Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Indecomposable Modules, Relative Projectivity and Radical Subgroups in Finite Groups

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Received: 11 July 2024; **Revised:** 16 August 2024; **Accepted:** 21 August 2024

Abstract: In the various blocks of a finite group G, irreducible characters sit with the indecomposable modules which afford them and such indecomposable modules in those blocks have got vertices and sources and in fact, every p-subgroup of G is a vertex of some indecomposable $\mathbb{F}G$-module. For any finite group G and a field $\mathbb{F}$ of characteristic p, where p is a prime that divides the order $|G|$ of G, every indecomposable $\mathbb{F}G$-module possesses a vertex and a source. Furthermore for a finite group G, kernels of the irreducible $\mathbb{F}G$-modules, vertices of the irreducible $\mathbb{F}G$-modules and defect groups of blocks of G all contain $O_p(G)$. Furthermore, the kernels of blocks of G are normal p'-subgroups of G which are contained in $O_{p'}(G)$, where $O_{p'}(G)$ is the kernel of the principal block of G. The kernels of blocks of G are related to the kernels of the indecomposable $\mathbb{F}G$-modules in those blocks. The object in this paper is to study characteristics and/or properties of vertices of indecomposable $\mathbb{F}G$-modules in relation to irreducible ordinary characters that they afford and sit with in blocks of G and furthermore study characteristics and/or properties that exist between kernels of irreducible $\mathbb{F}G$-modules, vertices of irreducible $\mathbb{F}G$-modules and defect groups of blocks of G and even establish as to when and how any and/or all of these would (if at all possible) coincide.

***Keywords*:** relative projectivity, indecomposable modules, vertices and sources, blocks of characters, irreducible ordinary and modular characters, kernels of modules and blocks, defect groups of blocks, radical subgroups

MSC: 20C15, 20C20, 20D15, 20D20

1. Introduction

Let G be a finite group, H a subgroup of G and M an indecomposable $\mathbb{F}G$-module such that M is H-projective. If $B(G)$ is a block of G, then there exists an irreducible $\mathbb{F}G$-module $N \in B(G)$ such that a defect group of $B(G)$ is a vertex of N. According to [1], if G is p-solvable and $M \in B(G)$ is an irreducible module, then a vertex of M contains up to conjugation the center of a defect group of $B(G)$.

According to [2, Lemma 1], for a block $B(G)$, if either a defect group of $B(G)$ is abelian or every irreducible character $\chi \in Irr(B(G))$ has height $h(\chi) = 0$, then $B(G)$ is a large vertex block (l. v. block). Knorr in [3] asserts that if M is an $\mathbb{F}G$-module with vertex $\mathfrak{V}$ affording $\chi \in Irr(G)$ and $B(G)$ is a block of G for which $\chi \in Irr(B(G))$, then there exists $D \in \delta(B(G))$ such that $\mathfrak{V} \leq D$.

DOI: https://doi.org/10.37256/cm.6120255290

By [4, Lemma 65.1], for H a normal subgroup of G, an $\mathbb{F}G$-module $V \in B_0(G)$ will satisfy that $V_H \in B_0(H)$, where $B_0(G)$ and $B_0(H)$ are the principal blocks of G and H respectively. Thus for H a normal subgroup of G, we have that $B_0(G)$ covers $B_0(H)$. By [5, Theorem 65.11] for an indecomposable $\mathbb{F}G$-module M, there exists a subgroup $K \leq G$ of G such that

1. M is K-projective,
2. if M is H-projective, then the subgroup H of G contains a conjugate of K in G.

For G a finite group with a block $B(G)$ whose defect group is D, any indecomposable module $M \in B(G)$ is D-projective. Moreover by [6] a defect group of a block of G is always the intersection of two Sylow p-subgroups of G and by [7] defect groups of blocks are to be found among the radical p-subgroups of G.

