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Research Article

Fractional View Analysis of Coupled Whitham-Broer-Kaup Equations Arising in Shallow Water with Caputo Derivative

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Abstract: This research explores the analysis of the nonlinear fractional systems described by the Whitham-Broer-Kaup equations using novel mathematical tools like the Aboodh transform iteration method and the Aboodh residual power series method in light of the Caputo operator theory. The Whitham-Broer-Kaup equations are critical for describing the nonlinear propagation of dispersive waves and have enormous practical relevance. In the application of the Aboodh transform iteration method and the Aboodh residual power series method, as well as the introduction of the Caputo operator, the modelling accuracy is improved by calculating the fractional derivatives. The study has proven the usefulness of these techniques in providing solutions to fractional nonlinear systems that are only approximate but are insightful concerning dynamic behaviour. The insertion of the Caputo operator in the modelling method gives it some sophistication, simulating the non-locality inherent in fractional calculus. Consequently, this study enriches mathematical models and computational approaches, bringing in solid tools for researchers who want to examine complex nonlinear systems with fractional nature.

***Keywords*:** Aboodh transform iteration method (ATIM), fractional nonlinear systems of Whitham-Broer-Kaup equations, Aboodh residual power series method (ARPSM), Caputo operator

MSC: 35R11, 35Q53, 26A33, 65M99, 76B15

1. Introduction

The mathematical framework of fractional calculus is very complicated. It questions the usual ideas of integration and differentiation by looking into the complex world of non-integer order derivatives and integrals. Including fractional orders expands the domain of classical calculus, enabling the investigation of complex systems exhibiting memory effects, long-range relationships, and unusual behaviours. The influence of this area of study may be seen in several other scientific domains, such as engineering [1], biology [2], physics [3], and finance [4], and it provides a complex approach to comprehending phenomena that challenge conventional calculus techniques. Using fractional integrals and derivatives allows one to study chaotic dynamics, non-local interactions, fractal geometries, and other fundamental properties of complexity. In signal processing [5, 6], material science [7], and stochastic processes [8, 9], among others, fragmental calculus plays an essential role in comprehending complicated events, identifying patterns, and creating novel ways to tackle problems.

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Nonlinear PDEs are a sophisticated and complicated type of mathematical modelling, and many scientific fields, including astrophysics and geophysics, utilize them to explain complex phenomena. Analytical solutions are hard to generate because of the system's magnifying complexity character and the nonlinear form of the equations describing the partial derivatives. However, noteworthy is that whereas linear PDEs have developed analytical solutions, the solution of nonlinear PDEs may require discretization methods, approximation techniques, or qualitative arguments [10–14].

A broad range of physical phenomena can be modelled using nonlinear PDE systems. They involve fluid dynamics, wave propagation, pattern formation, and population dynamics. Physics, biology, engineering, and even economics use several systems. These complexities call for advanced mathematical tools and knowledge, which entails a deep understanding of the processes at work and how each component influences the rest with their nonlinear interactions.

"How do we find an exact solution to FDEs?" is a reasonable question. Nonlinear fractional differential equations (FDEs) become highly critical to understanding and utilizing the mechanics of complex nonlinear physical phenomena. FDEs are studied in many subsets of physics, engineering, and mathematics applied in these fields. By modelling objects with either impulsive FDEs and time and space fractional advection-dispersion equations and fractional generalized Burgers fluids, a plethora of researchers and experts in the field have successfully solved these combined types of FDEs [15–19].

Researchers who work in this field have used efficient tools that enable them to find the correct numerical and analytical solutions for PDEs, such as the iterative Laplace transform method [20], differential transform method [21], homotopy perturbation method [22], variational iteration method [23], Laplace transform decomposition method [24], and Whitham, Broer, and Kaup [25–27] derived nonlinear WBK equations using the Boussinesq approximation:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\phi_1(\zeta,\ t)+a\phi_1(\zeta,\ t)\frac{\partial\phi_1}{\partial\zeta}(\zeta,\ t)+b\frac{\partial\phi_2}{\partial\zeta}(\zeta,\ t)+c\frac{\partial^2\phi_1}{\partial\zeta^2}(\zeta,\ t)=0,\\
&\frac{\partial}{\partial t}\phi_2(\zeta,\ t)+\phi_1(\zeta,\ t)\frac{\partial\phi_2}{\partial\zeta}(\zeta,\ t)+\phi_2(\zeta,\ t)\frac{\partial\phi_1}{\partial\zeta}(\zeta,\ t)+d\frac{\partial^3\phi_1}{\partial\zeta^3}(\zeta,\ t)-c\frac{\partial^2\phi_2}{\partial\zeta^2}(\zeta,\ t)=0,
\end{aligned}
\tag{1}
$$

where a, b, c, and d are constants composed of different diffusion powers. At the same time, $\phi_1(\zeta,\ t)$ and $\phi_2(\zeta,\ t)$ represent the fluids' horizontal velocities and heights, which vary significantly from equilibrium. In order to solve the WBK system of fractional order, Zheng and Wang [28] used an extended fractional Riccati subequation approach. In order to solve coupled systems, El-Borai et al. [29] used the exponential function method. Using the connected fractional reduced differential transform technique, the author [30] obtained approximate analytical solutions to the model. Other techniques that are used to find numerical solutions for the coupled system (1) are the finite difference approach [31], the exponential-function method [32], and others [33–35].

Omar Abu Arqub 2013 founded the RPSM. Demonstrating the semi-analytic style, the RPSM employs the Taylor series and the error function [36]. The given series algorithms can solve linear or nonlinear differential equations. The first success of RPSM in 2013 regards DE resolution in the fuzzy domain. The authors, Arqub et al. [37], had brought into effect a new set of RPSM algorithms to compute power series solutions to highly complicated DEs efficiently.

Moreover, Muhammad Arqub et al. [38] mentioned a new RPSM that handles fractional-order nonlinear boundary value difficulties. How to obtain insights into the solutions of fractional-order KdV-burgers problems was presented by El-Ajou et al. [39] using a new RPSM approach. Xu et al. first proposed giving the idea of the solution of the fourth and second-order PDEs by using fractional power series [40]. They merged RPSM with the least square method and thus proposed a numerical method that is both precise and accurate. Please see [41–44] for further clarifications.

To solve FDEs, both researchers relied upon a few methods. Step 1 is the Aboodh transform conversion of the initial equation [45]. Then, they changed equations and solved them one by one. Furthermore, using the inverse Aboodh transform, the first equation is resolved. The strategy is based on the homotopy perturbation methods combined with the Sumudu transform. We designed a new method based on the power series expansion, which needs no discretization, perturbation, or linearization for solving nonlinear and linear PDEs.

Conversely, in the case of RPSM, one has to go through several iterations to find a solution with fractional derivatives. However, finding coefficients is a piece of cake in this situation. The proposed technique uses the quick convergence series that may exactly give the approximate solution in closed form.

The most important mathematical discovery of the twentieth century can be regarded as Aboodh's new iterative method (ATIM) for fractional partial differential equations (FPDEs). The FPDEs with derivatives in the form of fractional numbers are notoriously hard to solve using standard methods because of their computational complexity and non-convergence. Our unique technology surpasses these limitations by minimizing the processing effort, increasing the precision rate, and steadily evolving approximation solutions. Over iterations, especially incorporating fractional derivatives, better solutions to intricate mathematical and physical problems have been found [46–48]. The concept of complex systems that can be well-defined and understood with fractional partial differential equations is now used to investigate issues in engineering, applied mathematics, and physics.

ATIM and ARPSM are two of the most typical ways of dealing with FPDEs. Another advantage is the analytical answers (and numerical solutions with reasonable accuracy) and the lack of need for discretization or linearization of FPDEs. This study will be implemented using two distinct methods, ATIM and ARPSM, to tackle the WBK system of nonlinear PDEs. With these two methods, nonlinear FDE problems have been solved.

The outline of this paper is as follows: In Section 2, we provide some basic concepts of fractional calculus used in our study. Section 3 presents the road map of the proposed method ARPSM with its implementation and discussion of the results. Section 4 illustrates the road map of the proposed method ATIM with its implementation and discussion of the results. Finally, Section 5 includes the conclusions of our study.

2. Basic concepts of fractional calculus

Definition 2.1 The function $v(\zeta,\, t)$ is uniquely ordered and piecewise continuous [49]:

$$A[v(\zeta,\, t)] = \frac{1}{\nu}\int_0^\infty v(\zeta,\, t)e^{-t\nu}dt = \Psi(\zeta,\, \nu),\;\; r_1 \le \nu \le r_2,$$

Inverse Aboodh transform may be expressed in the following way:

$$A^{-1}[\Psi(\zeta,\, \nu)] = \frac{1}{2\pi i}\int_{u-i\infty}^{u+i\infty} \Psi(\zeta,\, t)\nu e^{t\nu}dt = v(\zeta,\, t).$$

Where $\zeta = (\zeta_1,\, \zeta_2,\, \cdots,\, \zeta_p)$ and $p \in \mathbb{N}$.

Lemma 2.1 [50, 51] Let us define two functions of the form $v_1(\zeta,\, t)$ and $v_2(\zeta,\, t)$, both are exponential order and continuous on the interval $[0,\, \infty]$. Considering that $A[v_1(\zeta,\, t)] = \Psi_1(\zeta,\, t)$, $A[v_2(\zeta,\, t)] = \Psi_2(\zeta,\, t)$ and $\varpi_1,\, \varpi_2$ are constants:

1. $A\left[\varpi_1 v_1(\zeta,\, t) + \varpi_2 v_2(\zeta,\, t)\right] = \varpi_1\Psi_1(\zeta,\, \nu) + \varpi_2\Psi_2(\zeta,\, \nu)$,
2. $A^{-1}\left[\varpi_1\Psi_1(\zeta,\, t) + \varpi_2\Psi_2(\zeta,\, t)\right] = \varpi_1 v_1(\zeta,\, \nu) + \varpi_2 v_2(\zeta,\, \nu)$,
3. $A[J_t^p v(\zeta,\, t)] = \dfrac{\Psi(\zeta,\, \nu)}{\nu^p}$,
4. $A[D_t^p v(\zeta,\, t)] = \nu^p\Psi(\zeta,\, \nu) - \sum_{K=0}^{r-1}\dfrac{v^K(\zeta,\, 0)}{\nu^{K-p+2}}$, $r-1 < p \le r$, $r \in \mathbb{N}$.

Definition 2.2 [52] As stated by the Caputo, a function $v(\zeta,\, t)$ has a fractional order of order p as follows:

$$D_t^p v(\zeta,\, t) = J_t^{m-p} v^{(m)}(\zeta,\, t),\;\; r \ge 0,\; m-1 < p \le m,$$

where $\zeta = (\zeta_1,\ \zeta_2,\ \cdots,\ \zeta_p) \in \mathbb{R}^p$ and $m,\ p \in R,\ J_t^{m-p}$ is the R-L integral of $v(\zeta,\ t)$.

Definition 2.3 [53] The power series may be expressed as follows:

$$\sum_{r=0}^{\infty} \hbar_r(\zeta)(t-t_0)^{rp} = \hbar_0(t-t_0)^0 + \hbar_1(t-t_0)^p + \hbar_2(t-t_0)^{2p} + \cdots,$$

Let $p \in \mathbb{N}$ and $\zeta = (\zeta_1,\ \zeta_2,\ \cdots,\ \zeta_p) \in \mathbb{R}^p$ be the vector with real numbers $\zeta_1,\ \zeta_2,\ \cdots,\ \zeta_p$ as its elements. The MFPS t_0 is the identifying pattern for this series. $\hbar_r(\zeta)$ are the fractions that act as the coefficients for this series, and the variable is referred to as t.

Lemma 2.2 Consider the exponential order function $v(\zeta,\ t)$, the Aboodh transform (AT) is calculated by $A[v(\zeta,\ t)] = \Psi(\zeta,\ \nu)$. It follows that

$$A[D_t^{rp} v(\zeta,\ t)] = \nu^{rp}\Psi(\zeta,\ \nu) - \sum_{j=0}^{r-1} \nu^{p(r-j)-2} D_t^{jp} v(\zeta,\ 0),\ \ 0 < p \leq 1, \tag{2}$$

where $\zeta = (\zeta_1,\ \zeta_2,\ \cdots,\ \zeta_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$ and $D_t^{rp} = D_t^p.D_t^p.\cdots.D_t^p(r-times)$.

Proof. To validate Eq. (2), let's use the induction approach. When $r = 1$ is substituted into Eq. (2), the following result is obtained:

$$A[D_t^{2p} v(\zeta,\ t)] = \nu^{2p}\Psi(\zeta,\ \nu) - \nu^{2p-2} v(\zeta,\ 0) - \nu^{p-2} D_t^p v(\zeta,\ 0).$$

The validity of the equation for $r = 1$ is supported by Lemma 2.1, Part (4). After replacing $r = 2$ in Eq (2), we get:

$$A[D_r^{2p} v(\zeta,\ t)] = \nu^{2p}\Psi(\zeta,\ \nu) - \nu^{2p-2} v(\zeta,\ 0) - \nu^{p-2} D_t^p v(\zeta,\ 0). \tag{3}$$

Analyzing the left hand side of Eq. (3), we get:

$$\text{Left hand side} = A[D_t^{2p} v(\zeta,\ t)]. \tag{4}$$

There is a specific way to write Eq. (4) as follows:

$$\text{Left hand side} = A[D_t^{p} v(\zeta,\ t)]. \tag{5}$$

Let

$$z(\zeta,\ t) = D_t^p v(\zeta,\ t). \tag{6}$$

Therefore, Eq. (5) becomes:

$$\text{Left hand side} = A[D_t^p z(\zeta,\ t)]. \tag{7}$$

The fractional Caputo operator is

$$\text{Left hand side} = A[J^{1-p} z^{'}(\zeta,\ t)]. \tag{8}$$

Eq. (8) provides the fractional integral R-L of the Aboodh transformation

$$\text{Left hand side} = \frac{A[z^{'}(\zeta,\ t)]}{\nu^{1-p}}. \tag{9}$$

Eq. (9) is modified by using the Aboodh transform's property.

$$\text{Left hand side} = \nu^p Z(\zeta,\ \nu) - \frac{z(\zeta,\ 0)}{\nu^{2-p}}, \tag{10}$$

Referring to Eq. (6), we get:

$$Z(\zeta,\ \nu) = \nu^p \Psi(\zeta,\ \nu) - \frac{v(\zeta,\ 0)}{\nu^{2-p}},$$

$A[z(\zeta,\ t)] = Z(\zeta,\ \nu)$. Consequently, Eq. (10) is converted into

$$\text{Left hand side} = \nu^{2p}\Psi(\zeta,\ \nu) - \frac{v(\zeta,\ 0)}{\nu^{2-2p}} - \frac{D_t^p v(\zeta,\ 0)}{\nu^{2-p}}, \tag{11}$$

Eq. (2) and (11) are compatible. Let Eq. (2) is true for $r = 2$, so put $r = K$ in Eq. (2).

$$A[D_t^{Kp} v(\zeta,\ t)] = \nu^{Kp}\Psi(\zeta,\ \nu) - \sum_{j=0}^{K-1} \nu^{p(K-j)-2} D_t^{jp} D_t^{jp} v(\zeta,\ 0),\ \ 0 < p \leq 1. \tag{12}$$

we will show that for $r = K+1$ the Eq. (2) is true.

$$A[D_t^{(K+1)p} v(\zeta,\ t)] = \nu^{(K+1)p}\Psi(\zeta,\ \nu) - \sum_{j=0}^{K} \nu^{p((K+1)-j)-2} D_t^{jp} v(\zeta,\ 0). \tag{13}$$

In Eq. (13), on the left hand side, we get

$$\text{Left hand side} = A[D_t^{Kp}(D_t^{Kp})]. \tag{14}$$

Assume

$$D_t^{Kp} = g(\zeta,\ t).$$

From Eq. (14) we derived the following result:

$$\text{Left hand side} = A[D_t^{p} g(\zeta,\ t)]. \tag{15}$$

Eq. (15) may be stated as follows by combining the integral of R-L with the fractional Caputo derivative:

$$\text{Left hand side} = \nu^p A[D_t^{Kp} v(\zeta,\ t)] - \frac{g(\zeta,\ 0)}{\nu^{2-p}}. \tag{16}$$

Analyzing Eq. (16) with the help of Eq. (12), we get

$$\text{Left hand side} = \nu^{rp}\Psi(\zeta,\ \nu) - \sum_{j=0}^{r-1} \nu^{p(r-j)-2} D_t^{jp} v(\zeta,\ 0), \tag{17}$$

With the utilization of Eq. (17), we obtain

$$\text{Left hand side} = A[D_t^{rp} v(\zeta,\ 0)].$$

This establishes the validity of Eq. (2) for $r = K+1$. Therefore, Eq. (2) was shown to be true for all positive integers. □

Here is a new lemma that we present on the ARPSM multiple fractional Taylor's formula.
Lemma 2.3 Consider $v(\zeta,\ t)$ is ordered exponentially. $A[v(\zeta,\ t)] = \Psi(\zeta,\ \nu)$ is the AT of $v(\zeta,\ t)$ as a MFTS.

$$\Psi(\zeta,\ \nu) = \sum_{r=0}^{\infty} \frac{\hbar_r(\zeta)}{\nu^{rp+2}},\ \ \nu > 0, \tag{18}$$

where, $\zeta = (\zeta_1,\ \zeta_2,\ \cdots,\ \zeta_p) \in \mathbb{R}^p,\ p \in \mathbb{N}$.

