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Research Article

Inequalities of Coefficients and the Fekete-Szegö Problem Associated with λ-Pseudo Starlike Functions

Musthafa Ibrahim[1], Baskaran Sudharsanan[2], Alina Alb Lupaş[3*], Saravanan Gunasekar[4], Sibel Yalçin[5]

[1]College of Engineering, University of Buraimi, Al Buraimi, Oman
[2]Department of Mathematics, Agurchand Manmull Jain College, Meenambakkam, Chennai, 600061, Tamil Nadu, India
[3]Department of Mathematics and Computer Science, University of Oradea, 1 Universitatii Street, 410087 Oradea, Romania
[4]Department of Mathematics, PG and Research, Pachaiyappa's College, Chennai-600030, Tamil Nadu, India
[5]Department of Mathematics, Bursa Uludag University, 16059, Bursa, Turkey
E-mail: dalb@uoradea.ro

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Abstract: We introduce a new subclass of starlike functions, denoted by $\mathscr{RS}^*(\lambda,\ \Xi,\ \xi_1,\ \xi_2)$, that are influenced by the Janowski functions, which are well-known in the literature. Our main results are the coefficient estimates of the inverse function and the Fekete-Szegö inequality for this subclass. We also present some special cases of our results that are of interest.

***Keywords*:** analytic functions, inverse functions, fekete-szegö inequality, coefficient inequalities, λ-pseudo starlike functions, subordination

MSC: 30C45, 30C80

1. Introduction

We define $\mathscr{A}$ as the set of analytic functions in $\mathbb{U} = \{z : |z| < 1\}$, which can be represented as a series of the form

$$\mathfrak{f}(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

We also introduce $\mathscr{S}$, which represents all functions in $\mathscr{A}$ that are univalent in $\mathbb{U}$. It is a well-established fact that every function $\mathfrak{f}$ belonging to $\mathscr{S}$ has an inverse function $\mathfrak{f}^{-1}$ of the form

$$\mathfrak{f}^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n \tag{2}$$

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such that

$$\mathfrak{f}^{-1}(\mathfrak{f}(z)) = z \ (z \in \mathbb{U})$$

and

$$\mathfrak{f}\left(\mathfrak{f}^{-1}(w)\right) = w \ \left(w \in \mathbb{U}, \ |w| < r_0(f); \ r_0(f) \geq \frac{1}{4}\right).$$

Indeed, the inverse function $\mathfrak{f}^{-1}$ can be expressed as

$$\mathfrak{f}^{-1}(w) = w - a_2 w^2 + (2a_2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{3}$$

Lewin, in his work [1], introduced the concept of bi-univalent functions. These are analytic functions, denoted by $\mathfrak{f}$, in the unit disc $\mathbb{U}$, where both $\mathfrak{f}$ and its inverse $\mathfrak{f}^{-1}$ are univalent within $\mathbb{U}$. We use $\mathscr{BS}$ to represent the class of these bi-univalent functions.

Examples of functions that belong to the class $\mathscr{BS}$ include

$$\mathfrak{f}_1(z) = \frac{z}{1-z}, \ \mathfrak{f}_2(z) = -\log(1-z) \ \mathfrak{f}_3(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

where $\log(\cdot)$ is single-valued branch defined on $\mathbb{U}$. If the domain is a unit disc, Figures 1 and 2 show the mapping of $\mathfrak{f}_1$ and its inverse $\mathfrak{f}_1^{-1}$. However, the function $\frac{z}{1-z^2}$ is a member of the class $\mathscr{S}$ but does not belong to $\mathscr{BS}$.

In recent times, numerous researchers have introduced and explored various sub-classes of bi-univalent functions. For more details, refer to [2–7].

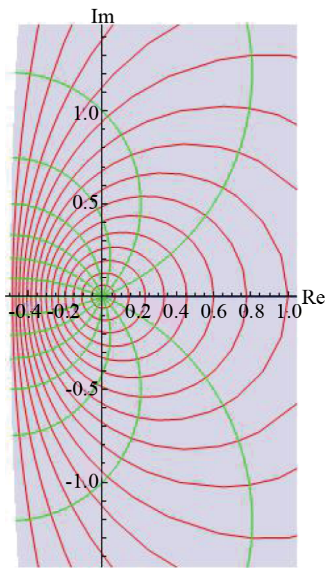

Figure 1. Representation of $\mathfrak{f}_1(z) = \frac{z}{1-z}$

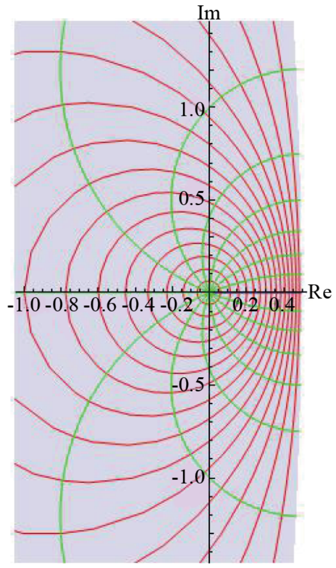

Figure 2. Representation of $\mathfrak{f}_1^{-1}(w) = \dfrac{w}{1+w}$

$\mathscr{P}$ refers to functions with a positive real part that can be represented by a power series of the form.

$$\mathrm{p}(z) = 1 + \sum_{n=1}^{\infty} \mathrm{p}_n z^n.$$

In their work, Ma and Minda [8] examined a function Ξ that belongs to $\mathscr{P}$ and satisfies the following conditions:

1. $\Xi(0) = 1,\ \Xi'(0) > 0$.
2. Ξ transforms the open unit disc $\mathbb{U}$ onto a starlike region with respect to 1 and symmetric with respect to the real axis.

