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Fractal Dimension Through Numerical Integration of Fractal Interpolation Functions

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Abstract: The initial objective of the paper is to propose an explicit relationship between the fractal dimension and fractal numerical integration of curves approximated through fractal interpolation from a discrete set of data points. Once the proposed relation is established, it is shown to be accurate by considering the data points of certain functions. The conventional box-counting dimension method has several drawbacks including the proper positioning of the boxes and determining the size of the boxes. The proposed relation becomes significant for such situations in providing the accurate determination of fractal dimension. Secondly, the paper aims to apply the derived relationship in the evaluation of two integral transforms of fractal interpolation functions, namely, the Fourier transform and the Laplace transform. The two integral transforms have been provided with alternate expressions, primarily, using the fractal dimension of fractal interpolation functions. Finally, considering these newly derived expressions and the proposed relation between fractal dimension and fractal numerical integration, this paper provides another method for the evaluation of the two integral transforms. This newly introduced method calculates the two integral transforms, via the fractal numerical integration of fractal interpolation functions.

Keywords: fractal dimension, numerical integration, fourier transform, laplace transform, fractal interpolation function

MSC: 28A80, 65D05, 44A10

1. Introduction

Fractal geometry has been introduced as an analog to the conventional Euclidean geometric techniques for the analysis and processing of objects of unpredictable nature. The geometric structures and characteristics with chaotic features are collectively defined as fractals [1]. According to the formal definition [2], a fractal is a set whose Hausdorff dimension exceeds the topological dimension. It becomes significant, especially for the analysis of many physical phenomena occurring in nature. For example, the traditional measures of length are incapable of finding the perimeter of Britain's coastline. Similarly, there are several instances in real life wherein the notion of fractal geometry and analysis is being applied. In the medical field, the images of our brain, nerves, and other organs show fractal-like structures. To determine, classify, and analyze such chaotic figures and datasets, specifically a new branch of fractal analysis has been introduced in [3], known as fractal interpolation. The technique of fractal interpolation relies on a finite set of contraction

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mappings defined on a complete metric space, commonly referred to as an iterated function system (IFS). The coefficients in the IFS are significant in the evaluation of certain fractal parameters. The fractal dimension is regarded as one of the significant fractal parameters that is used in measuring the complexity of signals. Fractal dimension indicates how densely an object occupies the space in which the object lies.

The concept of fractal dimension has been widely used in stock markets [4], medical fields [5] and several instances of art forms [6]. The analysis of the complexity of the biomedical images through fractal dimension helps to classify and characterize the physiological and pathological segments. For example, in [7], the changes in the white and grey brain matter of a patient affected with Alzheimer's disease are studied using fractal dimension. The fractal dimension has been used in the analysis of the Covid-19 outbreak and the forecasting of the future tendencies of the Coronavirus [8]. It is observed in [9] that the changes in the fractal pattern of EEG (electroencephalogram) signals directly influence the fractal dimension of animations. A comparison of the statistical, fractal, and multifractal algorithms in the determination of mechanical brain injury is performed in [10]. The detection of diseases in lung cavities is performed with fractal analysis in [11]. A fractal image of the lungs is created to construct a suitable mathematical model for the determination of diseases. A novel deep neural network architecture with a fractal structure is used in [12] to segment 2D and 3D brain tumor masks in neuroimaging data. [13] describes the fractal nature of the lungs and provides a review of the estimation of lung characteristics using fractal analysis. Another application of fractal dimension includes a different approach to solving the coronal heating problem as discussed in [14]. The efficiency of market index is analyzed using the fractal dimension, hurst exponent, and entropy approximation [15] for the countries like the USA, England, Japan, and Germany, including the emerging countries (BRICS). A novel strategy is proposed for the portfolio selection based on the fractal the market hypothesis (FMH) in [16]. The authors in [16] evaluate the fractal structures of the simulated portfolios using the Hurst exponent. Employing the Athens General Composite (ATS) index, the stock price dynamics within the Athens Stock Exchange (ASE) are modeled with fractal Brownian motion (fBm) in [17].

The correlation between the fractal dimension of the fluid and the characteristics of the velocity profile is examined in [18]. Considering the theory of fractal calculus, four independent fractal models of somatic growth in fishes have been proposed in [19]. Considering the product-like fractal measure, a phenomenological model is proposed [20] for the drainage of liquid as foam. A generalization of Moran's theorem that deals with the similarity dimension of strict self similar objects is provided in [21]. The effect of fractal dimension in the nature of the solution of the Grad-Shafranov equation is analyzed in [22]. A novel algorithm for the estimation of the fractal dimension of coastlines is introduced in [23] with the theory of FIF. The fractal dimension of the Thai dance gestures is investigated in [24]. The fractal dimension algorithm for the calculation of fluctuating continuous functions is introduced in [25]. A comprehensive introduction to the theory of fractal dimension is provided in [26, 27] provides an analysis of the Hopfield neural networks with fractal-fractional derivatives.

Numerous ideas and associated techniques have been developed so far to assess the fractal dimension of signals and structures. The concept of measures as a covering of certain subsets was initially introduced in [28]. Later on, Hausdorff put forward the idea of Hausdorff measure and dimension [29]. The properties and notions of Hausdorff measure and dimension were further developed in [2]. To make the calculation even simpler, [30] introduces the classical definition of box-counting dimension. Box-counting dimension is also referred to as entropy dimension, Kolmogorov entropy, metric dimension, capacity dimension, and logarithmic density. Reticular cell counting, a prominent method, was put forward in [31]. To overcome the flaws associated with the traditional differential box-counting method, an improvised version of differential box-counting (DBC) was later on introduced as Improved DBC [6]. Improved DBC too had its shortcomings. One of the earlier published works, Relative Improved DBC [32], aims at overcoming the shortcomings of Improved DBC. The estimation of fractal dimension using FIF is proposed in [3] and further discussed in [33].

Fractal numerical integration is another fractal parameter applied to evaluate of some Euclidean measures. Several numerical integration methods exist in the literature that correspond to the area under the interpolation functions. Simpson's rule, the midpoint rule, and the trapezoidal rule are the most widely used methods of numerical integration. In [34], a different approach to definite integral calculation utilizing triangles is suggested. However, these conventional techniques of numerical integration become infeasible in the case of non-Euclidean objects and functions. Thus, the technique of fractal numerical integration has evolved to numerically evaluate such chaotic functions. The benefits of fractal numerical integration involves the faster calculation of more accurate results. [35] Provides the formula for the classical integral of an affine FIF. The fractal numerical integration using linear FIF was initially proposed in [36] where the interpolating domain was closed intervals. Considering the rectangular interpolating domain, the numerical double integration formula using a bivariate FIF is given in [37]. Another numerical double integration formula for the data points defined over the triangular region is provided in [38]. The fractional integral of nonlinear FIF constructed with Rakotch contraction is investigated in [39]. The significance of the variable scaling factor in the flexibility of the fractal function and its Weyl-Marchaud fractional derivative is presented in [40].