Proof. The results of the fractional order analysis of Taylor's series are as follows:

$$v(\zeta,\ t) = \hbar_0(\zeta) + \hbar_1(\zeta)\frac{t^p}{\Gamma[p+1]} + \hbar_2(\zeta)\frac{t^{2p}}{\Gamma[2p+1]} + \cdots. \tag{19}$$

By applying the AT on Eq. (19), we obtain:

$$A\left[v(\zeta,\ t)\right]=A\left[\hbar_0(\zeta)\right]+A\left[\hbar_1(\zeta)\frac{t^p}{\Gamma[p+1]}\right]+A\left[\hbar_1(\zeta)\frac{t^{2p}}{\Gamma[2p+1]}\right]+\cdots$$

Our derivation is based on these AT properties.

$$A\left[v(\zeta,\ t)\right]=\hbar_0(\zeta)\frac{1}{\nu^2}+\hbar_1(\zeta)\frac{\Gamma[p+1]}{\Gamma[p+1]}\frac{1}{\nu^{p+2}}+\hbar_2(\zeta)\frac{\Gamma[2p+1]}{\Gamma[2p+1]}\frac{1}{\nu^{2p+2}}\ \cdots$$

We get Eq. (18), a new Taylor series in the AT. □

Lemma 2.4 The function $A[v(\zeta,\ t)]=\Psi(\zeta,\ \nu)$ is assumed to have an MFPS notation in the new form of Taylor's series (18).

$$\hbar_0(\zeta)=\lim_{\nu\to\infty}\nu^2\Psi(\zeta,\ \nu)=v(\zeta,\ 0). \tag{20}$$

Proof. Consider the Taylor's series:

$$\hbar_0(\zeta)=\nu^2\Psi(\zeta,\ \nu)-\frac{\hbar_1(\zeta)}{\nu^p}-\frac{\hbar_2(\zeta)}{\nu^{2p}}-\cdots \tag{21}$$

The result, represented by (21), is achieved by taking $\lim_{\nu\to\infty}$ on Eq. (20). □

Theorem 2.5 The MFPS representations of the functions $\Psi(\zeta,\ \nu)$ and $v(\zeta,\ t)$ are as follows:

$$\Psi(\zeta,\ \nu)=\sum_0^\infty\frac{\hbar_r(\zeta)}{\nu^{rp+2}},\ \ \nu>0,\ \ p\in\mathbb{N}$$

where $\zeta=(\zeta_1,\ \zeta_2,\ \cdots,\ \zeta_p)\in\mathbb{R}^p$, we have

$$\hbar_r(\zeta)=D_r^{rp}v(\zeta,\ 0),$$

where, $D_t^{rp}=D_t^p.D_t^p.\cdots.D_t^p(r-times)$.

Proof. The updated Taylor series gives us

$$\hbar_1(\zeta)=\nu^{p+2}\Psi(\zeta,\ \nu)-\nu^p\hbar_0(\zeta)-\frac{\hbar_2(\zeta)}{\nu^p}-\frac{\hbar_3(\zeta)}{\nu^{2p}}-\cdots \tag{22}$$

Solving Eq. (22) for $\lim_{\nu\to\infty}$, we get

$$\hbar_1(\zeta)=\lim_{\nu\to\infty}(\nu^{p+2}\Psi(\zeta,\ \nu)-\nu^p\hbar_0(\zeta))-\lim_{\nu\to\infty}\frac{\hbar_2(\zeta)}{\nu^p}-\lim_{\nu\to\infty}\frac{\hbar_3(\zeta)}{\nu^{2p}}-\cdots$$

The following equality is the result of taking the limit:

$$\hbar_1(\zeta) = \lim_{\nu\to\infty}(\nu^{p+2}\Psi(\zeta,\ \nu) - \nu^p\hbar_0(\zeta)). \tag{23}$$

The combination of Lemma 2.2 and Eq. (23) yields the following result:

$$\hbar_1(\zeta) = \lim_{\nu\to\infty}(\nu^2 A[D_t^p v(\zeta,\ t)](\nu)). \tag{24}$$

Furthermore, the outcome may be obtained by applying Eq. (24) along with Lemma 2.3.

$$\hbar_1(\zeta) = D_t^p v(\zeta,\ 0).$$

Using $\nu \to \infty$ and the modified Taylor's series, we achieve our result.

$$\hbar_2(\zeta) = \nu^{2p+2}\Psi(\zeta,\ \nu) - \nu^{2p}\hbar_0(\zeta) - \nu^p\hbar_1(\zeta) - \frac{\hbar_3(\zeta)}{\nu^p} - \cdots$$

Using the Lemma 2.3, we deduce:

$$\hbar_2(\zeta) = \lim_{\nu\to\infty}\nu^2(\nu^{2p}\Psi(\zeta,\ \nu) - \nu^{2p-2}\hbar_0(\zeta) - \nu^{p-2}\hbar_1(\zeta)). \tag{25}$$

To modify Eq. (25), Lemma 2.2 and 2.4 are used again.

$$\hbar_2(\zeta) = D_t^{2p} v(\zeta,\ 0).$$

A different version of Taylor's series is obtained by using the identical procedure on it.

$$\hbar_3(\zeta) = \lim_{\nu\to\infty}\nu^2(A[D_t^{2p} v(\zeta,\ p)](\nu)).$$

The final equation is obtained by applying Lemma 2.4.

$$\hbar_3(\zeta) = D_t^{3p} v(\zeta,\ 0).$$

In general we get

$$\hbar_r(\zeta) = D_t^{rp} v(\zeta,\ 0).$$

Hence proved. □

The following theorem gives our evidence of the necessary and sufficient conditions for the convergence of the modified Taylor formula.

Theorem 2.6 A novel fractional definition of Taylor's formula is given in Lemma 2.3 as: $A[v(\zeta,\ t)] = \Psi(\zeta,\ \nu)$. The new MFTS for $(0 < \nu \leq s)$ with $0 < p \leq 1$ is consistent with this inequality if $|\nu^a A[D_t^{(K+1)p} v(\zeta,\ t)]| \leq T$.

$$|R_K(\zeta,\ \nu)| \leq \frac{T}{\nu^{(K=1)p+2}},\ \ 0 < \nu \leq s.$$

Proof. Let us consider $(\zeta,\ t) D_t^{rp} v,\ r = 0,\ 1,\ 2,\ \cdots,\ K+1$, for which $0 < \nu \leq s$, is defined. Consider $|\nu^2 A [D_{t^{K+1}} v(\zeta,\ \tau)]| \leq T$ holds for $0 < \nu \leq s$ given the specific conditions. Analyze the equation using Taylor's series:

$$R_K(\zeta,\ \nu) = \Psi(\zeta, \nu) - \sum_{r=0}^{K} \frac{\hbar_r(\zeta)}{\nu^{rp+2}}. \tag{26}$$

When Theorem 2.5 is applied, Eq. (26) becomes:

$$R_K(\zeta,\ \nu) = \Psi(\zeta,\ \nu) - \sum_{r=0}^{K} \frac{D_t^{rp} v(\zeta,\ 0)}{\nu^{rp+2}}. \tag{27}$$

$\nu^{(K+1)a+2}$ should be multiply on both sides of Eq. (27):

$$\nu^{(K+1)p+2} R_K(\zeta,\ \nu) = \nu^2 (\nu^{(K+1)p} \Psi(\zeta,\ \nu) - \sum_{r=0}^{K} \nu^{(K+1-r)p-2} D_t^{rp} v(\zeta,\ 0)). \tag{28}$$

Using Lemma 2.2 on Eq. (28), we get:

$$\nu^{(K+1)p+2} R_K(\zeta,\ \nu) = \nu^2 A[D_t^{(K+1)p} v(\zeta,\ t)]. \tag{29}$$

Taking Eq. (29)'s absolute yields:

$$|\nu^{(K+1)p+2} R_K(\zeta,\ \nu)| = |\nu^2 A[D_t^{(K+1)p} v(\zeta,\ t)]|. \tag{30}$$

The following outcome is obtained by using the criteria provided in Eq. (30):

$$\frac{-T}{\nu^{(K+1)p+2}} \leq R_K(\zeta,\ \nu) \leq \frac{T}{\nu^{(K+1)p+2}}. \tag{31}$$

The intended result is obtained by Eq. (31).

$$|R_K(\zeta,\ \nu)| \le \frac{T}{\nu^{(K+1)p+2}}.$$

This leads to the discovery of the new series' convergence condition. □

3. Resolving PDEs using the ARPSM approach

We consider our general model in light of the ARPSM guiding principles.

Step 1 Construct a generalized form of the equation.

$$D_t^{qp} v(\zeta,\ t) + \vartheta(\zeta)N(v) - \zeta(\zeta,\ v) = 0, \tag{32}$$

Step 2 Eq. (32) may be simplified by applying the Aboodh transformation on both sides, thereby producing:

$$A[D_t^{qp} v(\zeta,\ t) + \vartheta(\zeta)N(v) - \zeta(\zeta,\ v)] = 0, \tag{33}$$

Let's use Lemma 2.2 to convert Eq. (33).

$$\Psi(\zeta,\ s) = \sum_{j=0}^{q-1} \frac{D_t^j v(\zeta,\ 0)}{s^{qp+2}} - \frac{\vartheta(\zeta)Y(s)}{s^{qp}} + \frac{F(\zeta,\ s)}{s^{qp}}, \tag{34}$$

where, $A[\Psi(\zeta,\ v)] = F(\zeta,\ s)$, $A[N(v)] = Y(s)$.

Step 3 To obtain the solution for Eq. (34) consider:

$$\Psi(\zeta,\ s) = \sum_{r=0}^{\infty} \frac{\hbar_r(\zeta)}{s^{rp+2}},\quad s > 0,$$

Step 4 Continue with these procedures:

$$\hbar_0(\zeta) = \lim_{s\to\infty} s^2\Psi(\zeta,\ s) = v(\zeta,\ 0),$$

Using Theorem 2.6 we get the subsequent result.

$$\hbar_1(\zeta) = D_t^p v(\zeta,\ 0),$$

$$\hbar_2(\zeta) = D_t^{2p} v(\zeta,\ 0),$$

$$\vdots$$

$$\hbar_w(\zeta) = D_t^{wp} v(\zeta,\ 0),$$

Step 5 Applying these procedures, we may get the K^{th}-truncated series as $\Psi(\zeta,\ s)$.

$$\Psi_K(\zeta,\ s) = \sum_{r=0}^{K} \frac{\hbar_r(\zeta)}{s^{rp+2}},\ \ s > 0,$$

$$\Psi_K(\zeta,\ s) = \frac{\hbar_0(\zeta)}{s^2} + \frac{\hbar_1(\zeta)}{s^{p+2}} + \cdots + \frac{\hbar_w(\zeta)}{s^{wp+2}} + \sum_{r=w+1}^{K} \frac{\hbar_r(\zeta)}{s^{rp+2}},$$

Step 6 It is necessary to estimate the residual function of Aboodh (ARF) of Eq. (34) separately from the K^{th}-truncated residual function of Aboodh in order to obtain:

$$ARes(\zeta,\ s) = \Psi(\zeta,\ s) - \sum_{j=0}^{q-1} \frac{D_t^j v(\zeta,\ 0)}{s^{jp+2}} + \frac{\vartheta(\zeta)Y(s)}{s^{jp}} - \frac{F(\zeta,\ s)}{s^{jp}},$$

and

$$ARes_K(\zeta,\ s) = \Psi_K(\zeta,\ s) - \sum_{j=0}^{q-1} \frac{D_t^j v(\zeta,\ 0)}{s^{jp+2}} + \frac{\vartheta(\zeta)Y(s)}{s^{jp}} - \frac{F(\zeta,\ s)}{s^{jp}}. \tag{35}$$

Step 7 Insert the expression of $\Psi_K(\zeta,\ s)$ into Eq. (35).

$$\begin{aligned} ARes_K(\zeta,\ s) = &\left(\frac{\hbar_0(\zeta)}{s^2} + \frac{\hbar_1(\zeta)}{s^{p+2}} + \cdots + \frac{\hbar_w(\zeta)}{s^{wp+2}} + \sum_{r=w+1}^{K} \frac{\hbar_r(\zeta)}{s^{rp+2}}\right) \\ &- \sum_{j=0}^{q-1} \frac{D_t^j v(\zeta,\ 0)}{s^{jp+2}} + \frac{\vartheta(\zeta)Y(s)}{s^{jp}} - \frac{F(\zeta,\ s)}{s^{jp}}. \end{aligned} \tag{36}$$

Step 8 Eq. (36) both sides is multiplied with s^{Kp+2}.

$$s^{Kp+2}ARes_K(\zeta,\ s) = s^{Kp+2}\Bigg(\frac{\hbar_0(\zeta)}{s^2} + \frac{\hbar_1(\zeta)}{s^{p+2}} + \cdots + \frac{\hbar_w(\zeta)}{s^{wp+2}} + \sum_{r=w+1}^{K} \frac{\hbar_r(\zeta)}{s^{rp+2}}$$

$$- \sum_{j=0}^{q-1} \frac{D_t^j v(\zeta,\ 0)}{s^{jp+2}} + \frac{\vartheta(\zeta)Y(s)}{s^{jp}} - \frac{F(\zeta,\ s)}{s^{jp}}\Bigg). \tag{37}$$

Step 9 This evaluation of Eq. (37) is being performed with regard to $\lim_{s\to\infty}$ on both sides.

$$\lim_{s\to\infty} s^{Kp+2}ARes_K(\zeta,\ s) = \lim_{s\to\infty} s^{Kp+2}\Bigg(\frac{\hbar_0(\zeta)}{s^2} + \frac{\hbar_1(\zeta)}{s^{p+2}} + \cdots + \frac{\hbar_w(\zeta)}{s^{wp+2}} + \sum_{r=w+1}^{K} \frac{\hbar_r(\zeta)}{s^{rp+2}}$$

$$- \sum_{j=0}^{q-1} \frac{D_t^j v(\zeta,\ 0)}{s^{jp+2}} + \frac{\vartheta(\zeta)Y(s)}{s^{jp}} - \frac{F(\zeta,\ s)}{s^{jp}}\Bigg).$$

Step 10 In order to get $\hbar_K(\zeta)$, we must resolve the following equation.

$$\lim_{s\to\infty}(s^{Kp+2}ARes_K(\zeta,\ s)) = 0,$$

Step 11 By putting $\hbar_K(\zeta)$ into the K-truncated series of $\Psi(\zeta,\ s)$, we can derive the K-approximate solution for Eq. (34).

Step 12 For $\Psi_K(\zeta,\ s)$, the K-approximate solution $v_K(\zeta,\ t)$ might be deduce by applying an inverse Aboodh transform.

3.1 *Example 1 via ARPSM*

Let us take the WBK system of the following form [54]:

$$D_t^p \phi_1(\zeta,\ t) + a\phi_1(\zeta,\ t)\frac{\partial \phi_1}{\partial \zeta}(\zeta,\ t) + b\frac{\partial \phi_2}{\partial \zeta}(\zeta,\ t) + c\frac{\partial^2 \phi_1}{\partial \zeta^2}(\zeta,\ t) = 0, \tag{38}$$

$$D_t^p \phi_2(\zeta,\ t) + \phi_1(\zeta,\ t)\frac{\partial \phi_2}{\partial \zeta}(\zeta,\ t) + \phi_2(\zeta,\ t)\frac{\partial \phi_1}{\partial \zeta}(\zeta,\ t) + d\frac{\partial^3 \phi_1}{\partial \zeta^3}(\zeta,\ t) - c\frac{\partial^2 \phi_2}{\partial \zeta^2}(\zeta,\ t) = 0, \tag{39}$$

where $0 < p \leq 1$.

Initial conditions are:

$$\phi_1(\zeta,\ 0) = \frac{\sinh(\zeta)}{2} + 1, \tag{40}$$

$$\phi_2(\zeta,\,0) = -\frac{4a^2\sinh(\zeta)+a^2\cosh^2(\zeta)+3a^2+4c\cosh(\zeta)-4\sinh(\zeta)-8}{8b}, \tag{41}$$

Using Eqs. (38) and (39) in conjunction with the Aboodh Transform (AT), we derive:

$$\phi_1(\zeta,\,t)-\frac{\frac{\sinh(\zeta)}{2}+1}{s^2}+\frac{a}{s^p}\mathcal{A}_t\left[\mathcal{A}_t^{-1}\phi_1(\zeta,\,t)\times\frac{\partial\mathcal{A}_t^{-1}\phi_1}{\partial\zeta}(\zeta,\,t)\right]+\frac{b}{s^p}\left[\frac{\partial\phi_2}{\partial\zeta}(\zeta,\,t)\right]+\frac{c}{s^p}\left[\frac{\partial^2\phi_1}{\partial\zeta^2}(\zeta,\,t)\right]=0, \tag{42}$$

$$\phi_2(\zeta,\,t)-\frac{-\frac{4a^2\sinh(\zeta)+a^2\cosh^2(\zeta)+3a^2+4c\cosh(\zeta)-4\sinh(\zeta)-8}{8b}}{s^2}$$

$$+\frac{1}{s^p}\mathcal{A}_t\left[\mathcal{A}_t^{-1}\phi_1(\zeta,\,t)\times\frac{\partial\mathcal{A}_t^{-1}\phi_2}{\partial\zeta}(\zeta,\,t)\right]$$

$$+\frac{1}{s^p}\mathcal{A}_t\left[\mathcal{A}_t^{-1}\phi_2(\zeta,\,t)\times\frac{\partial\mathcal{A}_t^{-1}\phi_1}{\partial\zeta}(\zeta,\,t)\right]+\frac{d}{s^p}\left[\frac{\partial^3\phi_1}{\partial\zeta^3}(\zeta,\,t)\right]-\frac{c}{s^p}\left[\frac{\partial^2\phi_2}{\partial\zeta^2}(\zeta,\,t)\right]=0. \tag{43}$$

The k^{th} truncated term series are

$$\phi_1(\zeta,\,s)=\frac{\frac{\sinh(\zeta)}{2}+1}{s^2}+\sum_{r=1}^{k}\frac{f_r(\zeta,\,s)}{s^{rp+1}},\quad r=1,\,2,\,3,\,4\,\cdots. \tag{44}$$

$$\phi_2(\zeta,\,s)=\frac{-\frac{4a^2\sinh(\zeta)+a^2\cosh^2(\zeta)+3a^2+4c\cosh(\zeta)-4\sinh(\zeta)-8}{8b}}{s^2}+\sum_{r=1}^{k}\frac{g_r(\zeta,\,s)}{s^{rp+1}},\quad r=1,\,2,\,3,\,4\,\cdots. \tag{45}$$

