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Research Article

# Strong Sandwich Results Involving the Riemann-Liouville Fractional Integral of an Extended $q$-Hypergeometric Function

**Alina Alb Lupaş**

Department of Mathematics and Computer Science, Faculty of Informatics and Sciences, University of Oradea, Oradea, Romania
E-mail: dalb@uoradea.ro

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**Abstract:** The classical theories of differential superordination and subordination have been extended to strong differential superordination and respectively, strong differential subordination. The two new theories have progressed well, revealing significant findings when various operators and specific hypergeometric functions have been included in the studies. The research revealed by this work expands the topic of the investigation by incorporating aspects of fractional calculus and quantum calculus. An extended version of $q$-hypergeometric function is introduced to correspond to the study of functions from the classes that were previously described and that are particularly defined for strong differential superordination and subordination theories. This work defines the Riemann-Liouville fractional integral applied to the extended $q$-hypergeometric function, used to get strong differential subordinations and superordination results. The theorems established for the strong differential superordination and subordination, establish the best subordinants and respectively the best dominants. Interesting corollaries are exposed for certain functions regarded as best subordinant or best dominant due to their particular geometric characteristics. Sandwich-type theorems and consequences conclude the study, stated to connect the outcomes obtained by applying the dual theories.

***Keywords***: Riemann-Liouville fractional integral, extended $q$-confluent hypergeometric function, strong differential subordination, strong differential superordination, best dominant, best subordinant

**MSC:** 30C45, 30A10, 33D05

## 1. Introduction

Romaguera and Antonino [1] employed the concept of strong differential subordination for the first time in their analysis of strong differential subordination of Briot-Bouquet. It appeared as an extension of Mocanu and Miller's classical concept of differential subordination [2, 3].

The notion emerged in 2009 [4], laying the groundwork for the field of strong differential subordination. The researchers in this theory expanded the concepts from the theory of differential subordination [5]. The strong differential superordination has a dual notion introduced in 2009 [6] beside the point established for classical field of differential superordination [7].

The next period showed the development of both theories. In [8] were given ways to obtain the best subordinant for the strong differential superordination, and in [9] are studied special cases of strong differential superordinations and


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subordinations. By linking various operators to the research, such as Liu-Srivastava operator [11], Sălăgean differential operator [10], Ruscheweyh operator [12], multiplier transformation [13, 14], combinations of Sălăgean and Ruscheweyh operators [15], the Komatu integral operator [16, 17], or differential operators [18, 19], strong differential subordinations could be further obtained. Citing recently released papers [20–23] shows that the topic is current and in the present.

In the early investigations, fractional calculus was strongly related to field of strong differential subordination [24], however this way of study was not continued. In Srivastava's recent paper [25], it is emphasized how the application of fractional calculus and quantum calculus to geometric function theory has redounded to its progress. A particular integro-differential complex operator that is related to the meromorphic functions in the punctured unit disk as well as to the Mittag-Leffler function was successfully constructed in the study reported in [26]. Several formulas for particular fractional differ-integral operators introduced applying Riemann-Liouville fractional integrals are provided by the study described in [27]. The study reported in [28] aims to conduct a qualitative analysis for a nonlinear Langevin integro-fractional differential equation. Fractional calculus is used in article [29] to create new relationships for the Mittag-Leffler functions with one, two, three, and four parameters. Thus, this work investigates new analytical features by involving fractional integration and differentiation for the Mittag-Leffler function generated by confluent hypergeometric functions. A novel function named Mittag-Leffler-confluent hypergeometric function is developed and studied in the work shown in [30] using the well-known tools of investigation that are the Mittag-Leffler function and the confluent hypergeometric function. Additionally, some analytical solutions to the integral equations are examined. In [31], distinctive variants of the Gamma and Kummer functions are introduced and then studied in terms of Mittag-Leffler functions. The analysis centers on the investigation of special functions in conjunction with fractional calculus.

It is well known that three distinct theories-classical differential superordinations and subordinations, fuzzy differential superordinations and subordinations, and strong differential superordinations and subordinations-are currently developing in the field of geometric function theory. The findings of this study attempt to revive the investigation of fractional operators and functions that are familiar to quantum calculus in the framework of theories of strong differential superordinations and subordinations. Current research that involve hypergeometric functions [32], use differential operators [33] to obtain sandwich results, or incorporate fractional calculus aspects [34] demonstrate the interest in developing theories of strong differential superordinations and subordinations. Aiming to continue the line of investigation linking aspects of fractional calculus and the theories of strong differential superordinations and subordinations, this paper also introduces quantum calculus aspects and uses an extended form of the $q$-hypergeometric function to deal with certain function classes presented in [35], particularly defined for the strong differential superordination and subordination theories. Furthermore, an operator defined by applying Riemann-Liouville fractional integral to this extended $q$-hypergeometric function [36] is used here having as inspiration the new fractional operator recently introduced and studied in [37] as the Riemann-Liouville fractional integral of $q$-hypergeometric function. When this operator was examined using the classical concepts of differential subordination and superordination, some significant additional knowledge emerged in [38]. Due to its adaptation to the classes specific to the theories of strong differential superordination and subordination, the operator employed in this research differs in form from the one described in [37].