According to [8] citing [5, Theorem 64.1], the group algebra $\mathbb{F}G$ of a finite group G over a field $\mathbb{F}$ of characteristic $p \neq 0$ has a finite number of isomorphism classes of indecomposable modules if and only if a Sylow p-subgroup of G is cyclic. Equally [8] contends that a block of $\mathbb{F}G$ has a finite number of isomorphism classes of indecomposable modules if and only if its defect group is cyclic.

By [9, Theorem III.2.13(i)], $0_p(G)$ is contained in the kernel of every irreducible $\mathbb{F}G$-module and by [9, Theorem III.4.12, Corollary III.4.13], $0_p(G)$ is contained in a vertex of every irreducible $\mathbb{F}G$-module. Any p-group which is contained in the kernel of an indecomposable module, by [9, Theorem III.4.12] that p-group is also contained in a vertex of that same module. By [9, Theorem III.2.13(ii)], $0_p(G)$ is the intersection of the kernels of all the irreducible $\mathbb{F}G$-modules.

If a finite group G contains a unique character of defect 1, then G will not possess a block of defect 1 as such a character will belong to a block with other characters of higher defect. In fact when G possesses a block of defect 1, then all the characters in that block have defect 1 themselves and by [4, Corollary 62.4], [9, Theorem IV.4.18] citing [10] all such characters have height 0.

Moreover every indecomposable module in a block of defect 1, will have as its vertex a cyclic group of prime order and a defect group of such a block will be a cyclic group of prime order as well. Thus by [2], such a block of defect 1 will be a large vertex block.

The object in this paper is to study characteristics and/or properties of vertices of indecomposable $\mathbb{F}G$-modules in relation to irreducible ordinary characters that they afford and sit with in blocks of G and furthermore study characteristics and/or properties that exist between kernels of irreducible $\mathbb{F}G$-modules, vertices of irreducible $\mathbb{F}G$-modules and defect groups of blocks of G and even establish as to when and how any and/or all of these would (if at all possible) coincide.

In §2 we give preliminaries where we define relative projectivity, a vertex and a source of a module, the kernel of a module and radical p-subgroups. In §3 we discuss indecomposable modules and we prove that vertices of indecomposable $\mathbb{F}G$-modules are always p-subgroups of G, an irreducible module in a block has vertex a defect group of the block if and only if it affords a character of the block defect etc.

In §4 we discuss kernels, vertices and radical subgroups wherein we prove that every p-subgroup of G is contained in a radical p-subgroup of G and that every normal p-subgroup of G is contained in all radical p-subgroups of G. In §5 we give concluding remarks which give some highlights of the study and in §6 we give a declaration of interest where the author declares that there is no clash of interest.

Throughout, all our groups are finite unless otherwise specified to the contrary and $\mathbb{F}$ will denote a field of characteristic p, where p is a prime that divides the order $|G|$ of G. By $Bl(G)$ we shall denote the set of all blocks of G, $B_0(G)$ will denote the principal block of G, $B(G)$ will generally denote a block of G and $\delta(B(G))$ will denote the set of all defect groups of $B(G)$. By $1_G \in G$ we shall denote the identity element of the finite group G.

2. Preliminaries

According to [9, Lemma IV.4.11], if $B(G)$ is a block of G with $H = ker(B(G))$, then H is a p'-group which is contained in the kernel of every module in $B(G)$. By [9, Corollary III.2.13(ii)], the intersection of the kernels of all the irreducible $\mathbb{F}G$-modules is $O_p(G)$. However by [11, Theorem 2.2, Lemma 2.3], for $D \in \delta(B(G))$ a defect group of $B(G)$, all modules in $B(G)$ are D-projective and there exists a simple module $M \in B(G)$ whose vertex is D.

Feit mentions in [9] that any ordinary/Brauer character is in a block $B(G)$ of G if it is afforded by a module in $B(G)$. By [9] any indecomposable module in a block of defect zero has the trivial vertex.