In addition, they made an assumption that $\Xi(z) = 1 + L_1 z + L_2 z^2 + \ldots$, where $L_1 \neq 0$. Then they introduced the following classes

$$\mathscr{S}^*(\Xi) := \left\{ \mathfrak{f} \in \mathscr{A} : \frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)} \prec \Xi(z) \right\}$$

and

$$\mathscr{C}(\Xi) := \left\{ \mathfrak{f} \in \mathscr{A} : 1 + \frac{z\mathfrak{f}''(z)}{\mathfrak{f}'(z)} \prec \Xi(z) \right\}.$$

The so-called Janowski starlike functions and Janowski convex functions (see [9]) are well-known special cases of $\mathscr{S}^*(\Xi)$ and $\mathscr{C}(\Xi)$ that have been deeply investigated by several researchers.

$$\mathscr{S}^*(\xi_1,\ \xi_2) := \left\{ \mathfrak{f} \in \mathscr{A} : \frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)} \prec \frac{1+\xi_1 z}{1+\xi_2 z},\ -1 \leq \xi_2 < \xi_1 \leq 1 \right\},$$

and

$$\mathscr{C}(\xi_1,\ \xi_2):=\left\{\mathfrak{f}\in\mathscr{A}:\ 1+\frac{z\mathfrak{f}''(z)}{\mathfrak{f}'(z)}\prec\frac{1+\xi_1 z}{1+\xi_2 z},\ -1\leq\xi_2<\xi_1\leq 1\right\}.$$

The subordination of the analytic function is shown here by $\prec$; its definition and characteristics can be found in any standard text. To represent $\aleph(\xi_1,\ \xi_2)$ for arbitrary fixed numbers $\xi_1,\ \xi_2,\ -1<\xi_1\leq 1,\ -1\leq\xi_2<\xi_1$, we use the family of functions $\mathrm{p}(z)\in\mathscr{P}$ that satisfy the given condition

$$\ell(z)\prec\frac{(1+\xi_1)\mathrm{p}(z)+1-\xi_1}{(1+\xi_2)\mathrm{p}(z)+1-\xi_2}.$$

This so-called Janowski class [9] was studied by several authors, see [10–12]. Extending the Janowski class of functions [9], Aouf [10, Eq. 1.4] defined the class $\ell(z)\in\mathscr{P}(\xi_1,\ \xi_2,\ \mathrm{p},\ \alpha)$ if and only if

$$\ell(z)=\frac{\mathrm{p}+[\mathrm{p}\xi_2+(\xi_1-\xi_2)(\mathrm{p}-\alpha)]\,w(z)}{[1+\xi_2 w(z)]},\quad(-1\leq\xi_2<\xi_1\leq 1, 0\leq\alpha<1),\tag{4}$$

where $w(z)$ is the a Schwarz function and $\mathrm{p}\in\mathbb{N}=\{1,\ 2,\ \ldots\}$. Recently, Breaz et al. [13, Eq. 4] used the following expression to study a new class of multivalent function

$$\aleph(\mathrm{p};\ \xi_1,\ \xi_2;\ \alpha;\ \Xi;\ z)=\frac{[(1+\xi_1)\mathrm{p}+\alpha(\xi_2-\xi_1)]\,\Xi(z)+[(1-\xi_1)\mathrm{p}-\alpha(\xi_2-\xi_1)]}{[(\xi_2+1)\Xi(z)+(1-\xi_2)]}.\tag{5}$$

$\aleph(\mathrm{p};\ \xi_1,\ \xi_2;\ \alpha;\ \Xi;\ z)$ is an extension of the class $\mathscr{P}(\xi_1,\ \xi_2,\ \mathrm{p},\ \alpha)$.

The conic domain's extremal functions are represented by the function $\hat{p}_{k,\ \varphi}(z)$, which is given by

$$\hat{p}_{k,\ \varphi}(z)=\begin{cases}\dfrac{1+(1-2\varphi)z}{1-z}, & \text{if } k=0\\ 1+\dfrac{2(1-\varphi)}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & \text{if } k=1\\ 1+\dfrac{2(1-\varphi)}{1-k^2}\sinh^2\left[\left(\dfrac{2}{\pi}\arccos k\right)arc\tanh\sqrt{z}\right], & \text{if } 0<k<1\\ 1+\dfrac{2(1-\varphi)}{1-k^2}\sin\left(\dfrac{\pi}{2R(t)}\displaystyle\int_0^{\frac{u(z)}{t}}\dfrac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right)+\dfrac{1}{k^2-1}, & \text{if } k>1\end{cases}\tag{6}$$