The initial objective of the present work is to derive the connection between the fractal dimension and fractal numerical integration of a linear FIF. The derived relationship can be used to calculate the fractal dimension of graphs of functions whenever the fractal numerical integral value is given. On the reverse part, using the fractal numerical integral value, it is easier to find the fractal dimension of signals.

The advantages and novelties of the proposed work can be enumerated as follows:

• This relationship becomes significant, especially for those data sets whose fractal dimension is difficult to determine. Normally, the box-counting method is considered a feasible technique to evaluate fractal dimensions. There are circumstances where the box-counting dimension fails to provide accurate results. The boxes occasionally fail to evenly cover the data; as a result, data loss may occur, and the calculated dimension value may not be proper.

• It is also necessary to have a large number of points for the accurate determination of fractal dimension through the box-counting method. The proposed relationship, however, helps to resolve such drawbacks to an extent.

Secondly, the above-mentioned relationship has been implemented in the evaluation of certain integral transforms of fractal interpolation functions (FIF). The paper mainly deals with the Fourier and Laplace integral transforms. The Fourier series expansion of fractal interpolation functions is investigated in [41]. This has been extended for nonperiodic functions in [3]. The Fourier transform of α-fractal functions is considered in [42]. Authors in [42] proposed a relation between fractal dimension and Fourier transform of fractal interpolation functions with a constant vertical scaling factor. The restrictive condition on the vertical scaling factor has been removed, and the relationship has been extended in [43], considering the function vertical scaling factor. The Laplace transforms of linear and non-linear fractal interpolation functions are discussed in [44] and [45] respectively.

The evaluation of the Fourier and Laplace transforms of FIF is implemented in two stages. The initial stage consists of the derivation of an explicit representation of the two transforms via the fractal dimension of FIF. This has been implemented, considering the recursive relation satisfied by the FIF. The proposed relation between the fractal dimension and fractal numerical integration is utilized in the second stage to generate the desired formulae for the Fourier and Laplace transforms in terms of the fractal numerical integration.

The paper is organized as follows. Following the introduction, the second section of the paper provides the basic definition of fractal interpolation functions. The formula of fractal dimension proposed by [46] is mentioned in this section. The considered numerical integration formula [36] is also specified here. This section further revises the definitions of the Fourier transform and Laplace transform. The third section of the paper derives the proposed relationship between fractal dimension and fractal numerical integration. The derived relationship is proved to be precise with the examples of certain Weierstrass functions in the same section. The relationship between fractal dimension and Fourier transform of a linear FIF with a constant vertical scaling factor, as discussed in [42] is revised in section four. The fifth section of the paper provides the formula for the Fourier transform, using the fractal numerical integration of FIF. A similar derivation of the Laplace transform of FIF, via fractal dimension, is done in section six. The seventh section discusses the derivation of the Laplace transform, using the fractal numerical integration of FIF. Summarizing the work done, the paper concludes in section eight.

2. Preliminaries

This section explains the definition of fractal interpolation functions, the usual formulae of fractal dimension, Laplace transforms and Fourier transforms.

Definition 1 The complete metric space $(\hat{X},\ \hat{d})$ with a finite number of contraction mappings $w_n:\ \hat{X}\to\hat{X}$ is known as a hyperbolic iterated function system (IFS). The contractivity factor of the IFS is $\hat{s}=\max\left\{\hat{s}_n: n=1,\ 2,\ \ldots,\ N\right\},$ where $\hat{s}_n$ is the contractivity factor of the map $w_n,$ for each $n=1,\ 2,\ \ldots,\ N.$

Definition 2 Consider a hyperbolic IFS $\left\{\hat{X};\ w_n,\ n=1,\ 2,\ \ldots,\ N\right\}$ with contractivity factor $\hat{s}.$ Then, the operator $W: H(\hat{X})\to H(\hat{X})$ defined by $W(B)=\bigcup_{n=1}^{N} w_n(B)$ is again a contraction on $H(\hat{X})$ with the same contractivity factor. Then, the unique fixed point C of W obtained by $\lim_{n\to\infty} W^{\circ n}(B)$ that satisfies $C=W(C)=\bigcup_{n=1}^{N} w_n(C)$ is known as the attractor of the IFS.

Definition 3 Consider the discrete set $D^{\mathbb{I}}=\left\{(p_n,\ q_n):\ n=0,\ 1,\ \ldots,\ N\right\}$ corresponding to a single variable function. The points $p_n,\ n=0,\ 1,\ \ldots,\ N$ are known as the input arguments which are linearly ordered by $p_0<p_1<\ldots<p_N$ and the output arguments $q_n,\ n=0,\ 1,\ \ldots,\ N$ are the function values at these points. Let $\mathbb{I}=[p_0,\ p_N]$ be the closed interval comprising all the input arguments and $\mathbb{I}_n$ denotes the subinterval $[p_{n-1},\ p_n].$ Define contractive homeomorphism $L_n^{\mathbb{I}}:\mathbb{I}\to\mathbb{I}_n$ such that

$$L_n^{I}(p_0)=p_{n-1},\ L_n^{\mathbb{I}}(p_N)=p_n,\ \text{ for }\ n=1,\ 2,\ \ldots,\ N. \tag{1}$$

The function $L_n^{\mathbb{I}}$ contracts the interpolating domain $\mathbb{I}$ horizontally. Define another function $F_n^{\mathbb{I}}:\mathbb{I}\times\mathbb{R}\to\mathbb{R},$ contractive in the second variable, satisfying the conditions

$$F_n^{\mathbb{I}}(p_0,\ q_0)=q_{n-1},\ F_n^{\mathbb{I}}(p_N,\ q_N)=q_n,\ \text{ for }\ n=1,\ 2,\ \ldots,\ N. \tag{2}$$

Then, the IFS for the data set is

$$W_n^{\mathbb{I}}(p,\ q)=\Big(L_n^{\mathbb{I}}(p),\ F_n^{\mathbb{I}}(p,\ q)\Big),\ n=1,\ 2,\ \ldots,\ N. \tag{3}$$