Definition 1 [3, 5, 7, 9, 12–16]

Let G be a group, H a subgroup of G and M an $\mathbb{F}G$-module for which there exists an $\mathbb{F}H$-module N such that M is a component of N^G. Then M is said to be H-projective. If M is an indecomposable $\mathbb{F}G$-module with K a subgroup of G such that M is K-projective and M is H-projective only for H a subgroup of G which contains a conjugate of K, then we call the subgroup K of G a vertex of M, which according to [16] is denote by $vx(M)$.

Let V be an indecomposable $\mathbb{F}G$-module with vertex P. Then there exists an indecomposable $\mathbb{F}P$-module L such that

(i) V is a component of L^G,

(ii) L is a component of V_P,

(iii) L has vertex P,

and the indecomposable $\mathbb{F}P$-module L is called a source of V, which according to [16] is denoted by $s(V)$.

Let V be an $\mathbb{F}G$-module. Define

$$ker(V) = \{g \in G \mid gv = v \quad \text{for all} \quad v \in V\}$$

and call $ker(V)$ the kernel of V, where V is said to be faithful if $ker(V)$ is trivial.

For $B(G)$ a block of G, the kernel of $B(G)$ denoted by $ker(B(G))$ is the intersection of the kernels of all the irreducible ordinary characters in $B(G)$.

A p-subgroup P of a group G such that $P = O_p(N_G(P))$ is called a radical p-subgroup of G.

Invariably, the kernel of an indecomposable $\mathbb{F}G$-module V is the kernel of the representation/character afforded by V. By [16] for G a group, H a subgroup of G and V an indecomposable $\mathbb{F}G$-module, V is H-projective if and only if a vertex of V is conjugate in G to a subgroup of H.

By [16, Theorem 4.2.5], if $([G\colon H],\ p) = 1$, then every $\mathbb{F}G$-module becomes H-projective and in particular for P a Sylow p-subgroup of G, every $\mathbb{F}G$-module becomes P-projective. By [9, Lemma II.2.12], $ker(V)$ is always a normal subgroup of G and for $B(G)$ a block of G, the kernel $ker(B(G))$ of $B(G)$ is a normal subgroup of G as well.

Lemma 1 Blocks with conjugate defect groups have the same defect.

Proof. Conjugate groups have the same order and so the result follows immediately. □

Lemma 2 A module is faithful if and only if affords a faithful character.

Proof. The kernel of a module coincides with the kernel of the character it affords and the result follows immediately. □

By [17, Exercise 19.1], a finite p-group P contains a faithful irreducible complex character if and only if $Z(P)$ is cyclic.

3. Indecomposable modules

Every p-subgroup of G is a vertex of some indecomposable $\mathbb{F}G$-module. By [16, Theorem 4.3.3, Lemma 4.3.5], [18, Theorem III.9.4], [19, Lemma 2] we have that every indecomposable module always possesses a vertex and a source. By [9, Corollary III.4.7], if H is a subgroup of G and W an indecomposable $\mathbb{F}H$-module, then W^G has an indecomposable component V such that V and W have a vertex and a source in common.

Lemma 3 For G a group, vertices of indecomposable $\mathbb{F}G$-modules are always p-subgroups of G.

Proof. Every indecomposable $\mathbb{F}G$-module sits in a block of G which has a p-subgroup of G as its defect group. Moreover the result follows immediately by [3], [4, Corollary 53.4], [9, Lemma III.4.4], [13, 14]. □

If M has vertex K, then any conjugate of K is also a vertex of M. Hence a vertex of an indecomposable module is uniquely determined up to conjugation in G. By [6, Corollary 4.18], [7, 1.2], [12], [16, Exercise 5.2.17], defect groups of blocks of any group G are to be found among the radical p-subgroups of G.

Moreover by [13], a vertex and a source will be trivial if p does not divide the order $|G|$ of G and so every indecomposable $\mathbb{F}G$-module is thus $\{1_G\}$-projective, where $\{1_G\}$ is the trivial subgroup of G.