The promising outcomes using the operator employed in [37] and [38] achieved in the framework of classical differential superordinations and subordinations were modified in order to correspond to the more recent theories of fuzzy differential superordination and subordination in [39]. By utilizing the specific operator introduced as the Riemann-Liouville fractional integral of an extended $q$-hypergeometric function, the present study's output is valuable as it contributes to the advancement of other correspondent theories of strong differential superordinations and subordinations. The results developed here are new and noteworthy due to their particular context of development and they are not applicable to either the fuzzy differential superordination and subordination theories or the classical differential superordination and subordination theories.

The main concepts utilized in the research are reviewed in Section 2, Preliminaries, along with a list of fundamental lemmas that were employed to prove the theorems stated as main results thay are presented in Section 3. Here, best dominants and best subordinants are obtained for strong differential superordinations, respectively subordinations, involving Riemann-Liouville fractional integral applied to extended $q$-hypergeometric function.

Considering particular functions with noteworthy geometric features as the best dominants and subordinants in the established theorems, interesting corollaries follow. Applications as sandwich-type theorems and related corollaries link the new findings from this study regarding the two dual theories.

## 2. Preliminaries

Consider $\mathscr{H}(\Delta\times\overline{\Delta})$ the class of analytic functions in $\Delta\times\overline{\Delta}$, with $\Delta=\{x\in\mathbb{C}: |x|<1\}$ and $\overline{\Delta}=\{x\in\mathbb{C}: |x|\leq 1\}$.

Special subclasses of $\mathscr{H}(\Delta\times\overline{\Delta})$ are defined in [35] regarding to the strong differential superordination and strong differential subordination theories:

$$\mathscr{A}^{*}_{n\xi}=\{h(x,\ \xi)=x+a_{n+1}(\xi)x^{n+1}+\cdots\in\mathscr{H}(\Delta\times\overline{\Delta})\},$$

with $\mathscr{A}^{*}_{1\xi}=\mathscr{A}^{*}_{\xi}$ and $a_j(\xi)$ holomorphic functions in $\overline{\Delta}$, $j\geq n+1\in\mathbb{N}$, and

$$\mathscr{H}^{*}[a,\ n,\ \xi]=\{h(x,\ \xi)=a+a_n(\xi)x^n+a_{n+1}(\xi)x^{n+1}+\cdots\in\mathscr{H}(\Delta\times\overline{\Delta})\},$$

with $a_j(\xi)$ holomorphic functions in $\overline{\Delta}$, $j\geq n\in\mathbb{N}$, $a\in\mathbb{C}$.

The notion of strong differential subordination described in [1] and studied in [4] and [35] is defined below. Denote the notion of "strong differential subordination" by "SDsub" to reduce the similarity rate of the paper.

**Definition 2.1** [4] Let $h_1(x,\ \xi),\ h_2(x,\ \xi)\in\mathscr{H}(\Delta\times\overline{\Delta})$. $h_1(x,\ \xi)$ is strongly subordinate to $h_2(x,\ \xi)$, denoted $h_1(x,\ \xi)\prec\prec h_2(x,\ \xi)$, if the analytic function $f$ with the properties $f(0)=0$, $|f(x)|<1$, $x\in\Delta$ is in existence and $h_1(x,\ \xi)=h_2(f(x),\ \xi)$, $\xi\in\overline{\Delta}$.

**Remark 2.1** [4] (i) When $h_2(x,\ \xi)$ is univalent in $\Delta$, $\forall\ \xi\in\overline{\Delta}$, Definition 2.1 is equivalent with $h_1\left(\Delta\times\overline{\Delta}\right)\subset h_2\left(\Delta\times\overline{\Delta}\right)$ and $h_1(0,\ \xi)=h_2(0,\ \xi)$, $\xi\in\overline{\Delta}$.

(ii) For the particular case $h_1(x,\ \xi)=h_1(x)$ and $h_2(x,\ \xi)=h_2(x)$, we have classical differential subordination.

To explore SDsub we need the lemma:

**Lemma 2.1** [40] Consider $w\in\mathscr{H}(\Delta\times\overline{\Delta})$ univalent and $f$, $g$ analytic functions in a domain $D\supset w\left(\Delta\times\overline{\Delta}\right)$ with the property $g(x)\neq 0$ for $x\in w\left(\Delta\times\overline{\Delta}\right)$. Define the functions $F(x,\ \xi)=xw'_x(x,\ \xi)g(w(x,\ \xi))$ and $G(x,\ \xi)=f(w(x,\ \xi))+F(x,\ \xi)$. In conditions:

1) $F$ is starlike univalent in $\Delta\times\overline{\Delta}$,

2) $Re\left(\dfrac{xG'_x(x,\ \xi)}{F(x,\ \xi)}\right)>0$, $(x,\ \xi)\in\Delta\times\overline{\Delta}$,

3) $u\in\mathscr{H}(\Delta\times\overline{\Delta})$ with the properties $u(0,\ \xi)=w(0,\ \xi)$ and $u\left(\Delta\times\overline{\Delta}\right)\subseteq D$, is a solution of the SDsub

$$f(u(x,\ \xi))+xu'_x(x,\ \xi)g(u(x,\ \xi))\prec\prec f(w(x,\ \xi))+xw'_x(x,\ \xi)g(w(x,\ \xi)),$$

then the SDsub holds

$$u(x,\ \xi)\prec\prec w(x,\ \xi),\quad (x,\ \xi)\in\Delta\times\overline{\Delta},$$

and $w$ is the best dominant.

