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Research Article

On Soft Submaximal and Soft Door Spaces

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Abstract: This paper is divided into two parts. The first part deals with soft submaximal spaces, where we present new theorems and some basic facts. Further, we successfully find a requirement that indicates the soft one-point compactification of a soft topological space is soft submaximal. In the second part, we study soft door spaces. We notice that every soft door space is a soft submaximal space, but a soft submaximal space need not be soft door. The class of soft door spaces is hereditary. We give couterexamples showing that this class is neither additive nor productive. We further show that images of soft door spaces under certain soft functions are also soft door spaces. After that, we characterize certain soft topological spaces in terms of soft limit points and the Krull dimension. At last, we discuss when the soft one-point compactification of a soft topological space is soft door.

Keywords: soft one-point compactification, soft door space, soft submaximal space, krull dimension of soft topological space

MSC: 54A40, 54A05, 54D30

1. Introduction

Different mathematical strategies have been offered to deal with the ambiguity and uncertainty present in practical challenges, such as those in medical science, engineering, social science, and economics. The fresh framework that Molodtsov [1] developed is known as a soft set, and its parameterizations are suitable for clearing up doubts and do not impose the limitations that accompanied the earlier tools. In multiple mathematical fields, Molodtsov personally implemented his idea. His theory was applied to other fields of mathematics by numerous researchers after him.

One field that specifically combines classical topology and soft set theory is soft topology. It is centered on the set of all soft sets and is motivated by the fundamental presumptions of classical topology. It is founded by Shabir and Naz [2]. Numerous researchers followed their direction and started to generalized various classical concepts in topology to soft settings, including soft extremally disconnected [3], soft compact [4], soft separable [5], soft nodec [6, 7], soft Lindelof [8], and soft connected spaces [9], and soft separation axioms [2, 10]. Moreover, other generalizations of soft open sets in soft topological spaces were suggested like the first or the second category soft sets [11, 12].

This paper is organized as follows: Section 3 discusses new findings and facts regarding soft submaximal spaces, including their relationship to soft T_0-spaces, the Krull dimension of soft topological spaces, Alexandroff soft topological

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spaces, and soft compact soft submaximal spaces. Moreover, the soft submaximality of soft one-point compactifications. We analyze soft door spaces in Section 4. It is observed that all soft door spaces are also soft submaximal spaces; however, a soft submaximal space does not necessarily have to be soft door. Soft door spaces form a hereditary class. We provide counterexamples demonstrating that this class is neither productive nor additive. We further show that, under specific soft functions, the image of soft door spaces are soft door spaces. Next, we define soft door spaces in terms of soft compact spaces and soft limit points. Finally, we address the connections between soft one-point compactification and soft door space. In Section 5, we give a summary of our findings.

2. Definitions and background

As a means of supporting our claims in the next parts, this section provides definitions and properties pertaining to soft set theory and soft topology. In what follows, we assume that $\mathcal{X}$ and $\mathbf{B}$ are, respectively, a given universal set and a given nonempty set of parameters.

Definition 1 [1] For any nonempty subset ϖ of $\mathbf{B}$, the soft set over $\mathcal{X}$ (along with ϖ) is an ordered pair (Z, ϖ) such that Z is a function from ϖ into the power set $\mathscr{P}(\mathcal{X})$ of $\mathcal{X}$. The collection of all soft sets over $\mathcal{X}$ along with ϖ is denoted by $SS(\mathcal{X})_\varpi$.

Remark 1 A soft set (Z, ϖ) over $\mathcal{X}$ can be identified with the indexed family $\{Z(\rho)\colon \rho \in \varpi\}$ of subsets of $\mathcal{X}$ and also with the function $Z = \{(\rho, Z(\rho))\colon \rho \in \varpi\}$.

A partial soft set (Z, ϖ) can be widened to a full soft set $(Z, \mathbf{B})$ by assigning $Z(\rho) = \emptyset$ for all $\rho \in \mathbf{B} - \varpi$.

