-\kappa)+\lambda(\lambda+\delta)^2E(S)^2(1-\kappa)+2\kappa\delta$$

$$(\lambda+\delta)E(S)-\kappa\delta(\lambda+\delta)^2E(S)^2+2(1-\kappa)(\lambda+\delta)^2E(S)V^*(\omega)+2(\lambda+\delta)^2E(V)(1-\kappa)$$

$$\left.-(1-\kappa)(\lambda+\delta)^3E(S)^2V^*(\omega)-2(1-\kappa)(\lambda+\delta)^3E(S)E(V)-(1-\kappa)(\lambda+\delta)^3E(V)^2\Big]\right)\Bigg\}$$

$$-3\Bigg\{\left(\frac{1-R_2^*(\delta)}{\delta}\right)\left([\left(\frac{1}{\omega^3}\right)\omega^3[1-R_1^*(\lambda)]\Big[(\lambda+\delta)^3(1-\kappa)E(S)^2V^*(\omega)-2(\lambda+\delta)^2(1-\kappa)\right.$$

$$E(S)V^*(\omega)-2(\lambda+\delta)^2(1-\kappa)E(V)+2(\lambda+\delta)^3(1-\kappa)E(S)E(V)+(\lambda+\delta)^3(1-\kappa)E(V)^2$$

$$+2\lambda(\lambda+\delta)E(S)(1-\kappa)-\lambda(\lambda+\delta)^2(1-\kappa)E(S)^2-2\delta\kappa(\lambda+\delta)E(S)+\kappa\delta(\lambda+\delta)^2E(S)^2\Big]$$

$$+2\omega^2(\lambda+\delta)^2(1-\kappa)[E(V)+[1-V^*(\omega)]]-2\omega^3(\lambda+\delta)^2(1-\kappa)E(S)[1-V^*(\omega)]$$

$$-2\omega^2(\lambda+\delta)^3(1-\kappa)E(S)[E(v)+[1-V^*(\omega)]]+\omega^3(\lambda+\delta)^3E(S)^2(1-\kappa)[1-V^*(\omega)]$$

$$\left.+(\lambda+\delta)(1-\kappa)(\omega(\lambda+\delta)^2E(V)[3\omega-\omega E(V)-1]+2[1-V^*(\omega)](\lambda+\delta)^2)\right)\Bigg\}-3\left((\lambda+\delta)^2E(S)^2\right)$$

$$\left((-E(S))\left(\frac{1}{\omega}\right)\Bigg\{-\omega(\lambda+\delta)^2E(V)+\omega[1-R_1^*(\lambda)][(\lambda+\delta)V^*(\omega)-\lambda]-\omega\delta(1-R_2^*(\delta))\right.$$

$$\left.+(\lambda+\delta)^2[E(V)+[1-V^*(\omega)]]\Bigg\}\right)+3\big((\lambda+\delta)E(S)\big)\left((-E(S))\left(\frac{1}{\omega^3}\right)\Bigg\{\omega^3(\lambda+\delta)^2E(V)[1-R_1^*(\lambda)]\right.$$

$$+\omega^3(\lambda+\delta)^3E(V)^2-\omega^3[1-R_1^*(\lambda)](\lambda+\delta)^2E(V)+(\lambda+\delta)\Big[\omega(\lambda+\delta)^2E(V)[3\omega-\omega E(V)-1]$$

$$+2[1-V^*(\omega)](\lambda+\delta)^2\Big]\Big\}\Big)-6\big((\lambda+\delta)W^{'}(1)(1-\rho)\big)-3\left(\frac{(\lambda+\delta)[1-V^*(\omega)]}{\omega}\right)$$

$$\Big[-2\kappa[1-R_1^*(\lambda)]+\kappa[1-R_1^*(\lambda)]E(S)(\lambda+\delta)-2(1-\kappa)[1-R_2^*(\delta)]+2E(S)(\lambda+\delta)$$

$$+2(1-\kappa)[1-R_2^*(\delta)]E(S)(\lambda+\delta)-(\lambda+\delta)^2E(S)^2\Big]$$

$$De_q^{'''}(1)=-3\big(-2\kappa[1-R_1^*(\lambda)]+\kappa[1-R_1^*(\lambda)]E(S)(\lambda+\delta)-2(1-\kappa)[1-R_2^*(\delta)]$$