Usually, the functions $L_n^{\mathbb{I}},\ F_n^{\mathbb{I}},$ satisfying the conditions (1) and (2) are defined as

$$L_n^{\mathbb{I}}(p)=l_{n1}^{\mathbb{I}}p+l_{n0}^{\mathbb{I}},\ F_n^{\mathbb{I}}(p,\ q)=\alpha_n^{\mathbb{I}}q+V_n^{\mathbb{I}}(p), \tag{4}$$

where $\alpha_n^{\mathbb{I}}$ is known as the freely chosen vertical scaling factor (VSF) that lies in $(-1,\ 1)$ with the corresponding scale vector $\overline{\alpha^{\mathbb{I}}}=(\alpha_1^{\mathbb{I}},\ \alpha_2^{\mathbb{I}},\ \ldots,\ \alpha_N^{\mathbb{I}})$ and $V_n^{\mathbb{I}}(p)=v_{n1}^{\mathbb{I}}p+v_{n0}^{\mathbb{I}}.$ The data set $D^{\mathbb{I}}$ along with the IFS (3) is known as the linear IFS, as long as $V_n^{\mathbb{I}}$ is considered as a linear function.

Lemma 1 Let $N>1$ be a positive integer. Consider the IFS (3) associated with the data set $D^{\mathbb{I}}$ where $0\le\alpha_n^{\mathbb{I}}<1,$ for $n=1,\ 2,\ \ldots,\ N.$ Then, the IFS is hyperbolic under a metric $\sigma^{\mathbb{I}},$ equivalent to the Euclidean metric on $\mathbb{R}^2.$ Moreover, there exists a nonempty set $G^{\mathbb{I}}$ in $\mathbb{R}^2$ which is compact and satisfies $G^{\mathbb{I}}=\bigcup_{n=1}^{N} W_n^{\mathbb{I}}(G^{\mathbb{I}}).$ The set $G^{\mathbb{I}}$ is unique.

Proof. Refer to theorem 2.1 in chapter 6 [46].

Definition 4 Let $\mathscr{F}^{\mathbb{I}}$ be a space of all functions $f^{\mathbb{I}}:\mathbb{I}\to\mathbb{R}$ that are continuous and satisfying the conditions $f^{\mathbb{I}}(p_0)=q_0,\ f^{\mathbb{I}}(p_N)=q_N.$ Then, $\mathscr{F}^{\mathbb{I}}$ is a complete metric space with the supremum metric. Define an operator $K^{\mathbb{I}}:\mathscr{F}^{\mathbb{I}}\to\mathscr{F}^{\mathbb{I}}$ by

$$(K^{\mathbb{I}} f^{\mathbb{I}})(p) = F_n^{\mathbb{I}}\left((L_n^{\mathbb{I}})^{-1}(p),\ f^{\mathbb{I}} \circ \left((L_n^{\mathbb{I}})^{-1}(p)\right)\right),\ p \in \mathbb{I}_n,\ \ n = 1,\ 2,\ \ldots,\ N \tag{5}$$

This operator $K^{\mathbb{I}}$ is known as the Read-Bajraktarevic (RB) operator.

Lemma 2 $K^{\mathbb{I}}$ is a contractive operator.

Proof. The contractivity of $K^{\mathbb{I}}$ is established in theorem 2.2 in chapter 6 [46].

Definition 5 The fixed point of the operator $K^{\mathbb{I}}$ is known as the fractal interpolation function (FIF) for the data set $D^{\mathbb{I}}$, denoted by $f^{\mathbb{I}}$, and $f^{\mathbb{I}}$ obeys the recursive relation

$$f^{\mathbb{I}}(p) = F_n^{\mathbb{I}}\left((L_n^{\mathbb{I}})^{-1}(p),\ f^{\mathbb{I}} \circ \left((L_n^{\mathbb{I}})^{-1}(p)\right)\right),\ p \in \mathbb{I}_n,\ \ n = 1,\ 2,\ \ldots,\ N \tag{6}$$

The FIF constructed with the linear IFS is defined as affine (linear) FIF.

Theorem 1 Consider the hyperbolic IFS as described in lemma 1 for $D^{\mathbb{I}}$. If $G^{\mathbb{I}}$ denotes the attractor of the IFS (3). Then $G^{\mathbb{I}}$ will be the graph of the FIF $f^{\mathbb{I}} : \mathbb{I} \to \mathbb{R}$.

Proof. Refer to theorem 2.2 in chapter 6 [46]. □

Definition 6 Example of Fractal Interpolation Function:

• Consider the data set

$$\{(0,\ 0),\ (0.075,\ -1.1465),\ (0.15,\ -0.406),\ (0.225,\ 0.5717),\ (0.3,\ 0.3864),$$

$$(0.375,\ -0.1238),\ (0.45,\ -0.0485),\ (0.525,\ -0.0044),\ (0.6, -0.3924),$$

$$(0.675,\ -0.2528),\ (0.75,\ 0.6832),\ (0.825,\ 0.8133),\ (0.9,\ -0.6141),$$

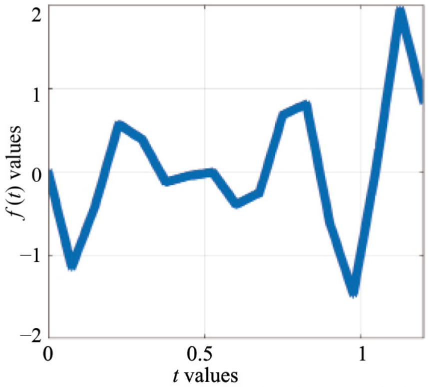
$$(0.975,\ -1.4692),\ (1.05,\ 0.0882),\ (1.125,\ 1.9464),\ (1.2,\ 0.8388)\}$$

associated with the function

$$-(1.4 - 3t)sin(18t)\ \text{ where }\ t \in [0,\ 1.2].$$

Let this data set be assigned with the IFS

$$\left\{(0.0625t,\ -1.155t),\ \ldots.,\ (0.0625t + 1.125,\ -0.549t + 1.946)\right\} \tag{7}$$



Attractor of the IFS (7) with the zero VSF, i.e, $\alpha_n^{\mathbb{I}} = 0$, for n = 1, 2, ..., N.

Figure 1. Attractor of the IFS (7) with VSF=0

Figure 1 shows the attractor obtained for this particular IFS (graph of the FIF for the considered data set) with the constant vertical scaling factor.