Given a soft set (Z, ϖ). The complement $(Z, \varpi)^c$ [13] of (Z, ϖ) is presented to be a soft set (Z^c, ϖ) having the property that $Z^c\colon \varpi \to \mathscr{P}(\mathcal{X})$ is a function such that $Z^c(\rho) = \mathcal{X} - Z(\rho)$ for each $\rho \in \varpi$.

We now recall the definitions of basic operations on $SS(\mathcal{X})_\varpi$.

Definition 2 [13–16] Let (Z, ϖ), (L, ϖ), $(M, \varpi) \in SS(\mathcal{X})_\varpi$. Then

(1) (Z, ϖ) is a soft subset of (L, ϖ), denoted by $(Z, \varpi) \widetilde{\subseteq} (L, \varpi)$, if $Z(\rho) \subseteq L(\rho)$ for each $\rho \in \varpi$.

(2) (Z, ϖ) equals (L, ϖ), denoted by $(Z, \varpi) = (L, \varpi)$, if $(Z, \varpi) \widetilde{\subseteq} (L, \varpi)$ and $(Z, \varpi) \widetilde{\supseteq} (L, \varpi)$.

(3) The soft intersection of (L, ϖ) and (M, ϖ) is defined to be the soft set (Z, ϖ) such that $Z(\rho) = L(\rho) \cap M(\rho)$ for each $\rho \in \varpi$ and is denoted by $(Z, \varpi) = (L, \varpi) \widetilde{\cap} (M, \varpi)$.

(4) The soft union of (L, ϖ) and (M, ϖ) is defined to be the soft set (Z, ϖ) such that $Z(\rho) = L(\rho) \cup M(\rho)$ for each $\rho \in \varpi$ and is denoted by $(Z, \varpi) = (L, \varpi) \widetilde{\cup} (M, \varpi)$.

(5) The soft difference between (Z, ϖ) and (L, ϖ) is defined to be the soft set $(M, \varpi) = (Z, \varpi) - (L, \varpi)$ such that $M(\rho) = Z(\rho) - L(\rho)$ for each $\rho \in \varpi$.

In much the same way, as in Definition 2, given an indexed collection $\{(Z_i, \varpi)\colon i \in I\}$ of soft sets, one can also define the soft union $\widetilde{\bigcup}_{i\in I}(Z_i, \varpi)$ and, if $I \neq \emptyset$, the soft intersection $\widetilde{\bigcap}_{i\in I}(Z_i, \varpi)$.

Definition 3 [17] A soft set (Z, ϖ) over $\mathcal{X}$ is named a soft point x_ρ, if there exists $x \in \mathcal{X}$ and $\rho \in \varpi$ such that $Z(\rho) = \{x\}$ and $Z(\varsigma) = \emptyset$ for all $\varsigma \in \varpi - \rho$. If x_ρ is a soft point over $\mathcal{X}$ along with ϖ and (Z, ϖ) is a soft set over $\mathcal{X}$, then the assertion $x_\rho \in (Z, \varpi)$ means that $x \in Z(\rho)$.

The family of all soft points over $\mathcal{X}$ along with ϖ is denoted by $SP(\mathcal{X})_\varpi$.

Definition 4 [16] Let $(Z, \varpi) \in SS(\mathcal{X})_\varpi$. Then (Z, ϖ) is said to be null along with ϖ, denoted by Φ_ϖ, if $Z(\rho) = \emptyset$ for each $\rho \in \varpi$. The null soft set $\Phi_{\mathbf{B}}$ is said to be the full null set. The soft set (Z, ϖ) is said to be absolute along with ϖ, denoted by $\mathcal{X}_\varpi$, if $Z(\rho) = \mathcal{X}$ for each $\rho \in \varpi$. The absolute soft set $\mathcal{X}_{\mathbf{B}}$ is said to be the full absolute soft set.

If x_ρ and y_ς are soft points over $\mathcal{X}$ along with ϖ, then we write $x_\rho \neq y_\varsigma$ and say that the soft points are distinct if (and only if) either $x \neq y$ or $\rho \neq \varsigma$. We say that soft sets (Z, ϖ), (L, ϖ) over $\mathcal{X}$ are disjoint if $(Z, \varpi) \widetilde{\cap} (L, \varpi) = \Phi_\varpi$.