where $u(z)=\dfrac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t\in(0,\ 1)$ and t is chosen such that $k=\cosh\left(\dfrac{\pi R'(t)}{4R(t)}\right)$, with $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ is the complementary integral of $R(t)$. Clearly, $\hat{p}_{k,\ \varphi}(z)$ is in $\mathscr{P}$ with the expansion of the form

$$\hat{p}_{k,\ \varphi}(z)=1+\tau_1 z+\tau_2 z^2+\cdots,\ (\tau_j=p_j(k,\ \varphi),\ j=1,\ 2,\ 3,\ \ldots),\tag{7}$$

we get

$$\tau_1=\begin{cases}\dfrac{8(1-\varphi)(\arccos k)^2}{\pi^2(1-k^2)}, & \text{if } 0\le k<1,\\ \dfrac{8(1-\varphi)}{\pi^2}, & \text{if } k=1\\ \dfrac{\pi^2(1-\varphi)}{4\sqrt{t}(k^2-1)R^2(t)(1+t)}, & \text{if } k>1.\end{cases} \tag{8}$$

Noor and Malik in [14] studied a class functions involving $\aleph\left(1;\,\xi_1,\,\xi_2;\,0;\,\hat{p}_{k,\,\varphi};\,z\right)$. The impact of Janowski functions on different subclasses was then investigated by a number of authors; see [13, 15–19].

Lemma 1 [8] If $\mathfrak{p}(z)=1+\sum\limits_{k=1}^{\infty}\mathfrak{p}_k z^k\in\mathscr{P}$, then

$$|\mathfrak{p}_k|\le 2,\ \forall k\ge 1 \text{ and } \left|\mathfrak{p}_2-\frac{\mathfrak{p}_1^2}{2}\right|\le 2-\frac{|\mathfrak{p}_1^2|}{2}.$$

Lemma 2 [8] If $\mathfrak{p}(z)=1+\sum\limits_{k=1}^{\infty}\mathfrak{p}_k z^k\in\mathscr{P}$, and v is complex number, then

$$\left|\mathfrak{p}_2-v\mathfrak{p}_1^2\right|\le 2\max\{1,\ |2v-1|\},$$

and the result is sharp for the functions

$$\mathfrak{p}_1(z)=\frac{1+z}{1-z} \text{ and } \mathfrak{p}_2(z)=\frac{1+z^2}{1-z^2}.$$

Inspired by [20] (also see [21, 22]), we examine a novel class of functions in this work by omitting the further strict condition that $\mathfrak{f}^{-1}$ be one-one.

Definition 1 For $0\le\lambda\le 1$, the class $\mathscr{RS}^*(\lambda,\ \Xi,\ \xi_1,\ \xi_2)$ consists of all analytic functions $\mathfrak{f}\in\mathscr{A}$ of the form (1) satisfying

$$\left[\mathfrak{f}'(z)\right]^{\lambda}\left[\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)}\right]^{1-\lambda}\prec\frac{(\xi_1+1)\Xi(z)-(\xi_1-1)}{(\xi_2+1)\Xi(z)-(\xi_2-1)} \tag{9}$$

where $\Xi(z)=1+L_1z+L_2z^2+\cdots\in\mathscr{P}$.

We will now present some of our class's special cases.

Remark 1

1. Let $\lambda=0$ in Definition ,lthen the class $\mathscr{RS}^*(\lambda,\ \Xi,\ \xi_1,\ \xi_2)$ reduces to class $\mathscr{S}^*(\Xi,\ \xi_1,\ \xi_2)$

$$\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)}\prec\frac{(\xi_1+1)\Xi(z)-(\xi_1-1)}{(\xi_2+1)\Xi(z)-(\xi_2-1)}.$$

2. Let $\lambda=1$ in Definition ,lthen the class $\mathscr{RS}^*(\lambda,\ \Xi,\ \xi_1,\ \xi_2)$ reduces to class $\mathscr{R}(\Xi,\ \xi_1,\ \xi_2)$

$$\mathfrak{f}'(z) \prec \frac{(\xi_1+1)\Xi(z)-(\xi_1-1)}{(\xi_2+1)\Xi(z)-(\xi_2-1)}. \tag{10}$$

3. Let $\lambda = 0$ and $\Xi(z) = p_{k,\ \varphi}(z)$ in Definition ,1then the class $\mathscr{RS}^*(\lambda,\ \Xi,\ \xi_1,\ \xi_2)$ reduces to classes $k-\mathscr{ST}$ [23].

2. Coefficient estimates of $\mathscr{RS}^*(\lambda, \Xi, \xi_1, \xi_2)$

In the first theorem, we will find the coefficient bounds for the function class $\mathscr{RS}^*(\lambda,\ \Xi,\ \xi_1,\ \xi_2)$.