The residual functions of Aboodh are

$$\begin{aligned}\mathcal{A}_tRes(\zeta,\,s)=\phi_1(\zeta,\,t)-\frac{\frac{\sinh(\zeta)}{2}+1}{s^2}+\frac{a}{s^p}\mathcal{A}_t\left[\mathcal{A}_t^{-1}\phi_1(\zeta,\,t)\times\frac{\partial\mathcal{A}_t^{-1}\phi_1}{\partial\zeta}(\zeta,\,t)\right]+\frac{b}{s^p}\left[\frac{\partial\phi_2}{\partial\zeta}(\zeta,\,t)\right]\\ +\frac{c}{s^p}\left[\frac{\partial^2\phi_1}{\partial\zeta^2}(\zeta,\,t)\right]=0,\end{aligned} \tag{46}$$

$$\mathcal{A}_t Res(\zeta,\ s) = \phi_2(\zeta,\ t) - \frac{-\dfrac{4a^2\sinh(\zeta)+a^2\cosh^2(\zeta)+3a^2+4c\cosh(\zeta)-4\sinh(\zeta)-8}{8b}}{s^2}$$

$$+\frac{1}{s^p}\mathcal{A}_t\left[\mathcal{A}_t^{-1}\phi_1(\zeta,\ t)\times\frac{\partial\mathcal{A}_t^{-1}\phi_2}{\partial\zeta}(\zeta,\ t)\right]+\frac{1}{s^p}\mathcal{A}_t\left[\mathcal{A}_t^{-1}\phi_2(\zeta,\ t)\times\frac{\partial\mathcal{A}_t^{-1}\phi_1}{\partial\zeta}(\zeta,\ t)\right] \tag{47}$$

$$+\frac{d}{s^p}\left[\frac{\partial^3\phi_1}{\partial\zeta^3}(\zeta,\ t)\right]-\frac{c}{s^p}\left[\frac{\partial^2\phi_2}{\partial\zeta^2}(\zeta,\ t)\right]=0,$$

k^{th}-ARFs are given as:

$$\mathcal{A}_t Res_k(\zeta,\ s) = \phi_{1k}(\zeta,\ t) - \frac{\dfrac{\sinh(\zeta)}{2}+1}{s^2}+\frac{a}{s^p}\mathcal{A}_t\left[\mathcal{A}_t^{-1}\phi_{1k}(\zeta,\ t)\times\frac{\partial\mathcal{A}_t^{-1}\phi_{1k}}{\partial\zeta}(\zeta,\ t)\right]+\frac{b}{s^p}\left[\frac{\partial\phi_{2k}}{\partial\zeta}(\zeta,\ t)\right]$$

$$+\frac{c}{s^p}\left[\frac{\partial^2\phi_{1k}}{\partial\zeta^2}(\zeta,\ t)\right]=0, \tag{48}$$

$$\mathcal{A}_t Res_k(\zeta,\ s) = \phi_{2k}(\zeta,\ t) - \frac{-\dfrac{4a^2\sinh(\zeta)+a^2\cosh^2(\zeta)+3a^2+4c\cosh(\zeta)-4\sinh(\zeta)-8}{8b}}{s^2}$$

$$+\frac{1}{s^p}\mathcal{A}_t\left[\mathcal{A}_t^{-1}\phi_{1k}(\zeta,\ t)\times\frac{\partial\mathcal{A}_t^{-1}\phi_{2k}}{\partial\zeta}(\zeta,\ t)\right]+\frac{1}{s^p}\mathcal{A}_t\left[\mathcal{A}_t^{-1}\phi_{2k}(\zeta,\ t)\times\frac{\partial\mathcal{A}_t^{-1}\phi_{1k}}{\partial\zeta}(\zeta,\ t)\right]$$

$$+\frac{d}{s^p}\left[\frac{\partial^3\phi_{1k}}{\partial\zeta^3}(\zeta,\ t)\right]-\frac{c}{s^p}\left[\frac{\partial^2\phi_{2k}}{\partial\zeta^2}(\zeta,\ t)\right]=0. \tag{49}$$

These are the steps to get $f_r(\zeta,\ s)$ and $g_r(\zeta,\ s)$: s^{rp+1} will be multiplied throughout the equation, insert the r^{th}-truncated series from Eq. (44) and (45) into the r^{th} residual function of Aboodh represented by Eq. (48) and (49). Then $\lim_{s\to\infty}(s^{rp+1}A_t Resv_1,\ r(\zeta,\ s))=0$ and $\lim_{s\to\infty}(s^{rp+1}A_t Resv_2,\ r(\zeta,\ s))=0$ for $r=1,\ 2,\ 3,\ \cdots$, are solve iteratively. Some of the terms are:

$$f_1(\zeta,\ s)=\frac{1}{4}\cosh(\zeta)\left(2\left(a^2-a-1\right)+(a-1)a\sinh(\zeta)\right),$$

$$g_1(\zeta,\ s)=\frac{1}{64b}\Big(\cosh(\zeta)\Big(45a^2-32\Big(bd+c^2+2\Big)\Big)-32\Big(a^2-2\Big)c\sinh(\zeta)-16\Big(a^2-1\Big)c\cosh(2\zeta) \tag{50}$$

$$+8\Big(3a^2-2\Big)\sinh(2\zeta)+3a^2\cosh(3\zeta)\Big),$$

$$f_2(\zeta,\ s)=\frac{1}{64}\Big(-3(2a+1)a^2\sinh(3\zeta)+\sinh(\zeta)\Big(a(32-15a(2a+1))+32\Big(bd+c^2+2\Big)\Big)$$

$$+32(a-1)c\sinh(2\zeta)+32(a-1)c\cosh(\zeta)-16(a-1)(a(2a+3)+2)\cosh(2\zeta)\Big),$$

$$g_2(\zeta,\ s)=-\frac{1}{64b}\Big(-13a^4\sinh(\zeta)-9a^4\sinh(3\zeta)-23a^4\cosh(2\zeta)-a^4\cosh(4\zeta)+13a^3\sinh(\zeta)$$

$$+9a^3\sinh(3\zeta)+23a^3\cosh(2\zeta)+a^3\cosh(4\zeta)+c\cosh(\zeta)\big(-75a^2+2a+32bd+32c^2+124\big)$$

$$+32a^2bd\sinh(\zeta)+64a^2bd\cosh(2\zeta)+32a^2c^2\sinh(\zeta)+64a^2c^2\cosh(2\zeta)-160a^2c\sinh(2\zeta) \tag{51}$$

$$-45a^2c\cosh(3\zeta)+84a^2\sinh(\zeta)+36a^2\sinh(3\zeta)+117a^2\cosh(2\zeta)+3a^2\cosh(4\zeta)-32abd\sinh(\zeta)$$

$$-64abd\cosh(2\zeta)+16ac\sinh(2\zeta)+6ac\cosh(3\zeta)-30a\sinh(\zeta)-6a\sinh(3\zeta)-32a\cosh(2\zeta)$$

$$-64bd\sinh(\zeta)-16bd\cosh(2\zeta)-96c^2\sinh(\zeta)-80c^2\cosh(2\zeta)+144c\sinh(2\zeta)+12c\cosh(3\zeta)$$

$$-92\sinh(\zeta)-12\sinh(3\zeta)-80\cosh(2\zeta)\Big).$$

and so on.

$f_r(\zeta,\ s)$ and $g_r(\zeta,\ s)$, for $r=1,\ 2,\ 3,\ \cdots$, are substituted in Eq. (44), (45), to obtain:

$$\phi_1(\zeta,\ s)=\frac{\frac{\sinh(\zeta)}{2}+1}{s}+\frac{1}{s^{p+1}}\left(\frac{1}{4}\cosh(\zeta)\left(2\left(a^2-a-1\right)+(a-1)a\sinh(\zeta)\right)\right)$$

$$+\frac{1}{s^{2p+1}}\left(\frac{1}{64}\Big(-3(2a+1)a^2\sinh(3\zeta)+\sinh(\zeta)\Big(a(32-15a(2a+1))+32\Big(bd+c^2+2\Big)\Big)\right. \tag{52}$$

$$\left.+32(a-1)c\sinh(2\zeta)+32(a-1)c\cosh(\zeta)-16(a-1)(a(2a+3)+2)\cosh(2\zeta)\Big)\right)+\cdots.$$

$$\phi_2(\zeta,\, s) = -\frac{\frac{4a^2\sinh(\zeta)+a^2\cosh^2(\zeta)+3a^2+4c\cosh(\zeta)-4\sinh(\zeta)-8}{8b}}{s} + \frac{1}{s^{p+1}}\Bigg(\frac{1}{64b}\Big(\cosh(\zeta)\Big(45a^2$$

$$-32\Big(bd+c^2+2\Big)\Big)-32\Big(a^2-2\Big)c\sinh(\zeta)-16\Big(a^2-1\Big)c\cosh(2\zeta)+8\Big(3a^2-2\Big)\sinh(2\zeta)$$

$$+3a^2\cosh(3\zeta)\Big)\Bigg)+\frac{1}{s^{2p+1}}\Bigg(-\frac{1}{64b}\Big(-13a^4\sinh(\zeta)-9a^4\sinh(3\zeta)-23a^4\cosh(2\zeta)-a^4\cosh(4\zeta)$$

$$+13a^3\sinh(\zeta)+9a^3\sinh(3\zeta)+23a^3\cosh(2\zeta)+a^3\cosh(4\zeta)+c\cosh(\zeta)\Big(-75a^2+2a+32bd+32c^2$$

$$+124\Big)+32a^2bd\sinh(\zeta)+64a^2bd\cosh(2\zeta)+32a^2c^2\sinh(\zeta)+64a^2c^2\cosh(2\zeta)-160a^2c\sinh(2\zeta)$$

$$-45a^2c\cosh(3\zeta)+84a^2\sinh(\zeta)+36a^2\sinh(3\zeta)+117a^2\cosh(2\zeta)+3a^2\cosh(4\zeta)-32abd\sinh(\zeta)$$

$$-64abd\cosh(2\zeta)+16ac\sinh(2\zeta)+6ac\cosh(3\zeta)-30a\sinh(\zeta)-6a\sinh(3\zeta)-32a\cosh(2\zeta)$$

$$-64bd\sinh(\zeta)-16bd\cosh(2\zeta)-96c^2\sinh(\zeta)-80c^2\cosh(2\zeta)+144c\sinh(2\zeta)+12c\cosh(3\zeta)$$

$$-92\sinh(\zeta)-12\sinh(3\zeta)-80\cosh(2\zeta)\Big)\Bigg)+\cdots. \tag{53}$$

Utilizing the Aboodh inverse transform results in

$$v_1(\zeta,\, t) = \frac{\sinh(\zeta)}{2}+1+\frac{t^p}{\Gamma(p+1)}\left(\frac{1}{4}\cosh(\zeta)\left(2\left(a^2-a-1\right)+(a-1)a\sinh(\zeta)\right)\right)$$

$$+\frac{t^{2p}}{\Gamma(2p+1)}\Bigg(\frac{1}{64}\Big(-3(2a+1)a^2\sinh(3\zeta)+\sinh(\zeta)\Big(a(32-15a(2a+1))+32\Big(bd+c^2+2\Big)\Big) \tag{54}$$

$$+32(a-1)c\sinh(2\zeta)+32(a-1)c\cosh(\zeta)-16(a-1)(a(2a+3)+2)\cosh(2\zeta)\Big)\Bigg)+\cdots.$$

$$v_2(\zeta,\ t) = -\frac{4a^2\sinh(\zeta)+a^2\cosh^2(\zeta)+3a^2+4c\cosh(\zeta)-4\sinh(\zeta)-8}{8b} + \frac{t^p}{\Gamma(p+1)}\Bigg(\frac{1}{64b}\Big(\cosh(\zeta)\Big(45a^2$$

$$-32\Big(bd+c^2+2\Big)\Big)-32\Big(a^2-2\Big)c\sinh(\zeta)-16\Big(a^2-1\Big)c\cosh(2\zeta)+8\Big(3a^2-2\Big)\sinh(2\zeta)$$

$$+3a^2\cosh(3\zeta)\Big)\Bigg)+\frac{t^{2p}}{\Gamma(2p+1)}\Bigg(-\frac{1}{64b}\Big(-13a^4\sinh(\zeta)-9a^4\sinh(3\zeta)-23a^4\cosh(2\zeta)-a^4\cosh(4\zeta)$$

$$+13a^3\sinh(\zeta)+9a^3\sinh(3\zeta)+23a^3\cosh(2\zeta)+a^3\cosh(4\zeta)+c\cosh(\zeta)\Big(-75a^2+2a+32bd+32c^2$$

$$+124\Big)+32a^2bd\sinh(\zeta)+64a^2bd\cosh(2\zeta)+32a^2c^2\sinh(\zeta)+64a^2c^2\cosh(2\zeta)-160a^2c\sinh(2\zeta)$$

$$-45a^2c\cosh(3\zeta)+84a^2\sinh(\zeta)+36a^2\sinh(3\zeta)+117a^2\cosh(2\zeta)+3a^2\cosh(4\zeta)-32abd\sinh(\zeta)$$

$$-64abd\cosh(2\zeta)+16ac\sinh(2\zeta)+6ac\cosh(3\zeta)-30a\sinh(\zeta)-6a\sinh(3\zeta)-32a\cosh(2\zeta)$$

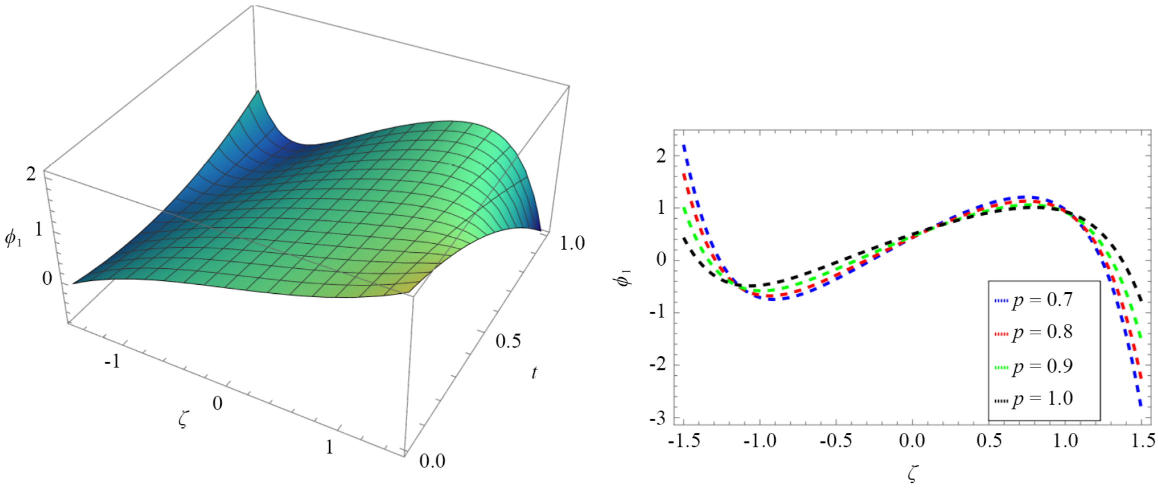
$$-64bd\sinh(\zeta)-16bd\cosh(2\zeta)-96c^2\sinh(\zeta)-80c^2\cosh(2\zeta)+144c\sinh(2\zeta)+12c\cosh(3\zeta)$$

$$-92\sinh(\zeta)-12\sinh(3\zeta)-80\cosh(2\zeta)\Big)\Bigg)+\cdots. \tag{55}$$

Table 1 presents a comparison of various fractional orders, for example, 1 of $\phi_1(\zeta,\ t)$ at $t = 0.1$ using the Aboodh residual power series method (ARPSM) applied to coupled Whitham-Broer-Kaup equations. The parameters a, b, c, and d are set to 1, 0.9, -0.5, and 1, respectively. Figure 1 illustrates the ARPSM solution for $p = 1$ in Figure 1 (a) and compares different values of p, for example, 1 of $\phi_1(\zeta,\ t)$ in Figure 1 (b) at the specified parameter values.

Table 1. Solution for different values of p of Example 1 of $\phi_1(\zeta,\ t)$ for $a = 1$, $b = 0.9$, $c = -0.5$, $d = 1$ and $t = 0.1$ using ARPSM

ζ	$\text{ARPSM}_{p=0.7}$	$\text{ARPSM}_{p=0.8}$	$\text{ARPSM}_{p=0.9}$	$\text{ARPSM}_{p=1.0}$
-1.0	0.235469	0.276993	0.309192	0.334074
-0.8	0.393945	0.433844	0.463942	0.48671
-0.6	0.536249	0.572441	0.599583	0.620018
-0.4	0.664326	0.696283	0.720447	0.73876
-0.2	0.781182	0.809169	0.830755	0.847369
0.0	0.890206	0.914917	0.934551	0.95
0.2	0.994823	1.01725	1.03572	1.05062
0.4	1.09828	1.11976	1.13804	1.15313
0.6	1.20344	1.22583	1.24524	1.26144
0.8	1.31237	1.33857	1.36099	1.37955
1.0	1.42569	1.46043	1.48882	1.51162



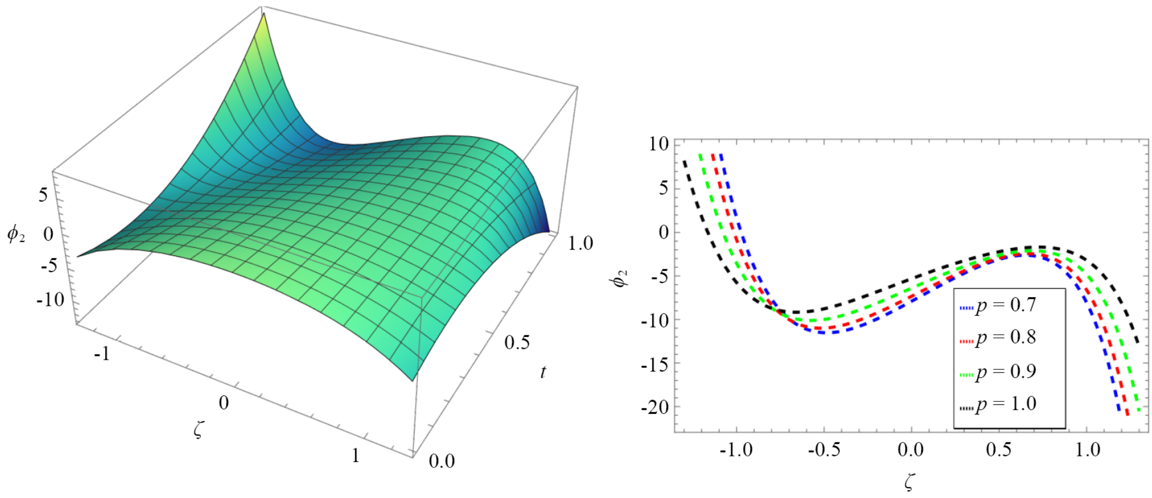
(a) ARPSM solution for $p = 1$ (b) Different fractional order

Figure 1. (a) ARPSM solution for $p = 1$, (b) comparison of different values of p of Example 1 of $\phi_1(\zeta, t)$ for $a = 1$, $b = 0.9$, $c = -0.5$, $d = 1$ and $t = 0.1$

Table 2 displays a similar comparison, but, for example, 1 of $\phi_2(\zeta, t)$ at $t = 0.1$ using ARPSM applied to the same coupled Whitham-Broer-Kaup equations. The parameters are kept consistent with those in Table 1. Figure 2 presents the ARPSM solution for $p = 1$ in Figure 2 (a) and compares various fractional orders, for example, 1 of $\phi_2(\zeta, t)$ in Figure 2 (b) under the specified parameter conditions. These tables and figures provide insights into the behavior of the solutions obtained using ARPSM for different values of p in the context of the coupled Whitham-Broer-Kaup equations.