Definition 5 [18] Let $(Z, \varpi) \in SS(\mathcal{X})_\varpi$. Then (Z, ϖ) is said to be a finite or a countable soft set if $Z(\rho)$ is finite or countable for each $\rho \in \varpi$. Otherwise, it is called infinite or uncountable.

Definition 6 [2] A soft topology over $\mathcal{X}$ is a subcollection ν of $SS(\mathcal{X})_\varpi$ that contains Φ_ϖ, $\mathcal{X}_\varpi$ and is closed under arbitrary unions and finite intersections of members of ν.

Definition 7 Let $\nu \widetilde{\subseteq} SS(\mathcal{X})_\varpi$ be a soft topology over $\mathcal{X}$. Then:

(1) The triplet $(\mathcal{X}, \nu, \varpi)$ is called a soft topological space over $\mathcal{X}$ (along with ϖ).

(2) If $(Z, \varpi) \in \nu$, (Z, ϖ) is called soft open in $(\mathcal{X}, \nu, \varpi)$.

(3) (Z, ϖ) is called soft closed in $(\mathcal{X}, \nu, \varpi)$ if $(Z, \varpi)^c \in \nu$. The family of all soft closed sets in $(\mathcal{X}, \nu, \varpi)$ is denoted by ν^c.

Definition 8 [19] Let $\mathscr{E} \widetilde{\subseteq} SS(\mathcal{X})_\varpi$ and let $\{\nu_i: i \in I\}$ be the class of all soft topologies over $\mathcal{X}$ along with ϖ such that $\mathscr{E} \widetilde{\subseteq} \nu_i$ for each i. If $\nu = \widetilde{\bigcap}_{i\in I} \nu_i$, ν is called "the soft topology generated by" $\mathscr{E}$.

Definition 9 [20] A soft basis (or soft base) for a soft topology ν is a collection $\mathscr{B} \widetilde{\subseteq} \nu$ such that each member of ν can be represented as a union of members of $\mathscr{B}$. It is called a countable soft base if $\mathscr{B}$ is countable.

In what follows, we assume that $(\mathcal{X}, \nu, \varpi)$ is a given soft topological space. Soft subsets over $\mathcal{X}$ along with ϖ are called soft subsets of $(\mathcal{X}, \nu, \varpi)$. That (Z, ϖ) is a soft subset of $(\mathcal{X}, \nu, \varpi)$ is denoted by $(Z, \varpi) \widetilde{\subseteq} (\mathcal{X}, \nu, \varpi)$.

Definition 10 [21] Let $(N, \varpi) \widetilde{\subseteq} (\mathcal{X}, \nu, \varpi)$. Then (N, ϖ) is called a soft neighborhood of $x_\rho \in SP(\mathcal{X})_\varpi$ if there exists $(W, \varpi) \in \nu(x_\rho)$ such that $x_\rho \in (W, \varpi) \widetilde{\subseteq} (N, \varpi)$, where $\nu(x_\rho)$ is the set of all members of ν that contain x_ρ.

Definition 11 [2] Let $(A, \varpi) \neq \Phi_\varpi$ be a soft subset of $(\mathcal{X}, \nu, \varpi)$. Then $(A, \nu_{(A,\varpi)}, \varpi)$ is defined to be a soft topological subspace of $(\mathcal{X}, \nu, \varpi)$, where $\nu_{(A,\varpi)} = \{(N, \varpi) \widetilde{\cap} (A, \varpi): (N, \varpi) \in \nu\}$ is a soft topology relative to (A, ϖ).

Lemma 1 [2] Let $(A, \nu_{(A,\varpi)}, \varpi)$ be a soft subspace of $(\mathcal{X}, \nu, \varpi)$ and let $(Z, \varpi) \widetilde{\subseteq} (A, \varpi) \in \nu$. Then $(Z, \varpi) \in \nu_{(A,\varpi)}$ if and only if $(Z, \varpi) \in \nu$.

Definition 12 Let $(Z, \varpi) \widetilde{\subseteq} (\mathcal{X}, \nu, \varpi)$.