Theorem 1 If the function $\mathfrak{f} \in \mathscr{RS}^*(\lambda,\ \Xi,\ \xi_1,\ \xi_2)$ is given by (1), then

$$|a_2| \leq \frac{|L_1|(\xi_1-\xi_2)}{2(1+\lambda)} \tag{11}$$

and

$$|a_3| \leq \frac{(\xi_1-\xi_2)|L_1|}{2(2+\lambda)} \max\{1,\ |2\delta-1|\}, \tag{12}$$

where

$$\delta = \frac{L_1(\xi_2+1)}{4} + \frac{\left(1-\frac{L_2}{L_1}\right)}{2} - \frac{(\xi_1-\xi_2)L_1(2+\lambda)(1-\lambda)}{8(1+\lambda)^2}.$$

Proof. Since $\mathfrak{f}(z) \in \mathscr{RS}^*(\lambda,\ \Xi,\ \xi_1,\ \xi_2)$, according to subordination relationship, there exists a Schwarz function $c(z)$ with $c(0) = 0$ and $|c(z)| < 1$, satisfying

$$\left[\mathfrak{f}'(z)\right]^{\lambda} \left[\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)}\right]^{1-\lambda} = \frac{(\xi_1+1)\Xi(c(z))-(\xi_1-1)}{(\xi_2+1)\Xi(c(z))-(\xi_2-1)}. \tag{13}$$

The left side of (13) will take the following form:

$$\left[\mathfrak{f}'(z)\right]^{\lambda} \left[\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)}\right]^{1-\lambda} = 1+(1+\lambda)a_2z+\frac{z^2}{2}(2+\lambda)\left[2a_3-(1-\lambda)a_2^2\right]+\cdots. \tag{14}$$

Let $\ell(z) \in \mathscr{P}$ be of the form

$$\ell(z) = 1+\sum_{n=1}^{\infty} \mathrm{p}_n z^n$$

and it is defined by

$$\ell(z)=\frac{1+c(z)}{1-c(z)}, \quad (z\in\mathbb{U}).$$

A simple computations gives

$$c(z)=\frac{\ell(z)-1}{\ell(z)+1}=\frac{1}{2}\mathrm{p}_1 z+\frac{1}{2}\left(\mathrm{p}_2-\frac{1}{2}\mathrm{p}_1^2\right)z^2+\frac{1}{2}\left(\mathrm{p}_3-\mathrm{p}_1\mathrm{p}_2+\frac{1}{4}\mathrm{p}_1^3\right)z^3+\cdots.$$

and considering

$$\begin{aligned}\frac{(\xi_1+1)\Xi(c(z))-(\xi_1-1)}{(\xi_2+1)\Xi(c(z))-(\xi_2-1)}=&1+\frac{L_1\mathrm{p}_1(\xi_1-\xi_2)z}{4}\\&+\frac{(\xi_1-\xi_2)L_1}{4}\left[\mathrm{p}_2-\mathrm{p}_1^2\left(\frac{(\xi_2+1)L_1+2\left(1-\frac{L_2}{L_1}\right)}{4}\right)\right]z^2+\cdots.\end{aligned} \tag{15}$$

Comparing the coefficients of z, z^2 between the equations (14) and (15), we obtain

$$a_2=\frac{L_1\mathrm{p}_1(\xi_1-\xi_2)}{4(1+\lambda)}, \tag{16}$$

$$a_3=\frac{(\xi_1-\xi_2)L_1}{4(2+\lambda)}\left[\mathrm{p}_2-\mathrm{p}_1^2\left(\frac{L_1(\xi_2+1)}{4}+\frac{\left(1-\frac{L_2}{L_1}\right)}{2}-\frac{(\xi_1-\xi_2)L_1(2+\lambda)(1-\lambda)}{8(1+\lambda)^2}\right)\right]. \tag{17}$$

Applying Lemma 1, we easily get

$$|a_2|\leq\frac{|L_1|(\xi_1-\xi_2)}{2(1+\lambda)}, \tag{18}$$

$$\begin{aligned}|a_3|&=\frac{(\xi_1-\xi_2)|L_1|}{4(2+\lambda)}\left|\mathrm{p}_2-\left(\frac{L_1(\xi_2+1)}{4}+\frac{\left(1-\frac{L_2}{L_1}\right)}{2}-\frac{(\xi_1-\xi_2)L_1(2+\lambda)(1-\lambda)}{8(1+\lambda)^2}\right)\mathrm{p}_1^2\right|\\&=\frac{(\xi_1-\xi_2)|L_1|}{4(2+\lambda)}\left|\mathrm{p}_2-\delta\mathrm{p}_1^2\right|\end{aligned} \tag{19}$$

where

$$\delta = \frac{L_1(\xi_2+1)}{4} + \frac{\left(1-\frac{L_2}{L_1}\right)}{2} - \frac{(\xi_1-\xi_2)L_1(2+\lambda)(1-\lambda)}{8(1+\lambda)^2}.$$

Now by applying Lemma 2, we get

$$|a_3| \leq \frac{(\xi_1-\xi_2)|L_1|}{2(2+\lambda)} \max\{1,\ |2\delta-1|\}. \tag{20}$$