The strong differential superordination is defined below. Also, denote the notion of "strong differential superordination" by "SDsup" to reduce the similarity rate of the paper.

**Definition 2.2** [6] Let $h_1(x, \xi), h_2(x, \xi) \in \mathscr{H}(\Delta \times \overline{\Delta})$. $h_1(x, \xi)$ is strongly superordinate to $h_2(x, \xi)$, denoted $h_2(x, \xi) \prec\prec h_1(x, \xi)$, if the analytic function $f$ with the properties $f(0) = 0$, $|f(x)| < 1$, $x \in \Delta$ is in existence and $h_2(x, \xi) = h_1(f(x), \xi), \xi \in \overline{\Delta}$.

**Remark 2.2** [6] (i) When $h_1(x, \xi)$ is univalent in $\Delta$, $\forall\ \xi \in \overline{\Delta}$, Definition 2.2 is equivalent with $h_2\left(\Delta \times \overline{\Delta}\right) \subset h_1\left(\Delta \times \overline{\Delta}\right)$ and $h_2(0, \xi) = h_1(0, \xi), \xi \in \overline{\Delta}$.

(ii) For the particular case $h_1(x, \xi) = h_1(x)$ and $h_2(x, \xi) = h_2(x)$, we have classical differential superordination.

**Definition 2.3** [41] Denote by $Q^* = \{f \in \mathscr{H}(\Delta \times \overline{\Delta}) : f$ injective $\overline{\Delta} \times \overline{\Delta} \backslash E(f, \xi),\ f'_x(z, \xi) \neq 0,\ z \in \partial\Delta \times \overline{\Delta} \backslash E(f, \xi)\}$, with $E(f, \xi) = \{z \in \partial\Delta : \lim_{x \to z} f(x, \xi) = \infty\}$ and $Q^*(a) = \{f : f \in Q^*,\ f(0, \xi) = a\}$.

To investigate SDSup we need the lemma:

**Lemma 2.2** [40] Consider $w \in \mathscr{H}(\Delta \times \overline{\Delta})$ univalent and $f, g$ analytic functions in a domain $D \supset w\left(\Delta \times \overline{\Delta}\right)$. In conditions:

1) $F(x, \xi) = xw'_x(x, \xi)\, g(w(x, \xi))$ is starlike univalent in $\Delta \times \overline{\Delta}$,

2) $Re\left(\dfrac{f'_x(w(x, \xi))}{g(w(x, \xi))}\right) > 0,\ (x, \xi) \in \Delta \times \overline{\Delta}$,

3) the function $f(u(t, \tau)) + tu'_t(t)\, g(u(t, \tau))$ is univalent in $\Delta \times \overline{\Delta}$,

4) $u(x, \xi) \in \mathscr{H}^*[w(0, \tau), 1, \tau] \cap Q^*$, with the property $u\left(\Delta \times \overline{\Delta}\right) \subseteq D$, satisfies the SDsup

$$f(w(x, \xi)) + xw'_x(x, \xi)\, g(w(x, \xi)) \prec\prec f(u(x, \xi)) + xu'_x(x, \xi)\, g(u(x, \xi)),$$

then the SDsup holds

$$w(x, \xi) \prec\prec u(x, \xi), \quad (x, \xi) \in \Delta \times \overline{\Delta},$$

and $w$ is the best subordinant.

In this article we explore several properties of the $q$-hypergeometric function studied also in [38]:

**Definition 2.4** [38] The extended $q$-hypergeometric function $\phi(a(\xi), b(\xi); q, x, \xi)$ is defined by

$$\phi(a(\xi), b(\xi); q, x, \xi) = \sum_{j=0}^{\infty} \frac{(a(\xi), q)_j}{(q, q)_j (b(\xi), q)_j} x^j,$$

where

$$(a(\xi), q)_j = \begin{cases} 1, & j = 0, \\ (1 - a(\xi))(1 - qa(\xi))\left(1 - q^2 a(\xi)\right) \ldots \left(1 - q^{j-1} a(\xi)\right), & j \in \mathbb{N}, \end{cases}$$

and $a(\xi), b(\xi)$ are holomorphic functions depending on the parameter $\xi \in \overline{\Delta}$, $0 < q < 1$.

We remind the definition of Riemann-Liouville fractional integral [42, 43] (denoted in this paper as RL-fr-int) applied to a function $f \in \mathscr{A}^*_\zeta$.

**Definition 2.5** [42, 43] The fractional integral of order $\alpha$ $(\alpha > 0)$ applied to the function $f \in \mathscr{H}(\Delta \times \overline{\Delta})$ is defined by

$$D_x^{-\alpha} f(x, \xi) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(y, \xi)}{(x-y)^{1-\alpha}} dy,$$

with condition $\log(x-y)$ to be real, when $(x-y) > 0$.