Table 2. Solution for different values of p of Example 1 of $\phi_2(\zeta, t)$ for $a = 1$, $b = 0.9$, $c = -0.5$, $d = 1$ and $t = 0.1$ using ARPSM

ζ	$\text{ARPSM}_{p=0.7}$	$\text{ARPSM}_{p=0.8}$	$\text{ARPSM}_{p=0.9}$	$\text{ARPSM}_{p=1.0}$
-1.0	0.548933	0.620522	0.674068	0.714297
-0.8	0.600848	0.66057	0.706811	0.742505
-0.6	0.634534	0.685478	0.726018	0.757953
-0.4	0.656518	0.700649	0.736748	0.765741
-0.2	0.670593	0.709512	0.742334	0.769243
0.0	0.678792	0.714074	0.744789	0.770486
0.2	0.681634	0.71512	0.745	0.770378
0.4	0.677881	0.712146	0.742764	0.76879
0.6	0.663685	0.702964	0.736616	0.764488
0.8	0.630695	0.682764	0.723362	0.754863
1.0	0.562133	0.64216	0.69709	0.735374



(a) ARPSM solution for $p = 1$

(b) Different fractional order

Figure 2. (a) ARPSM solution for $p = 1$, (b) comparison of different values of p of example 1 of $\phi_2(\zeta,\ t)$ for $a = 1$, $b = 0.9$, $c = -0.5$, $d = 1$ and $t = 0.1$

3.2 *Example 2 via ARPSM*

Let us take the WBK system of the following form:

$$D_t^p\phi_1(\zeta,\ t)+\phi_1(\zeta,\ t)\frac{\partial\phi_1}{\partial\zeta}(\zeta,\ t)+\frac{1}{2}\frac{\partial\phi_1}{\partial\zeta}(\zeta,\ t)+\frac{\partial\phi_2}{\partial\zeta}(\zeta,\ t)=0, \tag{56}$$

$$D_t^p\phi_2(\zeta,\ t)+\phi_1(\zeta,\ t)\frac{\partial\phi_2}{\partial\zeta}(\zeta,\ t)+\phi_2(\zeta,\ t)\frac{\partial\phi_1}{\partial\zeta}(\zeta,\ t)-\frac{1}{2}\frac{\partial^2\phi_2}{\partial\zeta^2}(\zeta,\ t)=0, \tag{57}$$

where $0 < p \leq 1$.

Initial conditions are:

$$\phi_1(\zeta,\ 0)=\chi-k\coth(k(\Theta+\zeta)), \tag{58}$$

$$\phi_2(\zeta,\ 0)=-k^2\text{csch}^2(k(\Theta+\zeta)), \tag{59}$$

and exact solution

$$\phi_1(\zeta,\ t)=\chi-k\coth(k(\Theta-t\chi+\zeta)). \tag{60}$$

$$\phi_2(\zeta,\ t)=-k^2\text{csch}^2(k(\Theta-t\chi+\zeta)). \tag{61}$$

Eq. (58) and (59) are used, and utilizing the AT on Eqs. (56) and (57):

$$\phi_1(\zeta,\, t)-\frac{\chi-k\coth(k(\Theta+\zeta))}{s^2}+\frac{1}{s^p}\mathcal{A}_t\left[\mathcal{A}_t^{-1}\phi_1(\zeta,\, t)\times\frac{\partial\mathcal{A}_t^{-1}\phi_1}{\partial\zeta}(\zeta,\, t)\right]$$

$$+\frac{1}{2s^p}\left[\frac{\partial\phi_1}{\partial\zeta}(\zeta,\, t)\right]+\frac{1}{s^p}\left[\frac{\partial\phi_2}{\partial\zeta}(\zeta,\, t)\right]=0, \tag{62}$$

$$\phi_2(\zeta,\, t)-\frac{-k^2\mathrm{csch}^2(k(\Theta+\zeta))}{s^2}+\frac{1}{s^p}\mathcal{A}_t\left[\mathcal{A}_t^{-1}\phi_1(\zeta,\, t)\times\frac{\partial\mathcal{A}_t^{-1}\phi_2}{\partial\zeta}(\zeta,\, t)\right]$$

$$+\mathcal{A}_t\left[\mathcal{A}_t^{-1}\phi_2(\zeta,\, t)\times\frac{\partial\mathcal{A}_t^{-1}\phi_1}{\partial\zeta}(\zeta,\, t)\right]-\frac{1}{2}\left[\frac{\partial^2\phi_2}{\partial\zeta^2}(\zeta,\, t)\right]=0. \tag{63}$$

The k^{th} truncated term series are

$$\phi_1(\zeta,\, s)=\frac{\chi-k\coth(k(\Theta+\zeta))}{s^2}+\sum_{r=1}^{k}\frac{f_r(\zeta,\, s)}{s^{rp+1}},\ \ r=1,\, 2,\, 3,\, 4\,\cdots. \tag{64}$$

$$\phi_2(\zeta,\, s)=\frac{-k^2\mathrm{csch}^2(k(\Theta+\zeta))}{s^2}+\sum_{r=1}^{k}\frac{g_r(\zeta,\, s)}{s^{rp+1}},\ \ r=1,\, 2,\, 3,\, 4\,\cdots. \tag{65}$$

Aboodh residual functions (ARFs) are

$$\mathcal{A}_tRes(\zeta,\, s)=\phi_1(\zeta,\, t)-\frac{\chi-k\coth(k(\Theta+\zeta))}{s^2}+\frac{1}{s^p}\mathcal{A}_t\left[\mathcal{A}_t^{-1}\phi_1(\zeta,\, t)\times\frac{\partial\mathcal{A}_t^{-1}\phi_1}{\partial\zeta}(\zeta,\, t)\right]$$

$$+\frac{1}{2s^p}\left[\frac{\partial\phi_1}{\partial\zeta}(\zeta,\, t)\right]+\frac{1}{s^p}\left[\frac{\partial\phi_2}{\partial\zeta}(\zeta,\, t)\right]=0 \tag{66}$$

$$\mathcal{A}_tRes(\zeta,\, s)=\phi_2(\zeta,\, t)-\frac{-k^2\mathrm{csch}^2(k(\Theta+\zeta))}{s^2}+\frac{1}{s^p}\mathcal{A}_t\left[\mathcal{A}_t^{-1}\phi_1(\zeta,\, t)\times\frac{\partial\mathcal{A}_t^{-1}\phi_2}{\partial\zeta}(\zeta,\, t)\right]$$

$$+\mathcal{A}_t\left[\mathcal{A}_t^{-1}\phi_2(\zeta,\, t)\times\frac{\partial\mathcal{A}_t^{-1}\phi_1}{\partial\zeta}(\zeta,\, t)\right]-\frac{1}{2}\left[\frac{\partial^2\phi_2}{\partial\zeta^2}(\zeta,\, t)\right]=0 \tag{67}$$

and the k^{th}-LRFs as:

$$\mathcal{A}_t Res_k(\zeta,\, s) = \phi_{1k}(\zeta,\, t) - \frac{\chi - k\coth(k(\Theta+\zeta))}{s^2} + \frac{1}{s^p}\mathcal{A}_t\left[\mathcal{A}_t^{-1}\phi_{1k}(\zeta,\, t) \times \frac{\partial \mathcal{A}_t^{-1}\phi_{1k}}{\partial \zeta}(\zeta,\, t)\right]$$

$$+ \frac{1}{2s^p}\left[\frac{\partial \phi_{1k}}{\partial \zeta}(\zeta,\, t)\right] + \frac{1}{s^p}\left[\frac{\partial \phi_{2k}}{\partial \zeta}(\zeta,\, t)\right] = 0 \tag{68}$$

$$\mathcal{A}_t Res_k(\zeta,\, s) = \phi_{2k}(\zeta,\, t) - \frac{-k^2\operatorname{csch}^2(k(\Theta+\zeta))}{s^2} + \frac{1}{s^p}\mathcal{A}_t\left[\mathcal{A}_t^{-1}\phi_{1k}(\zeta,\, t) \times \frac{\partial \mathcal{A}_t^{-1}\phi_{2k}}{\partial \zeta}(\zeta,\, t)\right]$$

$$+ \mathcal{A}_t\left[\mathcal{A}_t^{-1}\phi_{2k}(\zeta,\, t) \times \frac{\partial \mathcal{A}_t^{-1}\phi_{1k}}{\partial \zeta}(\zeta,\, t)\right] - \frac{1}{2}\left[\frac{\partial^2 \phi_{2k}}{\partial \zeta^2}(\zeta,\, t)\right] = 0 \tag{69}$$

These are the steps to get $f_r(\zeta,\ s)$ and $g_r(\zeta,\ s)$: s^{rp+1} will be multiplied throughout the equation, insert the r^{th}-truncated series from Eq. (64) and (65) into the r^{th} residual function of Aboodh represented by Eq. (68) and (69). Then $\lim_{s\to\infty}(s^{rp+1}A_t Resv_1,\ r(\zeta,\ s)) = 0$ and $\lim_{s\to\infty}(s^{rp+1}A_t Resv_2,\ r(\zeta,\ s)) = 0$ for $r = 1,\ 2,\ 3,\ \cdots$, are solve iteratively. Some of the terms are:

$$f_1(\zeta,\, s) = -\frac{1}{2}k^2\operatorname{csch}^2(k(\Theta+\zeta))(2k\coth(k(\Theta+\zeta)) + 2\chi + 1), \tag{70}$$

$$g_1(\zeta,\, s) = -2k^3\chi\coth(k(\Theta+\zeta))\operatorname{csch}^2(k(\Theta+\zeta)),$$

$$f_2(\zeta,\, s) = \frac{1}{2}k^3\operatorname{csch}^2(k(\Theta+\zeta))\Big(-\coth(k(\Theta+\zeta))\Big((2\chi+1)^2 - 4k^2\operatorname{csch}^2(k(\Theta+\zeta))\Big)$$

$$+ 4k^2\coth^3(k(\Theta+\zeta)) - 8k\chi\coth^2(k(\Theta+\zeta)) - 4k\chi\operatorname{csch}^2(k(\Theta+\zeta))\Big),$$

$$g_2(\zeta,\, s) = \frac{1}{2}k^4\operatorname{csch}^6(k(\Theta+\zeta))\Big(\Big(4k^2 - 2\chi^2\Big)\cosh(2k(\Theta+\zeta)) + 6k^2 - \chi^2\cosh(4k(\Theta+\zeta))$$

$$+ 2k\sinh(2k(\Theta+\zeta)) + 3\chi^2\Big). \tag{71}$$

and so on.

$f_r(\zeta,\ s)$ and $g_r(\zeta,\ s)$, for $r = 1,\ 2,\ 3,\ \cdots$, are substituted in Eq. (64), (65):

$$\phi_1(\zeta,\, s) = \frac{\chi - k\coth(k(\Theta - t\chi + \zeta))}{s} - \frac{\frac{1}{2}k^2\text{csch}^2(k(\Theta+\zeta))(2k\coth(k(\Theta+\zeta)) + 2\chi + 1)}{s^{p+1}}$$

$$+ \Big(\frac{1}{2}k^3\text{csch}^2(k(\Theta+\zeta))\Big(-\coth(k(\Theta+\zeta))\Big((2\chi+1)^2 - 4k^2\text{csch}^2(k(\Theta+\zeta))\Big)$$

$$+ 4k^2\coth^3(k(\Theta+\zeta)) - 8k\chi\coth^2(k(\Theta+\zeta)) - 4k\chi\text{csch}^2(k(\Theta+\zeta))\Big)\Big)/(s^{2p+1}) + \cdots. \tag{72}$$

$$\phi_2(\zeta,\, s) = \frac{-k^2\text{csch}^2(k(\Theta+\zeta))}{s} - \frac{2k^3\chi\coth(k(\Theta+\zeta))\text{csch}^2(k(\Theta+\zeta))}{s^{p+1}}$$

$$+ \Big(\frac{1}{2}k^4\text{csch}^6(k(\Theta+\zeta))\Big(\Big(4k^2 - 2\chi^2\Big)\cosh(2k(\Theta+\zeta)) + 6k^2 - \chi^2\cosh(4k(\Theta+\zeta))$$

$$+ 2k\sinh(2k(\Theta+\zeta)) + 3\chi^2\Big)\Big)/s^{2p+1} + \cdots. \tag{73}$$

Utilizing the Aboodh inverse transform results in

$$\phi_1(\zeta,\, t) = \chi - k\coth(k(\Theta - t\chi + \zeta)) - \frac{t^p}{\Gamma(p+1)}\Big(\frac{1}{2}k^2\text{csch}^2(k(\Theta+\zeta))(2k\coth(k(\Theta+\zeta)) + 2\chi + 1)\Big)$$

$$+ \frac{t^{2p}}{\Gamma(2p+1)}\Big(\frac{1}{2}k^3\text{csch}^2(k(\Theta+\zeta))\Big(-\coth(k(\Theta+\zeta))\Big((2\chi+1)^2 - 4k^2\text{csch}^2(k(\Theta+\zeta))\Big)$$

$$+ 4k^2\coth^3(k(\Theta+\zeta)) - 8k\chi\coth^2(k(\Theta+\zeta)) - 4k\chi\text{csch}^2(k(\Theta+\zeta))\Big)\Big) + \cdots. \tag{74}$$

$$\phi_2(\zeta,\, t) = -k^2\text{csch}^2(k(\Theta+\zeta)) - \frac{t^p}{\Gamma(p+1)}\Big(2k^3\chi\coth(k(\Theta+\zeta))\text{csch}^2(k(\Theta+\zeta))\Big)$$

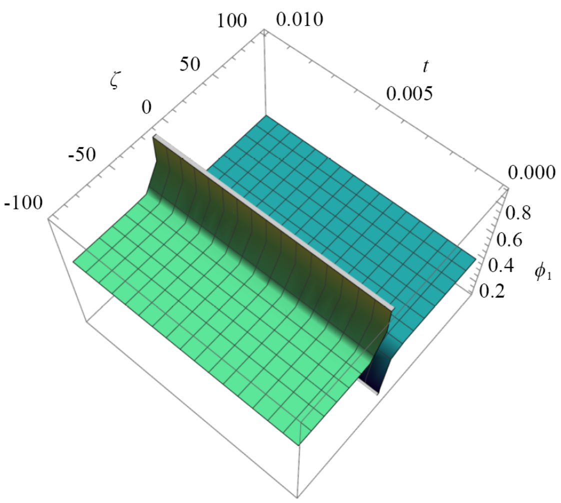
$$+ \frac{t^{2p}}{\Gamma(2p+1)}\Big(\frac{1}{2}k^4\text{csch}^6(k(\Theta+\zeta))\Big(\Big(4k^2 - 2\chi^2\Big)\cosh(2k(\Theta+\zeta)) + 6k^2 - \chi^2\cosh(4k(\Theta+\zeta))$$

$$+ 2k\sinh(2k(\Theta+\zeta)) + 3\chi^2\Big)\Big) + \cdots. \tag{75}$$

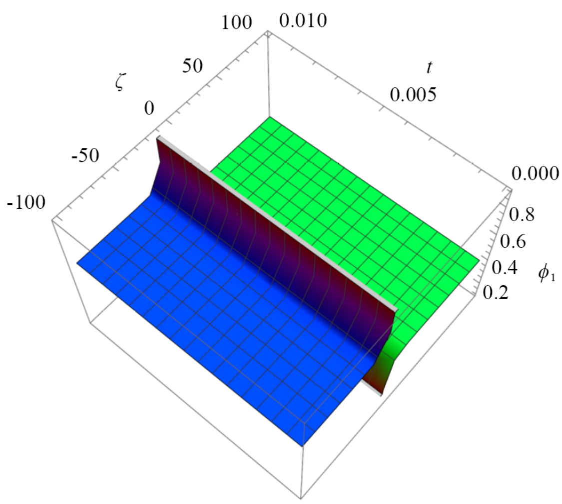
Table 3 presents a comparison of various fractional orders for $\phi_1(\zeta,\, t)$ in Example 2 of the coupled Whitham-Broer-Kaup equations at $t = 0.01$, $\chi = 0.5$, and $\Theta = 10$ using the Aboodh residual power series method (ARPSM). Figure 3 illustrates the results, with Figure 3 (a) depicting the ARPSM solution for $p = 1$, Figure 3 (b) showing the exact solution, and Figure 3 (c) presenting the comparison of various fractional orders for $\phi_1(\zeta,\, t)$.