(1) The soft closure [2] of (Z, ϖ), denoted by $cl_{\mathcal{X}}(Z, \varpi)$ (simply $cl(Z, \varpi)$), is defined by

$$cl(Z, \varpi) = \widetilde{\cap}\{(L, \varpi): (Z, \varpi) \widetilde{\subseteq} (L, \varpi), (L, \varpi) \in \nu^c\}.$$

(2) The soft interior [2] of (Z, ϖ), denoted by $int_{\mathcal{X}}(Z, \varpi)$ (simply $int(Z, \varpi)$), is defined by

$$int(Z, \varpi) = \widetilde{\cup}\{(G, \varpi): (G, \varpi) \widetilde{\subseteq} (Z, \varpi), (G, \varpi) \in \nu\}.$$

(3) The soft boundary of (Z, ϖ) [22, 23] is defined by $b(Z, \varpi) = cl(Z, \varpi) - int(Z, \varpi)$.

Lemma 2 [23] Let $(Z, \varpi) \widetilde{\subseteq} (\mathcal{X}, \nu, \varpi)$. Then

$$int((Z, \varpi)^c) = (cl(Z, \varpi))^c \text{ and } cl((Z, \varpi)^c) = (int(Z, \varpi))^c.$$

Definition 13 [20] Let $(H, \varpi) \widetilde{\subseteq} (\mathcal{X}, \nu, \varpi)$. A soft point $x_\rho \in SP(\mathcal{X})_\varpi$ can be referred to a soft limit point of (H, ϖ) if $(Z, \varpi) \widetilde{\cap} (H, \varpi) - \{x_\rho\} \neq \Phi_\varpi$ for all $(Z, \varpi) \in \nu(x_\rho)$. The set of all soft limit points of (H, ϖ) in $(\mathcal{X}, \nu, \varpi)$ is symbolized by $\mathscr{D}(H, \varpi)$.

Definition 14 [2] The soft topology space $(\mathcal{X}, \nu, \varpi)$ is called:

(1) a soft T_0-space if, for every pair x_ρ, y_ς of distinct soft points of $(\mathcal{X}, \nu, \varpi)$, there exists a soft open set (Z, ϖ) in $(\mathcal{X}, \nu, \varpi)$ such that either $x_\rho \in (Z, \varpi)$ and $y_\varsigma \notin (Z, \varpi)$ or $x_\rho \notin (Z, \varpi)$ and $y_\varsigma \in (Z, \varpi)$;

(2) a soft T_1-space if, for every pair x_ρ, y_ς of distinct soft points of $(\mathcal{X}, \nu, \varpi)$, there exists a soft open set (Z, ϖ) in $(\mathcal{X}, \nu, \varpi)$ such that $x_\rho \in (Z, \varpi)$ and $y_\varsigma \notin (Z, \varpi)$;

(3) a soft T_2-space (a soft Hausdorff space) if, for every pair x_ρ, y_ς of distinct soft points of $(\mathcal{X}, \nu, \varpi)$, there exist disjoint soft open sets (Z, ϖ), (W, ϖ) in $(\mathcal{X}, \nu, \varpi)$ such that $x_\rho \in (Z, \varpi)$ and $y_\varsigma \in (W, \varpi)$.

Definition 15 Let (Z, ϖ), $(W, \varpi) \widetilde{\subseteq} (\mathcal{X}, \nu, \varpi)$. Then (Z, ϖ) is called:

(1) soft nowhere dense [24] if $int(cl(Z, \varpi)) = \Phi_\varpi$;

(2) soft dense in (W, ϖ) [25, 24] if $(W, \varpi) \widetilde{\subseteq} cl(Z, \varpi)$;

(3) soft regular open [26] if $int(cl(Z, \varpi)) = (Z, \varpi)$;

(4) soft meager [11] if it a countable union of soft nowhere dense sets;

(5) soft codense [27] if $int(Z, \varpi) = \Phi_{\varpi}$;

(6) a residual soft set if it the complement of a soft meager set.