Proposition 1 Let M be H-projective with M being a component of L^G. If L is K-projective for some K a subgroup of H, then M is also K-projective.

Proof. There exists an $\mathbb{F}K$-module T such that L is a component of T^H so that M becomes a component of T^G. Furthermore this is [5, Lemma 65.5], [13, Fact 2.9]. □

Proposition 2 If G has a normal Sylow p-subgroup P, then every indecomposable $\mathbb{F}G$-module becomes P-projective.

Proof. The result follows immediately by [13, Theorem 2], [16, Theorem 4.2.5], [18, Theorem III. 9.4]. □

In fact for $P \in Syl_p(G)$, every indecomposable $\mathbb{F}G$-module becomes P-projective. By [8, Lemma 2.2], $P \in Syl_p(G)$ being normal with U an indecomposable $\mathbb{F}P$-module with vertex P, renders P to be the vertex of every indecomposable component of U^G.

Theorem 1 If G is a group and H is a subgroup of G that contains a Sylow p-subgroup of G, then every indecomposable $\mathbb{F}G$-module becomes H-projective.

Proof. This is [18, Theorem III. 9.2]. □

Proposition 3 Any indecomposable modules having a common source also have a common vertex.

Proof. By [15] every indecomposable module shares a vertex with its source. Moreover indecomposable modules with a common source are components of the same induced module. Hence the result follows immediately. □

Let M, M' be irreducible $\mathbb{F}G$-modules having $\mathfrak{V} \leq G$ as a common vertex and having sources L, L' respectively. Then by [4, Lemma 53.5], [5, Theorem 65.14], [9, Lemma III. 4.5], [13, Theorem 5], the sources L, L' of M, M' respectively are related as $\mathbb{F}\mathfrak{V}$-modules by $L' \cong x \bigotimes L$ for some $x \in N_G(\mathfrak{V})$.

Proposition 4 The trivial module always has a Sylow subgroup as its vertex.

Proof. The trivial module always affords the principal character which always has full defect and actually sits inside of the principal block whose defect group is always a Sylow subgroup. Moreover this is [13, Corollary 1]. Hence the result follows immediately completing the proof. □

Corollary 1 Any indecomposable $\mathbb{F}G$-module affording a linear character always has a Sylow subgroup as its vertex.

Proof. Any linear character $\chi \in Irr(G)$ is such that $(\chi(1_G),\, p) = 1$ making $\chi \in Irr(G)$ to have full defect by [16, Exercise 6.14]. Thus $\chi \in Irr(G)$ always sits in a block of full defect having a Sylow subgroup as its defect group. Hence the result follows immediately and the proof is complete. □

Corollary 2 Any indecomposable $\mathbb{F}G$-module of dimension 1 always has a Sylow subgroup as its vertex.

Proof. The result follows by [13, Theorem 9, Corollary 1]. □

Proposition 5 Conjugate modules sit in the same block.

Proof. We have by [9] that a character or Brauer character is in a block $B(G)$ if it is afforded by a module in $B(G)$ and according to [5, 16], conjugate characters sit in the same block. By [5, Corollary 30.14] conjugate modules afford the same character. Hence the result follows completing the proof. □

Proposition 6 An irreducible module in a block has vertex a defect group of the block if and only if it affords a character of the block defect.

Proof. Suppose that an irreducible module in a block has vertex a defect group of the block. The result follows in principal blocks, nonprincipal blocks containing linear characters and blocks of defect zero. The existence of such an irreducible module in a block, is guaranteed by [11, Theorem 2.2, Lemma 2.3]. By [3, Corollary 4.6, Remark 4.7(ii)], the result follows immediately. Conversely suppose that an irreducible module in a block affords a character of the block defect. The character of the block defect thus has height 0 and so the result follows by [3, Corollary 4.6, Remark 4.7(ii)], [15, Theorem 12.3.4], [16, Theorem 5.1.11(iv)]. □

Corollary 3 For G an abelian group, any indecomposable $\mathbb{F}G$-module has the unique Sylow subgroup of G as its vertex.