The investigation regards the RL-fr-int applied to the extended $q$-hypergeometric function defined in [36] by using Definitions 2.4 and 2.5.

**Definition 2.6** [36] The RL-fr-int applied to the extended $q$-confluent hypergeometric function is

$$\begin{aligned} D_x^{-\alpha} \phi(a(\xi), b(\xi); q, x, \xi) &= \frac{1}{\Gamma(\alpha)} \int_0^x \frac{\phi(a(\xi), b(\xi); q, y, \xi)}{(x-y)^{1-\alpha}} dy \\ &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{\infty} \frac{(a(\xi), q)_j}{(q, q)_j (b(\xi), q)_j} \int_0^x \frac{y^j}{(x-y)^{1-\alpha}} dy, \end{aligned} \tag{1}$$

with $a(\xi), b(\xi)$ holomorphic functions depending on the parameter $\xi \in \overline{\Delta}$, $0 < q < 1$, $\alpha > 0$.

After a simple calculation, it takes the following form

$$D_x^{-\alpha} \phi(a(\xi), b(\xi); q, x, \xi) = \sum_{j=0}^{\infty} \frac{(a(\xi), q)_j}{(q, q)_j (b(\xi), q)_j (j+1)_\alpha} x^{\alpha+j}, \tag{2}$$

and $D_x^{-\alpha} \phi(a(\xi), b(\xi); q, x, \xi) \in \mathscr{H}[0, \alpha, \xi]$.

The next section describes the outcome of the new research on SDsub and SDsup regarding RL-fr-int applied to extended $q$-hypergeometric function.

Throughout this paper, assume that $0 < q < 1$, $\alpha > 0$ and $a(\xi), b(\xi)$ are holomorphic functions.

## 3. Main results

The SDsub result obtained for the operator described by (2) is the next theorem:

**Theorem 3.1** Let $\left(\frac{D_x^{-\alpha} \phi(a(\xi), b(\xi); q, x, \xi)}{x}\right)^p \in \mathscr{H}\left(\Delta \times \overline{\Delta}\right)$ and a univalent function $w(x, \xi)$ in $U \times \overline{U}$ with the property $w(x, \xi) \neq 0$, $\forall x \in \Delta \backslash \{0\}$, $\xi \in \overline{\Delta}$. Assuming that the function $\frac{x w_x'(x, \xi)}{w(x, \xi)}$ is starlike univalent in $\Delta \times \overline{\Delta}$ and

$$\operatorname{Re}\left(1 + \frac{n}{s} w(x, \xi) + \frac{2r}{s} (w(x, \xi))^2 - \frac{x w_x'(x, \xi)}{w(x, \xi)} + \frac{x w_{x^2}''(x, \xi)}{w_x'(x, \xi)}\right) > 0, \tag{3}$$

for $m, n, r, s \in \mathbb{C}$, $s \neq 0$, $x \in \Delta \backslash \{0\}$, $\xi \in \overline{\Delta}$ and

$$H_{\alpha}^{q}(p, m, n, r, s; x, \xi) =: m + n\left[\frac{D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi)}{x}\right]^{p}$$
$$+ r\left[\frac{D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi)}{x}\right]^{2p} + sp\left[\frac{x(D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi))'_x}{D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi)} - 1\right]. \tag{4}$$

If $w$ is a solution of the strong subordination

$$H_{\alpha}^{q}(p, m, n, r, s; t, \tau) \prec\prec m + nw(x, \xi) + r(w(x, \xi))^2 + s\frac{xw'_x(x, \xi)}{w(x, \xi)}, \tag{5}$$

then $w$ is the best dominant of the strong subordination

$$\left(\frac{D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi)}{x}\right)^{p} \prec\prec w(x, \xi), \quad (x, \xi) \in \Delta \times \overline{\Delta}. \tag{6}$$

**Proof.** Considering the function $u(x, \xi) = \left(\frac{D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi)}{x}\right)^{p}$, $x \in \Delta\backslash\{0\}$, $\xi \in \overline{\Delta}$, differentiating it with respect to $x$, we get

$$u'_x(x, \xi) = p\left(\frac{D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi)}{x}\right)^{p-1}\left[\frac{(D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi))'_x}{x} - \frac{D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi)}{x^2}\right]$$

$$= p\left(\frac{D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi)}{x}\right)^{p-1}\frac{(D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi))'_x}{x} - \frac{p}{x}u(x, \xi)$$

and yields $\frac{xu'_x(x, \xi)}{u(x, \xi)} = p\left[\frac{x(D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi))'}{D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi)} - 1\right]$.

Define the analytic functions $f(z) = m + nz + rz^2$ and $g(z) = \frac{s}{z}$, with $g(z) \neq 0$, $z \in \mathbb{C}\backslash\{0\}$.

Define also the functions $F(x, \xi) = xw'_x(x, \xi)g(w(x, \xi)) = s\frac{xw'_x(x, \xi)}{w(x, \xi)}$ and $G(x, \xi) = f(w(x, \xi)) + F(x, \xi) = m + nw(x, \xi) + r(w(x, \xi))^2 + s\frac{xw'_x(x, \xi)}{w(x, \xi)}$.

