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Research Article

Application of Natural Transform Decomposition Method for Solution of Fractional Richards Equation

K. Raghavendar[1*], K. Pavani[1], K. Aruna[1], N. I. Okposo[2], M. Inc[3]

[1] Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology Vellore-632014, India
[2] Department of Mathematics, Delta State University, Abraka, Delta State, Nigeria
[3] Department of Mathematics, Firat University, 23119 Elazig, Turkiye
E-mail: raghavendar248@gmail.com

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Abstract: This study employs the natural transform decomposition method (NTDM) to examine analytical solutions of the nonlinear time-fractional Richards equation (TFRE). The NTDM is an innovative and attractive hybrid integral transform strategy that elegantly combines the Adomian decomposition method and the natural transform method. This solution strategy effectively generates rapidly convergent series-type solutions through an iterative process involving fewer calculations. The convergence and uniqueness of the solutions are presented. To demonstrate the efficiency of the considered solution method, two test cases of the TFRE are investigated within the framework of the Caputo-Fabrizio and Atangana-Baleanu-Caputo derivatives whose definitions incorporate nonsingular kernel functions. Numerical comparisons between the obtained approximate solutions, exact solutions, and those from existing related literature are presented to show the validity and accuracy of the technique. Graphical representations demonstrating the effect of varying non-integer, temporal, and spatial parameters on the behavior of the obtained model solutions are also presented. The results indicate that the execution of the method is straightforward and can be employed to explore complex physical systems governed by time-fractional nonlinear partial differential equations.

***Keywords*:** fractional calculus, time-fractional richards equation, caputo fabrizio derivative, atangana-baleanu-caputo derivative, water transport in unsaturated porous media

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1. Introduction

The study of subsoil flow is crucial in understanding Earth's critical zone where the intricate interactions between air, water, rocks, soil, and living organisms influence the condition of the natural habitat as well as the availability of life-preserving resources [1]. A typical example of subsoil flow in nature is infiltration. This phenomenon is often used in soil science and hydrology to describe the process by which water penetrates unsaturated porous media such as the soil under the driving forces of capillarity and gravity [1, 2]. Important factors affecting infiltration rate through unsaturated soil include soil surface conditions, hydraulic conductivity, vegetation cover and soil organic materials, soil slope geometry, soil porosity, and boundary and initial soil moisture conditions [3]. Apart from soil science and hydrology, infiltration

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also plays crucial roles in managing human activities involving waste management, irrigation, agricultural production, and geotechnical and geo-environmental engineering [1, 2, 4].

The standard mathematical model that describes the spatiotemporal dynamics of water infiltration through unsaturated porous media such as soil is the well-known Richards' equation whose formulation involves coupling the equation of continuity with the Darcy-Buckingham law [5]. Over the years, Richards equation has been the subject of extensive research investigations by applied and computational mathematicians, hydrologists, environmental and agricultural engineers as well as software developers [1]. In one dimension, Richards equation is expressed by the following quasilinear partial differential equation (PDE) [6, 7]

$$\frac{\partial \mathscr{G}}{\partial \zeta} = \frac{\partial}{\partial z}\left(\eta(\mathscr{G})\frac{\partial \mathscr{G}}{\partial z} - \beta(\mathscr{G})\right), \tag{1}$$

where ζ is the temporal variable denoting time (T), z is spatial variable denoting soil depth taken positive downward (L), $\mathscr{G} = \mathscr{G}(z,\ \zeta)$ represents the unsaturated soil moisture content, $\beta(\mathscr{G})$ denotes the hydraulic conductivity function (L/T) and $\eta(\mathscr{G})$ denotes soil-water diffusivity function (L^2/T) which can be viewed as nonlinear diffusion coefficient defined by

$$\eta(\mathscr{G}) = \beta(\mathscr{G})\frac{\partial \psi(\mathscr{G})}{\partial \mathscr{G}},$$

with ψ being the pressure head (L). Obtaining exact solutions to Richards equation is difficult due to its associated degeneracy and high nonlinearity attributed to the dependency of the model parameters on the dependent variable as well as the complexity in establishing the underlying soil-water physical relationships involving pressure-head, hydraulic conductivity and water-content [8]. To solve (1), the task of adequately estimating $\beta(\mathscr{G})$ and $\eta(\mathscr{G})$ must first be addressed as they are both dependent on the water content. Because porous media exhibit different material properties, various soil water retention functions such as the Campbell function [9], Fredlund function [10], Gardner function [11], Gardner-Russo function [12], Van Genuchten function [13] and Brooks-Corey function [14, 15] have been used to estimate $\beta(\mathscr{G})$ and $\eta(\mathscr{G})$. Among these, the Brooks-Corey function for which

$$\eta(\mathscr{G}) = \eta_0(q+1)\mathscr{G}^q \text{ and } \beta(\mathscr{G}) = \beta_0\mathscr{G}^l,\ l \geq 1,\ q \geq 0, \tag{2}$$

is commonly used [6, 14]. Here, η_0 and β_0 are soil parameters such as particle size, pore-size distribution, etc. Based on the Brooks-Corey's representation of β and η, various numerical and analytical solutions have been obtained for Richards equation. By setting $q = 0$ and $l = 2$ in (2), the Brooks-Corey's representation of (1) yields the classical Burgers equation [16] whereas for the special case $l = 1$ in (2), the Brooks-Corey's representation of (1) yields the Richards equation of order $(q,\ 1)$ [17] given as

$$\frac{\partial \mathscr{G}}{\partial \zeta} + b\frac{\partial \mathscr{G}^q}{\partial z} + c\frac{\partial^2 \mathscr{G}}{\partial z^2} = 0,\ b,\ c \neq 0,\ q \geq 1, \tag{3}$$

whose analytical solution is

$$\mathscr{G}(z,\ \zeta)=\left(\frac{d}{2b}\Big(1+\tanh\Big(\frac{d(q-1)}{2c}(z-d\zeta)\Big)\Big)\right)^{\frac{1}{q-1}}. \tag{4}$$

In recent years, mathematical tools such as fractional differential and integral operators have been widely used to reformulate and study many new and existing PDEs. These operators are general concepts in fractional calculus whose theory is a well-known generalization of the classical (or integer-order) integral and differential operators to those of fractional (or arbitrary or non-integer) orders. Unlike their integer-order counterparts, fractional differential operators admit many advantages. For instance, the high degree of freedom in their order of differentiation and non-locality property makes them more powerful and versatile tools for modeling real-world problems. Hence, many observed realities involving random walk, fractal dynamics as well as anomalous diffusion are adequately captured by derivatives of fractional orders. By the non-locality property, fractional differential operators are capable of manifesting hereditary and memory effects in the sense that subsequent states of a given dynamical system depend not only on its present state but also on its previous states [18]. Incorporating these non-local operators into PDEs generates a new class of differential equations known as fractional partial differential equations (FPDEs). Because of its versatility, fractional calculus has gained enormous importance in recent years because of its applications in diverse fields of science and engineering [19–21]. Particularly, FPDEs have wide applications in modeling problems arising in plasma physics, nano-scale flow and heat transfer, electrochemistry, biophysics, medical sciences, polymer physics, chaos, electric networks, fluid flow through porous media [22–27].