Proof. The result follows immediately by [3, Corollary 3.7]. □

Proposition 7 An irreducible module with a trivial vertex cannot sit in a block of positive defect.

Proof. The result follows by [3, Corollary 3.7] generalizing [1, Theorem 3.2]. □

Corollary 4 If $O_p(G) \neq \{1_G\}$, then all the irreducible $\mathbb{F}G$-modules have nontrivial vertices.

Proof. G will have no blocks of defect zero and hence the result follows by Proposition 7 above. □

An indecomposable module cannot afford characters of different defects as such a module would sit in different blocks, which is impossible. Moreover a representation and the character it affords always share a degree. Irreducible characters of height zero in blocks which thus give blocks their defects, are afforded by indecomposable $\mathbb{F}G$-modules whose vertices are the defect groups of those blocks.

If G is an abelian group, then every indecomposable $\mathbb{F}G$-module has the unique Sylow subgroup of G as its vertex and thus by [2] every block of G is a large vertex block.

4. Kernels, vertices and radical subgroups

According to [4, Theorm 65.4], [19, Theorm 1.4], the p-regular core $O_{p'}(G)$ of a group G is the intersection of the kernels of all the irreducible ordinary characters in the principal block of G. Generally, the kernel of a block is the intersection of the kernels of all the irreducible ordinary characters in that particular block.

Hence the p-regular core $O_{p'}(G)$ of a group G is the kernel of the principal block of G. Since a finite simple group G has only the principal character as a linear character, all the other characters will be nonlinear and thus have trivial kernels. In this regard therefore, all the blocks of a finite simple group G will have trivial kernels.

Lemma 4 If $B_0(G)$ contains an irreducible faithful module, then $O_{p'}(G)$ is trivial.

Proof. The kernels of all modules in $B_0(G)$ contain $O_{p'}(G)$ and hence the result follows. □

Lemma 5 Any block of G containing an irreducible faithful module has a trivial kernel.

Proof. Such a block would contain an irreducible faithful ordinary character and hence the result follows. □

Lemma 6 If θ is an irreducible representation of G, then $O_p(G) \subseteq ker(\theta)$.

Proof. This is [6, Lemma 2.32]. □

Proposition 8 Let G be a group and M an irreducible $\mathbb{F}G$-module.

(i) If the vertex of M is a normal subgroup of G, then $vx(M) = O_p(G)$.

(ii) If the kernel of M is a p-subgroup of G, then $ker(M) = O_p(G)$.

Proof. The result follows immediately by [9, Theorem III.2.13(i), Theorem III.4.12, Corollary III.4.13]. □

Proposition 9 Let G be a group.

(i) If G is an abelian p-group, $B(G)$ a block of G and M an irreducible $\mathbb{F}G$-module in $B(G)$, then $ker(M) = vx(M) = D \in \delta(B(G))$.

(ii) For the trivial $\mathbb{F}G$-module M, we have that $vx(M) \subseteq ker(M)$.

Proof. (i) G will be of deficiency class 0 and all $ker(M)$, $vx(M)$, $D \in \delta(B(G))$ contain $O_p(G)$. By [3, Corollary 3.7(i)], Proposition 8 above, the desired result follows immediately.

(ii) We have that $ker(M) = G$ and so by Proposition 7 above, the result follows. □

Proposition 10 Every p-subgroup of G is contained in a radical p-subgroup of G.

Proof. By [7, Lemma 1.3], any p-subgroup normalized by the normalizer of a radical p-subgroup is contained in that radical p-subgroup. According to Green deducing from [9, Corollary III.4.7], [13, Theorem 7] every p-subgroup of G is a vertex of some indecomposable $\mathbb{F}G$-module. However indecomposable $\mathbb{F}G$-modules sit in blocks which have defect groups. By [18, Theorem IV.13.5], we have that a vertex of any indecomposable $\mathbb{F}G$-module in a block is contained in a defect group of that block and defect groups being radical subgroups of G by [18, Theorem IV.13.6(3)], the desired result follows. □

Moreover every p-subgroup of G is contained in a Sylow p-subgroup of G and Sylow p-subgroups of G are radical by [7, 1.1].