Table 3. Solution for different values of p of Example 2 of $\phi_1(\zeta,\ t)$ for $t = 0.01$, $\chi = 0.5$ and $\Theta = 10$ using ARPSM

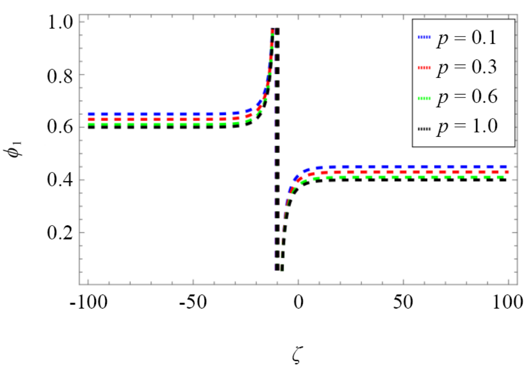
ζ	$\text{ARPSM}_{p=0.6}$	$\text{ARPSM}_{p=0.8}$	$\text{ARPSM}_{p=1.0}$	Exact	$\text{Error}_{p=0.6}$	$\text{Error}_{p=0.8}$	$\text{Error}_{p=1}$
-1.0	0.354925	0.358312	0.359849	0.360346	0.00542066	0.00203399	4.97053×10^{-4}
-0.8	0.357082	0.360274	0.361725	0.362195	0.00511259	0.00192008	4.6935×10^{-4}
-0.6	0.359118	0.36213	0.363501	0.363944	0.00482625	0.00181405	4.43552×10^{-4}
-0.4	0.361041	0.363886	0.365182	0.365601	0.00455972	0.00171523	4.19496×10^{-4}
-0.2	0.36286	0.365548	0.366774	0.367171	0.00431128	0.001623	3.97035×10^{-4}
0.0	0.364581	0.367123	0.368284	0.36866	0.0040794	0.00153681	3.76036×10^{-4}
0.2	0.36621	0.368617	0.369717	0.370073	0.00386271	0.00145616	3.56381×10^{-4}
0.4	0.367754	0.370034	0.371076	0.371414	0.00365996	0.00138063	3.37964×10^{-4}
0.6	0.369218	0.371378	0.372367	0.372688	0.00347004	0.00130979	3.20687×10^{-4}
0.8	0.370607	0.372655	0.373594	0.373899	0.00329194	0.0012433	3.04464×10^{-4}
1.0	0.371925	0.373869	0.374761	0.37505	0.00312476	0.00118082	2.89215×10^{-4}



(a) ARPSM solution for $p = 1$



(b) Exact solution

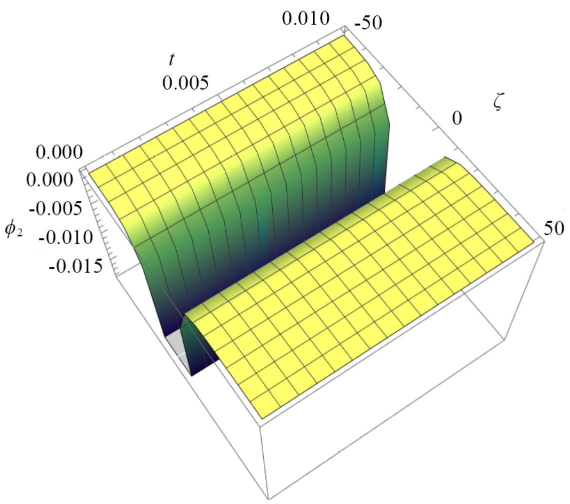


(c) Different fractional order

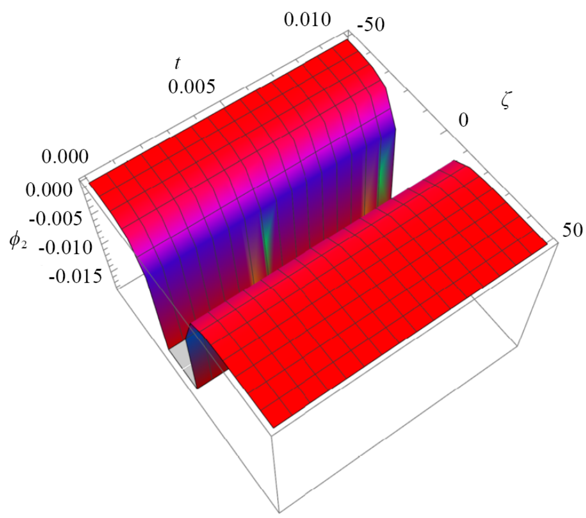
Figure 3. (a) ARPSM solution for $p = 1$, (b) Exact solution, (c) solutions for different values of p of Example 2 of $\phi_1(\zeta,\ t)$ for $t = 0.01$, $\chi = 0.5$ and $\Theta = 10$

Table 4. Solution for different values of p of Example 2 of $\phi_2(\zeta,\ t)$ for $t = 0.01$, $\chi = 0.5$ and $\Theta = 10$ using ARPSM

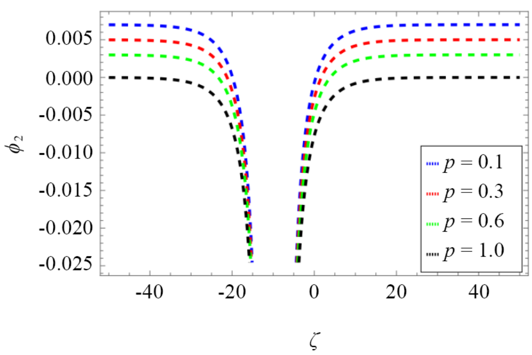
ζ	ARPSM$_{p=0.6}$	ARPSM$_{p=0.8}$	ARPSM$_{p=1.0}$	Exact	Error$_{p=0.6}$	Error$_{p=0.8}$	Error$_{p=1}$
-1.0	-0.00996764	-0.0097054	-0.00951657	-0.0095033	0.000464345	0.000202098	1.32766×10^{-5}
-0.8	-0.00942358	-0.00917896	-0.00900273	-0.00899034	0.00043324	0.00018862	1.23939×10^{-5}
-0.6	-0.00891592	-0.00868745	-0.00852277	-0.00851119	0.000404739	0.000176265	1.15845×10^{-5}
-0.4	-0.00844164	-0.00822798	-0.0080739	-0.00806306	0.000378574	0.000164919	1.0841×10^{-5}
-0.2	-0.00799797	-0.00779794	-0.00765362	-0.00764346	0.000354513	0.000154482	1.01569×10^{-5}
0.0	-0.00758248	-0.007395	-0.00725966	-0.00725013	0.000332348	0.000144864	9.52628×10^{-6}
0.2	-0.00719293	-0.00701702	-0.00688998	-0.00688103	0.000311898	0.000135986	8.94412×10^{-6}
0.4	-0.00682732	-0.0066621	-0.00654273	-0.00653432	0.000293001	0.00012778	8.40584×10^{-6}
0.6	-0.00648384	-0.00632851	-0.00621623	-0.00620832	0.000275511	0.000120183	7.90740×10^{-6}
0.8	-0.00616083	-0.00601466	-0.00590897	-0.00590152	0.000259303	0.000113139	7.44519×10^{-6}
1.0	-0.00585679	-0.00571913	-0.00561955	-0.00561253	0.00024426	0.000106601	7.01601×10^{-6}



(a) ARPSM solution for $p = 1$



(b) Exact solution



(c) Different fractional order

Figure 4. (a) ARPSM solution for $p = 1$, (b) Exact solution, (c) solutions for different values of p of Example 2 of $\phi_2(\zeta,\ t)$ for $t = 0.01$, $\chi = 0.5$ and $\Theta = 10$

Table 4 compares $\phi_2(\zeta,\, t)$ in Example 2 of the coupled Whitham-Broer-Kaup equations under the same conditions. Correspondingly, Figure 4 provides visual representations, with Figure 4 (a) demonstrating the ARPSM solution for $p = 1$, Figure 4 (b) depicting the exact solution, and Figure 4 (c) showcasing the comparison of various fractional orders for $\phi_2(\zeta,\, t)$. These tables and figures offer insights into the behavior of the solutions obtained using ARPSM and facilitate comparisons with the exact solutions, highlighting the method's efficacy across different fractional orders.

4. Aboodh transform iterative approach methodology

Consider the general PDE of fractional order.

$$D_t^p v(\zeta,\, t) = \Phi\Big(v(\zeta,\, t),\, D_\zeta^v v(\zeta,\, t),\, D_\zeta^{2v} v(\zeta,\, t),\, D_\zeta^{3v} v(\zeta,\, t)\Big),\;\; 0 < p,\, v \leq 1, \tag{76}$$

Having the IC's

$$v^{(k)}(\zeta,\, 0) = h_k,\;\; k = 0,\, 1,\, 2,\, \cdots,\, m-1, \tag{77}$$

Consider $v(\zeta,\, t)$ is unknown function and $\Phi\Big(v(\zeta,\, t),\, D_\zeta^v v(\zeta,\, t),\, D_\zeta^{2v} v(\zeta,\, t),\, D_\zeta^{3v} v(\zeta,\, t)\Big)$ can be linear or nonlinear operator of $v(\zeta,\, t)$, $D_\zeta^v v(\zeta,\, t)$, $D_\zeta^{2v} v(\zeta,\, t)$ and $D_\zeta^{3v} v(\zeta,\, t)$. When AT is applied on Eq. (76) we deduce:

$$A[v(\zeta,\, t)] = \frac{1}{s^p}\left(\sum_{k=0}^{m-1} \frac{v^{(k)}(\zeta,\, 0)}{s^{2-p+k}} + A\Big[\Phi\Big(v(\zeta,\, t),\, D_\zeta^v v(\zeta,\, t),\, D_\zeta^{2v} v(\zeta,\, t),\, D_\zeta^{3v} v(\zeta,\, t)\Big)\Big]\right), \tag{78}$$

The IAT gives us:

$$v(\zeta,\, t) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1} \frac{v^{(k)}(\zeta,\, 0)}{s^{2-p+k}} + A\Big[\Phi\Big(v(\zeta,\, t),\, D_\zeta^v v(\zeta,\, t),\, D_\zeta^{2v} v(\zeta,\, t),\, D_\zeta^{3v} v(\zeta,\, t)\Big)\Big]\right)\right]. \tag{79}$$

An infinite series solution is produced by iteratively applying the AT.

$$v(\zeta,\, t) = \sum_{i=0}^{\infty} v_i. \tag{80}$$

In this case, $\Phi\Big(v,\, D_\zeta^v v,\, D_\zeta^{2v} v,\, D_\zeta^{3v} v\Big)$ are operator, which can be decomposed into the following:

$$\Phi\Big(v,\ D_\zeta^{v}v,\ D_\zeta^{2v}v,\ D_\zeta^{3v}v\Big)=\Phi\Big(v_0,\ D_\zeta^{v}v_0,\ D_\zeta^{2v}v_0,\ D_\zeta^{3v}v_0\Big)+\sum_{i=0}^{\infty}\left(\Phi\Big(\sum_{k=0}^{i}\Big(v_k,\ D_\zeta^{v}v_k,\ D_\zeta^{2v}v_k,\ D_\zeta^{3v}v_k\Big)\Big)\right.$$

$$\left.-\Phi\Big(\sum_{k=1}^{i-1}\Big(v_k,\ D_\zeta^{v}v_k,\ D_\zeta^{2v}v_k,\ D_\zeta^{3v}v_k\Big)\Big)\right). \tag{81}$$

Eq. (81) and (80) are substituted in Eq. (79), to obtain:

$$\begin{aligned}\sum_{i=0}^{\infty}v_i(\zeta,\ t)=&A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{v^{(k)}(\zeta,\ 0)}{s^{2-p+k}}+A\left[\Phi(v_0,\ D_\zeta^{v}v_0,\ D_\zeta^{2v}v_0,\ D_\zeta^{3v}v_0)\right]\right)\right]\\&+A^{-1}\left[\frac{1}{s^p}\left(A\left[\sum_{i=0}^{\infty}\left(\Phi\sum_{k=0}^{i}(v_k,\ D_\zeta^{v}v_k,\ D_\zeta^{2v}v_k,\ D_\zeta^{3v}v_k)\right)\right]\right)\right]\\&-A^{-1}\left[\frac{1}{s^p}\left(A\left[\left(\Phi\sum_{k=1}^{i-1}(v_k,\ D_\zeta^{v}v_k,\ D_\zeta^{2v}v_k,\ D_\zeta^{3v}v_k)\right)\right]\right)\right]\end{aligned} \tag{82}$$

$$v_0(\zeta,\ t)=A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{v^{(k)}(\zeta,\ 0)}{s^{2-p+k}}\right)\right],$$

$$v_1(\zeta,\ t)=A^{-1}\left[\frac{1}{s^p}\left(A\left[\Phi(v_0,\ D_\zeta^{v}v_0,\ D_\zeta^{2v}v_0,\ D_\zeta^{3v}v_0)\right]\right)\right],$$

$$\vdots$$

$$\begin{aligned}v_{m+1}(\zeta,\ t)=&A^{-1}\left[\frac{1}{s^p}\left(A\left[\sum_{i=0}^{\infty}\left(\Phi\sum_{k=0}^{i}(v_k,\ D_\zeta^{v}v_k,\ D_\zeta^{2v}v_k,\ D_\zeta^{3v}v_k)\right)\right]\right)\right]\\&-A^{-1}\left[\frac{1}{s^p}\left(A\left[\left(\Phi\sum_{k=1}^{i-1}(v_k,\ D_\zeta^{v}v_k,\ D_\zeta^{2v}v_k,\ D_\zeta^{3v}v_k)\right)\right]\right)\right],\ \ m=1,\ 2,\ \cdots.\end{aligned} \tag{83}$$

The m-term approximation of Eq. (76) is given as:

$$v(\zeta,\ t)=\sum_{i=0}^{m-1}v_i. \tag{84}$$

4.1 *Example 1 via ATIM*

$$D_t^p v_1(\zeta,\, t) = -a\phi_1(\zeta,\, t)\frac{\partial \phi_1}{\partial \zeta}(\zeta,\, t) - b\frac{\partial \phi_2}{\partial \zeta}(\zeta,\, t) - c\frac{\partial^2 \phi_1}{\partial \zeta^2}(\zeta,\, t), \tag{85}$$

$$D_t^p v_2(\zeta,\, t) = -\phi_1(\zeta,\, t)\frac{\partial \phi_2}{\partial \zeta}(\zeta,\, t) - \phi_2(\zeta,\, t)\frac{\partial \phi_1}{\partial \zeta}(\zeta,\, t) - d\frac{\partial^3 \phi_1}{\partial \zeta^3}(\zeta,\, t) + c\frac{\partial^2 \phi_2}{\partial \zeta^2}(\zeta,\, t), \quad \text{where} \quad 0 < p \leq 1 \tag{86}$$

Initial conditions are:

$$v_1(\zeta,\, 0) = \frac{\sinh(\zeta)}{2} + 1, \tag{87}$$

$$v_2(\zeta,\, 0) = -\frac{4a^2\sinh(\zeta) + a^2\cosh^2(\zeta) + 3a^2 + 4c\cosh(\zeta) - 4\sinh(\zeta) - 8}{8b}, \tag{88}$$

The following equations emerge as a result of applying the AT to both sides of Eqs. (85) and (86).

$$A[D_t^p v_1(\zeta,\, t)] = \frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{v_1^{(k)}(\zeta,\, 0)}{s^{2-p+k}} + A\Big[-a\phi_1(\zeta,\, t)\frac{\partial \phi_1}{\partial \zeta}(\zeta,\, t) - b\frac{\partial \phi_2}{\partial \zeta}(\zeta,\, t) - c\frac{\partial^2 \phi_1}{\partial \zeta^2}(\zeta,\, t)\Big]\right) \tag{89}$$

$$A[D_t^p v_2(\zeta,\, t)] = \frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{v_2^{(k)}(\zeta,\, 0)}{s^{2-p+k}} + A\Big[-\phi_1(\zeta,\, t)\frac{\partial \phi_2}{\partial \zeta}(\zeta,\, t) - \phi_2(\zeta,\, t)\frac{\partial \phi_1}{\partial \zeta}(\zeta,\, t)\right.$$

$$\left. - d\frac{\partial^3 \phi_1}{\partial \zeta^3}(\zeta,\, t) + c\frac{\partial^2 \phi_2}{\partial \zeta^2}(\zeta,\, t)\Big]\right) \tag{90}$$

After solving equations (89) and (90) using the IAT, we have:

$$v_1(\zeta,\, t) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{v_1^{(k)}(\zeta,\, 0)}{s^{2-p+k}} + A\Big[-a\phi_1(\zeta,\, t)\frac{\partial \phi_1}{\partial \zeta}(\zeta,\, t) - b\frac{\partial \phi_2}{\partial \zeta}(\zeta,\, t) - c\frac{\partial^2 \phi_1}{\partial \zeta^2}(\zeta,\, t)\Big]\right)\right] \tag{91}$$

$$v_2(\zeta,\, t) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{v_2^{(k)}(\zeta,\, 0)}{s^{2-p+k}} + A\Big[-\phi_1(\zeta,\, t)\frac{\partial \phi_2}{\partial \zeta}(\zeta,\, t) - \phi_2(\zeta,\, t)\frac{\partial \phi_1}{\partial \zeta}(\zeta,\, t) - d\frac{\partial^3 \phi_1}{\partial \zeta^3}(\zeta,\, t)\right.\right.$$

$$\left.\left. + c\frac{\partial^2 \phi_2}{\partial \zeta^2}(\zeta,\, t)\Big]\right)\right] \tag{92}$$

By iteratively using the AT, we obtain:

$$
\begin{aligned}
(v_1)_0(\zeta,\, t) &= A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{v_1^{(k)}(\zeta,\, 0)}{s^{2-p+k}}\right)\right]\\
&= A^{-1}\left[\frac{v_1(\zeta,\, 0)}{s^2}\right]\\
&= \frac{\sinh(\zeta)}{2}+1,\\
(v_2)_0(\zeta,\, t) &= A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{v_2^{(k)}(\zeta,\, 0)}{s^{2-p+k}}\right)\right]\\
&= A^{-1}\left[\frac{v_2(\zeta,\, 0)}{s^2}\right]\\
&= -\frac{4a^2\sinh(\zeta)+a^2\cosh^2(\zeta)+3a^2+4c\cosh(\zeta)-4\sinh(\zeta)-8}{8b}.
\end{aligned}
$$

After substituted the integral of RL into Eqs. (85) and (86), we get the same result.

$$
v_1(\zeta,\, t) = \frac{\sinh(\zeta)}{2}+1+A\left[-a\phi_1(\zeta,\, t)\frac{\partial\phi_1}{\partial\zeta}(\zeta,\, t)-b\frac{\partial\phi_2}{\partial\zeta}(\zeta,\, t)-c\frac{\partial^2\phi_1}{\partial\zeta^2}(\zeta,\, t)\right] \tag{93}
$$

$$
\begin{aligned}
v_2(\zeta,\, t) = &-\frac{4a^2\sinh(\zeta)+a^2\cosh^2(\zeta)+3a^2+4c\cosh(\zeta)-4\sinh(\zeta)-8}{8b}\\
&+A\left[-\phi_1(\zeta,\, t)\frac{\partial\phi_2}{\partial\zeta}(\zeta,\, t)-\phi_2(\zeta,\, t)\frac{\partial\phi_1}{\partial\zeta}(\zeta,\, t)-d\frac{\partial^3\phi_1}{\partial\zeta^3}(\zeta,\, t)+c\frac{\partial^2\phi_2}{\partial\zeta^2}(\zeta,\, t)\right]
\end{aligned} \tag{94}
$$