We will chech the conditions from Lemma 2.1. It is evidently that $F(x, \xi)$ is starlike univalent.

Differentiating the function $G$ with respect to $x$ we get

$$G'_x(x, \xi) = s + w'_x(x, \xi) + 2rw(x, \xi)w'_x(x, \xi) + s\frac{\left(w'_x(x, \xi) + xw''_{x^2}(x, \xi)\right)w(x, \xi) - x(w'_x(x, \xi))^2}{(w(x, \xi))^2}$$

and

$$\frac{xG'_x(x,\ \xi)}{F(x,\ \xi)} = \frac{xG'_x(x,\ \xi)}{s\frac{xw'_x(x,\ \xi)}{w(x,\ \xi)}} = 1+\frac{n}{s}w(x,\ \xi)+\frac{2r}{s}(w(x,\ \xi))^2-\frac{xw'_x(x,\ \xi)}{w(x,\ \xi)}+\frac{xw''_{x^2}(x,\ \xi)}{w'_x(x,\ \xi)}.$$

The second condition

$$\text{Re}\left(\frac{xG'_x(x,\ \xi)}{F(x,\ \xi)}\right) = \text{Re}\left(1+\frac{n}{s}w(x,\ \xi)+\frac{2r}{s}(w(x,\ \xi))^2-\frac{xw'_x(x,\ \xi)}{w(x,\ \xi)}+\frac{xw''_{x^2}(x,\ \xi)}{w'_x(x,\ \xi)}\right) > 0$$

is true form the relation (3).

We get the function

$$m+nu(x,\ \xi)+r(u(x,\ \xi))^2+s\frac{xu'_x(x,\ \xi)}{u(x,\ \xi)}$$

$$=m+n\left[\frac{D_x^{-\alpha}\phi(a(\xi),\ b(\xi);\ q,\ x,\ \xi)}{x}\right]^p+r\left[\frac{D_x^{-\alpha}\phi(a(\xi),\ b(\xi);\ q,\ x,\ \xi)}{x}\right]^{2p}$$

$$+sp\left[\frac{x(D_x^{-\alpha}\phi(a(\xi),\ b(\xi);\ q,\ x,\ \xi))'_x}{D_x^{-\alpha}\phi(a(\xi),\ b(\xi);\ q,\ x,\ \xi)}-1\right].$$

SDsub 3.3 take the form $m+nu(x,\ \xi)+r(u(x,\ \xi))^2+s\frac{xu'_x(x,\ \xi)}{u(x,\ \xi)} \prec\prec m+nw(x,\ \xi)+r(w(x,\ \xi))^2+s\frac{xw'_x(x,\ \xi)}{w(x,\ \xi)}$.

The conditions from Lemma 2.1 being fulfilled, we obtain $u(x,\ \xi) \prec\prec w(x,\ \xi)$, $(x,\ \xi) \in \Delta\times\overline{\Delta}$, written as $\left(\frac{D_x^{-\alpha}\phi(a(\xi),\ b(\xi);\ q,\ x,\ \xi)}{x}\right)^p \prec\prec w(x,\ \xi)$ with $w$ the best dominant. □

Considering in Theorem 3.1 as best dominant the particular function $w(x,\ \xi)=\frac{Mx+\xi}{Nx+\xi}$, $(x,\ \xi)\in\Delta\times\overline{\Delta}$, we get the following special case:

**Corollary 3.2** Assuming that relation (3) takes place, if the SDsub

$$H_\alpha^q(p,\ m,\ n,\ r,\ s;\ x,\ \xi) \prec\prec m+n\frac{Mx+\xi}{Nx+\xi}+r\left(\frac{Mx+\xi}{Nx+\xi}\right)^2+s\frac{(M-N)x\xi}{(Mx+\xi)(Nx+\xi)},$$

is fulfilled for $m,\ n,\ r,\ s\in\mathbb{C}$, $s\neq 0$, $-1\leq N<M\leq 1$, and the function $H_\alpha^q(p,\ m,\ n,\ r,\ s;\ x,\ \xi)$ is defined by relation (4), then $\frac{Mx+\xi}{Nx+\xi}$ is the best dominant for the SDsub

$$\left(\frac{D_x^{-\alpha}\phi(a(\xi),\ b(\xi);\ q,\ x,\ \xi)}{x}\right)^p \prec\prec \frac{Mx+\xi}{Nx+\xi},\quad (x,\ \xi)\in\Delta\times\overline{\Delta}.$$

Also, considering in Theorem 3.1 as best dominant the particular function $w(x, \xi) = \left(\frac{\xi + x}{\xi - x}\right)^k$, $(x, \xi) \in \Delta \times \overline{\Delta}$, we get the following special case:

**Corollary 3.3** Assuming that relation (3) takes place, if the SDsub

$$H_\alpha^q(p, m, n, r, s; x, \xi) \prec\prec m + n\left(\frac{\xi + x}{\xi - x}\right)^k + r\left(\frac{\xi + x}{\xi - x}\right)^{2k} + s\frac{2kx\xi}{\xi^2 - x^2},$$

is fulfilled for $m, n, r, s \in \mathbb{C}$, $s \neq 0$, $0 < k \leq 1$, and the function $H_\alpha^q(p, m, n, r, s; x, \xi)$ is defined by relation (4), then $\left(\frac{\xi + x}{\xi - x}\right)^k$ is the best dominant for the SDsub

$$\left(\frac{D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi)}{x}\right)^p \prec\prec \left(\frac{\xi + x}{\xi - x}\right)^k, \ (x, \xi) \in \Delta \times \overline{\Delta}.$$

The SDsup result obtained for the operator described by (2) is the next theorem:

**Theorem 3.4** Consider $w \in \mathscr{H}(\Delta \times \overline{\Delta})$ univalent with the properties $w(x, \xi) \neq 0$ and $\frac{xw_x'(x, \xi)}{w(x, \xi)}$ is starlike univalent. Suppose that

$$\operatorname{Re}\left(\frac{2r}{s}(w(x, \xi))^2 + \frac{n}{s}w(x, \xi)\right) > 0, \ \text{for } n, r, s \in \mathbb{C}, \ s \neq 0. \tag{7}$$

If $\left(\frac{D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi)}{x}\right)^p \in \mathscr{H}[w(0, \xi), (\alpha - 1)p, \xi] \cap Q^*$, the function $H_\alpha^q(p, m, n, r, s; x, \xi)$ defined by the relation (4) is univalent in $\Delta \times \overline{\Delta}$, then the SDsup

$$m + nw(x, \xi) + r(w(x, \xi))^2 + s\frac{xw_x'(x, \xi)}{w(x, \xi)} \prec\prec H_\alpha^q(p, m, n, r, s; x, \xi) \tag{8}$$

is endowed for $m, n, r, s \in \mathbb{C}$, $s \neq 0$, then $w$ is the best subordinant for the following SDsup

$$w(x, \xi) \prec\prec \left(\frac{D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi)}{x}\right)^p, \ (x, \xi) \in \Delta \times \overline{\Delta}. \tag{9}$$

**Proof.** Considering again the function $u(x, \xi) = \left(\frac{D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi)}{x}\right)^p$, $(x, \xi) \in (\Delta \backslash \{0\}) \times \overline{\Delta}$, and the analytic functions $f(z) = m + nz + rz^2$ and $g(z) = \frac{s}{z}$, with $g(z) \neq 0$, $z \in \mathbb{C} \backslash \{0\}$, we verify the conditions from Lemma 2.2.

Taking account that $\frac{f_x'(w(x, \xi))}{g(w(x, \xi))} = \frac{w_x'(x, \xi)[n + 2rw(x, \xi)]w(x, \xi)}{s}$, it follows that

$$\operatorname{Re}\left(\frac{f_x'(w(x, \xi))}{g(w(x, \xi))}\right) = \operatorname{Re}\left(\frac{2r}{s}(w(x, \xi))^2 + \frac{n}{s}w(x, \xi)\right) > 0, \text{ for } n, r, s \in \mathbb{C}, \ s \neq 0,$$

by relation (7).

SDsup (8) can be written as

$$m+nw(x,\ \xi)+r(w(x,\ \xi))^2+s\frac{xw'_x(x,\ \xi)}{w(x,\ \xi)}\prec\prec m+nu(x,\ \xi)+r(u(x,\ \xi))^2+s\frac{xu'_x(x,\ \xi)}{u(x,\ \xi)}.$$

The conditions from Lemma 2.2 being fulfilled, we get

$$w(x,\ \xi)\prec\prec u(x,\ \xi)=\left(\frac{D_x^{-\alpha}\phi(a(\xi),\ b(\xi);\ q,\ x,\ \xi)}{x}\right)^p,\quad (x,\ \xi)\in\Delta\times\overline{\Delta},$$

and $w$ is the best subordinant. □

Considering in Theorem 3.4 as best subordinant the particular function $w(x,\ \xi)=\dfrac{Mx+\xi}{Nx+\xi}$, $(x,\ \xi)\in\Delta\times\overline{\Delta}$, we get the following special case:

**Corollary 3.5** Assuming that relation (7) takes place and $\left(\dfrac{D_x^{-\alpha}\phi(a(\xi),\ b(\xi);\ q,\ x,\ \xi)}{x}\right)^p\in\mathscr{H}[w(0,\ \xi),(\alpha-1)p,\ \xi]\cap Q^*$, if the SDsup

$$m+n\frac{Mx+\xi}{Nx+\xi}+r\left(\frac{Mx+\xi}{Nx+\xi}\right)^2+s\frac{(M-N)x\xi}{(Mx+\xi)(Nx+\xi)}\prec\prec H_\alpha^q(p,\ m,\ n,\ r,\ s;\ x,\ \xi),$$

is fulfilled for $m,\ n,\ r,\ s\in\mathbb{C}$, $s\neq 0$, $-1\leq N<M\leq 1$, and the function $H_\alpha^q(p,\ m,\ n,\ r,\ s;\ x,\ \xi)$ is defined by relation (4), then $\dfrac{Mx+\xi}{Nx+\xi}$ is the best subordinant for the SDsup

$$\frac{Mx+\xi}{Nx+\xi}\prec\prec\left(\frac{D_x^{-\alpha}\phi(a(\xi),\ b(\xi);\ q,\ x,\ \xi)}{x}\right)^p,\quad (x,\ \xi)\in\Delta\times\Delta.$$