The process of determining the exact solutions of FPDEs can pose significant difficulties. Hence, many authors have relied on a variety of semi-analytical methods to obtain series-type solutions to this class of problems. Some of these solution methods include the Adomian decomposition method (ADM) [28], integral transform method (ITM) [29], variational transform method (VIM) [30], Laplace-Adomian decomposition method (LADM) [31], homotopy perturbation method (HPM) [32], q-homotopy analysis method (q-HAM) [33], q-homotopy analysis transform method (q-HATM) [34], natural transform decomposition method (NDTM) [35, 36], reduced differential transform method (RDTM) [37], Local Fractional Sumudu Decomposition Method [38], Sumudu variational iteration method (SVIM) [39] and local fractional natural decomposition method [40]. Among the above-mentioned techniques, the NTDM [41] whose methodology systematically combines the well-known NTM and ADM is considered as a very powerful, efficient, and simple semi-analytical tool for solving many types of non-linear systems of FPDEs [42, 43]. This solution technique is known to effectively address various types of nonlinearities and iteratively handle fractional derivatives. Unlike many numerical methods, it does not incorporate round-off errors since it does not require any form of linearization, perturbation, or discretization. Its methodology reduces enormous computational time and yields either exact or approximate solutions in the form of rapidly convergent series. Furthermore, compared to the HPM which involves a time-consuming process of solving functional equations in each iteration, and the VIM with its inherent difficulty in finding the Lagrange multiplier, the NDTM is simple. Hence, it is a fast and excellent solution technique for a wide range of linear and non-linear FPDEs.

In this work, the NTDM is used to investigate series-type solutions for the following one-dimensional $(q,\ 1)$-order time fractional Richards equation:

$$\begin{cases}\dfrac{\partial^{\kappa}\mathscr{G}}{\partial\zeta^{\kappa}}+b\dfrac{\partial\mathscr{G}^{q}}{\partial z}+c\dfrac{\partial^{2}\mathscr{G}}{\partial z^{2}}=0,\ b,\ c\neq 0,\ q\geq 1,\\[2ex] \mathscr{G}(z,\ 0)=\mathscr{G}_{0}(z),\end{cases} \tag{5}$$

where κ $(0<\kappa\leq 1)$ is the order of the fractional derivative. When $\kappa=1$, many existing research investigations have employed various iterative and semi-analytical solution methods for different versions of the classical Richards equation (see, for instance [6, 44–48]). Other approaches such as Fourier transformation and separation of variables have also been applied to some linear or linearized versions with finite boundaries [49, 50]. Here, the problem (5) is considered with respect to the Caputo-Fabrizio-Caputo (CFC) and Atangana-Baleanu-Caputo (ABC) derivatives. Unlike the Caputo derivative, the CFC and ABC fractional differential operators are non-singular and incorporate the exponential and Mittag-Leffler kernel functions in their respective definitions. Although, the problem (5) has been considered in [24] and [25] within the framework of the Caputo fractional derivative, the novelty of this work lies in the utilization of the NTDM on the TFRE (5) with respect to the CFC and ABC fractional derivatives, which to the best of our knowledge, is the first in literature. We summarize the content of the rest part of the paper as follows: Section 2 provides an overview of several fundamental definitions relevant to this work. In Section 3, the solution procedure of the NTDM applied to the TFRE is discussed with respect to the CFC and ABC fractional derivatives. In Section 4 the NTDM is implemented to obtain series-type solutions for two different cases of the TFRE. Section 5 provides the results and discussion of the obtained solution. Finally, we draw a conclusion of this work in Section 6 based on the obtained findings.

2. Fundamental definitions

This section will provide some fundamental definitions of fractional derivatives and natural transforms [22, 23].

Definition 1 [51] The CFC fractional derivative of order κ is given by

$$ {}^{\mathrm{CFC}}D_{\zeta}^{\kappa}\mathscr{G}(\zeta)=\frac{1}{1-\kappa}\int_{0}^{\zeta}\mathscr{G}'(\tau)\exp\Big(\frac{-\kappa(\zeta-\tau)}{1-\kappa}\Big)d\tau,\ \zeta\geq 0. \tag{6}$$

Definition 2 [52] Let $V\in H^{1}(0,\ T),\ T>0,\ \kappa\in[0,\ 1]$. The ABC fractional derivative of order κ is given as follows

$$ {}^{\mathrm{ABC}}D_{\zeta}^{\kappa}\mathscr{G}(\zeta)=\frac{\mathbb{M}(\kappa)}{1-\kappa}\int_{0}^{\zeta}\mathscr{G}'(\tau)E_{\kappa}\Big(\frac{-\kappa(\zeta-\tau)^{\kappa}}{1-\kappa}\Big)d\tau. \tag{7}$$

Where $\mathbb{M}(\kappa)$ is the normalization function satisfying $\mathbb{M}(0)=\mathbb{M}(1)=1$ and E_{κ} is the Mittag-Leffler function.

Definition 3 The NT of $\mathscr{G}(\zeta)$ is given as

$$ N^{+}[\mathscr{G}(\zeta)]=R(s,\ u)=\frac{1}{u}\int_{0}^{\infty}e^{(\frac{-s\zeta}{u})}\mathscr{G}(\zeta)d\zeta,\ u,\ s>0. \tag{8}$$

Definition 4 The NT of the CFC fractional derivative is defined as

$$ N^{+}[{}_{0}^{\mathrm{CFC}}D_{\zeta}^{\kappa}\mathscr{G}(\zeta)]=\frac{1}{M(v,\ \kappa,\ s)}\big(N^{+}[\mathscr{G}(\zeta)]-\frac{1}{s}\mathscr{G}(0)\big), \tag{9}$$

where $M(v,\ \kappa,\ s)=1-\kappa+\kappa\left(\frac{v}{s}\right)$.

Definition 5 The NT of the ABC fractional derivative is defined as

$$ N^{+}[{}_{0}^{\mathrm{ABC}}D_{\zeta}^{\kappa}\mathscr{G}(\zeta)]=\frac{\mathbb{M}(\kappa)}{P(v,\ \kappa,\ s)}\big(N^{+}[\mathscr{G}(\zeta)]-\frac{1}{s}\mathscr{G}(0)\big), \tag{10}$$

here $P(v,\ \kappa,\ s)=1-\kappa+\kappa\left(\frac{v}{s}\right)^{\kappa}$.

3. Methodology

In this section, the NTDM is employed to demonstrate its applicability on the time-fractional Richard's equations (5) by considering the cases with the CFC and ABC fractional derivatives.

3.1 *NTDM solution for TFRE with CFC fractional derivative*

Considering the TFRE (5) with fractional derivative in the CFC sense, an application of the NT yields

$$\frac{1}{M(v,\ \kappa,\ s)}\left[N^{+}(\mathscr{G}(z,\ \zeta))-\frac{\mathscr{G}(z,\ 0)}{s}\right]=-N^{+}[b\,\mathscr{G}_z^q-c\,\mathscr{G}_{zz}]. \tag{11}$$

Taking the inverse NT on (11) gives

$$\mathscr{G}(z,\ \zeta)=N^{-1}\left[\frac{\mathscr{G}(z,\ 0)}{s}-M(v,\ \kappa,\ s)N^{+}\left[b\,\mathscr{G}_z^q-c\,\mathscr{G}_{zz}\right]\right]. \tag{12}$$

Next, we represent the solution of (12) by the infinite series

$$\mathscr{G}(z,\ \zeta)=\sum_{m=0}^{\infty}\mathscr{G}_m(z,\ \zeta), \tag{13}$$

while the nonlinear term is expressed by the decomposition series

$$\mathscr{G}_z^q=\sum_{m=0}^{\infty}A_m, \tag{14}$$

with A_m being the Adomian polynomials. Substituting (13) and (14) into (12) gives

$$\sum_{m=0}^{\infty}\mathscr{G}_m(z,\ \zeta)=N^{-1}\left[\frac{\mathscr{G}(z,\ 0)}{s}\right]-N^{-1}\left[M(v,\ \kappa,\ s)N^{+}\left[b\sum_{m=0}^{\infty}A_m-c\sum_{m=0}^{\infty}(\mathscr{G}_m)_{zz}\right]\right]. \tag{15}$$