Corollary 5 Every normal p-subgroup of G is contained in all radical p-subgroups of G.

Proof. By [20, Problem 5.1] we have that every normal p-subgroup of G is contained in $O_p(G)$ and by [7, Proposition 1.4] we have that $O_p(G)$ is contained in all radical p-subgroups of G. Hence the result follows immediately completing the proof. □

We have that every normal p-subgroup of G is contained in all Sylow p-subgroups of G which are radical by [7, 1.1]. By [18, Theorem IV.13.6(2)] every normal p-subgroup of G is contained in defect groups of all the blocks of G and by [6, Corollary 4.18], [7, 1.2], [16, Exercise 5.2.17], [21, Theorem 5], defect groups of blocks of G are maximal normal p-subgroups of their normalizers in G and are thus radical.

Furthermore by [21, Theorem 5], a defect group of a block of G is a Sylow p-subgroup of its normalizer in G. According to [4, Theorem 53.9(ii)] a normal p-subgroup of G is contained in the kernel of every irreducible module and also in a vertex of every irreducible module. Thus kernels of irreducible $\mathbb{F}G$-modules, vertices of irreducible $\mathbb{F}G$-modules and defect groups of blocks of G all contain $O_p(G)$.

Proposition 11 Let G be a group.

(i) If G is abelian, then G has a unique radical p-subgroup.

(ii) If G is a p-group, then G has a unique radical p-subgroup.

(iii) If G has a block with a normal defect group D, then $D = O_p(G)$.

Proof. (i) G abelian has a unique Sylow subgroup S such that by [7, Corollary 1.5] we have $S = O_p(G)$ and by [7, Theorem 1.4] $O_p(G)$ is the unique minimal radical p-subgroup of G.

(ii) G a p-group gives that $O_p(G) = G$.

(iii) D a defect group of a block of G makes D a radical p-subgroup of G by [6, Corollary 4.18], [7, 1.2], [9, Theorem III.8.15], [16, Exercise 5.2.17] and its normality renders $D = O_p(G)$ by [7, Corollary 1.5]. □

Proposition 11(ii) above asserts that a p-group is actually radical in itself. We have by [6, Corollary 4.18], [7, 1.2], [9, Theorem III.8.15], [16, Exercise 5.2.17] that a p-subgroup P of G which is a defect group of some block of G, is actually radical.

By Proposition 10 above, P is contained in a radical p-subgroup of G and moreover P is a vertex of some indecomposable module that sits in some block of G such that P is contained in a defect group D of that block. We also have that $Z(D) \subseteq P \subseteq D$.

Proposition 12 An irreducible $\mathbb{F}G$-module whose vertex is a radical p-subgroup of G, affords a character of the block defect.

Proof. The radical p-subgroup of G which is a vertex of an irreducible $\mathbb{F}G$-module is actually a defect group of the block of G which contains that irreducible $\mathbb{F}G$-module. Hence the result follows immediately by Proposition 6 above and the proof is complete. □

Proposition 13 If $O_p(G) \neq \{1_G\}$, then there is no faithful irreducible $\mathbb{F}G$-module.

Proof. Since $O_p(G)$ is the intersection of the kernels of all the irreducible $\mathbb{F}G$-modules, the result follows. □

By [6, Theorem 6.10], [16, Theorem 8.1], [22, Proposition 3B] we have for any $\chi \in Irr(B(G))$ that

$$ker(B(G)) = O_{p'}(ker(\chi)) = ker(\chi) \cap O_{p'}(G)$$

and in particular we have that $ker(B_0(G)) = O_{p'}(G)$.