The following terms are obtained in accordance with the ATIM procedure:

$$
\begin{aligned}
v_{10}(\zeta,\, t) &= \frac{\sinh(\zeta)}{2}+1,\\
v_{20}(\zeta,\, t) &= -\frac{4a^2\sinh(\zeta)+a^2\cosh^2(\zeta)+3a^2+4c\cosh(\zeta)-4\sinh(\zeta)-8}{8b},
\end{aligned} \tag{95}
$$

$$v_{11}(\zeta,\ t)=\frac{t^{p}\cosh(\zeta)((a-1)a\sinh(\zeta)+2(a-1)a-2)}{4\Gamma(p+1)},$$

$$v_{21}(\zeta,\ t)=\Big(t^{p}\Big(\cosh(\zeta)\Big(45a^{2}-32\Big(bd+c^{2}+2\Big)\Big)-32\Big(a^{2}-2\Big)c\sinh(\zeta)$$

$$-16\Big(a^{2}-1\Big)c\cosh(2\zeta)+8\Big(3a^{2}-2\Big)\sinh(2\zeta)+3a^{2}\cosh(3\zeta)\Big)\Big)/\Big(64bp\Gamma(p)\Big), \tag{96}$$

$$v_{12}(\zeta,\ t)=\frac{1}{64}t^{2p}\Big(\Big(-3(2a+1)a^{2}\sinh(3\zeta)+\sinh(\zeta)\Big(a(32-15a(2a+1))+32\Big(bd+c^{2}+2\Big)\Big)$$

$$+32(a-1)c\sinh(2\zeta)+32(a-1)c\cosh(\zeta)-16(a-1)(a(2a+3)+2)\cosh(2\zeta)\Big)/\Big(\Gamma(2p+1)\Big)$$

$$-\Big(a4^{p+1}t^{p}\Gamma\Big(p+\frac{1}{2}\Big)\cosh(\zeta)((a-1)a\sinh(\zeta)+2(a-1)a-2)((a-1)a(2\sinh(\zeta)+\cosh(2\zeta))$$

$$-2\sinh(\zeta))\Big)/\Big(\sqrt{\pi}\Gamma(p+1)\Gamma(3p+1)\Big)\Big).$$

$$v_{22}(\zeta,\ t)=\frac{t^{2p}}{1{,}024b}\Big(\Big(8\Big(((a-1)a-3)a^{2}\cosh(4\zeta)+\sinh(\zeta)\Big(a\Big(a\Big(13(a-1)a-4\Big(8bd+8c^{2}+21\Big)\Big)$$

$$+32bd+30\Big)+64bd+96c^{2}+92\Big)-c\cosh(\zeta)\Big(a(2-75a)+4\Big(8bd+8c^{2}+31\Big)\Big)$$

$$+\cosh(2\zeta)\Big(a\Big(a\Big(23(a-1)a-64bd-64c^{2}-117\Big)+64bd+32\Big)+16\Big(bd+5c^{2}+5\Big)\Big)$$

$$+16(a-1)(10a+9)c\sinh(2\zeta)+3(a(15a-2)-4)c\cosh(3\zeta)+3(a(3a((a-1)a-4)+2)$$

$$+4)\sinh(3\zeta)\Big)\Big)/\Big(p\Gamma(2p)\Big)+\Big(4^{p}t^{p}\Gamma\Big(p+\frac{1}{2}\Big)\Big(32(a-1)\Big(a^{3}-2a-2\Big)c\sinh(\zeta)+96(a-1)\Big(3a^{3}$$

$$-6a-2\Big)c\sinh(3\zeta)-15(a-1)a^{3}\cosh(5\zeta)-128((a-1)a-1)\sinh(2\zeta)\Big(3a^{2}-2\Big(bd+c^{2}+2\Big)\Big)$$

$$+256((a-1)a-1)\Big(a^{2}-2\Big)c\cosh(2\zeta)-16(3a-4)\Big(3a^{2}+a-1\Big)a\sinh(4\zeta)-2\cosh(\zeta)\Big(a\Big(a\Big(69(a$$

$$-1)a-16\Big(bd+c^{2}+7\Big)\Big)+16\Big(bd+c^{2}+4\Big)\Big)+32\Big)-3\cosh(3\zeta)\Big(a\Big(a\Big(141(a-1)a-32\Big(bd+c^{2}+7\Big)\Big)$$

$$+32\Big(bd+c^{2}+4\Big)\Big)+64\Big)+64(a-1)^{2}(a+1)ac\cosh(4\zeta)\Big)\Big)\Big(\sqrt{\pi}\Gamma(p+1)\Gamma(3p+1)\Big)\Big). \tag{97}$$

ATIM final solution are given below:

$$v_1(\zeta,\, t) = v_{10}(\zeta,\, t) + v_{11}(\zeta,\, t) + v_{12}(\zeta,\, t) + \cdots. \tag{98}$$

$$v_2(\zeta,\, t) = v_{20}(\zeta,\, t) + v_{21}(\zeta,\, t) + v_{22}(\zeta,\, t) + \cdots. \tag{99}$$

$$\begin{aligned} v(\zeta,\, t) = {} & \frac{\sinh(\zeta)}{2} + 1 + \frac{t^p \cosh(\zeta)((a-1)a\sinh(\zeta) + 2(a-1)a-2)}{4\Gamma(p+1)} \\ & + \frac{1}{64} t^{2p} \Big(\Big(-3(2a+1)a^2 \sinh(3\zeta) + \sinh(\zeta) \Big(a(32 - 15a(2a+1)) + 32 \Big(bd + c^2 + 2 \Big) \Big) \\ & + 32(a-1)c\sinh(2\zeta) + 32(a-1)c\cosh(\zeta) - 16(a-1)(a(2a+3)+2)\cosh(2\zeta) \Big) / \Big(\Gamma(2p+1) \Big) \\ & - \Big(a4^{p+1} t^p \Gamma\Big(p + \frac{1}{2}\Big) \cosh(\zeta)((a-1)a\sinh(\zeta) + 2(a-1)a-2)((a-1)a(2\sinh(\zeta) + \cosh(2\zeta)) \\ & - 2\sinh(\zeta)) \Big) / \Big(\sqrt{\pi}\Gamma(p+1)\Gamma(3p+1) \Big) \Big) + \cdots. \end{aligned} \tag{100}$$

$$\begin{aligned} w(\zeta,\, t) = {} & -\frac{4a^2\sinh(\zeta) + a^2\cosh^2(\zeta) + 3a^2 + 4c\cosh(\zeta) - 4\sinh(\zeta) - 8}{8b} + \Big(t^p \Big(\cosh(\zeta) \Big(45a^2 - 32 \Big(bd \\ & + c^2 + 2 \Big) \Big) - 32 \Big(a^2 - 2 \Big) c\sinh(\zeta) - 16 \Big(a^2 - 1 \Big) c\cosh(2\zeta) + 8 \Big(3a^2 - 2 \Big) \sinh(2\zeta) + 3a^2\cosh(3\zeta) \Big) \Big) / \\ & \Big(64bp\Gamma(p) \Big) + \frac{t^{2p}}{1,024b} \Big(\Big(8 \Big(((a-1)a-3)a^2\cosh(4\zeta) + \sinh(\zeta) \Big(a \Big(a \Big(13(a-1)a - 4 \Big(8bd + 8c^2 + 21 \Big) \Big) \\ & + 32bd + 30 \Big) + 64bd + 96c^2 + 92 \Big) - c\cosh(\zeta) \Big(a(2 - 75a) + 4 \Big(8bd + 8c^2 + 31 \Big) \Big) \\ & + \cosh(2\zeta) \Big(a \Big(a \Big(23(a-1)a - 64bd - 64c^2 - 117 \Big) + 64bd + 32 \Big) + 16 \Big(bd + 5c^2 + 5 \Big) \Big) \\ & + 16(a-1)(10a+9)c\sinh(2\zeta) + 3(a(15a-2)-4)c\cosh(3\zeta) + 3(a(3a((a-1)a-4)+2) \\ & + 4)\sinh(3\zeta) \Big) \Big) / \Big(p\Gamma(2p) \Big) + \Big(4^p t^p \Gamma\Big(p + \frac{1}{2}\Big) \Big(32(a-1) \Big(a^3 - 2a - 2 \Big) c\sinh(\zeta) + 96(a-1) \Big(3a^3 \\ & - 6a - 2 \Big) c\sinh(3\zeta) - 15(a-1)a^3\cosh(5\zeta) - 128((a-1)a-1)\sinh(2\zeta) \Big(3a^2 - 2 \Big(bd + c^2 + 2 \Big) \Big) \end{aligned}$$

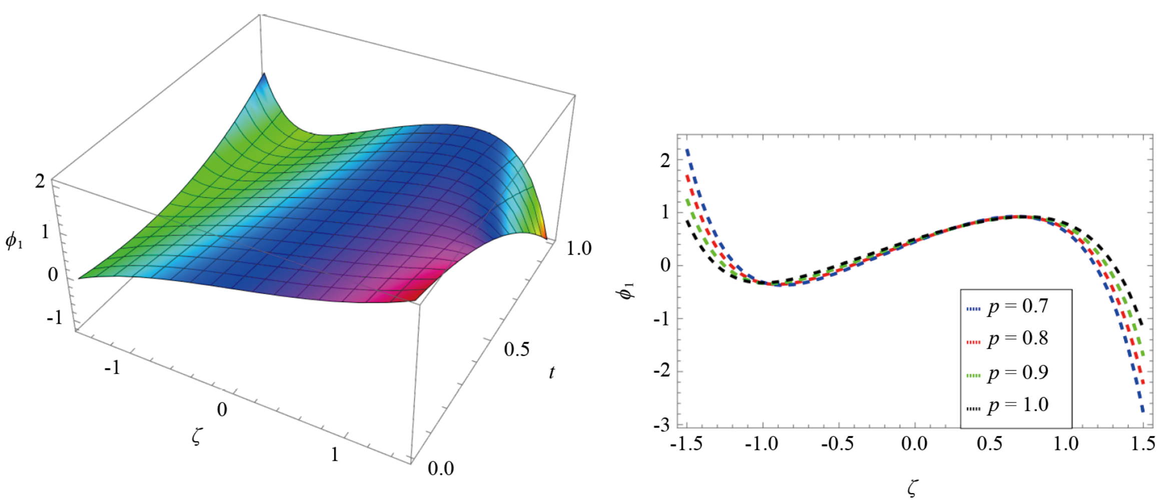
$$+256((a-1)a-1)\left(a^2-2\right)c\cosh(2\zeta)-16(3a-4)\left(3a^2+a-1\right)a\sinh(4\zeta)-2\cosh(\zeta)\Big(a\Big(a\Big(69(a$$

$$-1)a-16\left(bd+c^2+7\right)\Big)+16\left(bd+c^2+4\right)\Big)+32\Big)-3\cosh(3\zeta)\Big(a\Big(a\Big(141(a-1)a-32\left(bd+c^2+7\right)\Big)$$

$$+32\left(bd+c^2+4\right)\Big)+64\Big)+64(a-1)^2(a+1)ac\cosh(4\zeta)\Big)\Big)\left(\sqrt{\pi}\Gamma(p+1)\Gamma(3p+1)\right)\Big)+\cdots. \quad (101)$$

Table 5. Solution for different values of p of Example 1 of $\phi_1(\zeta, t)$ for $a=1$, $b=0.9$, $c=-0.5$, $d=1$ and $t=0.1$ using ATIM

ζ	$\text{ATIM}_{p=0.7}$	$\text{ATIM}_{p=0.8}$	$\text{ATIM}_{p=0.9}$	$\text{ATIM}_{p=1.0}$
-1.0	0.237934	0.277991	0.309585	0.334225
-0.8	0.39556	0.434497	0.4642	0.486809
-0.6	0.537275	0.572856	0.599746	0.620081
-0.4	0.66493	0.696527	0.720543	0.738797
-0.2	0.781462	0.809282	0.8308	0.847386
0.0	0.890206	0.914917	0.934551	0.95
0.2	0.994544	1.01714	1.03568	1.05061
0.4	1.09768	1.11951	1.13795	1.1531
0.6	1.20241	1.22542	1.24508	1.26137
0.8	1.31075	1.33792	1.36073	1.37945
1.0	1.42322	1.45943	1.48843	1.51147



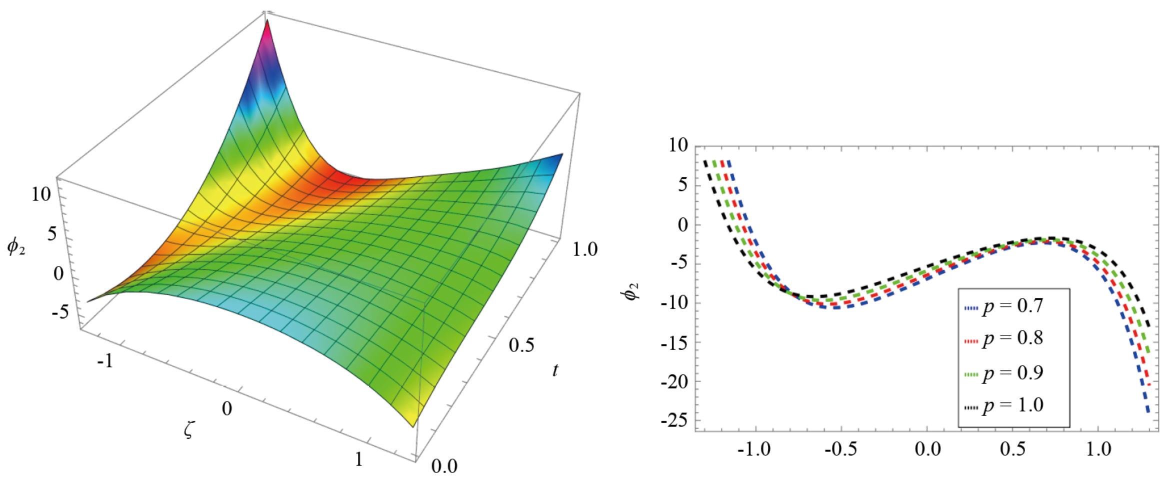
(a) ATIM solution for $p=1$ (b) Different fractional order

Figure 5. (a) ATIM solution for $p=1$, (b) comparison of different values of p of Example 1 of $\phi_1(\zeta, t)$ for $a=1$, $b=0.9$, $c=-0.5$, $d=1$ and $t=0.1$

Table 5 presents a comparison of various fractional orders for the numerical iterative transform method (ATIM) solution of Example 1, focusing on the variable $\phi_1(\zeta, t)$ at $t = 0.1$, with parameters $a = 1$, $b = 0.9$, $c = -0.5$, and $d = 1$ of the coupled Whitham-Broer-Kaup equations. Figure 5 complements this analysis, where Figure 5 (a) displays the ATIM solution for $p = 1$, while Figure 5 (b) illustrates the comparison of various fractional orders, for example, 1 of $\phi_1(\zeta, t)$ at $t = 0.1$, with the same parameter values.

Table 6. Solution for different values of p of Example 1 of $\phi_2(\zeta, t)$ for $a = 1$, $b = 0.9$, $c = -0.5$, $d = 1$ and $t = 0.1$ using ATIM

ζ	$\text{ATIM}_{p=0.7}$	$\text{ATIM}_{p=0.8}$	$\text{ATIM}_{p=0.9}$	$\text{ATIM}_{p=1.0}$
-1.0	0.5534	0.62233	0.67478	0.714571
-0.8	0.604748	0.662148	0.707433	0.742744
-0.6	0.637348	0.686617	0.726467	0.758125
-0.4	0.658247	0.701349	0.737023	0.765847
-0.2	0.671364	0.709824	0.742457	0.76929
0.0	0.678716	0.714043	0.744777	0.770482
0.2	0.680795	0.714781	0.744866	0.770326
0.4	0.676389	0.711542	0.742526	0.768698
0.6	0.661831	0.702214	0.73632	0.764374
0.8	0.629313	0.682205	0.723141	0.754779
1.0	0.563404	0.642674	0.697293	0.735452



(a) ATIM solution for $p = 1$

(b) Different fractional order

Figure 6. (a) ATIM solution for $p = 1$, (b) comparison of different values of p of Example 1 of $\phi_2(\zeta, t)$ for $a = 1$, $b = 0.9$, $c = -0.5$, $d = 1$ and $t = 0.1$

Similarly, Table 6 provides a comparison of various fractional orders for the ATIM solution of Example 1, focusing on the variable $\phi_2(\zeta, t)$ at $t = 0.1$, with parameters $a = 1$, $b = 0.9$, $c = -0.5$, and $d = 1$ of the coupled Whitham-Broer-Kaup equations. Figure 6 complements this analysis, where Figure 6 (a) displays the ATIM solution for $p = 1$, and Figure 6 (b) illustrates the comparison of various fractional orders, for example, 1 of $\phi_2(\zeta, t)$ at $t = 0.1$, with the same parameter values. These tables and figures provide valuable insights into the behavior of the solutions under different fractional orders, facilitating a comprehensive understanding of the coupled Whitham-Broer-Kaup equations.

4.2 *Example 2 via ATIM*

$$D_t^p \phi_1(\zeta,\, t) = -\phi_1(\zeta,\, t)\frac{\partial \phi_1}{\partial \zeta}(\zeta,\, t) - \frac{1}{2}\frac{\partial \phi_1}{\partial \zeta}(\zeta,\, t) - \frac{\partial \phi_2}{\partial \zeta}(\zeta,\, t), \tag{102}$$

$$D_t^p \phi_2(\zeta,\, t) = -\phi_1(\zeta,\, t)\frac{\partial \phi_2}{\partial \zeta}(\zeta,\, t) - \phi_2(\zeta,\, t)\frac{\partial \phi_1}{\partial \zeta}(\zeta,\, t) + \frac{1}{2}\frac{\partial^2 \phi_2}{\partial \zeta^2}(\zeta,\, t), \tag{103}$$

where $0 < p \leq 1$.