From (15) we have the following iterations

$$^{CFC}\mathscr{G}_0(z,\ \zeta) = N^{-1}\left[\frac{\mathscr{G}(z,\ 0)}{s}\right],$$

$$^{CFC}\mathscr{G}_1(z,\ \zeta) = -N^{-1}\left[M(v,\ \kappa,\ s)N^{+}\left[b\,A_0 - c\,\mathscr{G}_{0zz}\right]\right],$$

$$^{CFC}\mathscr{G}_2(z,\ \zeta) = -N^{-1}\left[M(v,\ \kappa,\ s)N^{+}\left[b\,A_1 - c\,\mathscr{G}_{1zz}\right]\right],$$

$$\vdots$$

$$^{CFC}\mathscr{G}_{m+1}(z,\ \zeta) = -N^{-1}\left[M(v,\ \kappa,\ s)N^{+}\left[b\,A_m - c\,(\mathscr{G}_m)_{zz}\right]\right],\ m \geq 3. \tag{16}$$

In view (13), the NTDM yields

$$^{CFC}\mathscr{G}(z,\ \zeta) = {}^{CFC}\mathscr{G}_0(z,\ \zeta) + {}^{CFC}\mathscr{G}_1(z,\ \zeta) + {}^{CFC}\mathscr{G}_2(z,\ \zeta) + \dots, \tag{17}$$

as the series solution for the TFRE (5) with fractional derivative in the CFC sense.

Theorem 1 [23] When $0 < (\sigma_1 + \sigma_2)(1 - \kappa + \kappa\zeta) < 1$, then $NTDM_{\mathrm{CFC}}$ solution of the TFRE (5) exists uniquely.

Theorem 2 The $NTDM_{\mathrm{CFC}}$ solution of the TFRE (5) is convergent.

3.2 *NTDM solution for TFRE with ABC fractional derivative*

Considering the TFRE (5) with fractional derivative in the ABC sense, an application of the NT yields

$$\frac{\mathbb{M}(\kappa)}{P(v,\ \kappa,\ s)}\left[N^{+}(\mathscr{G}(z,\ \zeta)) - \frac{\mathscr{G}(z,\ 0)}{s}\right] = -N^{+}[b\,\mathscr{G}_z^q - c\,\mathscr{G}_{zz}]. \tag{18}$$

Employing the inverse NT on (18) gives

$$\mathscr{G}(z,\ \zeta) = N^{-1}\left[\frac{\mathscr{G}(z,\ 0)}{s} - \frac{P(v,\ \kappa,\ s)}{\mathbb{M}(\kappa)}N^{+}\left[b\,\mathscr{G}_z^q - c\,\mathscr{G}_{zz}\right]\right]. \tag{19}$$

Next, we substitute (13) and (14) into (18) to get

$$\sum_{m=0}^{\infty}\mathscr{G}_m(z,\ \zeta) = N^{-1}\left[\frac{\mathscr{G}(z,\ 0)}{s}\right] - N^{-1}\left[\frac{P(v,\ \kappa,\ s)}{\mathbb{M}(\kappa)}N^{+}\left[b\sum_{m=0}^{\infty}A_m - c\sum_{m=0}^{\infty}(\mathscr{G}_m)_{zz}\right]\right]. \tag{20}$$

From (20), we have the following iterations

$$ {}^{\mathrm{ABC}}\mathscr{G}_0(z,\ \zeta) = N^{-1}\left[\frac{\mathscr{G}(z,\ 0)}{s}\right], $$

$$ {}^{\mathrm{ABC}}\mathscr{G}_1(z,\ \zeta) = -N^{-1}\left[\frac{P(v,\ \kappa,\ s)}{\mathbb{M}(\kappa)}N^{+}\left[b\,A_0 - c\,\mathscr{G}_{0zz}\right]\right], $$

$$ {}^{\mathrm{ABC}}\mathscr{G}_2(z,\ \zeta) = -N^{-1}\left[\frac{P(v,\ \kappa,\ s)}{\mathbb{M}(\kappa)}N^{+}\left[b\,A_1 - c\,\mathscr{G}_{1zz}\right]\right], $$

$$ \vdots $$

$$ {}^{\mathrm{ABC}}\mathscr{G}_{m+1}(z,\ \zeta) = -N^{-1}\left[\frac{P(v,\ \kappa,\ s)}{\mathbb{M}(\kappa)}N^{+}\left[b\,A_m - c\,(\mathscr{G}_m)_{zz}\right]\right],\ m \geq 3. \tag{21} $$

In view (13), the NTDM yields

$$ {}^{\mathrm{ABC}}\mathscr{G}(z,\ \zeta) = {}^{\mathrm{ABC}}\mathscr{G}_0(z,\ \zeta) + {}^{\mathrm{ABC}}\mathscr{G}_1(z,\ \zeta) + {}^{\mathrm{ABC}}\mathscr{G}_2(z,\ \zeta) + \dots, \tag{22} $$

as the series solution for the TFRE (5) with fractional derivative in the ABC sense.

Theorem 3 [23] When $0 < (\sigma_1 + \sigma_2)(1 - \kappa + \kappa\frac{\zeta^\kappa}{\Gamma(\kappa+1)}) < 1$, the $NTDM_{\mathrm{ABC}}$ solution of the TFRE (5) exists uniquely.

Theorem 4 [23] The $NTDM_{\mathrm{ABC}}$ solution of the TFRE (5) is convergent.

4. Implementation of the NTDM on the TFRE

Here, we obtain series-type solutions for two cases of the TFRE within the context of the CFC and ABC fractional derivatives.

4.1 *Case 1*

Assume in (1) that $\beta = \frac{\mathscr{G}^3}{3}cmh^{-1}$ and $\eta = 1cm^2h^{-1}$. By setting $q = 3$, $b = d = \frac{1}{q}$ and $c = -1$ in (3) and (4), we consider the following TFRE with associated initial data:

$$ \begin{cases} D_\zeta^\kappa \mathscr{G} + \mathscr{G}^2\mathscr{G}_z - \mathscr{G}_{zz} = 0,\ 0 < \kappa \leq 1, \\ \mathscr{G}(z,\ 0) = \sqrt{\frac{1}{2}\Big(1 + \tanh\Big(-\frac{z}{3}\Big)\Big)}. \end{cases} \tag{23} $$

The exact solution of (23) is

$$\mathscr{G}(z,\ \zeta)=\sqrt{\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{z}{3}-\frac{\zeta}{9}\right)}. \tag{24}$$

$NTDM_{\mathrm{CFC}}$: By considering (23) with fractional derivative in the CF sense, the NTDM solution steps leading to (16) yields the following solution components

$$^{\mathrm{CFC}}\mathscr{G}_0(z,\ \zeta)=\sqrt{\frac{1}{2}\Big(1+\tanh\Big(-\frac{z}{3}\Big)\Big)},$$

$$^{\mathrm{CFC}}\mathscr{G}_1(z,\ \zeta)=\frac{2\sqrt{2}\left(2\tanh\left(\frac{z}{3}\right)-2\right)\left(\tanh^2\left(\frac{z}{3}\right)-1\right)\left(\frac{1}{144}-\frac{\kappa}{144}+\frac{\kappa\ \zeta}{144}\right)}{\left(1-\tanh\left(\frac{x}{3}\right)\right)^{3/2}},$$

$$^{\mathrm{CFC}}\mathscr{G}_2(z,\ \zeta)=\frac{\sqrt{2}\sqrt{1-\tanh\left(\frac{z}{3}\right)}\left(2\tanh\left(\frac{z}{3}\right)+3\tanh^2\left(\frac{z}{3}\right)-1\right)\left(2-4\kappa+2\kappa^2+4\kappa\zeta-4\kappa^2\zeta+\kappa^2\zeta^2\right)}{1,296},$$

and so on. Hence, the approximate series solution $^{\mathrm{CFC}}\mathscr{G}(z,\ \zeta)$ for TFRE (23) in the sense of the CFC fractional derivative is obtained according to (17).