Definition 7 [46] A fractal dimension is an index that measures the complexity of fractal patterns or sets by dividing the change in scale by the change in detail. It is not necessarily an integer. The most popular method of determining the fractal dimension of curves is through the box-counting algorithm. The object for which the dimension is to be estimated is covered with boxes of a certain size. Then, the ratio of the number of boxes to the box size, as the size tends to 0, is regarded as the box-counting dimension. Other methods include Hausdorff dimension, Differentialbox-counting method, Reticular cell counting method, and so on. A different method of dimension estimation with FIF is explained below.

Consider the data set $D^{\mathbb{I}} = \{(p_n, q_n) : n = 0, 1, 2, ..., N\}$ and the associated IFS

$$W_n^{\mathbb{I}}(p, q) = \left(L_n^{\mathbb{I}}(p), F_n^{\mathbb{I}}(p, q)\right), n = 1, 2, ..., N. \tag{8}$$

where the functions $L_n^{\mathbb{I}}$ and $F_n^{\mathbb{I}}$ are defined as in (4), satisfying the respective endpoint conditions (1) and (2). Choose the vertical scaling factor $\alpha_n^{\mathbb{I}}$ such that $0 \leq \alpha_n^{\mathbb{I}} < 1$. If $\alpha_n^{\mathbb{I}}$ satisfies the inequality

$$\sum_{n=1}^{N} |\alpha_n^{\mathbb{I}}| > 1 \tag{9}$$

and the interpolation points do not lie on the same line, then the fractal dimension of the graph of FIF generated by this IFS is the unique solution of the equation

$$\sum_{n=1}^{N} |\alpha_n^{\mathbb{I}}|(l_{n1}^{\mathbb{I}})^{D-1} = 1 \tag{10}$$

otherwise, the fractal dimension is 1.

Definition 8 Suppose the interpolation points are equally spaced, which implies $l_{n1}^{\mathbb{I}} = \frac{1}{N}$, for $n = 1, 2, ..., N$. If the condition (9) holds, then the fractal dimension D is obtained as

$$D = 1 + \frac{log\left(\sum_{n=1}^{N} |\alpha_n^{\mathbb{I}}|\right)}{log(N)}. \tag{11}$$

The following theorem provides one of the methods to select the value of the vertical scaling factor satisfying the condition (9) [47].

Definition 9 Consider the data set $D^{\mathbb{I}} = \{(p_n, q_n) : n = 0, 1, 2, ..., N\}$ where each p_n is equally spaced and the associated IFS (3) with the linear map $V_n^{\mathbb{I}}$. Let the vertical scaling factor be such that $\alpha_n^{\mathbb{I}} = \alpha$, for $n = 1, 2, ..., N$, where $\frac{1}{N} < \alpha < 1$. Moreover, if the interpolation points are not collinear, then the fractal dimension of the graph of FIF can be calculated as

$$D = 2 + log_N \alpha. \tag{12}$$

Definition 10 The numerical integration formula for the single variable FIF as proposed in [36] is

$$M^{\mathbb{I}} = \frac{N_0}{1 - \sum_{n=1}^{n} \alpha_n^{\mathbb{I}} l_{n1}^{\mathbb{I}}}, \text{ where } N_0 = \sum_{n=1}^{N} \int_{\mathbb{I}_n} V_n^{\mathbb{I}} \circ (L_n^{\mathbb{I}})^{-1}(p) dp. \tag{13}$$

Definition 11 Fourier transform of a function $\mathfrak{g} : \mathbb{R} \to \mathbb{C}$ is defined according to [42] as

$$\hat{\mathfrak{g}}(w) = \int_{-\infty}^{\infty} e^{2\pi i w t} \mathfrak{g}(t) dt. \tag{14}$$

Definition 12 Laplace transform of a function $\mathfrak{g}$ defined for every real number is given as [48]

$$G(s) = \int_{0}^{\infty} \mathfrak{g}(t) e^{-st} dt. \tag{15}$$

As mentioned earlier, this section introduced the basic definitions and associated results for the better comprehension of the article.

3. Relation between fractal dimension and fractal numerical integration

This section derives the relationship between fractal dimension and fractal numerical integration.

Theorem 2 Consider the data set $D^{\mathbb{I}}$ where each p_n is equally spaced, so that the horizontal scaling factor is constant, $l_{n1}^{\mathbb{I}} = \frac{1}{N}$, for $n = 1, 2, ..., N$. Consider the IFS (3) associated with this data set where the vertical scaling factor $\alpha_n^{\mathbb{I}}$ lies in $[0, 1)$. If (9) holds and the interpolation points are not collinear, the fractal numerical integration $M^{\mathbb{I}}$ can be derived in terms of the dimension D as

$$M^{\mathbb{I}} = \frac{N_0}{1 - \sum\limits_{n=1}^{N} \alpha_n^{\mathbb{I}} N^{-D} \sum\limits_{n=1}^{N} |\alpha_n^{\mathbb{I}}|}, \text{ where } N_0 = \sum_{n=1}^{N} \int_{\mathbb{I}_n} V_n^{\mathbb{I}} \circ (L_n^{\mathbb{I}})^{-1}(p)dp. \tag{16}$$

Conversely, the fractal dimension can be calculated in terms of fractal numerical integration as

$$D = \frac{log\left(M^{\mathbb{I}} \sum\limits_{n=1}^{N} \alpha_n^{\mathbb{I}} \sum\limits_{n=1}^{N} |\alpha_n^{\mathbb{I}}|\right) - log\left(M^{\mathbb{I}} - N_0\right)}{log\left(N\right)}. \tag{17}$$

Proof. For the data set $D^{\mathbb{I}}$ with the equally spaced p_n values, the fractal numerical integral formula becomes

$$M^{\mathbb{I}} = \frac{N_0}{1 - \sum\limits_{n=1}^{N} \frac{\alpha_n^{\mathbb{I}}}{N}}$$

$$= \frac{N_0 N}{N - \sum\limits_{n=1}^{N} \alpha_n^{\mathbb{I}}}. \tag{18}$$

When $l_{n1}^{\mathbb{I}} = \frac{1}{N}$, $\forall n = 1, 2, ..., N$, (10) becomes

$$\sum_{n=1}^{N} \frac{|\alpha_n^{\mathbb{I}}|}{N^{D-1}} = 1$$

which implies

$$N^{1-D} = \frac{1}{\sum\limits_{n=1}^{N} |\alpha_n^{\mathbb{I}}|}.$$

Since N can be written as $N = N^{1-D}N^D$, (18) changes to

$$\frac{N_0 N}{N - \sum\limits_{n=1}^{N} \alpha_n^{\mathbb{I}}} = \frac{N_0 N^{1-D} N^D}{N^{1-D}N^D - \sum\limits_{n=1}^{N} \alpha_n^{\mathbb{I}}}.$$