Initial conditions are:

$$v_1(\zeta,\, 0) = \chi - k\coth(k(\Theta + \zeta)), \tag{104}$$

$$v_2(\zeta,\, 0) = -k^2\mathrm{csch}^2(k(\Theta + \zeta)), \tag{105}$$

After solving Eqs. (102) and (103) using the IAT, we get:

$$A[D_t^p v_1(\zeta,\, t)] = \frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{v_1^{(k)}(\zeta,\, 0)}{s^{2-p+k}} + A\left[-\phi_1(\zeta,\, t)\frac{\partial \phi_1}{\partial \zeta}(\zeta,\, t) - \frac{1}{2}\frac{\partial \phi_1}{\partial \zeta}(\zeta,\, t) - \frac{\partial \phi_2}{\partial \zeta}(\zeta,\, t)\right]\right) \tag{106}$$

$$A[D_t^p v_2(\zeta,\, t)] = \frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{v_2^{(k)}(\zeta,\, 0)}{s^{2-p+k}} + A\left[-\phi_1(\zeta,\, t)\frac{\partial \phi_2}{\partial \zeta}(\zeta,\, t) - \phi_2(\zeta,\, t)\frac{\partial \phi_1}{\partial \zeta}(\zeta,\, t) + \frac{1}{2}\frac{\partial^2 \phi_2}{\partial \zeta^2}(\zeta,\, t)\right]\right) \tag{107}$$

These equations are obtained by using the IAT on Eqs. (106) and (107).

$$v_1(\zeta,\, t) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{v_1^{(k)}(\zeta,\, 0)}{s^{2-p+k}} + A\left[-\phi_1(\zeta,\, t)\frac{\partial \phi_1}{\partial \zeta}(\zeta,\, t) - \frac{1}{2}\frac{\partial \phi_1}{\partial \zeta}(\zeta,\, t) - \frac{\partial \phi_2}{\partial \zeta}(\zeta,\, t)\right]\right)\right] \tag{108}$$

$$v_2(\zeta,\, t) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{v_2^{(k)}(\zeta,\, 0)}{s^{2-p+k}} + A\left[-\phi_1(\zeta,\, t)\frac{\partial \phi_2}{\partial \zeta}(\zeta,\, t) - \phi_2(\zeta,\, t)\frac{\partial \phi_1}{\partial \zeta}(\zeta,\, t) + \frac{1}{2}\frac{\partial^2 \phi_2}{\partial \zeta^2}(\zeta,\, t)\right]\right)\right] \tag{109}$$

Using the AT iteratively yields:

$$(v_1)_0(\zeta,\, t) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{v_1^{(k)}(\zeta,\, 0)}{s^{2-p+k}}\right)\right]$$

$$= A^{-1}\left[\frac{v_1(\zeta,\, 0)}{s^2}\right]$$

$$= \chi - k\coth(k(\Theta+\zeta)),$$

$$(v_2)_0(\zeta,\, t) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{v_2^{(k)}(\zeta,\, 0)}{s^{2-p+k}}\right)\right]$$

$$= A^{-1}\left[\frac{v_2(\zeta,\, 0)}{s^2}\right]$$

$$= -k^2\text{csch}^2(k(\Theta+\zeta)).$$

Eqs. (102) and (103) are modified to obtain the same form by using the integral of RL.

$$v_1(\zeta,\, t) = \chi - k\coth(k(\Theta+\zeta)) + A\left[-\phi_1(\zeta,\, t)\frac{\partial\phi_1}{\partial\zeta}(\zeta,\, t) - \frac{1}{2}\frac{\partial\phi_1}{\partial\zeta}(\zeta,\, t) - \frac{\partial\phi_2}{\partial\zeta}(\zeta,\, t)\right] \tag{110}$$

$$v_2(\zeta,\, t) = -k^2\text{csch}^2(k(\Theta+\zeta)) + A\left[-\phi_1(\zeta,\, t)\frac{\partial\phi_2}{\partial\zeta}(\zeta,\, t) - \phi_2(\zeta,\, t)\frac{\partial\phi_1}{\partial\zeta}(\zeta,\, t) + \frac{1}{2}\frac{\partial^2\phi_2}{\partial\zeta^2}(\zeta,\, t)\right] \tag{111}$$

The following terms are obtained in accordance with the ATIM procedure:

$$v_{10}(\zeta,\, t) = \chi - k\coth(k(\Theta+\zeta)),$$

$$v_{20}(\zeta,\, t) = -k^2\text{csch}^2(k(\Theta+\zeta)),$$

$$v_{11}(\zeta,\, t) = -\frac{k^2t^p\text{csch}^2(k(\Theta+\zeta))(2k\coth(k(\Theta+\zeta))+2\chi+1)}{2\Gamma(p+1)},$$

$$v_{21}(\zeta,\, t) = -\frac{2k^3\chi t^p\coth(k(\Theta+\zeta))\text{csch}^2(k(\Theta+\zeta))}{\Gamma(p+1)},$$

$$v_{12}(\zeta,\, t) = \frac{1}{4}k^3 t^{2p}\operatorname{csch}^2(k(\Theta+\zeta))\Bigg(\Bigg(k^2 2^{2p+1} t^p \Gamma\left(p+\frac{1}{2}\right)\operatorname{csch}^2(k(\Theta+\zeta))(2k\coth(k(\Theta+\zeta))$$

$$+2\chi+1)\Big(\coth(k(\Theta+\zeta))(2k\coth(k(\Theta+\zeta))+2\chi+1)$$

$$+k\operatorname{csch}^2(k(\Theta+\zeta))\Big)\Bigg)\Big/\Big(\sqrt{\pi}\Gamma(p+1)\Gamma(3p+1)\Big)$$

$$+\Big(2\coth(k(\Theta+\zeta))\Big(8k^2\operatorname{csch}^2(k(\Theta+\zeta))+4k^2-(2\chi+1)^2\Big)$$

$$-8k\chi\Big(3\operatorname{csch}^2(k(\Theta+\zeta))+2\Big)\Big)\Big/\Big(\Gamma(2p+1)\Big)\Bigg),$$

$$v_{22}(\zeta,\, t) = \frac{1}{2}k^4 t^{2p}\operatorname{csch}^6(k(\Theta+\zeta))\Bigg(\Bigg(k^2 2^{2p+1}\chi t^p \Gamma\left(p+\frac{1}{2}\right)((4\chi+2)\cosh(2k(\Theta+\zeta))$$

$$+4k(\sinh(2k(\Theta+\zeta))+3\coth(k(\Theta+\zeta)))+6\chi+3)\Bigg)\Big/\Big(\sqrt{\pi}\Gamma(p+1)\Gamma(3p+1)\Big)$$

$$+\Big(\Big(4k^2-2\chi^2\Big)\cosh(2k(\Theta+\zeta))+6k^2-\chi^2\cosh(4k(\Theta+\zeta))$$

$$+2k\sinh(2k(\Theta+\zeta))+3\chi^2\Big)\Big/\Big(\Gamma(2p+1)\Big)\Bigg). \tag{112}$$

ATIM solution is given below:

$$v_1(\zeta,\, t) = v_{10}(\zeta,\, t)+v_{11}(\zeta,\, t)+v_{12}(\zeta,\, t)+\cdots. \tag{113}$$

$$v_2(\zeta,\, t) = v_{20}(\zeta,\, t)+v_{21}(\zeta,\, t)+v_{22}(\zeta,\, t)+\cdots. \tag{114}$$

$$v_1(\zeta,\, t) = \chi - k\coth(k(\Theta+\zeta)) - \frac{k^2 t^p\operatorname{csch}^2(k(\Theta+\zeta))(2k\coth(k(\Theta+\zeta))+2\chi+1)}{2\Gamma(p+1)}$$

$$+\frac{1}{4}k^3 t^{2p}\operatorname{csch}^2(k(\Theta+\zeta))\Bigg(\Bigg(k^2 2^{2p+1} t^p \Gamma\left(p+\frac{1}{2}\right)\operatorname{csch}^2(k(\Theta+\zeta))(2k\coth(k(\Theta+\zeta))$$

$$+2\chi+1)\Big(\coth(k(\Theta+\zeta))(2k\coth(k(\Theta+\zeta))+2\chi+1)$$

$$+k\operatorname{csch}^2(k(\Theta+\zeta))\Big)\Bigg)\Big/\Big(\sqrt{\pi}\Gamma(p+1)\Gamma(3p+1)\Big)$$

$$+\Big(2\coth(k(\Theta+\zeta))\Big(8k^2\mathrm{csch}^2(k(\Theta+\zeta))+4k^2-(2\chi+1)^2\Big)$$

$$-8k\chi\Big(3\mathrm{csch}^2(k(\Theta+\zeta))+2\Big)\Big)/\Big(\Gamma(2p+1)\Big)\Big)+\cdots. \tag{115}$$

$$v_2(\zeta,\,t)=-k^2\mathrm{csch}^2(k(\Theta+\zeta))-\frac{2k^3\chi t^p\coth(k(\Theta+\zeta))\mathrm{csch}^2(k(\Theta+\zeta))}{\Gamma(p+1)}$$

$$+\frac{1}{2}k^4t^{2p}\mathrm{csch}^6(k(\Theta+\zeta))\Big(\Big(k^2 2^{2p+1}\chi t^p\Gamma\Big(p+\frac{1}{2}\Big)((4\chi+2)\cosh(2k(\Theta+\zeta))$$

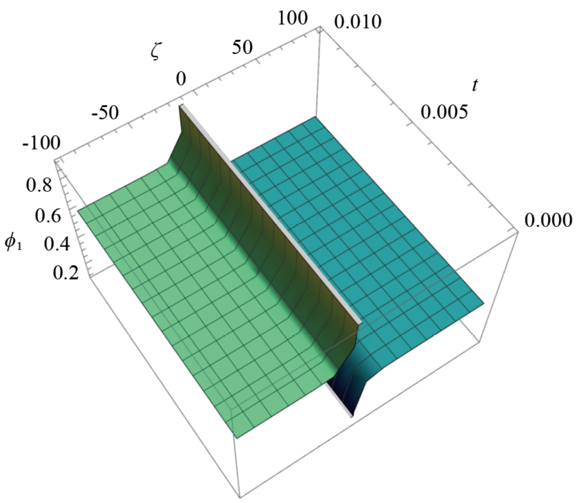
$$+4k(\sinh(2k(\Theta+\zeta))+3\coth(k(\Theta+\zeta)))+6\chi+3)\Big)/\Big(\sqrt{\pi}\Gamma(p+1)\Gamma(3p+1)\Big)$$

$$+\Big(\Big(4k^2-2\chi^2\Big)\cosh(2k(\Theta+\zeta))+6k^2-\chi^2\cosh(4k(\Theta+\zeta))+2k\sinh(2k(\Theta+\zeta))$$

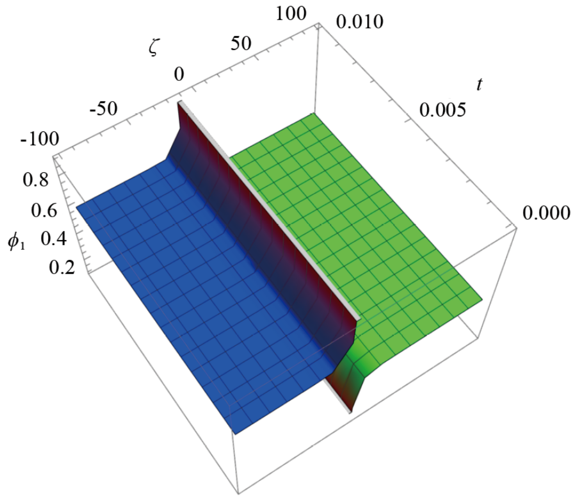
$$+3\chi^2\Big)/\Big(\Gamma(2p+1)\Big)\Big)+\cdots. \tag{116}$$

Table 7. Solution for different values of p of Example 2 of $\phi_1(\zeta,\,t)$ for $t=0.01$, $\chi=0.5$ and $\Theta=10$ using ATIM

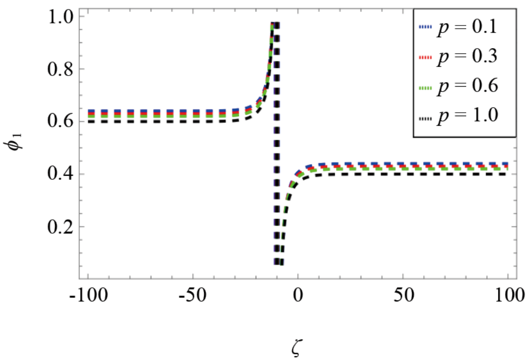
ζ	$\mathrm{ATIM}_{p=0.6}$	$\mathrm{ATIM}_{p=0.8}$	$\mathrm{ATIM}_{p=1.0}$	Exact	$\mathrm{Error}_{p=0.6}$	$\mathrm{Error}_{p=0.8}$	$\mathrm{Error}_{p=1}$
-1.0	0.357153	0.359619	0.360285	0.360346	0.00319301	0.000727165	6.08174×10^{-5}
-0.8	0.359182	0.361508	0.362137	0.362195	0.00301305	0.000686601	5.73682×10^{-5}
-0.6	0.361099	0.363295	0.36389	0.363944	0.00284565	0.000648831	5.41616×10^{-5}
-0.4	0.362911	0.364987	0.36555	0.365601	0.00268971	0.000613613	5.11761×10^{-5}
-0.2	0.364627	0.366591	0.367123	0.367171	0.00254425	0.000580733	4.83927×10^{-5}
0.0	0.366252	0.36811	0.368614	0.36866	0.00240839	0.000549996	4.57943×10^{-5}
0.2	0.367792	0.369552	0.37003	0.370073	0.00228134	0.000521228	4.33656×10^{-5}
0.4	0.369252	0.37092	0.371373	0.371414	0.00216239	0.000494273	4.10928×10^{-5}
0.6	0.370637	0.372219	0.372649	0.372688	0.0020509	0.000468989	3.89635×10^{-5}
0.8	0.371952	0.373454	0.373862	0.373899	0.00194629	0.000445248	3.69664×10^{-5}
1.0	0.373202	0.374627	0.375015	0.37505	0.00184804	0.000422935	3.50915×10^{-5}



(a) ATIM solution for $p = 1$



(b) Exact solution

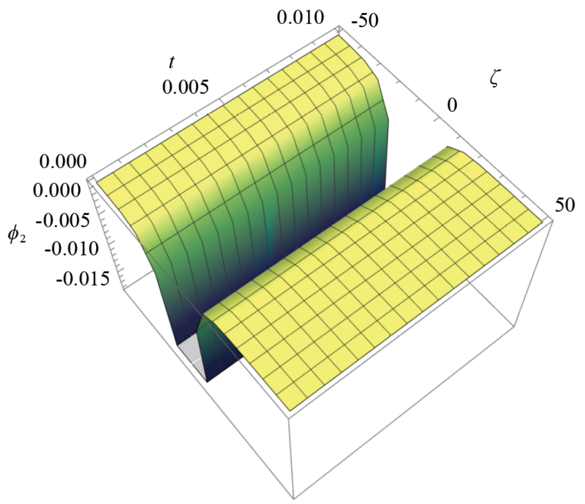


(c) Different fractional order

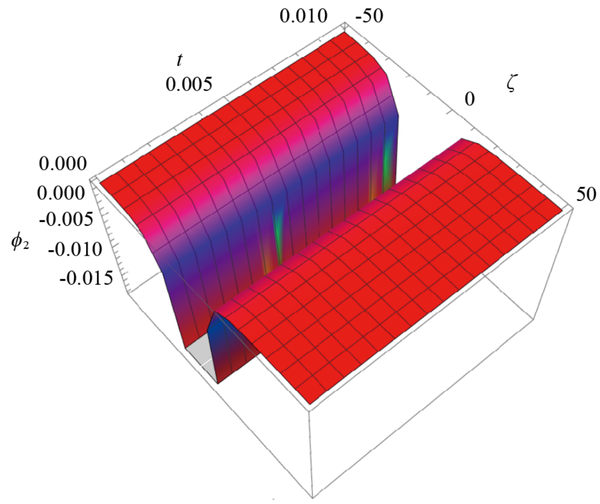
Figure 7. (a) ATIM solution for $p = 1$, (b) Exact solution, (c) solutions for different values of p of Example 2 of $\phi_1(\zeta, t)$ for $t = 0.01$, $\chi = 0.5$ and $\Theta = 10$

Table 8. Solution for different values of p of Example 2 of $\phi_2(\zeta, t)$ for $t = 0.01$, $\chi = 0.5$ and $\Theta = 10$ using ATIM

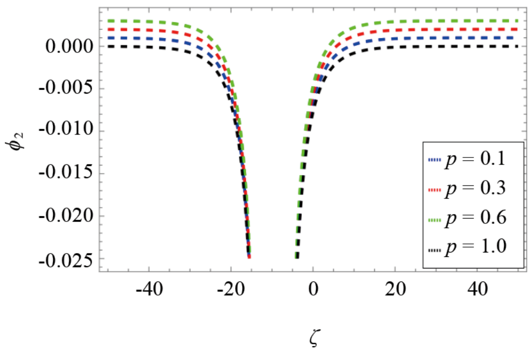
ζ	$\text{ATIM}_{p=0.6}$	$\text{ATIM}_{p=0.8}$	$\text{ATIM}_{p=1.0}$	Exact	$\text{Error}_{p=0.6}$	$\text{Error}_{p=0.8}$	$\text{Error}_{p=1}$
-1.0	-0.00987459	-0.00964149	-0.0095033	-0.0095033	0.000371295	0.000138197	1.66263×10^{-9}
-0.8	-0.00933681	-0.00911933	-0.00899034	-0.00899034	0.000346468	0.000128989	1.46276×10^{-9}
-0.6	-0.0088349	-0.00863173	-0.00851118	-0.00851119	0.000323714	0.000120548	1.28973×10^{-9}
-0.4	-0.00836589	-0.00817586	-0.00806306	-0.00806306	0.000302823	0.000112796	1.13951×10^{-9}
-0.2	-0.00792707	-0.00774913	-0.00764346	-0.00764346	0.000283608	0.000105664	1.00876×10^{-9}
0.0	-0.00751604	-0.00734922	-0.00725013	-0.00725013	0.000265905	0.0000990908	8.94684×10^{-10}
0.2	-0.0071306	-0.00697406	-0.00688103	-0.00688103	0.000249569	0.0000930237	7.94908×10^{-10}
0.4	-0.00676879	-0.00662174	-0.00653432	-0.00653432	0.000234471	0.0000874148	7.07448×10^{-10}
0.6	-0.00642882	-0.00629055	-0.00620832	-0.00620832	0.000220496	0.0000822219	6.30619×10^{-10}
0.8	-0.00610907	-0.00597893	-0.00590152	-0.00590152	0.000207543	0.0000774072	5.62992×10^{-10}
1.0	-0.00580805	-0.00568547	-0.00561253	-0.00561253	0.00019552	0.0000729372	5.03347×10^{-10}



(a) ATIM solution for $p = 1$



(b) Exact solution



(c) Different fractional order

Figure 8. (a) ATIM solution for $p = 1$, (b) Exact solution, (c) solutions for different values of p of Example 2 of $\phi_1(\zeta, t)$ for $t = 0.01$, $\chi = 0.5$ and $\Theta = 10$

Table 9. Various fractional order comparison for $t = 0.1$, $a = 1$, $b = 0.9$, $c = -0.5$ and $d = 1$ of Example 1 for $\phi_1(\zeta, t)$

ζ	$\text{ARPSM}_{p=0.8}$	$\text{ATIM}_{p=0.8}$	$\text{ARPSM}_{p=0.9}$	$\text{ATIM}_{p=0.9}$	$\text{ARPSM}_{p=1.0}$	$\text{ATIM}_{p=1.0}$
-1.0	0.276993	0.277991	0.309192	0.309585	0.334074	0.334225
-0.8	0.433844	0.434497	0.463942	0.4642	0.48671	0.486809
-0.6	0.572441	0.572856	0.599583	0.599746	0.620018	0.620081
-0.4	0.696283	0.696527	0.720447	0.720543	0.73876	0.738797
-0.2	0.809169	0.809282	0.830755	0.8308	0.847369	0.847386
0.0	0.914917	0.914917	0.934551	0.934551	0.95	0.95
0.2	1.01725	1.01714	1.03572	1.03568	1.05062	1.05061
0.4	1.11976	1.11951	1.13804	1.13795	1.15313	1.1531
0.6	1.22583	1.22542	1.24524	1.24508	1.26144	1.26137
0.8	1.33857	1.33792	1.36099	1.36073	1.37955	1.37945
1.0	1.46043	1.45943	1.48882	1.48843	1.51162	1.51147