□

Corollary 1 If the function $\mathfrak{f} \in \mathscr{S}^*(\Xi,\ \xi_1,\ \xi_2)$ is given by (1), then

$$|a_2| \leq \frac{|L_1|(\xi_1-\xi_2)}{2} \tag{21}$$

and

$$|a_3| \leq \frac{(\xi_1-\xi_2)|L_1|}{4} \max\{1,\ |2\delta_s-1|\}, \tag{22}$$

where

$$\delta_s = \frac{L_1(\xi_2+1)}{4} + \frac{\left(1-\frac{L_2}{L_1}\right)}{2} - \frac{(\xi_1-\xi_2)L_1}{4}.$$

The result is sharp for the function

$$\hat{\mathfrak{f}}_1(z) = ze^{\int_0^z \left(\frac{(\xi_1+1)\Xi(t)-\xi_1+1}{(\xi_2+1)\Xi(t)-\xi_2+1}-1\right)t^{-1}dt}.$$

Figure 6 shows the image of $\mathbb{U}$ under $\hat{\mathfrak{f}}_1(z)$ with suitable choice of parameters. In Figure 3, we choose $\xi_1 = 1,\ \xi_2 = -1,\ \Xi(z) = 1+z$. In Figure 4, we choose $\xi_1 = 1,\ \xi_2 = -1,\ \Xi(z) = 1+\frac{4}{5}z+\frac{1}{5}z^4$. In Figure 5, we choose $\xi_1 = 1,\ \xi_2 = -1$, $\Xi(z) = 1+\frac{5}{6}z+\frac{1}{6}z^5$.

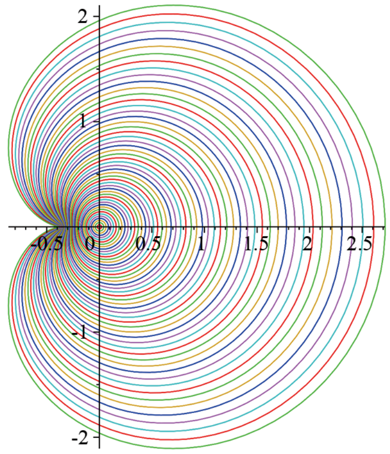

Figure 3. ze^z

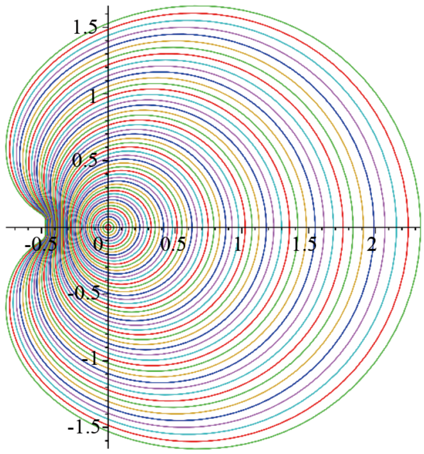

Figure 4. $ze^{\frac{4}{5}z+\frac{1}{20}z^4}$

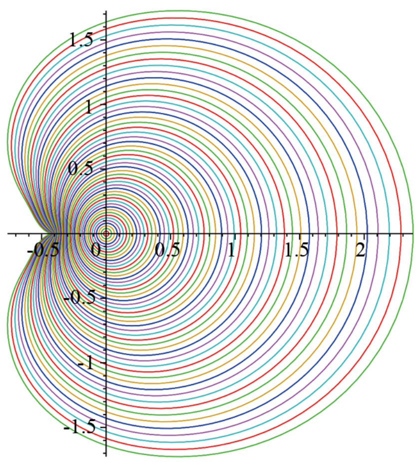

Figure 5. $ze^{\frac{5}{6}z+\frac{1}{30}z^5}$

The images of $\mathbb{U}$ under $\hat{\mathfrak{f}}_1(z)$ with suitable choice of parameters.

Corollary 2 If the function $\mathfrak{f} \in \mathscr{R}(\Xi, \xi_1, \xi_2)$ is given by (1), then

$$|a_2| \leq \frac{|L_1|(\xi_1 - \xi_2)}{4} \tag{23}$$

and

$$|a_3| \leq \frac{(\xi_1 - \xi_2)|L_1|}{6} \max\{1, |2\delta_r - 1|\}, \tag{24}$$

where

$$\delta_r = \frac{L_1(\xi_2 + 1)}{4} + \frac{\left(1 - \frac{L_2}{L_1}\right)}{2}.$$

The result is sharp for the function

$$\hat{\mathfrak{f}}_2(z) = \int_0^z \frac{(\xi_1 + 1)\Xi(t) - \xi_1 + 1}{(\xi_2 + 1)\Xi(t) - \xi_2 + 1} dt.$$

Figure 10 shows the images of $\mathbb{U}$ under $\hat{\mathfrak{f}}_2(z)$ with suitable choice of parameters. In Figure 7, we choose $\xi_1 = 1, \xi_2 = -1$, $\Xi(z) = 1 + z$. In Figure 8, we choose $\xi_1 = 1, \xi_2 = -1$, $\Xi(z) = 1 + \frac{4}{5}z + \frac{1}{5}z^4$. In Figure 9, we choose $\xi_1 = 1, \xi_2 = -1$, $\Xi(z) = 1 + \frac{5}{6}z + \frac{1}{6}z^5$.