$$+2(\lambda+\delta)E(S)+2(1-\kappa)[1-R_2^*(\delta)](\lambda+\delta)E(S)-(\lambda+\delta)^2E(S)^2\big)$$

where

$$\rho=\kappa\{1-R_1^*(\lambda)-(\lambda+\delta)E(S)+1\}-(1-\kappa)\{1-R_2^*(\delta)-(\lambda+\delta)E(S)+1\}$$

$$W^{'}(1)=\frac{(\lambda+\delta)[E(V)+[1-V^*(\omega)]]}{\omega}$$

(ii) With regard to φ, (31) and putting $\varphi=1$ produces the expected no. of patients in the system (L_s)

$$L_s=K_s^{'}(1)=\lim_{\varphi\to 1}\frac{d}{d\varphi}K_s(\varphi)$$

$$=R_0\left[\frac{Ne_s^{'''}(1)De_q^{''}(1)-De_q^{'''}(1)Ne_q^{''}(1)}{3(De_q^{''}(1))^2}\right] \tag{34}$$

$$Ne_s^{'''}(1)=Nr_q^{'''}(1)+6(\lambda+\delta)E(S)\bigg((-E(S))\bigg\{\left(\frac{1}{\omega}\right)(\lambda+\delta)^2[E(V)+[1-V^*(\omega)]$$

$$-E(S)[\omega[1-R_1^*(\lambda)][-(\lambda+\delta)^2E(V)-\lambda]-\omega[1-R_2^*(\delta)]\delta\bigg\}\bigg)-6(1-\rho)$$

$$\left(\frac{(\lambda+\delta)[1-V^*(\omega)]}{\omega}\right)$$

(iii) Little's formula is utilized to predict the projected time the patient will spend in the system (W_s) and the queue (W_q). Specifically, $W_s = \frac{L_s}{\lambda}$ and $W_q = \frac{L_q}{\lambda}$.

4.3 *The busy cycle and the average busy period*

$$R_0 = \frac{A(T_0)}{A(T_y)+A(T_0)}; A(T_y) = \frac{1}{\lambda}\left(\frac{1}{R_0}-1\right);$$

$$A(T_\varphi) = \frac{1}{\lambda R_0} = A(T_0)+A(T_y). \tag{35}$$

where T_0 indicates the amount of time the system spent empty. Because the time gap between the arrivals of two patients is exponential. With λ as the parameter, we have $A(T_0) = (1/\lambda)$. By placing it into (35) and applying the previously discovered information, we may obtain (30).

$$A(T_y) = \frac{1}{\lambda} \times \left\{\frac{Ne(\varphi)}{De(\varphi)} - 1\right\} \tag{36}$$

$$Ne(\varphi) = \Big[1-\kappa\{1-R_1^*(\lambda)-(\lambda+\delta)E(S)+1\}-(1-\kappa)\{1-R_2^*(\delta)-(\lambda+\delta)E(S)+1\}\Big]$$

$$\left[1+\left(\frac{(\lambda+\delta)[1-V^*(\omega)}{\omega}\right)\right]+\left(\frac{1-R_1^*(\lambda)}{\lambda}\right)\Bigg[\left(\frac{1}{\omega}\right)\Big(\omega\kappa(\lambda+\delta)[1-V^*(\omega)]-\omega\kappa(\lambda+\delta)^2E(S)$$

$$[1-V^*(\omega)]+\kappa(\lambda+\delta)^2[E(V)+(1-V^*(\omega))]+\omega(\lambda+\delta)V^*(\omega)-\omega(\lambda+\delta)^2E(V)-\omega\lambda+\omega$$

$$[1-R_2^*(\delta)]\Big\{\lambda(1-\kappa)-\lambda(\lambda+\delta)(1-\kappa)E(S)-\kappa\delta+\kappa\delta(\lambda+\delta)E(S)-(1-\kappa)V^*(\omega)(\lambda+\delta)$$

$$+(\lambda+\delta)^2E(S)V^*(\omega)(1-\kappa)+(1-\kappa)(\lambda+\delta)^2E(V)\Big\}\Big)\Bigg]+\left(\frac{1-R_2^*(\delta)}{\delta}\right)\Bigg[\left(\frac{1}{\omega}\right)\Big(\omega\Big(1-R_1^*(\lambda)\Big)\Big\{(\lambda+\delta)$$