Table 10. Various fractional order comparison for $t = 0.1$, $a = 1$, $b = 0.9$, $c = -0.5$ and $d = 1$ of Example 1 for $\phi_2(\zeta, t)$

ζ	$\text{ARPSM}_{p=0.8}$	$\text{ATIM}_{p=0.8}$	$\text{ARPSM}_{p=0.9}$	$\text{ATIM}_{p=0.9}$	$\text{ARPSM}_{p=1.0}$	$\text{ATIM}_{p=1.0}$
-1.0	0.620522	0.62233	0.674068	0.67478	0.714297	0.714571
-0.8	0.66057	0.662148	0.706811	0.707433	0.742505	0.742744
-0.6	0.685478	0.686617	0.726018	0.726467	0.757953	0.758125
-0.4	0.700649	0.701349	0.736748	0.737023	0.765741	0.765847
-0.2	0.709512	0.709824	0.742334	0.742457	0.769243	0.76929
0.0	0.714074	0.714043	0.744789	0.744777	0.770486	0.770482
0.2	0.71512	0.714781	0.745	0.744866	0.770378	0.770326
0.4	0.712146	0.711542	0.742764	0.742526	0.76879	0.768698
0.6	0.702964	0.702214	0.736616	0.73632	0.764488	0.764374
0.8	0.682764	0.682205	0.723362	0.723141	0.754863	0.754779
1.0	0.64216	0.642674	0.69709	0.697293	0.735374	0.735452

Table 11. The error comparison for $t = 0.01$, $\chi = 0.5$ and $\Theta = 10$ of the proposed methods of Example 2 for $\phi_1(\zeta, t)$

ζ	Exact	$\text{ARPSM}_{p=1}$	$\text{ATIM}_{p=1}$	$\text{Error}_{\text{ARPSM}}$	$\text{Error}_{\text{ATIM}}$
-1.0	0.360346	0.359849	0.360285	4.97053×10^{-4}	6.08174×10^{-5}
-0.8	0.362195	0.361725	0.362137	4.6935×10^{-4}	5.73682×10^{-5}
-0.6	0.363944	0.363501	0.36389	4.43552×10^{-4}	5.41616×10^{-5}
-0.4	0.365601	0.365182	0.36555	4.19496×10^{-4}	5.11761×10^{-5}
-0.2	0.367171	0.366774	0.367123	3.97035×10^{-4}	4.83927×10^{-5}
0.0	0.36866	0.368284	0.368614	3.76036×10^{-4}	4.57943×10^{-5}
0.2	0.370073	0.369717	0.37003	3.56381×10^{-4}	4.33656×10^{-5}
0.4	0.371414	0.371076	0.371373	3.37964×10^{-4}	4.10928×10^{-5}
0.6	0.372688	0.372367	0.372649	3.20687×10^{-4}	3.89635×10^{-5}
0.8	0.373899	0.373594	0.373862	3.04464×10^{-4}	3.69664×10^{-5}
1.0	0.37505	0.374761	0.375015	2.89215×10^{-4}	3.50915×10^{-5}

Table 12. The error comparison for $t = 0.01$, $\chi = 0.5$ and $\Theta = 10$ of the proposed methods of Example 2 for $\phi_2(\zeta, t)$

ζ	Exact	$\text{ARPSM}_{p=1}$	$\text{ATIM}_{p=1}$	$\text{Error}_{\text{ARPSM}}$	$\text{Error}_{\text{ATIM}}$
-1.0	-0.0095033	-0.00951657	-0.0095033	1.32766×10^{-5}	1.66263×10^{-9}
-0.8	-0.00899034	-0.00900273	-0.00899034	1.23939×10^{-5}	1.46276×10^{-9}
-0.6	-0.00851119	-0.00852277	-0.00851118	1.15845×10^{-5}	1.28973×10^{-9}
-0.4	-0.00806306	-0.0080739	-0.00806306	1.0841×10^{-5}	1.13951×10^{-9}
-0.2	-0.00764346	-0.00765362	-0.00764346	1.01569×10^{-5}	1.00876×10^{-9}
0.0	-0.00725013	-0.00725966	-0.00725013	9.526284×10^{-6}	8.94684×10^{-10}
0.2	-0.00688103	-0.00688998	-0.00688103	8.94412×10^{-6}	7.94908×10^{-10}
0.4	-0.00653432	-0.00654273	-0.00653432	8.40584×10^{-6}	7.07448×10^{-10}
0.6	-0.00620832	-0.00621623	-0.00620832	7.90740×10^{-6}	6.30619×10^{-10}
0.8	-0.00590152	-0.00590897	-0.00590152	7.44519×10^{-6}	5.62992×10^{-10}
1.0	-0.00561253	-0.00561955	-0.00561253	7.016015×10^{-6}	5.03347×10^{-10}

Table 7 presents a comparison of various fractional orders for $\phi_1(\zeta, t)$ of Example 2 at $t = 0.01$, $\chi = 0.5$, and $\Theta = 10$ using the Aboodh transform iteration method (ATIM) for coupled Whitham-Broer-Kaup equations. Figure 7 illustrates the ATIM solution for $p = 1$ in (a), the exact solution in (b), and various fractional order comparisons for $\phi_1(\zeta, t)$ in (c). Table 8 provides a comparison of various fractional orders for $\phi_2(\zeta, t)$ of Example 2 at $t = 0.01$, $\chi = 0.5$, and $\Theta = 10$ using ATIM for coupled Whitham-Broer-Kaup equations. Figure 8 displays the ATIM solution for $p = 1$ in (a), the exact solution in (b), and various fractional order comparisons for $\phi_2(\zeta, t)$ in (c). Table 9 presents a comparison of various fractional orders for $\phi_1(\zeta, t)$ of example 1 at $t = 0.1$, $a = 1$, $b = 0.9$, $c = -0.5$, and $d = 1$ for coupled Whitham-Broer-Kaup equations. Table 10 provides a comparison of various fractional orders for $\phi_2(\zeta, t)$ of example 1 at $t = 0.1$, $a = 1$, $b = 0.9$, $c = -0.5$, and $d = 1$ for coupled Whitham-Broer-Kaup equations. Tables 11 and 12 present error comparisons for $\phi_1(\zeta, t)$ and $\phi_2(\zeta, t)$ of Example 2 at $t = 0.01$, $\chi = 0.5$, and $\Theta = 10$ using the proposed methods for coupled Whitham-Broer-Kaup equations.

5. Conclusion

In conclusion, this study has delved into the analysis of fractional nonlinear systems described by the Whitham-Broer-Kaup equations, employing advanced mathematical methodologies such as the Aboodh transform iteration method (ATIM) and the Aboodh residual power series method (ARPSM) within the context of the Caputo operator. Applying these methods has proven effective in providing accurate and efficient solutions to the intricate dynamics of the Whitham-Broer-Kaup equations. By incorporating the Caputo operator, the research captures the non-local nature of fractional calculus, enhancing the precision of the modeling approach. The findings underscore the significance of these mathematical tools in tackling complex nonlinear systems prevalent in diverse scientific domains. This study contributes to the theoretical understanding of fractional nonlinear systems and provides practical insights for researchers and practitioners in applied mathematics and computational modeling. The successful application of the ATIM and ARPSM methods, coupled with the Caputo operator, positions them as valuable assets in addressing challenges posed by fractional components in nonlinear systems.

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Conflict of interest

The author declares no competing financial interest.

References

[1] Tenreiro Machado JA, Silva MF, Barbosa RS, Jesus IS, Reis CM, Marcos MG, et al. Some applications of fractional calculus in engineering. *Mathematical Problems in Engineering*. 2010; 2010: 639801.

[2] Ionescu C, Lopes A, Copot D, Machado JT, Bates JH. The role of fractional calculus in modeling biological phenomena: A review. *Communications in Nonlinear Science and Numerical Simulation*. 2017; 51: 141-159.

[3] Vivek S, Vijayakumar V. Discussion on optimal feedback control for stochastic fractional differential system by hemivariational inequalities. *Journal of Control and Decision*. 2024; 2024: 1-14.

[4] Dhayal R, Malik M, Abbas S, Debbouche A. Optimal controls for second-order stochastic differential equations driven by mixed-fractional Brownian motion with impulses. *Mathematical Methods in the Applied Sciences*. 2020; 43(7): 4107-4124.

[5] Ma YK, Kavitha K, Shukla A, Vijayakumar V, Nisar KS. An analysis on the optimal control and approximate controllability for Hilfer fractional neutral integro-differential systems with finite delay. *Optimal Control Applications and Methods*. 2024; 45(3): 1086-1107.
[6] Dhayal R, Malik M. Stability analysis of damped fractional stochastic differential systems with Poisson jumps: an successive approximation approach. *International Journal of Systems Science*. 2024; 2024: 1-13.
[7] Sun H, Zhang Y, Baleanu D, Chen W, Chen Y. A new collection of real world applications of fractional calculus in science and engineering. *Communications in Nonlinear Science and Numerical Simulation*. 2018; 64: 213-231.
[8] El-Sayed AMA. On the stochastic fractional calculus operators. *Journal of Fractional Calculus and Applications*. 2015; 6(1): 101-109.
[9] Anastassiou GA. Foundation of stochastic fractional calculus with fractional approximation of stochastic processes. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales Serie A Matemáticas*. 2020; 114(2): 89.
[10] Alhejaili W, Az-Zo'bi E, El-Tantawy SA. On the analytical soliton approximations to fractional forced Korteweg-de Vries equation arising in fluids and Plasmas using two novel techniques. *Communications in Theoretical Physics*. 2024; 76: 085001.
[11] Noor S, Alyousef HA, Shafee A, El-Tantawy SA. A novel analytical technique for analyzing the (3 + 1)-dimensional fractional calogero-bogoyavlenskii-schiff equation: investigating solitary/shock waves and many others physical phenomena. *Physica Scripta*. 2024; 99(6): 065257.
[12] Alshehry AS, Yasmin H, Ali A, Khan I. Fractional-order view analysis of Fishers and foam drainage equations within Aboodh transform. *Engineering Computations*. 2024; 41(3): 489-515.
[13] Noor S, Albalawi W, Al-Sawalha MM, Ismaeel SM. Mathematical frameworks for investigating fractional nonlinear coupled Korteweg-de Vries and Burger's equations. *Frontiers in Physics*. 2024; 12: 1374452.
[14] Noor S, Albalawi W, Shah R, Al-Sawalha MM, Ismaeel SM, El-Tantawy SA. On the approximations to fractional nonlinear damped Burger's-type equations that arise in fluids and plasmas using Aboodh residual power series and Aboodh transform iteration methods. *Frontiers in Physics*. 2024; 12: 1374481.
[15] Yasmin H, Alshehry AS, Ganie AH, Mahnashi AM. Perturbed Gerdjikov-Ivanov equation: Soliton solutions via Backlund transformation. *Optik*. 2024; 298: 171576.
[16] Yasmin H, Alderremy AA, Hamid Ganie A, Aly S. Iterative solution of the fractional Wu-Zhang equation under Caputo derivative operator. *Frontiers in Physics*. 2024; 12: 1333990.
[17] Alshammari S, Moaddy K, Alshammari M, Alsheekhhussain Z, Al-Sawalha MM, Yar M. Analysis of solitary wave solutions in the fractional-order Kundu-Eckhaus system. *Scientific Reports*. 2024; 14: 3688.
[18] El-Tantawy SA, Matoog RT, Alrowaily AW, Ismaeel SM. On the shock wave approximation to fractional generalized Burger-Fisher equations using the residual power series transform method. *Physics of Fluids*. 2024; 36(2): 023105.
[19] Yasmin H, Alshehry AS, Ganie AH, Shafee A. Noise effect on soliton phenomena in fractional stochastic Kraenkel-Manna-Merle system arising in ferromagnetic materials. *Scientific Reports*. 2024; 14(1): 1810.
[20] Areshi M, Khan A, Shah R, Nonlaopon K. Analytical investigation of fractional-order Newell-Whitehead-Segel equations via a novel transform. *AIMS Mathematics*. 2022; 7(4): 6936-6958.
[21] Qin Y, Khan A, Ali I, Al Qurashi M, Khan H, Shah R, et al. An efficient analytical approach for the solution of certain fractional-order dynamical systems. *Energies*. 2020; 13(11): 2725.
[22] Mophou GM. Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations. *Applied Mathematics and Computation*. 2011; 217(19): 7579-7587.
[23] Bhanotar SA, Kaabar MK. Analytical solutions for the nonlinear partial differential equations using the conformable triple Laplace transform decomposition method. *International Journal of Differential Equations*. 2021; 2021: 1-18.
[24] Yasmin H, Naveed I. A comparative study of the fractional coupled burgers and Hirota-Satsuma KdV equations via analytical techniques. *Symmetry*. 2022; 14(7): 1364.
[25] Whitham GB. Variational methods and applications to water waves. *Proceedings of the Royal Society of London Series A Mathematical and Physical Sciences*. 1967; 299(1456): 6-25.
[26] Xie F, Yan Z, Zhang H. Explicit and exact traveling wave solutions of Whitham-Broer-Kaup shallow water equations. *Physics Letters A*. 2001; 285(1-2): 76-80.
[27] Kaup D. A higher-order water-wave equation and the method for solving it. *Progress of Theoretical Physics*. 1975; 54(2): 396-408.
[28] Alshammari M, Iqbal N, Mohammed WW, Botmart T. The solution of fractional-order system of KdV equations with exponential-decay kernel. *Results in Physics*. 2022; 38: 105615.

[29] El-Borai MM, El-Sayed WG, Al-Masroub RM. Exact solutions for time fractional coupled Whithan-Broer-Kaup equations via exp-function method. *International Research Journal of Engineering and Technology*. 2015; 2(6): 307-315.

[30] Saha Ray S. A novel method for travelling wave solutions of fractional Whitham-Broer-Kaup, fractional modified Boussinesq and fractional approximate long wave equations in shallow water. *Mathematical Methods in the Applied Sciences*. 2015; 38(7): 1352-1368.

[31] Cui M. Compact finite difference method for the fractional diffusion equation. *Journal of Computational Physics*. 2009; 228(20): 7792-7804.

[32] Zheng B. Exp-function method for solving fractional partial differential equations. *The Scientific World Journal*. 2013; 2013: 465723.

[33] Yasmin H, Iqbal N. Analysis of fractional-order system of one-dimensional Keller-Segel equations: a modified analytical method. *Symmetry*. 2022; 14(7): 1321.

[34] Hassan HN, El-Tawil MA. A new technique of using homotopy analysis method for solving high-order nonlinear differential equations. *Mathematical Methods in the Applied Sciences*. 2011; 34(6): 728-742.

[35] Zhang Z, Yong X, Chen Y. Symmetry analysis for Whitham-Broer-Kaup equations. *Journal of Nonlinear Mathematical Physics*. 2008; 15(4): 383-397.

[36] Arqub OA. Series solution of fuzzy differential equations under strongly generalized differentiability. *Journal of Advanced Research in Applied Mathematics*. 2013; 5(1): 31-52.

[37] Abu Arqub O, Abo-Hammour Z, Al-Badarneh R, Momani S. A reliable analytical method for solving higher-order initial value problems. *Discrete Dynamics in Nature and Society*. 2013; 2013: 673829.

[38] Arqub OA, El-Ajou A, Zhour ZA, Momani S. Multiple solutions of nonlinear boundary value problems of fractional order: a new analytic iterative technique. *Entropy*. 2014; 16(1): 471-493.

[39] El-Ajou A, Arqub OA, Momani S. Approximate analytical solution of the nonlinear fractional KdV-Burgers equation: a new iterative algorithm. *Journal of Computational Physics*. 2015; 293: 81-95.

[40] Xu F, Gao Y, Yang X, Zhang H. Construction of fractional power series solutions to fractional Boussinesq equations using residual power series method. *Mathematical Problems in Engineering*. 2016; 2016: 5492535.

[41] Zhang J, Wei Z, Li L, Zhou C. Least-squares residual power series method for the time-fractional differential equations. *Complexity*. 2019; 2019: 1-15.

[42] Jaradat I, Alquran M, Abdel-Muhsen R. An analytical framework of 2D diffusion, wave-like, telegraph, and Burgers' models with twofold Caputo derivatives ordering. *Nonlinear Dynamics*. 2018; 93: 1911-1922.

[43] Jaradat I, Alquran M, Al-Khaled K. An analytical study of physical models with inherited temporal and spatial memory. *The European Physical Journal Plus*. 2018; 133: 1-11.

[44] Alquran M, Al-Khaled K, Sivasundaram S, Jaradat HM. Mathematical and numerical study of existence of bifurcations of the generalized fractional Burgers-Huxley equation. *Nonlinear Studies*. 2017; 24(1): 235-244.

[45] Zhang MF, Liu YQ, Zhou XS. Efficient homotopy perturbation method for fractional non-linear equations using Sumudu transform. *Thermal Science*. 2015; 19(4): 1167-1171.

[46] Ojo GO, Mahmudov NI. Aboodh transform iterative method for spatial diffusion of a biological population with fractional-order. *Mathematics*. 2021; 9(2): 155.

[47] Awuya MA, Ojo GO, Mahmudov NI. Solution of space-time fractional differential equations using Aboodh transform iterative method. *Journal of Mathematics*. 2022; 2022: 4861588.

[48] Awuya MA, Subasi D. Aboodh transform iterative method for solving fractional partial differential equation with Mittag-Leffler Kernel. *Symmetry*. 2021; 13(11): 2055.

[49] Aboodh KS. The new integral transform "Aboodh Transform". *Global journal of pure and Applied mathematics*. 2013; 9(1): 35-43.

[50] Aggarwal S, Chauhan R. A comparative study of Mohand and Aboodh transforms. *International Journal of Research in Advent Technology*. 2019; 7(1): 520-529.

[51] Benattia ME, Belghaba K. Application of the Aboodh transform for solving fractional delay differential equations. *Universal Journal of Mathematics and Applications*. 2020; 3(3): 93-101.

[52] Delgado BB, Macias-Diaz JE. On the general solutions of some non-homogeneous Div-curl systems with Riemann-Liouville and Caputo fractional derivatives. *Fractal and Fractional*. 2021; 5(3): 117.

[53] Alshammari S, Al-Smadi M, Hashim I, Alias MA. Residual power series technique for simulating fractional Bagley-Torvik problems emerging in applied physics. *Applied Sciences*. 2019; 9(23): 5029.

[54] Liu JG, Yang XJ, Feng YY, Geng LL. Numerical solutions and conservation laws of the time fractional coupled WBK-type system. *Mathematical Methods in the Applied Sciences*. 2021; 44(5): 4105-4116.