$\mathrm{NTDM_{ABC}}$: By considering (23) with fractional derivative in the ABC sense, the NTDM solution steps leading to (21) yields the following solution components

$$^{\mathrm{ABC}}\mathscr{G}_0(z,\ \zeta)=\sqrt{\frac{1}{2}\Big(1+\tanh\Big(-\frac{z}{3}\Big)\Big)},$$

$$^{\mathrm{ABC}}\mathscr{G}_1(z,\ \zeta)=\frac{\sqrt{2}\sqrt{1-\tanh\left(\frac{z}{3}\right)}\left(\tanh\left(\frac{z}{3}\right)+1\right)\left(1-\kappa+\kappa\zeta^{\kappa}\right)}{36\Gamma(1+\kappa)},$$

$$^{\mathrm{ABC}}\mathscr{G}_2(z,\ \zeta)=-\frac{\left(2\mathrm{e}^{-\frac{2z}{3}}-1\right)}{1,296\Gamma(1+\kappa)\cosh\left(\frac{z}{3}\right)^4\Gamma(1+2\kappa)\left(\frac{1}{\mathrm{e}^{\frac{2z}{3}}+1}\right)^{3/2}}\Big(\Gamma(1+2\kappa)\,\Gamma(1+\kappa)-2\kappa\Gamma(1+\kappa),$$

$$\Gamma(1+2\kappa)+2\kappa\zeta^{\kappa}\Gamma(1+2\kappa)+\kappa^2\Gamma(1+\kappa)\,\Gamma(1+2\kappa)-2\kappa^2\zeta^{\kappa}\Gamma(1+2\kappa)+\kappa^2\zeta^{2\kappa}\Gamma(1+\kappa)\Big),$$

and so on. Hence, the approximate series solution $^{\mathrm{ABC}}\mathscr{G}(z,\ \zeta)$ for TFRE (23) in the sense of the ABC fractional derivative is obtained according to (22).

4.2 *Case 2*

Assume in (1) that $\beta = \frac{\mathscr{G}^4}{4} cmh^{-1}$ and $\eta = 1cm^2h^{-1}$. By setting $q = 4$, $b = d = \frac{1}{4}$ and $c = -1$ in (3) and (4), we consider the following TFRE with associated initial data:

$$\begin{cases} D_{\zeta}^{\kappa}\mathscr{G} + \mathscr{G}^3\mathscr{G}_z - \mathscr{G}_{zz} = 0,\ 0 < \kappa \le 1, \\ \mathscr{G}(z,\ 0) = \sqrt[3]{\frac{1}{2}\left(1 + \tanh\left(\frac{-3z}{8}\right)\right)}. \end{cases} \tag{25}$$

The exact solution of (25) is

$$\mathscr{G}(z,\ \zeta) = \left(\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{3z}{8} - \frac{3\zeta}{32}\right)\right)^{\frac{1}{3}}. \tag{26}$$

NTDM$_{\text{CFC}}$: By considering (25) with fractional derivative in the CFC sense, the NTDM solution steps leading to (16) yields the following iterates:

$$^{\text{CFC}}\mathscr{G}_0(z,\ \zeta) = \sqrt[3]{\frac{1}{2}\left(1 + \tanh\left(\frac{-3z}{8}\right)\right)},$$

$$^{\text{CFC}}\mathscr{G}_1(z,\ \zeta) = \frac{\sqrt[3]{\frac{1}{2} - \frac{\tanh\left(\frac{3z}{8}\right)}{2}}\left(\tanh\left(\frac{3z}{8}\right) + 1\right)\left(2\tanh\left(\frac{3z}{8}\right) + 4\sqrt[3]{\frac{1}{2} - \frac{\tanh\left(\frac{3z}{8}\right)}{2}}^{3} - 1\right)(1 - \kappa + \kappa\zeta)}{32},$$

$$\text{CFC}_{G_2}(z,\ \zeta) = \frac{\sqrt[3]{\frac{1}{2} - \frac{\tanh\left(\frac{3z}{8}\right)}{2}\left(\tanh\left(\frac{3z}{8} + 1\right)\right)}}{1{,}024}\left(\sqrt[3]{\frac{1}{2} - \frac{\tanh\left(\frac{3z}{8}\right)}{2}}^{3}\left(\tanh\left(\frac{3z}{8}\right)\left(4 + 110\tanh\left(\frac{3z}{8}\right) - 34\right)\right)\right.$$

$$\left. + \sqrt[3]{\frac{1}{2} - \frac{\tanh\left(\frac{3z}{8}\right)}{2}}^{6}\left(32 + 80\tanh\left(\frac{3z}{8}\right)\right) + \tanh\left(\frac{3z}{8}\right)\left(-21 - 21\tanh\left(\frac{3z}{8}\right) + 35\tanh^2\left(\frac{3z}{8}\right)\right) + 8\right)$$

$$\left(2\left((1-\kappa)^2 - 2\kappa\zeta(1+\kappa)\right) + \kappa^2\zeta^2\right).$$

and so on. Finally, the approximate series solution ${}^{CFC}\mathscr{G}(z,\ \zeta)$ for TFRE (25) concerning the CFC fractional derivative is obtained according to (17).

NTDM$_{\text{ABC}}$: By considering (25) with fractional derivative in the ABC sense, the NTDM solution steps leading to (21) yields the following solution components

$$ {}^{ABC}\mathscr{G}_0(z,\ \zeta) = \sqrt[3]{\frac{1}{2}\left(1+\tanh\left(\frac{-3z}{8}\right)\right)}, $$

$$ {}^{ABC}\mathscr{G}_1(z,\ \zeta) = \frac{\sqrt[3]{\frac{1}{2}-\frac{\tanh\left(\frac{3z}{8}\right)}{2}}\left(\tanh(\frac{3z}{8})+1\right)\left(2\tanh(\frac{3z}{8})+4\sqrt[3]{\frac{1}{2}-\frac{\tanh\left(\frac{3z}{8}\right)}{2}}^{3}-1\right)(1-\kappa+\kappa\zeta^{\kappa})}{32\Gamma(1+\kappa)}, $$

$$ {}^{ABC}\mathscr{G}_2(z,\ \zeta) = \frac{\sqrt[3]{\frac{1}{2}-\frac{\tanh\left(\frac{3z}{8}\right)}{2}}\left(\tanh(\frac{3z}{8})+1\right)}{512\Gamma(1+\kappa)\Gamma(1+2\kappa)}\left((1-\kappa)^2+2\kappa\zeta^{\kappa}(1-\kappa)\Gamma(1+2\kappa)+\kappa^2\zeta^{2\kappa}\Gamma(1+\kappa)\right) $$

$$ \left(\sqrt[3]{\frac{1}{2}-\frac{\tanh\left(\frac{3z}{8}\right)}{2}}^{3}\left(\tanh\left(\frac{3z}{8}\right)\left(4+110\tanh\left(\frac{3z}{8}\right)\right)-34\right)+\sqrt[3]{\frac{1}{2}-\frac{\tanh\left(\frac{3z}{8}\right)}{2}}^{6}\left(32+80\tanh\left(\frac{3z}{8}\right)\right)\right. $$

$$ \left.+\tanh\left(\frac{3z}{8}\right)\left(-21-21\tanh\left(\frac{3z}{8}\right)+35\tanh^2\left(\frac{3z}{8}\right)\right)+8\right), $$

and so on. Finally, the approximate series solution ${}^{ABC}\mathscr{G}(z,\ \zeta)$ for TFRE (25) concerning the ABC fractional derivative is obtained according to (22).