$$(1-\kappa)V^*(\omega)-(\lambda+\delta)^2(1-\kappa)E(S)V^*(\omega)-(\lambda+\delta)^2(1-\kappa)E(V)-\lambda(1-\kappa)+\lambda(\lambda+\delta)(1-\kappa)$$

$$E(S)+\kappa\delta-\kappa\delta(\lambda+\delta)E(S)\Big\}+\omega(\lambda+\delta)(1-\kappa)[1-V^*(\omega)]-\omega(\lambda+\delta)^2(1-\kappa)E(S)[1-V^*(\omega)]$$

$$+(\lambda+\delta)^2(1-\kappa)[E(V)+[1-V^*(\omega)]]-\omega\delta\Big)\Bigg]-E(S)\Bigg[\left(\frac{1}{\omega}\right)\Big((\lambda+\delta)^2[E(V)+[1-V^*(\omega)]$$

$$-E(S)[\omega[1-R_1^*(\lambda)][-(\lambda+\delta)^2E(V)-\lambda]-\omega[1-R_2^*(\delta)]\delta\Big)\Bigg]$$

$$De(\varphi) = \left[1 - \kappa\{1 - R_1^*(\lambda) - (\lambda+\delta)E(S) + 1\} - (1-\kappa)\{1 - R_2^*(\delta) - (\lambda+\delta)E(S) + 1\}\right]$$

$$A(T_\varphi) = \frac{1}{\lambda} \times \left\{\frac{Ne(\varphi)}{De(\varphi)}\right\} \tag{37}$$

5. Special cases

This section examines a few real-world uses for our approach that are consistent with recent studies.

Case (i): No ordinary orbit and No premium orbit.

Let $R_1^*(\lambda) = R_2^*(\delta) = 1$ and our approach reduces to an $M/G/1$ RQ with WVs. Here are the results that coincide with Gao et al. [28].

$$K_s(\varphi) = R_0 \frac{Ne_s(\varphi)}{De_s(\varphi)}$$

$$Ne_s(\varphi) = \varphi(1-\varphi)(\varphi - \kappa\varphi S^*(C_s(\varphi)) - \varphi(1-\kappa)S^*(C_s(\varphi)))\left(\frac{(\lambda+\delta)(1-V^*(A_v(\varphi)))}{A_v(\varphi)}\right)$$

$$+ \frac{\varphi(1-S^*(C_s(\varphi)))}{C_s(\varphi)}\{(\lambda+\delta)V^*(A_v(\varphi)) - \lambda\} - \delta + (\lambda+\delta)\left[\frac{\omega[1-V^*(A_v(\varphi))]}{A_v(\varphi)}\right]$$

$$De_s(\varphi) = (1-\varphi)(\varphi - \kappa\varphi S^*(C_s(\varphi)) - \varphi(1-\kappa)S^*(C_s(\varphi)))$$

where,

$$R_0 = \left[\frac{Ne(R_0)}{De(R_0)}\right]$$

$$Ne(R_0) = (1 - \kappa\{-(\lambda+\delta)E(S) + 1\} - (1-\kappa)\{-(\lambda+\delta)E(S) + 1\})$$

$$De(R_0) = \left[1 - \kappa\{-(\lambda+\delta)E(S) + 1\} - (1-\kappa)\{-(\lambda+\delta)E(S) + 1\}\right]\left[1 + \left(\frac{(\lambda+\delta)[1-V^*(\omega)]}{\omega}\right)\right]$$

$$- E(S)\left[(\lambda+\delta)^2[E(V) + [1 - V^*(\omega)]]\right]$$

Case (ii): No premium orbit and No working vacation.

Let $R_2^*(\delta) = 1$; and $\omega = 0$. Our approach reduces to an $M/G/1$ RQ with general retrial times. Here are the results that coincide with Gómez-Corral [29].