Dividing by N^D, we get

$$M^{\mathbb{I}} = \frac{N_0 N^{1-D}}{N^{1-D} - \sum_{n=1}^{N} \alpha_n^{\mathbb{I}} N^{-D}} \tag{19}$$

Substituting $N^{1-D} = \frac{1}{\sum_{n=1}^{N} |\alpha_n^{\mathbb{I}}|}$, (19) becomes

$$M^{\mathbb{I}} = \frac{\frac{N_0}{\sum_{n=1}^{N} |\alpha_n^{\mathbb{I}}|}}{\frac{1}{\sum_{n=1}^{N} |\alpha_n^{\mathbb{I}}|} - \sum_{n=1}^{N} \alpha_n^{\mathbb{I}} N^{-D}}$$

$$= \frac{\frac{N_0}{\sum_{n=1}^{N} |\alpha_n^{\mathbb{I}}|}}{\frac{1 - \sum_{n=1}^{N} \alpha_n^{\mathbb{I}} N^{-D} \sum_{n=1}^{N} |\alpha_n^{\mathbb{I}}|}{\sum_{n=1}^{N} |\alpha_n^{\mathbb{I}}|}}$$

which implies

$$M^{\mathbb{I}} = \frac{N_0}{1 - \sum_{n=1}^{N} \alpha_n^{\mathbb{I}} N^{-D} \sum_{n=1}^{N} |\alpha_n^{\mathbb{I}}|}. \tag{20}$$

Similarly, to arrive at the fractal dimension formula in terms of the fractal numerical integration formula, consider (20). Then,

$$\frac{N_0}{M^{\mathbb{I}}} = 1 - \sum_{n=1}^{N} \alpha_n^{\mathbb{I}} N^{-D} \sum_{n=1}^{N} |\alpha_n^{\mathbb{I}}|$$

From this,

$$\sum_{n=1}^{N} \alpha_n^{\mathbb{I}} N^{-D} \sum_{n=1}^{N} |\alpha_n^{\mathbb{I}}| = 1 - \frac{N_0}{M^{\mathbb{I}}} = \frac{(M^{\mathbb{I}} - N_0)}{M^{\mathbb{I}}}.$$

Now,

$$N^{-D} = \frac{(M^{\mathbb{I}} - N_0)}{M^{\mathbb{I}}} \frac{1}{\sum\limits_{n=1}^{N} \alpha_n^{\mathbb{I}} \sum\limits_{n=1}^{N} |\alpha_n^{\mathbb{I}}|}$$

Taking the logarithm

$$-Dlog(N) = log\left(\frac{M^{\mathbb{I}} - N_0}{M^{\mathbb{I}} \sum\limits_{n=1}^{N} \alpha_n^{\mathbb{I}} \sum\limits_{n=1}^{N} |\alpha_n^{\mathbb{I}}|}\right)$$

$$= log(M^{\mathbb{I}} - N_0) - log\left(M^{\mathbb{I}} \sum_{n=1}^{N} \alpha_n^{\mathbb{I}} \sum_{n=1}^{N} |\alpha_n^{\mathbb{I}}|\right)$$

Dividing by $-log(N)$,

$$D = \frac{log\left(M^{\mathbb{I}} \sum\limits_{n=1}^{N} \alpha_n^{\mathbb{I}} \sum\limits_{n=1}^{N} |\alpha_n^{\mathbb{I}}|\right) - log(M^{\mathbb{I}} - N_0)}{log(N)}$$

Hence the proof. □

3.1 *Examples and illustrations*

Example 1 Consider the function

$$f(p) = 1 + (3p^3 + 2p^2 - 0.7) \sum_{k=1}^{\infty} \frac{cos(10^k \pi p)}{2^k}, \text{ where } p \in [-1, 1].$$

The numerical integral value and the fractal dimension calculated using the formulae (13), (16) and (11), (17) respectively are given in Table 1 with different number of subintervals of $[-1, 1]$.

Table 1. Fractal numerical integral and fractal dimension results for Example 1

N	$M^{\mathbb{I}}$ (13)	D (11)	$M^{\mathbb{I}}$ (16)	D (17)
4	2.1694	1.4098	2.1694	1.4098
8	2.1834	1.4626	2.1834	1.4626
16	2.0135	1.4829	2.0135	1.4829
32	2.0117	1.4950	2.0117	1.4950
64	2.0040	1.3541	2.0040	1.3541

Table 2. Fractal numerical integral and fractal dimension results for Example 2

N	$M^{\mathbb{I}}$ (13)	D (11)	$M^{\mathbb{I}}$ (16)	D (17)
16	3.4	1.6577	3.4	1.6577
32	3.4	1.8303	3.4	1.8303
64	3.4	1.7501	3.4	1.7501
128	3.4	1.5811	3.4	1.5811
256	3.4	1.7617	3.4	1.7617

Example 2 Consider the function

$$f(p) = 3p^2 + 2p + 0.7 - 5\sum_{k=1}^{\infty} \frac{sin(6^k \pi p)}{2^k}, \text{ where } p \in [-1,\ 1].$$