5. Results and discussion

This section presents a discussion on the approximate solutions and graphical representations obtained via the NTDM for the TFRE. The numerical results in Table 1 and Table 3 compare absolute errors obtained via the NTDM, q-HATM [24] and RDTM [25] for Case 1 and Case 2, respectively, of the considered TFRE. These error analysis demonstrate that the results obtained through the presented show good convergence. As indicated in Tables 2 and Table 4, the numerical investigations were conducted for distinct values of the time, space, and fractional order variables for Case 1 and Case 2, respectively. For Case 1 of the TFRE, Table 2 displays comparisons between the obtained numerical and exact solutions for the CFC and ABC versions of the TFRE (23) at different values of the fractional parameter κ. The absolute errors with respect to each of the considered fractional derivatives in Case 1 are also presented in Table 2 when $\kappa = 1$. Similarly, Table 4 displays comparisons between the obtained numerical solutions and exact solutions for the CFC and ABC versions of the TFRE (25) in Case 2 at different values of the fractional parameter κ. The absolute errors with respect to each of the considered fractional derivatives in Case 2 are also presented in Table 4 when $\kappa = 1$.

Plots in Figure 1 and Figure 3 show the solution profiles in 2D for the TFRE in Case 1 and Case 2, respectively, with respect to ζ $(0 \le \zeta \le 1)$, distinct values of the fractional parameter index κ and fixed depth z of the porous media.

Apart from the similarities in solution behaviors demonstrated by these 2D plots for the CFC and ABC versions of the TFRE, these plots also depict the convergence of the obtained approximate solutions to the exact solutions as $\kappa \longrightarrow 1$ for each case. Furthermore, it can be observed that as the fractional order increases (that is, as $\kappa \uparrow 1$), the unsaturated soil moisture content $\mathscr{G}(z,\ \zeta)$ also increases and approaches its exact solution at $\kappa = 1$. The physical implication is that the amount of moisture content in the porous media continues to increase until it approaches saturation as $\kappa \uparrow 1$). Figure 2 and Figure 4 consist of 3D plots demonstrating the surface profiles obtained via the considered solution method for Case 1 and Case 2, respectively, of the TFRE taking z and ζ such that $-10 \leq z \leq 10$ and $0 \leq \zeta \leq 1$ and with various fractional order values. The methodology presented in this paper demonstrates that as the fractional order approaches the classical scenario, the obtained solutions converge to the exact solution in each case. Also, the obtained numerical solutions and graphical representations demonstrate significant agreement between the fractional derivatives considered in this work. These findings further confirm the efficacy and accuracy of the NDTM. Hence, the physical representations of our results have the potential to provide valuable information for further research investigation into nonlinear phenomena describing subsurface flow in unsaturated porous media.

Table 1. Absolute error for Case 1 with $\kappa = 1$

z	ζ	${}^{CFC}\mathscr{G}$	${}^{ABC}\mathscr{G}$	q-HATM [24]	RDTM [25]
	0.25	2.3378E-12	2.3378E-12	1.0118E-10	9.6481E-11
2.5	0.5	1.5292E-10	1.5292E-10	3.1597E-09	2.1261E-10
	0.75	1.7797E-09	1.7797E-09	2.3376E-08	1.6551E-09
	1	1.0213E-08	1.0213E-08	9.5791E-08	1.0104E-08
	0.25	7.2849E-13	7.2849E-13	5.4285E-11	1.5356E-10
5	0.5	4.6944E-11	4.6944E-11	1.7608E-09	2.1380E-10
	0.75	5.3836E-10	5.3836E-10	1.3553E-08	6.8580E-10
	1	3.0451E-09	3.0451E-09	5.7887E-08	3.3583E-09
	0.25	6.3971E-14	6.3971E-14	2.5716E-12	3.1736E-10
7.5	0.5	4.1643E-12	4.1643E-12	8.0173E-11	1.5434E-10
	0.75	4.8249E-11	4.8249E-11	5.9219E-10	2.3957E-10
	1	2.7577E-10	2.7577E-10	2.4230E-09	1.2878E-10
	0.25	1.2385E-14	1.2385E-14	4.1617E-12	1.6858E-09
10	0.5	7.9049E-13	7.9049E-13	1.3383E-10	1.3367E-09
	0.75	8.9779E-12	8.9779E-12	1.0199E-09	1.7278E-09
	1	5.0288E-11	5.0288E-11	4.3117E-09	9.0444E-10

Table 2. Approximate solutions and absolute error for (23) in the CFC and ABC sense at different values of z, ζ and κ

z	ζ	${}^{CFC}\mathscr{G}$ at $\kappa = 0.50$	${}^{CFC}\mathscr{G}$ at $\kappa = 0.75$	${}^{CFC}\mathscr{G}$ at $\kappa = 1$	$\mathscr{G}_{Exact}$	Abs. Err. at $\kappa = 1$
	0.25	4.2255E-01	4.1522E-01	4.0796E-01	4.0796E-01	2.3378E-12
2.5	0.5	4.2744E-01	4.2248E-01	4.1747E-01	4.1747E-01	1.5292E-10
	0.75	4.3236E-01	4.2979E-01	4.2711E-01	4.2711E-01	1.7797E-09
	1	4.3729E-01	4.3716E-01	4.3687E-01	4.3687E-01	1.0213E-08
	0.25	1.9880E-01	1.9465E-01	1.9063E-01	1.9063E-01	7.2849E-13
5	0.5	2.0159E-01	1.9868E-01	1.9580E-01	1.9580E-01	4.6944E-11
	0.75	2.0441E-01	2.0278E-01	2.0110E-01	2.0110E-01	5.3836E-10
	1	2.0726E-01	2.0696E-01	2.0652E-01	2.0652E-01	3.0451E-09
	0.25	8.7851E-02	8.5937E-02	8.4098E-02	8.4098E-02	6.3971E-14
7.5	0.5	8.9139E-02	8.7782E-02	8.6449E-02	8.6449E-02	4.1643E-12
	0.75	9.0444E-02	8.9666E-02	8.8866E-02	8.8866E-02	4.8249E-11
	1	9.1768E-02	9.1589E-02	9.1348E-02	9.1348E-02	2.7577E-10
	0.25	3.8303E-02	3.7461E-02	3.6654E-02	3.6654E-02	1.2385E-14
10	0.5	3.8869E-02	3.8272E-02	3.7685E-02	3.7685E-02	7.9049E-13
	0.75	3.9444E-02	3.9099E-02	3.8745E-02	3.8745E-02	8.9779E-12
	1	4.0027E-02	3.9944E-02	3.9835E-02	3.9835E-02	5.0288E-11
		${}^{ABC}\mathscr{G}$ at $\kappa = 0.50$	${}^{ABC}\mathscr{G}$ at $\kappa = 0.75$	${}^{ABC}\mathscr{G}$ at $\kappa = 1$	$\mathscr{G}_{Exact}$	Abs. Err. at $\kappa = 1$
	0.25	4.2872E-01	4.1914E-01	4.0796E-01	4.0796E-01	2.3378E-12
2.5	0.5	4.3333E-01	4.2681E-01	4.1747E-01	4.1747E-01	1.5292E-10
	0.75	4.3687E-01	4.3359E-01	4.2711E-01	4.2711E-01	1.7797E-09
	1	4.3987E-01	4.3986E-01	4.3687E-01	4.3687E-01	1.0213E-08
	0.25	2.0235E-01	1.9684E-01	1.9063E-01	1.9063E-01	7.2849E-13
5	0.5	2.0504E-01	2.0115E-01	1.9580E-01	1.9580E-01	4.6944E-11
	0.75	2.0712E-01	2.0501E-01	2.0110E-01	2.0110E-01	5.3836E-10
	1	2.0890E-01	2.0863E-01	2.0652E-01	2.0652E-01	3.0451E-09
	0.25	8.9498E-02	8.6939E-02	8.4098E-02	8.4098E-02	6.3971E-14
7.5	0.5	9.0749E-02	8.8920E-02	8.6449E-02	8.6449E-02	4.1643E-12
	0.75	9.1726E-02	9.0703E-02	8.8866E-02	8.8866E-02	4.8249E-11
	1	9.2562E-02	9.2378E-02	9.1348E-02	9.1348E-02	2.7577E-10
	0.25	3.9028E-02	3.7901E-02	3.6654E-02	3.6654E-02	1.2385E-14
10	0.5	3.9580E-02	3.8772E-02	3.7685E-02	3.7685E-02	7.9049E-13
	0.75	4.0011E-02	3.9556E-02	3.8745E-02	3.8745E-02	8.9779E-12
	1	4.0380E-02	4.0293E-02	3.9835E-02	3.9835E-02	5.0288E-11