$$K_s(\varphi) = \frac{Ne_s(\varphi)}{De_s(\varphi)}$$

$$Ne_s(\varphi) = (1-\varphi)\left\{\left(\frac{1-R_1^*(\lambda)}{\lambda}\right)\{\lambda\varphi(1-\kappa)S^*(C_s(\varphi)) - \kappa\varphi S^*(C_s(\varphi))(\delta) - \varphi(1-\kappa)S^*C_s(\varphi)(\lambda+\delta)\right.$$

$$\left. V^*(\lambda+\delta)(1-\varphi)\} + \varphi V^*(\lambda+\delta)^2(1-\varphi) - \lambda\varphi\right\} + \varphi(1-\varphi)(\varphi - \kappa\varphi[R_1^*(\lambda)$$

$$+\varphi(1-R_1^*(\lambda))]S^*(C_s(\varphi)) - \varphi(1-\kappa)S^*C_s(\varphi)\left(\frac{(\lambda+\delta)(1-V^*(\lambda+\delta)(1-\varphi))}{(\lambda+\delta)(1-\varphi)}\right)$$

$$+\frac{\varphi(1-S^*(C_s(\varphi)))}{C_s(\varphi)}[R_1^*(\lambda)+\varphi(1-R_1^*(\lambda))]\{V^*(\lambda+\delta)^2(1-\varphi)-\lambda\} - \delta$$

$$De_s(\varphi) = (1-\varphi)(\varphi - \kappa\varphi[R_1^*(\lambda)+\varphi(1-R_1^*(\lambda))]S^*(C_s(\varphi))) - \varphi(1-\kappa)S^*(C_s(\varphi))$$

where,

$$R_0 = \left[\frac{Ne(R_0)}{De(R_0)}\right]$$

$$Ne(R_0) = 1-\kappa\{1-R_1^*(\lambda)-E(S)(\lambda+\delta)+1\} - (1-\kappa)\{1-E(S)(\lambda+\delta)+1\}$$

$$De(R_0) = 1-\kappa\{1-E(S)(\lambda+\delta)\} - (1-\kappa)\{1-E(S)(\lambda+\delta)\} - E(S)[(\lambda+\delta)^2E(V)]$$

6. Numerical results

Using MATLAB, this section illustrates the many parameters that may be applied to system behavior measurements. The exponential distributions of service times, working vacation periods, premium retrial times, and ordinary retrial times are analyzed. The numerical measurements that are required to satisfy the stability criteria are selected through the use of a random selection process. The values that are computed in Tables 1, 2, and 3 are as follows: the server is currently idle (R_0), the average system size (L_s), the average queue size (L_q), the server is currently idle during ordinary and premium retrial times $(R_1(1)$ and $R_2(1))$, the mean waiting time in the queue (W_q), during working vacation time $(\psi_\omega(1))$, and busy time $(\mu(1))$ in the queueing model that we have computed.

Table 1 displays that ordinary retrial rate a_1 hikes, R_0, L_q, $R_1(1)$, $\psi_\omega(1)$ and W_q are diminishes.

Table 2 displays that premium retrial rate a_2 hikes, R_0, L_q, $R_1(1)$, $\psi_\omega(1)$ and W_q are diminishes.

Table 3 displays that working vacation rate ψ_ω escalates, R_0, L_q, L_s, $\psi_\omega(1)$, $\mu(1)$ and W_q are diminishes and $R_2(1)$ is escalates.

Table 1. L_q and R_0 for various Ordinary retrial rate (a_1) for the principles of $\kappa = 0.5,\ \lambda = 0.1,\ \omega = 1.7,\ \delta = 0.5,\ \mu = 5,\ \psi = 6$

Retrial rate a_1	R_0	$R_1(1)$	$\psi_\omega(1)$	L_q	W_q
1	1.2324	2.1191	0.0652	4.6159	46.1589
1.2	1.1996	2.0505	0.0635	4.4950	44.9504
1.4	1.1669	1.9809	0.0618	4.3742	43.7416
1.5	1.1505	1.9458	0.0609	4.3128	43.1283
1.9	1.0851	1.8032	0.0574	4.0683	40.6834
2	1.0687	1.7670	0.0566	4.0105	40.1047
2.3	1.0197	1.6573	0.0540	3.8299	38.2989

Table 2. L_q and R_0 for various Premium retrial rate (a_2) for the principles of $\kappa = 0.5,\ \lambda = 2.1,\ \omega = 1.7,\ \delta = 5,\ \mu = 0.5,\ \psi = 6$