Definition 16 [28, 29] Let $SS(\mathcal{X})_{\varpi_1}$, $SS(\Upsilon)_{\varpi_2}$ be two families of soft sets over different sets of parameters, and let p: $\mathcal{X} \to \Upsilon$, q: $\varpi_1 \to \varpi_2$ be functions. The image of $(Z, \varpi_1) \in SS(\mathcal{X})_{\varpi_1}$ under h_{pq}: $SS(\mathcal{X})_{\varpi_1} \to SS(\Upsilon)_{\varpi_2}$ is $h_{pq}(Z, \varpi_1) = (h_{pq}(Z), q(\varpi_1))$ in $SS(\Upsilon)_{\varpi_2}$ such that

$$h_{pq}(Z)(\rho_2) = \begin{cases} \bigcup\limits_{\rho_1 \in q^{-1}(\rho_2) \cap \varpi_1} p(Z(\rho_1)), & q^{-1}(\rho_2) \cap \varpi_1 \neq \emptyset \\ \\ \emptyset, & \text{otherwise,} \end{cases}$$

for every $\rho_2 \in \varpi_2$.

The inverse image of $(W, \varpi_2) \in SS(\Upsilon)_{\varpi_2}$ under h_{pq} is $h_{pq}^{-1}(W, \varpi_2) = (h_{pq}^{-1}(W), q^{-1}(\varpi_2))$ such that

$$h_{pq}^{-1}(W)(\rho_1) = \begin{cases} p^{-1}(W(q(\rho_1))), & q(\rho_1) \in \varpi_2 \\ \\ \emptyset, & \text{otherwise,} \end{cases}$$

for every $\rho_1 \in \varpi_1$.

The bijectivity (resp. surjectivity, injectivity) of the soft function h_{pq} relies on bijectivity (resp. surjectivity, injectivity) of p and q.

Definition 17 [29] Let $(\mathcal{X}, \nu_1, \varpi_1)$, $(\Upsilon, \nu_2, \varpi_2)$ be soft topological spaces and let p: $\mathcal{X} \to \Upsilon$, q: $\varpi_1 \to \varpi_2$ be functions. A soft function h_{pq}: $SS(\mathcal{X})_{\varpi_1} \to SS(\Upsilon)_{\varpi_2}$ is said to be:

(1) soft pq-continuous if $h_{pq}^{-1}(Z, \varpi_2) \in \nu_1$ for each $(Z, \varpi_2) \in \nu_2$;

(2) soft pq-open if $h_{pq}(Z, \varpi_1) \in \nu_2$ for each $(Z, \varpi_1) \in \nu_1$.

Note that the above concepts were introduced in a different way in [30]. However, both methods are related in this manner: Given soft topological spaces $(\mathcal{X}, \nu_1, \varpi)$, $(\Upsilon, \nu_2, \varpi)$, for a mapping p: $\mathcal{X} \to \Upsilon$, if q: $\varpi \to \varpi$ is the identity mapping, then h_{pq} is soft pq-continuous in the sense of [29] if and only if p is soft continuous from $(\mathcal{X}, \nu_1, \varpi)$ to $(\Upsilon, \nu_2, \varpi)$ in the sense of [30].

Definition 18 [4] For a fixed parametric set ϖ, suppose $\{(\mathcal{X}_i, \nu_i, \varpi)$: $i \in I\}$ is a collection of soft topological spaces. The initial soft topology over $\mathcal{X}$ generated by the family $\{p_i$: $i \in I\}$ is called the product soft topology ν on $\mathcal{X} = \prod_{i \in I} \mathcal{X}_i$, where p_i: $(\mathcal{X}, \nu, \varpi) \to (\mathcal{X}_i, \nu_i, \varpi)$ are the projection soft functions for all i.

Lemma 3 [4] For a fixed parametric set ϖ, suppose $\{(\mathcal{X}_i, \nu_i, \varpi)$: $i \in I\}$ is a collection of soft topological spaces. For all i, the projection soft functions p_i: $(\prod \mathcal{X}_i, \prod \nu_i, \varpi)_{i \in I} \to (\mathcal{X}_i, \nu_i, \varpi)$ are soft open.