Table 3. Absolute error for Case 2 with $\kappa = 1$

z	ζ	${}^{\mathrm{CFC}}\mathscr{G}$	${}^{\mathrm{ABC}}\mathscr{G}$	q-HATM [24]	RDTM [25]
	0.25	4.4617E-13	4.4617E-13	1.8695E-11	2.4900E-10
2.5	0.5	2.9041E-11	2.9041E-11	5.8346E-10	2.0000E-11
	0.75	3.3638E-10	3.3638E-10	4.3148E-09	3.5400E-09
	1	1.9215E-09	1.9215E-09	1.7678E-08	1.7110E-08
	0.25	1.0888E-13	1.0888E-13	1.1607E-11	2.0600E-10
5	0.5	7.0062E-12	7.0062E-12	3.7495E-10	1.7100E-10
	0.75	8.0239E-11	8.0239E-11	2.8743E-09	2.4400E-09
	1	4.5327E-10	4.5327E-10	1.2227E-08	1.1750E-08
	0.25	1.1248E-14	1.1248E-14	2.0994E-13	5.5440E-09
7.5	0.5	7.2728E-13	7.2728E-13	7.5436E-12	5.3050E-09
	0.75	8.3700E-12	8.3700E-12	6.0801E-11	6.4280E-09
	1	4.7517E-11	4.7517E-11	2.6947E-10	5.4700E-09
	0.25	4.1209E-16	4.1209E-16	2.1387E-10	1.0859E-08
10	0.5	2.5893E-14	2.5893E-14	3.3635E-09	1.4878E-08
	0.75	2.8934E-13	2.8934E-13	1.6901E-08	1.4459E-08
	1	1.5936E-12	1.5936E-12	5.3149E-08	1.0306E-08

Table 4. Approximate solutions and absolute error for (25) in the CFC and ABC sense at different values of z, ζ and κ

z	ζ	${}^{\mathrm{CFC}}\mathscr{G}$ at $\kappa = 0.50$	${}^{\mathrm{CFC}}\mathscr{G}$ at $\kappa = 0.75$	${}^{\mathrm{CFC}}\mathscr{G}$ at $\kappa = 1$	$\mathscr{G}_{\mathtt{Exact}}$	$\mathtt{Abs.Err.}$ at $\kappa = 1$
	0.25	5.2792E-01	5.2262E-01	5.1734E-01	5.1734E-01	4.4617E-13
2.5	0.5	5.3146E-01	5.2790E-01	5.2433E-01	5.2433E-01	2.9041E-11
	0.75	5.3500E-01	5.3321E-01	5.3136E-01	5.3136E-01	3.3638E-10
	1	5.3855E-01	5.3853E-01	5.3844E-01	5.3844E-01	1.9215E-09
	0.25	2.9552E-01	2.9206E-01	2.8866E-01	2.8866E-01	1.0888E-13
5	0.5	2.9784E-01	2.9546E-01	2.9310E-01	2.9310E-01	7.0062E-12
	0.75	3.0017E-01	2.9890E-01	2.9759E-01	2.9759E-01	8.0239E-11
	1	3.0252E-01	3.0237E-01	3.0215E-01	3.0215E-01	4.5327E-10
	0.25	1.5936E-01	1.5745E-01	1.5557E-01	1.5557E-01	1.1248E-14
7.5	0.5	1.6065E-01	1.5933E-01	1.5801E-01	1.5801E-01	7.2728E-13
	0.75	1.6194E-01	1.6122E-01	1.6049E-01	1.6049E-01	8.3700E-12
	1	1.6325E-01	1.6315E-01	1.6301E-01	1.6301E-01	4.7517E-11
	0.25	8.5401E-02	8.4370E-02	8.3362E-02	8.3362E-02	4.1209E-16
10	0.5	8.6092E-02	8.5379E-02	8.4674E-02	8.4674E-02	2.5893E-14
	0.75	8.6788E-02	8.6401E-02	8.6006E-02	8.6006E-02	2.8934E-13
	1	8.7490E-02	8.7435E-02	8.7360E-02	8.7360E-02	1.5936E-12
		${}^{\mathrm{ABC}}\mathscr{G}$ at $\kappa = 0.50$	${}^{\mathrm{ABC}}\mathscr{G}$ at $\kappa = 0.75$	${}^{\mathrm{ABC}}\mathscr{G}$ at $\kappa = 1$	$\mathscr{G}_{\mathtt{Exact}}$	$\mathtt{Abs.Err.}$ at $\kappa = 1$
	0.25	5.3237E-01	5.3699E-01	5.1734E-01	5.1734E-01	4.4617E-13
2.5	0.5	5.3569E-01	5.3803E-01	5.2433E-01	5.2433E-01	2.9041E-11
	0.75	5.3823E-01	5.3873E-01	5.3136E-01	5.3136E-01	3.3638E-10
	1	5.4038E-01	5.3928E-01	5.3844E-01	5.3844E-01	1.9215E-09
	0.25	2.9846E-01	3.0158E-01	2.8866E-01	2.8866E-01	1.0888E-13
5	0.5	3.0066E-01	3.0229E-01	2.9310E-01	2.9310E-01	7.0062E-12
	0.75	3.0237E-01	3.0277E-01	2.9759E-01	2.9759E-01	8.0239E-11
	1	3.0382E-01	3.0314E-01	3.0215E-01	3.0215E-01	4.5327E-10
	0.25	1.6099E-01	1.6273E-01	1.5557E-01	1.5557E-01	1.1248E-14
7.5	0.5	1.6222E-01	1.6313E-01	1.5801E-01	1.5801E-01	7.2728E-13
	0.75	1.6317E-01	1.6340E-01	1.6049E-01	1.6049E-01	8.3700E-12
	1	1.6398E-01	1.6360E-01	1.6301E-01	1.6301E-01	4.7517E-11
	0.25	8.6278E-02	8.7216E-02	8.3362E-02	8.3362E-02	4.1209E-16
10	0.5	8.6938E-02	8.7429E-02	8.4674E-02	8.4674E-02	2.5893E-14
	0.75	8.7450E-02	8.7573E-02	8.6006E-02	8.6006E-02	2.8934E-13
	1	8.7886E-02	8.7684E-02	8.7360E-02	8.7360E-02	1.5936E-12