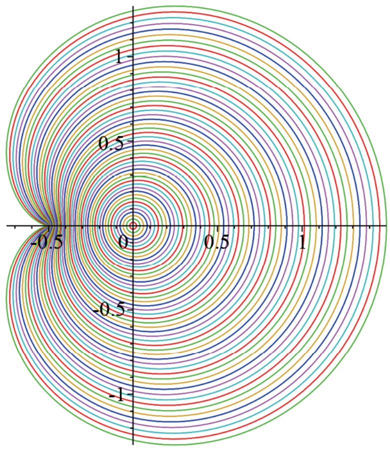

Figure 6. $z + \frac{1}{2}z^2$

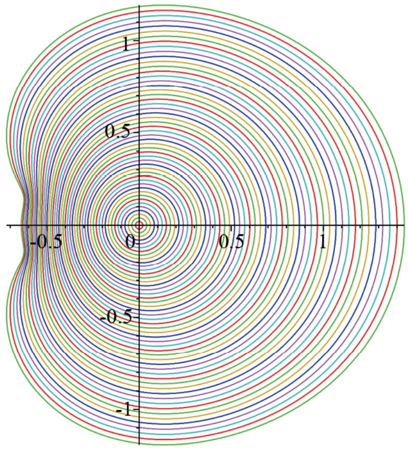

Figure 7. $z+\frac{2}{5}z^2+\frac{1}{25}z^5$

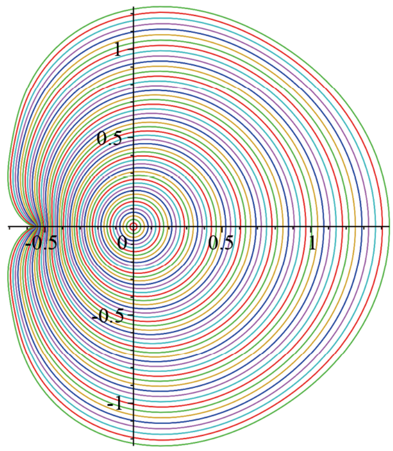

Figure 8. $z+\frac{5}{12}z^2+\frac{1}{36}z^6$

The images of $\mathbb{U}$ under $\hat{\mathfrak{f}}_2(z)$ with suitable choice of parameters.

3. Coefficient inequalities for the function f^{-1}

Theorem 2 If the function $\mathfrak{f} \in \mathscr{R}\mathscr{S}^*(\lambda,\ \Xi,\ \xi_1,\ \xi_2)$ is given by (1), then for the coefficients of $\mathfrak{f}^{-1}(w)$ the following estimates hold:

$$|d_2| \leq \frac{|L_1|(\xi_1-\xi_2)}{2(1+\lambda)} \tag{25}$$

and

$$|d_3| \leq \frac{(\xi_1 - \xi_2)|L_1|}{2(2+\lambda)} \max\{1, \ |2\delta_1 - 1|\}, \tag{26}$$

where

$$\delta_1 = \frac{L_1(\xi_2+1)}{4} + \frac{\left(1 - \frac{L_2}{L_1}\right)}{2} + \frac{(\xi_1 - \xi_2)L_1(2+\lambda)(3+\lambda)}{8(1+\lambda)^2}.$$

Proof. From equations (2) and (3), we get the following relations

$$d_2 = -a_2,$$

$$d_3 = 2a_2^2 - a_3. \tag{27}$$

From relations (16), (17), and (27), we have

$$d_2 = -\frac{L_1 \mathrm{p}_1(\xi_1 - \xi_2)}{4(1+\lambda)}, \tag{28}$$

$$d_3 = \frac{L_1^2 p_1^2 (\xi_1 - \xi_2)^2}{8(1+\lambda)^2} - \frac{(\xi_1 - \xi_2)L_1}{4(2+\lambda)} \left[p_2 - p_1^2 \left(\frac{L_1(\xi_2+1)}{4} + \frac{\left(1 - \frac{L_2}{L_1}\right)}{2} - \frac{(\xi_1 - \xi_2)L_1(2+\lambda)(1-\lambda)}{8(1+\lambda)^2} \right) \right]. \tag{29}$$

On simple computations,

$$d_3 = -\frac{(\xi_1 - \xi_2)L_1}{4(2+\lambda)} \left[\mathrm{p}_2 - \mathrm{p}_1^2 \left(\frac{L_1(\xi_2+1)}{4} + \frac{\left(1 - \frac{L_2}{L_1}\right)}{2} + \frac{(\xi_1 - \xi_2)L_1(2+\lambda)(3+\lambda)}{8(1+\lambda)^2} \right) \right]. \tag{30}$$

Taking modulus on both sides and applying Lemma 2 on the right hand side of (28) and (30), one can obtain the result as in (25) and (26). □