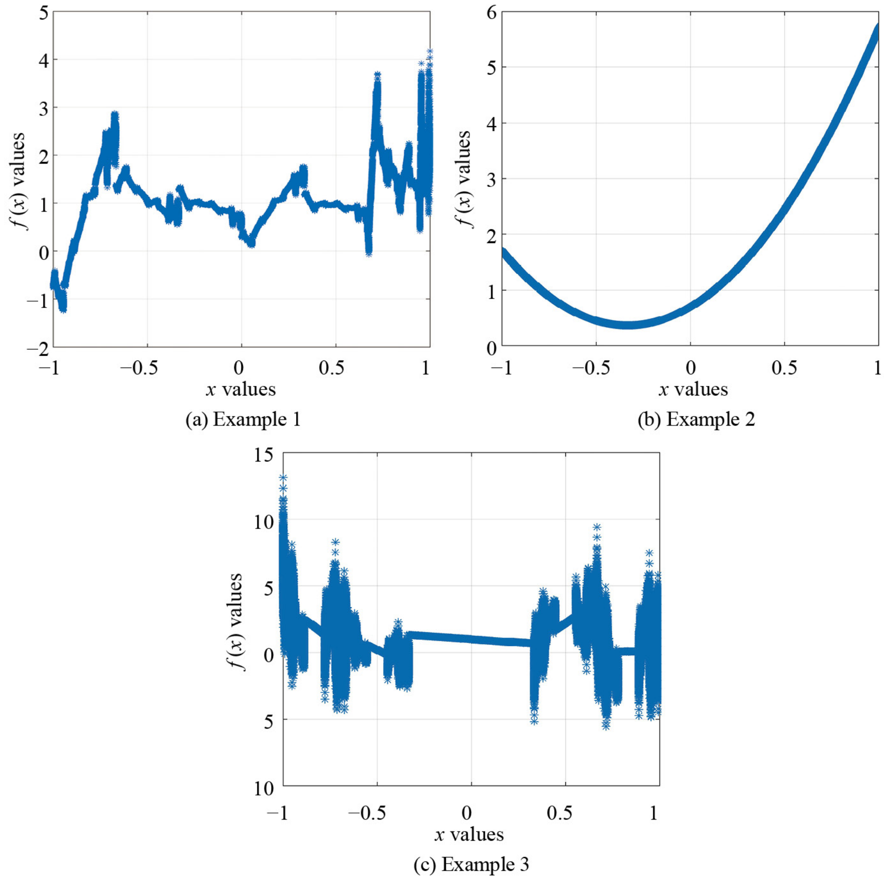


Figure 2. Attractors of Examples 1, 2 and 3 produced using linear IFS

Table 2 provides the numerical integral and the fractal dimension value calculated using the newly derived formulae (16) and (17) and they are compared with the results obtained by their usual formulae.

Example 3 Consider the function

$$f(p) = 1 + (3p^4 + 2p) \sum_{k=1}^{\infty} \frac{cos(8^k \pi p)}{2^k}, \text{ where } p \in [-1, 1].$$

Table 3. Fractal numerical integral and fractal dimension results for Example 3

N	$M^{\mathbb{I}}$ (13)	D (11)	$M^{\mathbb{I}}$ (16)	D (17)
4	3.0933	0.4771	3.0933	0.4771
8	2.2327	1.4530	2.2327	1.4530
16	2.5992	0.8437	2.5992	0.8437
32	2.6157	1.0423	2.6157	1.0423
64	2.0203	1.5832	2.0203	1.5832

Similar to the above examples, Table 3 shows the numerical integral results and the fractal dimension values for the function considered in Example 3.

The graphs of the considered examples are plotted in Figure 2. These graphs are obtained as the attractors of the linear IFS defined for the data sets generated from Examples 1, 2 and 3. All these computations are done using MATLAB R2018a.

All the examples considered here correspond to the test functions and are used to validate the newly formulated equations for dimension and fractal numerical integration.

Analyzing the tables, it is observed that the dimension values obtained using the newly proposed formula coincide with the dimension results with the conventional formula. Similarly, the numerical integral values computed using fractal dimension match with the integral results calculated with fractal interpolation functions.

4. Fourier transform via fractal dimension

This section revises the alternate formula for the Fourier transform of FIF as given in [42], whenever its fractal dimension is known.

Theorem 3 Consider the data set $D^{\mathbb{I}}$ where $p_0 = 0$, $p_N = 1$ and each p_n is equally spaced, so that the horizontal scaling factor is constant, $l_{n1}^{\mathbb{I}} = \frac{1}{N}$, for $n = 1, 2, ..., N$. Consider the IFS (3) associated with this data set with a constant vertical scaling factor α chosen in $[0, 1)$. If (9) holds and the interpolation points are not collinear, the Fourier transform of the FIF can be derived in terms of its dimension as

$$(\hat{f}^{\mathbb{I}})(w) = \hat{Q}(w) + z(w) \sum_{u=1}^{\infty} \frac{1}{N^{(3-D)u}} \frac{\hat{Q}(w/N^u)}{z(w/N^u)}, \tag{21}$$

where $\hat{Q}(w) = \sum_{n=1}^{N} \int_{\mathbb{I}_n} e^{2\pi wip} \left(V_n^{\mathbb{I}} \circ \left((L_n^{\mathbb{I}})^{-1}(p) \right) \right) dp$, and $z(w) = e^{2\pi iw} - 1$.

Proof. Refer to the derivation in [42]. □

5. Fourier transform via fractal numerical integration

There has been considerable research reported in the areas related to the Fourier transform of FIF [49–51]. In [42], the Fourier transform of a linear FIF is derived with a constant vertical scaling factor. A different expression for the Fourier transform of linear FIF is also derived in the same paper using its fractal dimension values. The formulations derived in [42] have been generalized with function scaling factors in [43]. Considering constant vertical scaling factors, this section proposes a new strategy to evaluate the Fourier transform of a linear FIF constructed with the IFS (3), in terms of its fractal numerical integral values. Initially, the expression for the Fourier transform of a linear FIF with fractal dimension is considered. Then, the proposed relation between the fractal dimension and the fractal numerical integration is employed to obtain the desired expression.

Theorem 4 Consider the data set $D^{\mathbb{I}}$ where $p_0 = 0$, $p_N = 1$ and each p_n is equally spaced, so that the horizontal scaling factor is constant, $l_{n1}^{\mathbb{I}} = \frac{1}{N}$, for $n = 1, 2, ..., N$. Consider the IFS (3) associated with this data set with a constant vertical scaling factor α chosen in $[0, 1)$. If (9) holds and the interpolation points are not collinear, the Fourier transform of the FIF can be derived in terms of its fractal numerical integration as

$$(\hat{f^{\mathbb{I}}})(w) = \hat{Q}(w) + z(w)\sum_{u=1}^{\infty}\frac{1}{N^{Ru}}\frac{\hat{Q}(w/N^u)}{z(w/N^u)}, \tag{22}$$

where

$$\hat{Q}(w) = \sum_{n=1}^{N}\int_{\mathbb{I}_n} e^{2\pi wip}\left(V_n^{\mathbb{I}} \circ \left((L_n^{\mathbb{I}})^{-1}(p)\right)\right)dp,\ z(w) = e^{2\pi iw} - 1$$

and

$$R = 1 + \frac{log(M^{\mathbb{I}} - N_0) - log(M^{\mathbb{I}}) - 2log(\alpha)}{log(N)}.$$

Proof. Consider the following expression for the fractal dimension of a FIF, in terms of its fractal numerical integration, as given in (17).

$$D = \frac{log\left(M^{\mathbb{I}}\sum_{n=1}^{N}\alpha_n^{\mathbb{I}}\sum_{n=1}^{N}|\alpha_n^{\mathbb{I}}|\right) - log\left(M^{\mathbb{I}} - N_0\right)}{log\left(N\right)}$$