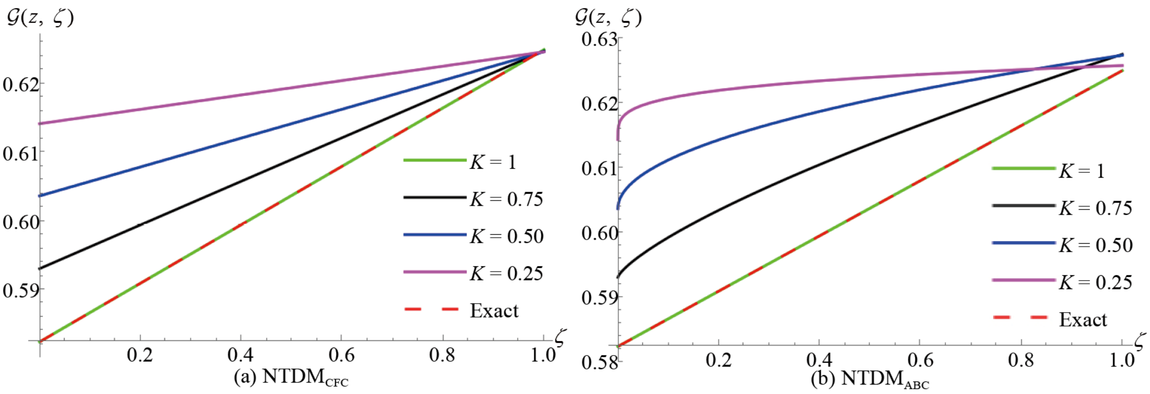
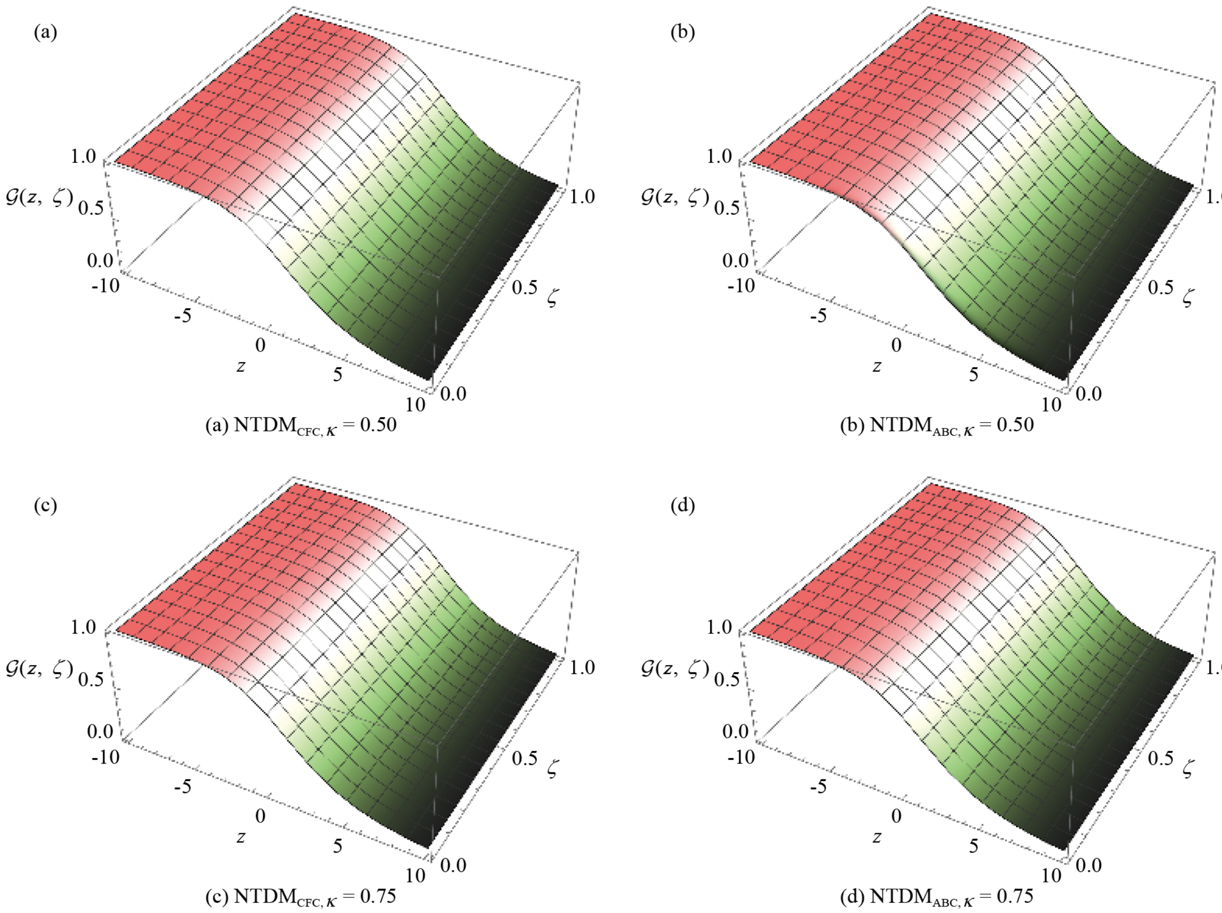


Figure 1. 2D solution profiles for Case 1 at $z = 1$ and different values of κ



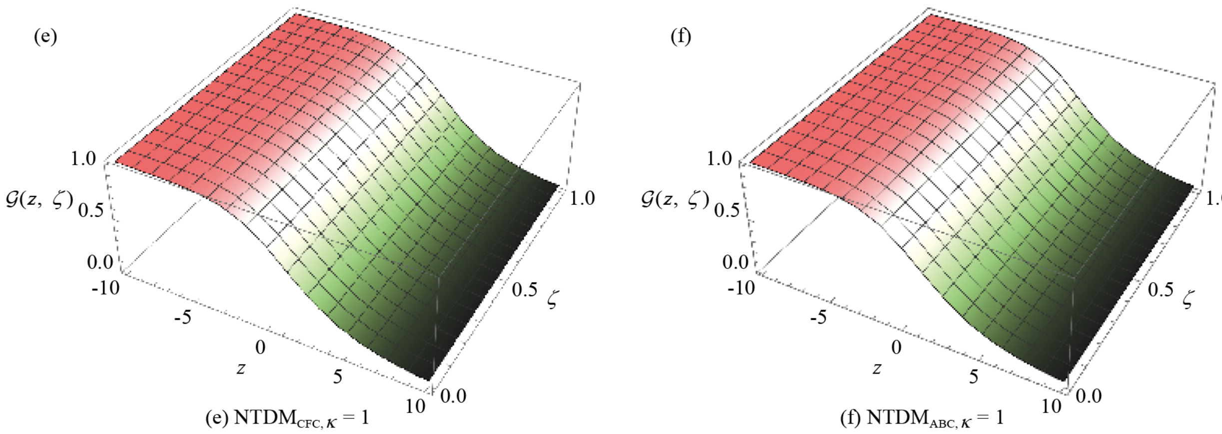


Figure 2. Surface plot of approximate solutions for $\mathscr{G}(z, \zeta)$ in Case 1 at distinct values of κ