Retrial rate a_2	R_0	$R_2(1)$	$\psi_\omega(1)$	L_q	W_q
1	6.5594	12.7380	2.7360	96.5114	45.9578
1.2	6.3098	10.4066	2.6589	90.9567	43.3127
1.4	6.0601	8.2619	2.5818	90.2727	42.9870
1.5	5.8105	6.2934	2.5047	88.2150	42.0071
1.9	5.5609	4.4916	2.4276	83.7210	39.8712
2	5.3112	2.8475	2.3505	80.0156	38.1026
2.3	5.0616	1.3529	2.2734	77.9615	37.1245

Table 3. R_0 and L_q for various Working vacation probabilities (ψ_ω) for the principles of $\kappa = 0.7,\ \lambda = 0.4,\ \omega = 1.1,\ \delta = 0.2,\ \mu = 3,\ \psi = 8$

Working Vacation ψ_ω	R_0	$R_2(1)$	$\mu(1)$	$\psi_\omega(1)$	L_q	L_s	W_q
0.3	1.2586	0.0125	0.4625	0.4600	3.5261	4.1016	8.8153
0.4	1.2057	0.0565	0.3904	0.3683	3.3400	4.0083	8.3501
0.5	1.1527	0.0965	0.3230	0.2829	2.8326	3.3828	7.0815
0.6	1.0998	0.1327	0.2601	0.2040	2.3586	2.7987	5.8957
0.7	1.0468	0.1650	0.2019	0.1313	1.9171	2.2558	4.7928
0.8	0.9939	0.1933	0.1484	0.0651	1.5091	1.7542	3.7727
0.9	0.9409	0.2178	0.0994	0.0051	1.1342	1.2939	2.8355

Display the three-dimensional graph that illustrates the performance metrics for the system, taking into account the influence of the parameters ω, δ, λ, μ, ψ, and κ. Figures (5), (6), and (7) are displayed. It is evident from the surface, as depicted in Figure (5), that the escalation of the ordinary retrial rate (a_1), (L_q), and (W_q) decreases. As shown in Figure (6), we discovered that the values of (L_q) and (W_q) decrease when the premium retrial rate a_2 is increased. As shown in

Figure 7, we discovered that the values of (L_q) and (W_q) decrease when the working vacation rate (ψ_ω) increases. Now that we have obtained the influence of the parameters ω, δ, λ, μ, ψ, and κ, Figures (3), (4), and (5), we may proceed to the next step. Provide a visual representation of the two-dimensional graph that illustrates the performance metrics for the framework. The escalation of the ordinary retrial rate (a_1), (W_q) is seen to decrease in Figure (3). As seen in Figure (3), the values of (W_q) decrease as the premium retrial rate a_2 increases. According to the findings shown in Figure (4), we discovered that the values of (L_q) and (W_q) decrease when the working vacation rate ψ_ω increases. It is possible to correctly identify the influence of features concerning the system's assessment standards using the presented numerical results, which guarantees that these results are equivalent to situations that occur in the actual world.