Considering the non-negative, constant vertical scaling factor α, $\sum_{n=1}^{N}|\alpha_n^{\mathbb{I}}| = \sum_{n=1}^{N}|\alpha| = \sum_{n=1}^{N}\alpha = N\alpha$. Then, the above expression becomes,

$$D = \frac{log(M^{\mathbb{I}}\alpha^2N^2) - log(M^{\mathbb{I}} - N_0)}{log(N)}$$

$$= \frac{log(M^{\mathbb{I}}) + log((\alpha N)^2) - log(M^{\mathbb{I}} - N_0)}{log(N)}$$

$$= \frac{log(M^{\mathbb{I}}) + 2log(\alpha) + 2log(N) - log(M^{\mathbb{I}} - N_0)}{log(N)}$$

Dividing by $log(N)$,

$$D = 2 + \frac{log(M^{\mathbb{I}}) + 2log(\alpha) - log(M^{\mathbb{I}} - N_0)}{log(N)}. \tag{23}$$

Subtracting this expression of D from 3 and mulitplying by the variable u,

$$(3-D)u = \left(1 + \frac{log(M^{\mathbb{I}} - N_0) - log(M^{\mathbb{I}}) - 2log(\alpha)}{log(N)}\right)u$$

$$= Ru, \text{ where } R = 1 + \frac{log(M^{\mathbb{I}} - N_0) - log(M^{\mathbb{I}}) - 2log(\alpha)}{log(N)}.$$

Substituting this expression for $(3-D)u$ in (21), the formula for the Fourier transform becomes

$$(\hat{f^{\mathbb{I}}})(w) = \hat{Q}(w) + z(w)\sum_{u=1}^{\infty}\frac{1}{N^{Ru}}\frac{\hat{Q}(w/N^u)}{z(w/N^u)},$$

where

$$\hat{Q}(w) = \sum_{n=1}^{N}\int_{\mathbb{I}_n} e^{2\pi wip}\left(V_n^{\mathbb{I}} \circ \left((L_n^{\mathbb{I}})^{-1}(p)\right)\right) dp, \; z(w) = e^{2\pi iw} - 1$$

and

$$R = 1 + \frac{log(M^{\mathbb{I}} - N_0) - log(M^{\mathbb{I}}) - 2log(\alpha)}{log(N)}.$$

6. Laplace transform via fractal dimension

This section proposes the formula for the Laplace transform using the fractal dimension of FIF.

Theorem 5 Consider the data set $D^{\mathbb{I}}$ where $p_0 = 0,\ p_N = 1$ and each p_n is equally spaced, so that the horizontal scaling factor is constant, $l^{\mathbb{I}}_{n1} = \frac{1}{N}$, for $n = 1,\ 2,\ \ldots,\ N$. Consider the IFS (3) associated with this data set with a constant vertical scaling factor α chosen in $[0,\ 1)$. If (9) holds and the interpolation points are not collinear, the Laplace transform of the FIF can be derived in terms of its dimension as

$$F(s) = V(s) + z(s)\sum_{u=1}^{\infty}\frac{1}{N^{(3-D)u}}\frac{V(s/N^u)}{z(s/N^u)}, \tag{24}$$

where $V(s) = \sum_{n=1}^{N}\int_{\mathbb{I}_n} e^{-sp}\left(V^{\mathbb{I}}_n \circ \left((L^{\mathbb{I}}_n)^{-1}(p)\right)\right)dp$, and $z(s) = e^{-s} - 1$.

Proof. By definition, the Laplace transform of FIF is given by

$$F(s) = \int_0^{\infty} f^{\mathbb{I}}(p)e^{-sp}dp.$$

Since $f^{\mathbb{I}}$ satisfies the recursive relation,

$$F(s) = \sum_{n=1}^{N}\int_{\mathbb{I}_n}\left(\alpha^{\mathbb{I}}_n f^{\mathbb{I}} \circ \left((L^{\mathbb{I}}_n)^{-1}(p')\right) + V^{\mathbb{I}}_n \circ \left((L^{\mathbb{I}}_n)^{-1}(p')\right)\right)e^{-sp'}\ dp'.$$

Let $V(s) = \sum_{n=1}^{N}\int_{\mathbb{I}_n} V^{\mathbb{I}}_n \circ \left((L^{\mathbb{I}}_n)^{-1}(p')\right)e^{-sp'}\ dp'$. Put $p = (L^{\mathbb{I}}_n)^{-1}(p')$.

Then, $p' = L^{\mathbb{I}}_n(p) = l^{\mathbb{I}}_{n1}p + l^{\mathbb{I}}_{n0} = \frac{p}{N} + l^{\mathbb{I}}_{n0}$.

Then, $V(s)$ becomes

$$V(s) = \sum_{n=1}^{N}\int_{\mathbb{I}}\frac{1}{N}V^{\mathbb{I}}_n(p)e^{-s(\frac{p}{N}+l^{\mathbb{I}}_{n0})}\ dp$$

$$= \frac{1}{N}\sum_{n=1}^{N}e^{-sl^{\mathbb{I}}_{n0}}\int_{\mathbb{I}}V^{\mathbb{I}}_n(p)e^{-s(\frac{p}{N})}\ dp$$

$$= \frac{1}{N}\sum_{n=1}^{N}e^{-sl^{\mathbb{I}}_{n0}}V^{\mathbb{I}}_n\left(\frac{s}{N}\right).$$

Similarly,

$$\sum_{n=1}^{N}\int_{\mathbb{I}_n}\alpha^{\mathbb{I}}_n f^{\mathbb{I}} \circ \left((L^{\mathbb{I}}_n)^{-1}(p)\right)dp$$

turns out to be

$$\frac{1}{N}\sum_{n=1}^{N}\alpha_n^{\mathbb{I}}e^{-sl_{n0}^{\mathbb{I}}}\int_{\mathbb{I}}f^{\mathbb{I}}(p)e^{-s(\frac{p}{N})}dp=\frac{1}{N}\sum_{n=1}^{N}\alpha_n^{\mathbb{I}}e^{-sl_{n0}^{\mathbb{I}}}F\left(\frac{s}{N}\right).$$

Therefore,

$$F(s)=V(s)+\frac{1}{N}\sum_{n=1}^{N}\alpha_n^{\mathbb{I}}e^{-sl_{n0}^{\mathbb{I}}}F\left(\frac{s}{N}\right)$$

$$=V(s)+\frac{1}{N}w(s)F\left(\frac{s}{N}\right),\ \text{where}\ \ w(s)=\sum_{n=1}^{N}\alpha_n^{\mathbb{I}}e^{-sl_{n0}^{\mathbb{I}}}.$$