Corollary 3 If the function $\mathfrak{f} \in \mathscr{S}^*(\Xi, \ \xi_1, \ \xi_2)$ and $\mathfrak{f}^{-1}$ is given by (2), then

$$|d_2| \leq \frac{|L_1|(\xi_1 - \xi_2)}{2}$$

and

$$|d_3| \leq \frac{(\xi_1 - \xi_2)|L_1|}{4} \max\{1, \ |2\delta_{1s} - 1|\},$$

where

$$\delta_{1s} = \frac{L_1(\xi_2+1)}{4} + \frac{\left(1-\frac{L_2}{L_1}\right)}{2} + \frac{3(\xi_1-\xi_2)L_1}{4}.$$

Corollary 4 If the function $\mathfrak{f} \in \mathscr{R}(\Xi,\ \xi_1,\ \xi_2)$ and $\mathfrak{f}^{-1}$ is given by (2), then

$$|d_2| \leq \frac{|L_1|(\xi_1-\xi_2)}{4}$$

and

$$|d_3| \leq \frac{(\xi_1-\xi_2)|L_1|}{6} \max\{1,\ |2\delta_{1r}-1|\},$$

where

$$\delta_{1r} = \frac{L_1(\xi_2+1)}{4} + \frac{\left(1-\frac{L_2}{L_1}\right)}{2} + \frac{3L_1(\xi_1-\xi_2)}{8}.$$

An intriguing generalization of a class of starlike functions is the so-called class of starlike functions associated with the vertical domain, described as follows.

Definition 2 [24] $\mathfrak{f} \in \mathscr{A}$ is said to be in $\mathscr{S}(\eta_1,\ \eta_2)$ if it satisfies

$$\eta_1 < \mathrm{Re}\left(\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)}\right) < \eta_2,\ z \in \mathbb{U}. \tag{31}$$

where $0 \leq \eta_1 < 1 < \eta_2$.

Let $\lambda = 0$, $\xi_1 = 1$, $\xi_2 = -1$ and

$$\Xi(z) = 1 + \frac{\eta_2-\eta_1}{\pi} i \log\left(\frac{1-e^{2\pi i((1-\eta_1)/(\eta_2-\eta_1))}z}{1-z}\right) \tag{32}$$

in Theorem 3.

Corollary 5 [25] Let $\mathfrak{f} \in \mathscr{S}^*(\eta_1, \eta_2)$, then the coefficient estimates of the inverse function are

$$|d_2| \leq \frac{2(\eta_2-\eta_1)}{\pi} \sin\left(\frac{\pi(1-\eta_1)}{\eta_2-\eta_1}\right)$$

and

$$|d_3| \leq \frac{2(\eta_2-\eta_1)}{\pi}\sin\left(\frac{\pi(1-\eta_1)}{\eta_2-\eta_1}\right)\max\left\{1,\ \left|\frac{1}{2}-3\frac{\eta_2-\eta_1}{\pi}i+\left(\frac{1}{2}+3\frac{\eta_2-\eta_1}{\pi}i\right)e^{2\pi i\frac{1-\eta_1}{\eta_2-\eta_1}}\right|\right\}.$$

4. Fekete-Szegö inequality for the function of $\mathscr{RS}^*(\lambda, \Xi, \xi_1, \xi_2)$

The Fekete-Szegö problem solution will be provided for the functions that belong to the classes defined in the first section.

Theorem 3 Let $\mathfrak{f} \in \mathscr{RS}^*(\lambda,\ \Xi,\ \xi_1,\ \xi_2)$ given by (1). Then for all $\mu \in \mathbb{C}$ we have

$$|\ a_3-\mu a_2^2\ |\leq \frac{(\xi_1-\xi_2)\,|L_1|}{2(2+\lambda)}\max\{1,\ |2\tau-1|\} \tag{33}$$

with

$$\tau=\frac{1}{4}\left(L_1(\xi_2+1)+2\left(1-\frac{L_2}{L_1}\right)-\rho_1-\mu\rho_2\right) \tag{34}$$

where

$$\rho_1=\frac{(\xi_1-\xi_2)L_1(2+\lambda)(1-\lambda)}{2(1+\lambda)^2}$$

and

$$\rho_2=\frac{(2+\lambda)L_1(\xi_1-\xi_2)}{2(1+\lambda)^2}.$$

Proof. If $\mathfrak{f} \in \mathscr{RS}^*(\lambda,\ \Xi,\ \xi_1,\ \xi_2)$, in the view of relation (16) and (17), for $\mu \in \mathbb{C}$ we have

$$a_3-\mu a_2^2=\frac{(\xi_1-\xi_2)L_1}{4(2+\lambda)}\left[\mathrm{p}_2-\mathrm{p}_1^2\left(\frac{L_1(\xi_2+1)}{4}+\frac{\left(1-\frac{L_2}{L_1}\right)}{2}-\frac{(\xi_1-\xi_2)L_1(2+\lambda)(1-\lambda)}{8(1+\lambda)^2}\right)\right]-\mu\frac{L_1^2\mathrm{p}_1^2(\xi_1-\xi_2)^2}{16(1+\lambda)^2} \tag{35}$$

$$=\frac{(\xi_1-\xi_2)L_1}{4(2+\lambda)}\left[\mathrm{p}_2-\frac{1}{4}\mathrm{p}_1^2\left(L_1(\xi_2+1)+2\left(1-\frac{L_2}{L_1}\right)-\rho_1-\mu\rho_2\right)\right]. \tag{36}$$