5. Concluding remarks

In finite groups, p-subgroups play a very important role in their study e.g. every p-subgroup of a finite group G is a vertex of some indecomposable $\mathbb{F}G$-module, for $P \in Syl_p(G)$ every indecomposable $\mathbb{F}G$-module becomes P-projective etc. Every normal p-subgroup of G is contained in all Sylow p-subgroups of G, is contained in defect groups of all the blocks of G, is contained in the kernel of every irreducible $\mathbb{F}G$-module, is contained in a vertex of every irreducible $\mathbb{F}G$-module, is contained in all radical p-subgroups of G. Thus kernels of irreducible $\mathbb{F}G$-modules, vertices of irreducible $\mathbb{F}G$-modules, defect groups of blocks of G, Sylow p-subgroups of G, radical p-subgroups of G, all contain $O_p(G)$.

Conflict of interest

The author hereby declares that there is no conflict of interest.

References

[1] Hamernik W, Michler G. On vertices of simple modules in p-solvable groups. *Mitteilungen aus dem Mathem Seminar Giessen*. 1976; 121: 147-172.

[2] Berger TR, Knorr R. On Brauer's height 0 conjecture. *Nagoya Mathematical Journal*. 1988; 109: 109-116.

[3] Knorr R. On the vertices of irreducible modules. *Annals of Mathematics*. 1979; 110(3): 487-499.

[4] Dornhoff L. *Group Representation Theory Part B: Modular Representation Theory*. New York: Marcel Dekker, Inc.; 1972.

[5] Curtis CW, Reiner I. *Representation Theory of Finite Groups and Associative Algebras*. New York: Interscience Publishers, a division of John Wiley and Sons; 1962.

[6] Navarro G. *Characters and Blocks of Finite Groups*. Cambridge: Cambridge University Press; 1998.

[7] Dade EC. Counting characters in blocks, I. *Inventiones Mathematicae*. 1992; 109: 187-210.

[8] Hamernik W. Indecomposable modules with cyclic vertex. *Mathematische Zeitschrift*. 1975; 142: 87-90.

[9] Feit W. *The Representation Theory of Finite Groups*. Amsterdam: North-Holland; 1982.

[10] Brauer R, Feit W. On the number of irreducible characters of finite groups in a given block. *Proceedings of the National Academy of Sciences of the United States of America*. 1959; 45(3): 361-365.

[11] Knorr R. Blocks, vertices and normal subgroups. *Mathematische Zeitschrift*. 1976; 148: 54-60.

[12] Alperin JL, Fong P. Weights for symmetric and general linear groups. *Journal of Algebra*. 1990; 131(1): 2-22.

[13] Green JA. On the indecomposable representations of a finite group. *Mathematische Zeitschrift*. 1959; 70: 430-445.

[14] Green JA. Blocks of modular representations. *Mathematische Zeitschrift*. 1962; 79: 100-115.

[15] Karpilovsky G. *Group Representations, Volume 5*. Amsterdam: North-Holland Mathematics Studies; 1996.

[16] Nagao H, Tsushima Y. *Representations of Finite Groups*. San Diego: Academic Press, Inc.; 1988.

[17] Dornhoff L. *Group Representation Theory Part A: Ordinary Representation Theory*. New York: Marcel Dekker, Inc.; 1971.

[18] Alperin JL. *Local Representation Theory*. Cambridge: Cambridge University Press; 1986.

[19] Brauer R. Some applications of the theory of blocks of characters of finite groups I. *Journal of Algebra*. 1964; 1(2): 152-167.

[20] Isaacs IM. *Algebra: A Graduate Course*. California: Brookes/Cole Publishing Company; 1994.

[21] Osima M. Notes on blocks of group characters. *Mathematical Journal of Okayama University*. 1955; 4: 175-188.

[22] Brauer R. Some applications of the theory of blocks of characters of finite groups IV. *Journal of Algebra*. 1971; 17(4): 489-521.