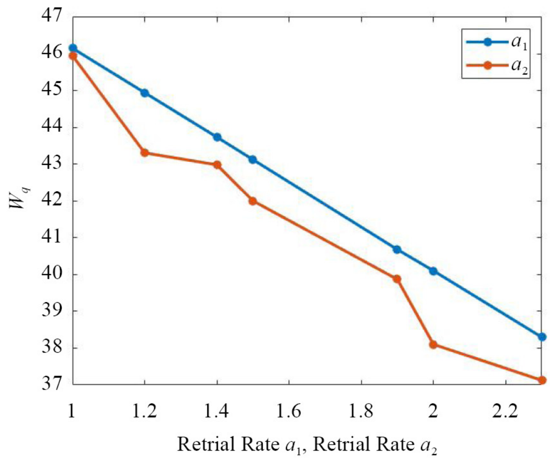

Figure 3. Ordinary retrial rate a_1 verses premium retrial rate a_2

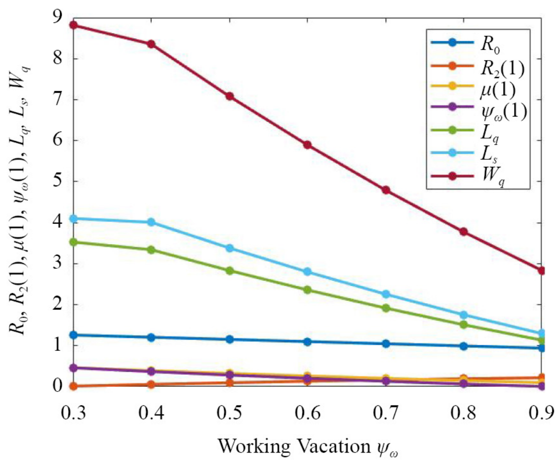

Figure 4. R_0 verses working vacation rate ψ_ω

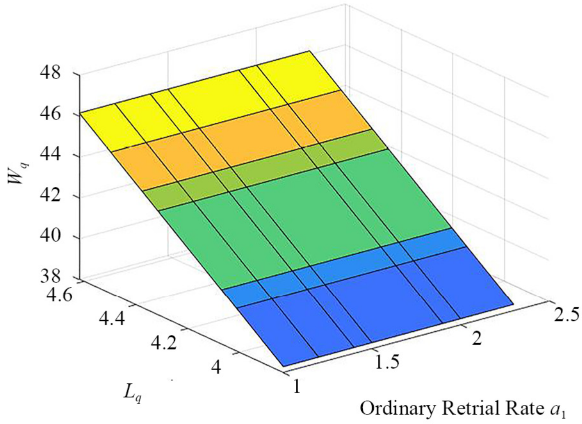

Figure 5. W_q, L_q verses ordinary retrial rate a_1

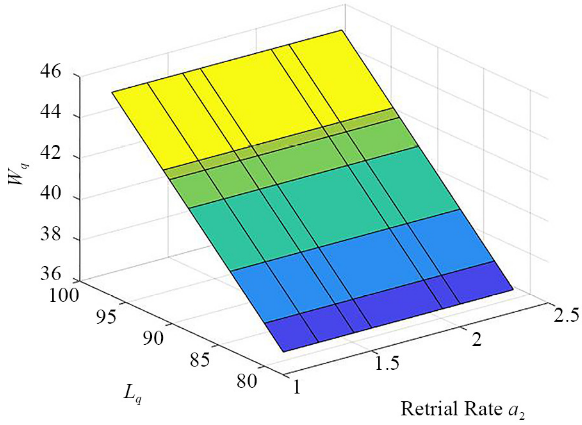

Figure 6. W_q, L_q verses premium retrial rate a_2

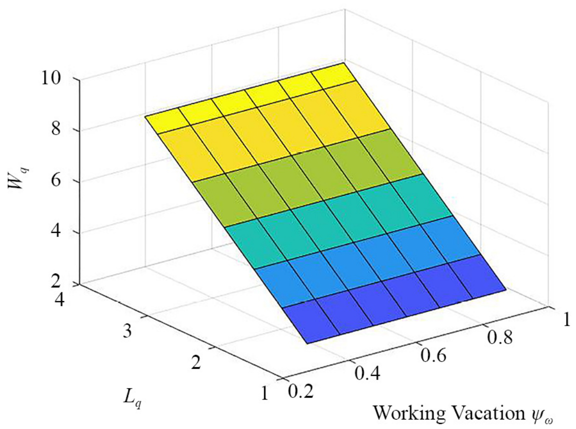

Figure 7. W_q, L_q verses working vacation rate ψ_ω

7. Conclusion

The article details a study focusing on a double-orbit retrial queue model that considers two types of patients within its framework. This model extends the traditional retrial queue concept by incorporating the notion of double orbit, essential for characterizing queue dynamics and reliability in scenarios involving an unreliable single server. The study employs a non-Markovian approach to develop the mathematical model, which allows for analyzing both queue behavior and system reliability. Key performance metrics, such as queue length and system characteristics, are derived using probability-generating functions (PGF). These metrics are influenced by the number of patients in the system and those in the queue. Numerical simulations are conducted to explore the different system characteristics that impact performance. Specifically, the paper evaluates the potential application of the double-orbit retrial queuing model in hospital management systems. Premium orbit patients wait in the queue for less time than ordinary patients. This decrease in waiting time is critical since the shorter the waiting period, the lesser the risk to patients. This queueing paradigm is highly useful in a variety of real-world applications, particularly in healthcare and pandemic scenarios. The analytical methods used are validated through steady-state results. In conclusion, the paper presents a detailed investigation into the double-orbit retrial queue model with working vacation, emphasizing its utility in understanding and optimizing queueing systems with unreliable servers, particularly in healthcare environments.

Conflict of interest

The authors declare no competing financial interest.

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