Also, considering in Theorem 3.4 as best subordinant the particular function $w(x,\ \xi)=\left(\dfrac{\xi+x}{\xi-x}\right)^k$, $(x,\ \xi)\in\Delta\times\overline{\Delta}$, we get the following special case:

**Corollary 3.6** Assuming that relation (7) takes place and $\left(\dfrac{D_x^{-\alpha}\phi(a(\xi),\ b(\xi);\ q,\ x,\ \xi)}{x}\right)^p\in\mathscr{H}[w(0,\ \xi),(\alpha-1)p,\ \xi]\cap Q^*$, if the SDsup

$$m+n\left(\frac{\xi+x}{\xi-x}\right)^k+r\left(\frac{\xi+x}{\xi-x}\right)^{2k}+s\frac{2kx\xi}{\xi^2-x^2}\prec\prec H_\alpha^q(p,\ m,\ n,\ r,\ s;\ x,\ \xi),$$

is fulfilled for $m,\ n,\ r,\ s\in\mathbb{C}$, $s\neq 0$, $0<k\leq 1$, and the function $H_\alpha^q(p,\ m,\ n,\ r,\ s;\ x,\ \xi)$ is defined by relation (4), then $\left(\dfrac{\xi+x}{\xi-x}\right)^k$ is the best subordinant for the SDsup

$$\left(\frac{\xi+x}{\xi-x}\right)^k\prec\prec\left(\frac{D_x^{-\alpha}\phi(a(\xi),\ b(\xi);\ q,\ x,\ \xi)}{x}\right)^p,\quad (x,\ \xi)\in\Delta\times\overline{\Delta}.$$

Looking at Theorems 3.1 and 3.4 together, they generate a sandwich-type result.

**Theorem 3.7** Consider $w_1, w_2 \in \mathscr{H}\left(\Delta \times \overline{\Delta}\right)$ univalent with the properties $w_1(x, \xi) \neq 0$, $w_2(x, \xi) \neq 0$, $\forall (x, \xi) \in \Delta \times \overline{\Delta}$. Assuming the functions $\frac{x(w_1)'_x(x, \xi)}{w_1(x, \xi)}$, $\frac{x(w_2)'_x(x, \xi)}{w_2(x, \xi)}$ are starlike univalent in $\Delta \times \overline{\Delta}$ and $w_1$ satisfies relation (3) and $w_2$ satisfies relation (7) if $\left(\frac{D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi)}{x}\right)^p \in \mathscr{H}[w(0, \xi), (\alpha-1)p, \xi] \cap Q^*$, the function $H_\alpha^q(p, m, n, r, s; x, \xi)$ defined in (3.2) is univalent in $\Delta \times \overline{\Delta}$ and the sandwich-type result

$$m + nw_1(x, \xi) + r(w_1(x, \xi))^2 + s\frac{x(w_1)'_x(x, \xi)}{w_1(x, \xi)} \prec\prec H_\alpha^q(p, m, n, r, s; x, \xi)$$

$$\prec\prec m + nw_2(x, \xi) + r(w_2(x, \xi))^2 + s\frac{x(w_2)'_x(x, \xi)}{w_2(x, \xi)},$$

is endowed for $m, n, r, s \in \mathbb{C}$, $s \neq 0$, then $w_1$ and $w_2$ are respectively the best subordinant and the best dominant for the following sandwich-type result

$$w_1(x, \xi) \prec\prec \left(\frac{D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi)}{x}\right)^p \prec\prec w_2(x, \xi), \quad (x, \xi) \in \Delta \times \overline{\Delta}.$$

For the special case when $w_1(x, \xi) = \frac{M_1x+\xi}{N_1x+\xi}$ and $w_2(x, \xi) = \frac{M_2x+\xi}{N_2x+\xi}$, $(x, \xi) \in \Delta \times \overline{\Delta}$, considering together the Corollaries 3.2 and 3.5, we get the following special case:

**Corollary 3.8** Assuming that relations (3.1) and (3.5) take place and $\left(\frac{D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi)}{x}\right)^p \in \mathscr{H}[w(0, \xi),$ $(\alpha-1)p, \xi] \cap Q^*$, if the sandwich-type result

$$m + n\frac{M_1x+\xi}{N_1x+\xi} + r\left(\frac{M_1x+\xi}{N_1x+\xi}\right)^2 + s\frac{(M_1-N_1)x\xi}{(M_1x+\xi)(N_1x+\xi)}$$

$$\prec\prec H_\alpha^q(p, m, n, r, s; x, \xi) \prec\prec m + n\frac{M_2x+\xi}{N_2x+\xi} + r\left(\frac{M_2x+\xi}{N_2x+\xi}\right)^2 + s\frac{(M_2-N_2)x\xi}{(M_2x+\xi)(N_2x+\xi)},$$