Substituting the similar equation in place of $F\left(\frac{s}{N}\right)$,

$$F(s)=V(s)+\frac{1}{N}w(s)V\left(\frac{s}{N}\right)+\frac{1}{N^2}w(s)w\left(\frac{s}{N}\right)F\left(\frac{s}{N^2}\right).$$

Proceeding similarly, the following expression for $F(s)$ is obtained

$$F(s)=V(s)+\sum_{u=1}^{\infty}V\left(\frac{s}{N^u}\right)\frac{1}{N^u}\prod_{j=0}^{u-1}w\left(\frac{s}{N^j}\right). \tag{25}$$

Since the data set considered here spans over $[0,\ 1]$, and $l_{n1}^{\mathbb{I}}=\frac{1}{N}$, $\forall n=1,\ 2,\ ...,\ N$, the coefficient $l_{n0}^{\mathbb{I}}=\frac{n-1}{N}$, $\forall n=1,\ 2,\ ...,\ N$. Considering the constant vertical scaling factor α, $w(s)$ can be written as

$$w(s)=\alpha\sum_{n=1}^{N}e^{-\frac{s(n-1)}{N}}.$$

$\sum_{n=1}^{N}e^{-\frac{s(n-1)}{N}}$ is a geometric series with first term 1 and common ratio $e^{-\frac{s}{N}}$.

Therefore, its sum will be $\frac{e^{-s}-1}{e^{-\frac{s}{N}}-1}$. Then,

$$w(s)=\alpha\sum_{n=1}^{N}e^{-\frac{s(n-1)}{N}}$$

$$=\alpha\frac{z(s)}{z(\frac{s}{N})},\ \text{where}\ \ z(s)=e^{-s}-1.$$

Therefore, (25) changes to

$$F(s) = V(s) + \sum_{u=1}^{\infty} V\left(\frac{s}{N^u}\right)\frac{(\alpha)^u}{N^u}\prod_{j=0}^{u-1}\frac{z\left(\frac{s}{N^j}\right)}{z\left(\frac{s}{N^{j+1}}\right)}.$$

It is to be noted that

$$\prod_{j=0}^{u-1}\frac{z\left(\frac{s}{N^j}\right)}{z\left(\frac{s}{N^{j+1}}\right)} = \frac{z(s)}{z\left(\frac{s}{N^u}\right)}.$$

Thus,

$$F(s) = V(s) + z(s)\sum_{u=1}^{\infty}\frac{(\alpha)^u}{N^u}\frac{V\left(\frac{s}{N^u}\right)}{z\left(\frac{s}{N^u}\right)}.$$

The formula (11) for the fractal dimension of FIF with constant vertical scaling factor α will be $D = 2 + \frac{log(\alpha)}{log(N)}$, which implies $\alpha = N^{D-2}$. Hence, $\frac{(\alpha)^u}{N^u} = N^{(D-3)u}$. Therefore, the formula for the Laplace transform in terms of the fractal dimension is given by

$$F(s) = V(s) + z(s)\sum_{u=1}^{\infty}\frac{1}{N^{(D-3)u}}\frac{V\left(\frac{s}{N^u}\right)}{z\left(\frac{s}{N^u}\right)}.$$

□

7. Laplace transform via fractal numerical integration

This section provides the Laplace transform of FIF using the fractal numerical integration formula.

Theorem 6 Consider the data set $D^{\mathbb{I}}$ where $p_0 = 0$, $p_N = 1$ and each p_n is equally spaced, so that the horizontal scaling factor is constant, $l_{n1}^{\mathbb{I}} = \frac{1}{N}$, for $n = 1, 2, \ldots, N$. Consider the IFS (3) associated with this data set with a constant vertical scaling factor α chosen in $[0, 1)$. If (9) holds and the interpolation points are not collinear, the Laplace transform of the FIF can be derived in terms of its fractal numerical integration as

$$F(s) = V(s) + z(s)\sum_{u=1}^{\infty}\frac{1}{N^{Ru}}\frac{V\left(\frac{s}{N^u}\right)}{z\left(\frac{s}{N^u}\right)}, \text{ where } R = 1 + \frac{log(M^{\mathbb{I}} - N_0) - log(M^{\mathbb{I}}) - 2log(\alpha)}{log(N)}. \quad (26)$$

Proof. Consider the formula (23). Substitute (23) in (24),

$$F(s) = V(s) + z(s) \sum_{u=1}^{\infty} \frac{1}{N^{Ru}} \frac{V\left(\frac{s}{N^u}\right)}{z\left(\frac{s}{N^u}\right)}, \text{ where } R = 1 + \frac{log(M^{\mathbb{I}} - N_0) - log(M^{\mathbb{I}}) - 2log(\alpha)}{log(N)}.$$

□

The formulae (21), (22), (24) and (26) can be considered as the numerical methods of finding the respective Fourier and Laplace transforms of the functions.

8. Conclusion

A direct relation between fractal dimension and fractal numerical integration of FIF is explored in this paper. In place of an explicit formulation, a FIF is always represented by a recursive formula. This recursive formula has paramount importance, especially in the derivation of fractal numerical integration formulas. Considering certain restrictive conditions on the vertical scaling factor, the formulation of fractal numerical integration in terms of fractal dimension and vice versa is provided in this paper. The notion of fractal dimension is very useful in classifying the physiological and pathological segments of biomedical images. Estimating dimensions through the conventional box-counting method often causes the problem of proper positioning of the boxes. The selection of improper box size may sometimes lead to incorrect measures. The formulation proposed between fractal dimension and fractal numerical integration is quite useful for those circumstances wherein the direct evaluation of the two parameters is difficult to produce. The proposed formulation is then verified through a couple of examples. Observing the examples, it is possible to conclude the convergence of the generated formulae. The recursive formula of the FIF is again utilized in generating new expressions for the Fourier and Laplace transforms of FIF in terms of its fractal dimension. Finally, the derived relationship between fractal dimension and fractal numerical integration is applied in providing quite novel formulae for the two integral transforms. Using these new formulae, it is now easier to calculate the Fourier and Laplace transforms of FIF whenever its fractal numerical integral value is known. One may consider the future directions of this work as the generalization of the formulation derived here to higher dimensional interpolating regions and the derivation of other transforms using fractal dimension or fractal numerical integration.

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Conflict of interest

The authors declare no competing financial interest.

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