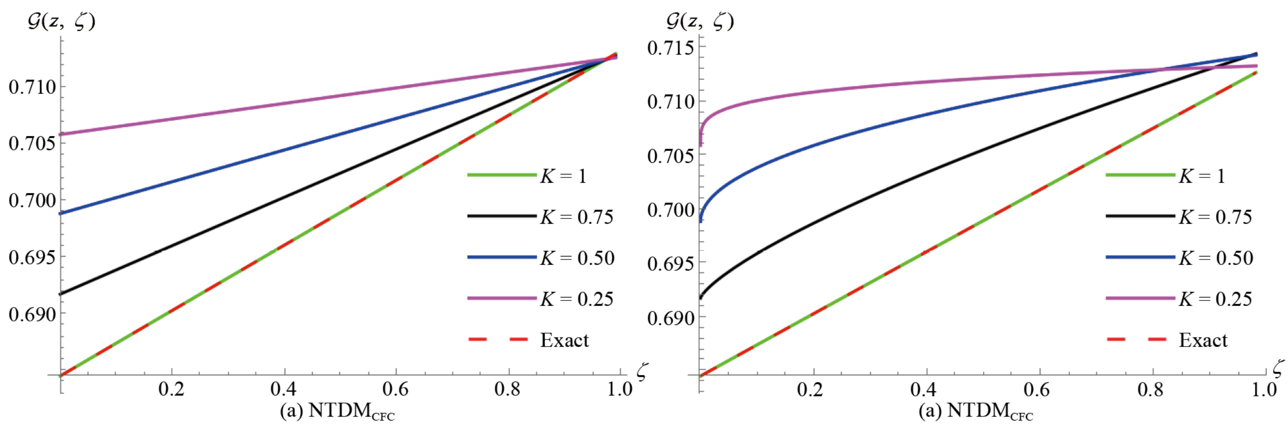
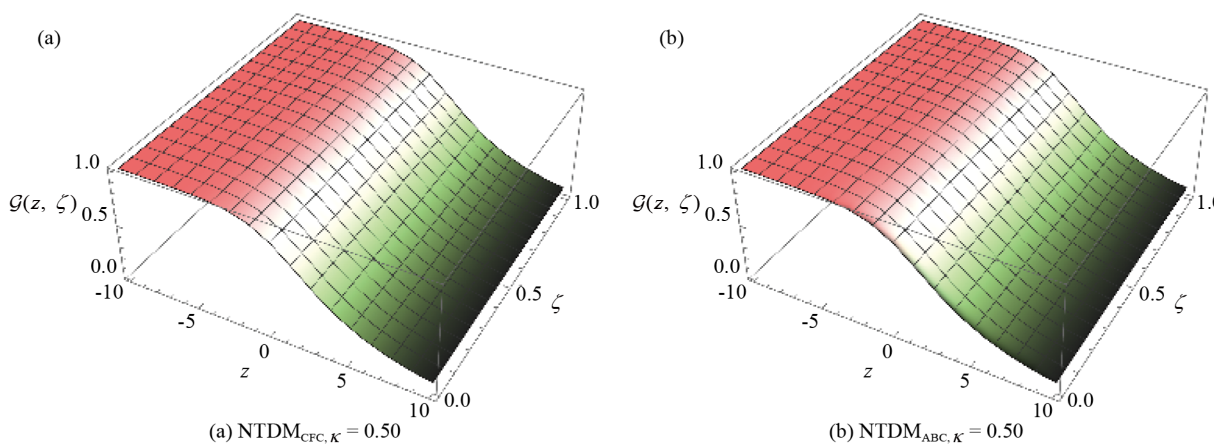
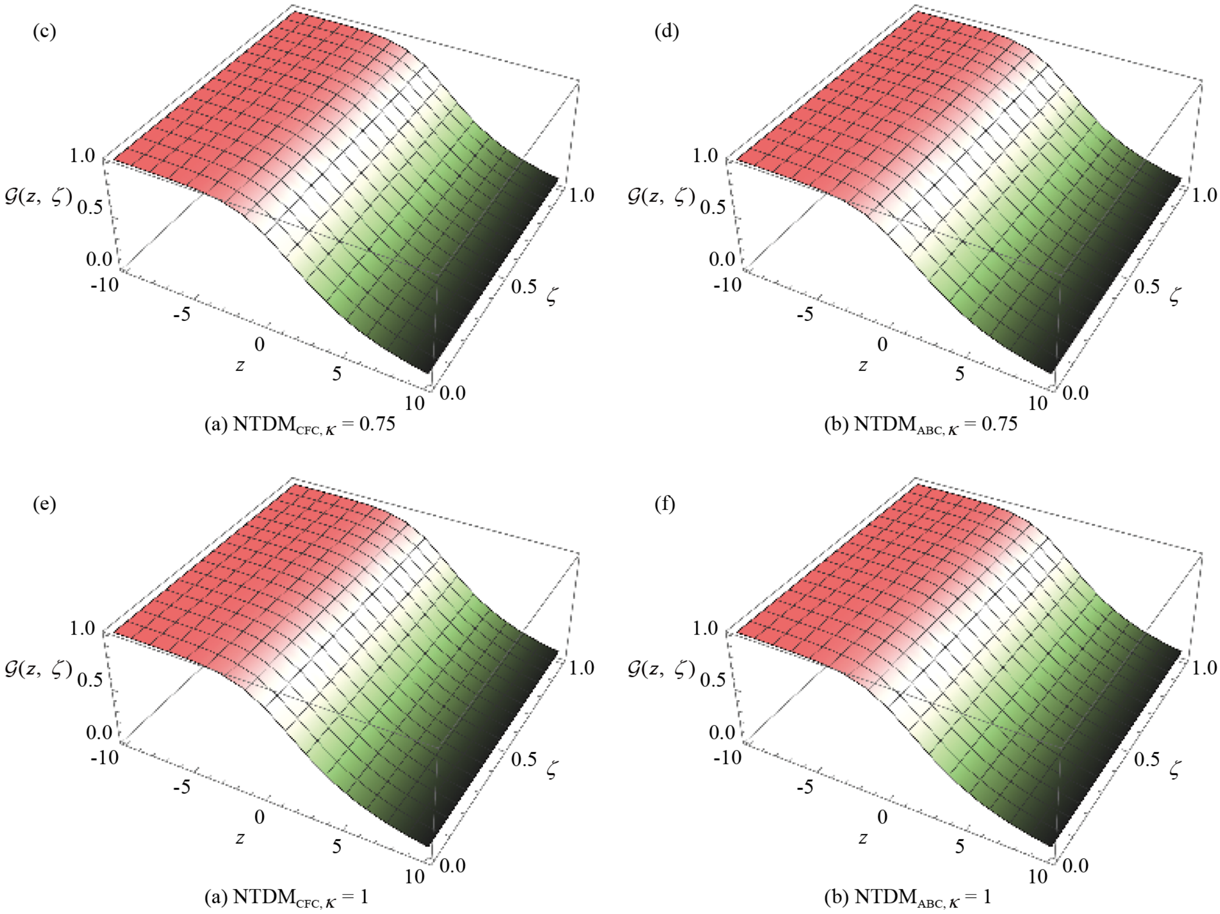


Figure 3. 2D solution profile for Case 2 at $z = 1$ and different values of κ





(a) $\text{NTDM}_{\text{CFC}, \kappa} = 0.75$ (b) $\text{NTDM}_{\text{ABC}, \kappa} = 0.75$

(a) $\text{NTDM}_{\text{CFC}, \kappa} = 1$ (b) $\text{NTDM}_{\text{ABC}, \kappa} = 1$

Figure 4. Surface plot of approximate solutions for $\mathscr{G}(z, \zeta)$ in Case 2 at distinct values of κ

6. Conclusion

In this work, we used NTDM to estimate approximate solutions to two cases of the TFRE (5) with CFC and ABC fractional operators. Results were obtained for different fractional orders at different depths z and time ζ in both cases. The effectiveness of the proposed approach is determined by comparing the errors between the exact solutions and the obtained NTDM solutions. Error estimates are also presented. The obtained solutions reveal that for each of the considered cases, the solutions obtained for the two fractional derivatives have excellent proximity to the exact solution at $\kappa = 1$. In addition, we observe that as the fractional order increases, the obtained solutions approach the exact solution. Numerical computations were conducted to determine the results for different orders of the fractional derivative. The graphs illustrate the effectiveness and feasibility of the considered approach. Moreover, this methodology can be applied to analyze possible solutions for a diverse range of issues that arise in different research domains.

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Conflict of interest

The authors declare no competing financial interest.

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