is fulfilled for $m, n, r, s \in \mathbb{C}$, $s \neq 0$, $-1 \leq N_2 < N_1 < M_1 < M_2 \leq 1$, and the function $H_\alpha^q(p, m, n, r, s; x, \xi)$ is defined by relation (4), then $\frac{M_1x+\xi}{N_1x+\xi}$ and $\frac{M_2x+\xi}{N_2x+\xi}$ are respectively the best subordinant and the best dominant for the following sandwich-type result

$$\frac{M_1x+\xi}{N_1x+\xi} \prec\prec \left(\frac{D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi)}{x}\right)^p \prec\prec \frac{M_2x+\xi}{N_2x+\xi}.$$

For the special case when $w_1(x, \xi) = \left(\frac{\xi+x}{\xi-x}\right)^{k_1}$ and $w_2(x, \xi) = \left(\frac{\xi+x}{\xi-x}\right)^{k_2}$, $(x, \xi) \in \Delta \times \overline{\Delta}$, considering together the Corollaries 3.3 and 3.6, we get the following special case:

**Corollary 3.9** Assuming that relations (3) and (7) take place and $\left(\frac{D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi)}{x}\right)^p \in \mathscr{H}[w(0, \xi), (\alpha-1)p, \xi] \cap Q^*$, if the sandwich-type result

$$m+n\left(\frac{\xi+x}{\xi-x}\right)^{k_1}+r\left(\frac{\xi+x}{\xi-x}\right)^{2k_1}+s\frac{2k_1x\xi}{\xi^2-x^2}$$

$$\prec\prec H_\alpha^q(p, m, n, r, s; x, \xi) \prec\prec m+n\left(\frac{\xi+x}{\xi-x}\right)^{k_2}+r\left(\frac{\xi+x}{\xi-x}\right)^{2k_2}+s\frac{2k_2x\xi}{\xi^2-x^2},$$

is fulfilled for $m, n, r, s \in \mathbb{C}$, $s \neq 0$, $-1 \leq N_2 < N_1 < M_1 < M_2 \leq 1$, and the function $H_\alpha^q(p, m, n, r, s; x, \xi)$ is defined by relation (3), then $\left(\frac{\xi+x}{\xi-x}\right)^{k_1}$ and $\left(\frac{\xi+x}{\xi-x}\right)^{k_2}$ are respectively the best subordinant and the best dominant for the following sandwich-type result

$$\left(\frac{\xi+x}{\xi-x}\right)^{k_1} \prec\prec \left(\frac{D_x^{-\alpha}\phi(a(\xi), b(\xi); q, x, \xi)}{x}\right)^p \prec\prec \left(\frac{\xi+x}{\xi-x}\right)^{k_2}.$$

## 4. Conclusion

Stimulated by the inspiring outcomes of studies pertaining to geometric function theory that incorporate aspects of quantum calculus and fractional calculus, the theories of SDsub and its dual, SDsup, embed such aspects in an attempt of this work to revive a study started in [24] but not pursued up to this point. The novel aspects of this research's conclusion consist in the definition of the RL-fr-int of the extended $q$-hypergeometric function, stated in Definition 2.6 and provided in relations (1) and (2), and in the way it is applied to derive new SDsub results and the dual new SDsup results. The scope of the investigation is to provide means for developing the theories of SDsub and SDsup following the approach involving dominants of the SDsub and subordinants of the SDsup, respectively. The focus of such an approach is on providing means of finding the best dominant and the best subordinant, respectively. In every exposed theorem, the best dominants and best subordinants are established, respectively. The purpose of finding the best dominant in the case of SDsub or the best subordinant in the case of SDsup, is to use particular functions so that interesting geometric properties are further derived. The novelty of the results presented here resides in the use of the RL-fr-int for stating the new outcome involving the established ways of finding the best dominant and the best subordinant for the SDsub and SDsup, respectively. When functions distinguished by their geometric properties are substituted as best dominant or best subordinant in the theorems, significant corollaries are derived. The new results of the research concerning the two dual theories of SDsub and SDsup considered in this paper are connected by sandwich-type theorems and corollaries.

The aim of the work is to suggest a new direction for the study of SDsub and its dual, SDsup, that integrate quantum calculus and fractional calculus. By involving the concepts discussed in this article to other hypergeometric functions and operators defined with them, further fascinate operators could be draw.

Taking into account the geometrical features obtained from the conclusions presented in the corollaries, future research may lead to the introduction of new subclasses of functions utilizing the RL-fr-int of the extended $q$-hypergeometric function, as pointed out in [44].

Using $q$-Riemann-Liouville fractional integral instead of the standard RL-fr-int in conjunction with $q$-hypergeometric function may result in an interesting operator. Strong and fuzzy differential subordination and superordination theories, which are more recent extensions, or traditional differential subordination and superordination theories could be used to investigate this operator in more detail. Using the $q$-Riemann-Liouville fractional integral, similar operators have recently been investigated in [45] or [46].

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## Conflict of interest

The authors declare no competing financial interest.

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