Taking absolute value on both sides, we get

$$|\ a_3-\mu a_2^2\ |=\frac{(\xi_1-\xi_2)\,|L_1|}{4(2+\lambda)}\left|\mathrm{p}_2-\tau\mathrm{p}_1^2\right|. \tag{37}$$

Using Lemma 2, we get

$$| a_3 - \mu a_2^2 | \leq \frac{(\xi_1 - \xi_2) |L_1|}{2(2+\lambda)} \max\{1, \ |2\tau - 1|\}.$$

□

Corollary 6 Let $\mathfrak{f} \in \mathscr{S}^*(\Xi, \ \xi_1, \ \xi_2)$ be given by (1). Then for all $\mu \in \mathbb{C}$ we have

$$| a_3 - \mu a_2^2 | \leq \frac{(\xi_1 - \xi_2) |L_1|}{4} \max\{1, \ |2\tau_s - 1|\}$$

with

$$\tau_s = \frac{1}{4}\left(L_1(\xi_2+1) + 2\left(1 - \frac{L_2}{L_1}\right) - (\xi_1 - \xi_2) L_1 (1+\mu)\right).$$

Corollary 7 Let $\mathfrak{f} \in \mathscr{R}(\Xi, \ \xi_1, \ \xi_2)$ be given by (1). Then for all $\mu \in \mathbb{C}$ we have

$$| a_3 - \mu a_2^2 | \leq \frac{(\xi_1 - \xi_2) |L_1|}{6} \max\{1, \ |2\tau_r - 1|\}$$

with

$$\tau_r = \frac{1}{4}\left(L_1(\xi_2+1) + 2\left(1 - \frac{L_2}{L_1}\right) - \frac{3}{8}\mu L_1 (\xi_1 - \xi_2)\right).$$

Letting $\lambda = 0$, $\xi_1 = 1$, $\xi_2 = -1$ and $\Xi(z)$ be of the form (32) in Theorem 3, then we have the following result obtained by Sim and Kwon [25].

Corollary 8 [25] Let $\mathfrak{f} \in \mathscr{S}^*(\eta_1, \ \eta_2)$. Then, for any μ,

$$\left|a_3 - \mu a_2^2\right| \leq \frac{\eta_2 - \eta_1}{\pi} \sin\left(\frac{\pi(1-\eta_1)}{\eta_2 - \eta_1}\right)$$

$$\max\left\{1; \ \left|\frac{1}{2} + (1-2\mu)\frac{\eta_2 - \eta_1}{\pi} i + \left(\frac{1}{2} - (1-2\mu)\frac{\eta_2 - \eta_1}{\pi} i\right) e^{2\pi i \frac{1-\eta_1}{\eta_2 - \eta_1}}\right|\right\}.$$

Letting $\lambda = 0$, $\xi_1 = 1$, and $\xi_2 = -1$ in Theorem 3, then we have the following result obtained by Tu and Xiong [26].

Corollary 9 [26] Suppose $\mathfrak{f}(z) \in \mathscr{S}^*(\Xi)$ $(z \in \mathbb{U})$. Then

$$\left|a_3 - \mu a_2^2\right| \leq \frac{L_1}{2} \max\left\{1; \left|L_1 + \frac{L_2}{L_1} - 2\mu L_1\right|\right\} \quad (\mu \in \mathbb{C}).$$

The inequality is sharp for the function given by

$$\mathfrak{f}(z)=\begin{cases} z\exp\int_0^z\left[\Xi(t)-1\right]\frac{1}{t}dt, & \text{if } \left|L_1+\frac{L_2}{L_1}-2\mu L_1\right|\geq 1\\ z\exp\int_0^z\left[\Xi(t^2)-1\right]\frac{1}{t}dt, & \text{if } \left|L_1+\frac{L_2}{L_1}-2\mu L_1\right|\leq 1. \end{cases}$$

5. Conclusion

In this paper, we have introduced a new subclass of starlike functions, denoted by $\mathscr{R}\mathscr{S}^*(\lambda,\ \Xi,\ \xi_1,\ \xi_2)$, that are influenced by the Janowski functions. We have obtained the coefficient estimates of the inverse function and the Fekete-Szegö result for this subclass. We have also presented some special cases of our results that generalize some existing results in the literature. Some possible directions for future work are

To investigate other properties of the class $\mathscr{R}\mathscr{S}^*(\lambda,\ \Xi,\ \xi_1,\ \xi_2)$, such as growth, distortion, rotation, and radius problems.

To consider other subclasses of starlike functions that are defined by using different superordinate functions or different differential operators.

To study the applications of the class $\mathscr{R}\mathscr{S}^*(\lambda,\ \Xi,\ \xi_1,\ \xi_2)$ to other branches of mathematics, such as differential equations, harmonic analysis, and complex dynamics.

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Conflict of interest

The authors declare no competing financial interest.

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