Definition 19 [31] For a fixed parametric set ϖ, suppose $\{(\mathcal{X}_i, \nu_i, \varpi)$: $i \in I\}$ is a pairwise disjoint collection of soft topological spaces. The soft topology ν on $\widetilde{\bigcup}_{i \in I}(\mathcal{X}_i, \varpi)$ produced by a soft basis $\mathscr{A} = \{(Z, \varpi) \widetilde{\subseteq} \widetilde{\bigcup}_{i \in I}(\mathcal{X}_i, \varpi)$: $(Z, \varpi) \in \nu_i$ for some $i\}$ is called a sum of $\{(\mathcal{X}_i, \nu_i, \varpi)$: $i \in I\}$ and symbolized by $(\bigoplus_{i \in I} \mathcal{X}_i, \nu, \varpi)$.

Definition 20 [32] A soft topological space $(\mathcal{X}, \nu, \varpi)$ is called an "Alexandroff soft topological space" if ν is closed under any soft intersection.

Definition 21 [33] A soft topological space $(\mathcal{X}, \nu, \varpi)$ is called "soft compact" if every cover of $\mathcal{X}_{\varpi}$ by soft open sets in $(\mathcal{X}, \nu, \varpi)$ has a finite subcover.

Lemma 4 [33] Assume $(\mathcal{X}, \nu, \varpi)$ is a soft compact space and $(Z, \varpi) \in \nu^c$. Then (Z, ϖ) is soft compact.

Definition 22 [34] Let $(\mathcal{X}, \nu, \varpi)$ be a soft topological space that is soft non-compact and let $y \notin \mathcal{X}$. Set $\mathcal{X}^* = \mathcal{X} \cup \{y\}$. The soft one-point compactification of $(\mathcal{X}, \nu, \varpi)$ is a soft topological space $(\mathcal{X}^*, \nu^*, \varpi)$, where ν^* is defined by $\nu^* =$

$\nu \,\widetilde{\cup}\, \{(Z, \varpi)\colon y_\rho \in (Z, \varpi),\ (Z, \varpi)^c \in CK_\nu(\mathcal{X}_\varpi)\}$ such that $(Z, \varpi) \in SS(\mathcal{X}^*_\varpi)$ and $CK_\nu(\mathcal{X}_\varpi)$ is the class of "soft compact and soft closed" sets in $(\mathcal{X}, \nu, \varpi)$.

3. On soft submaximal spaces

At this part, we begin by revisiting the notion of soft submaximal spaces, accompanied by some new findings.

Definition 23 [35, 36] Let $(\mathcal{X}, \nu, \varpi)$ be soft topological space. Then $(\mathcal{X}, \nu, \varpi)$ is said to be soft submaximal if all soft dense subsets of $(\mathcal{X}, \nu, \varpi)$ are in ν.

Lemma 5 Let $(\mathcal{X}, \nu, \varpi)$ be a soft topological space and let $(Z, \varpi) \,\widetilde{\subseteq}\, (\mathcal{X}, \nu, \varpi)$. Then (Z, ϖ) has no soft limit points if and only if (Z, ϖ) is soft closed and soft discrete.

Proof. Let $x \in \mathcal{X}$ and $\rho \in \varpi$. If $x_\rho \in (Z, \varpi)$ and x_ρ is not a soft limit point of (Z, ϖ), then there is a soft open set (G, ϖ) containing x_ρ such that $(Z, \varpi) \,\widetilde{\cap}\, (G, \varpi) = \{x_\rho\}$. This implies that if no soft point of (Z, ϖ) is a soft limit point of (Z, ϖ), then (Z, ϖ) is soft discrete. Assuming that $\mathscr{D}(Z, \varpi) = \Phi_\varpi$, we deduce that (Z, ϖ) is soft closed because $cl(Z, \varpi) = (Z, \varpi) \,\widetilde{\cup}\, \mathscr{D}(Z, \varpi) = (Z, \varpi)$.

Conversely, if (Z, ϖ) is soft closed and soft discrete, then it indeed does not contain any soft limit point: if x_ρ is one, since (Z, ϖ) is soft closed, x_ρ must be in (Z, ϖ), but this contradicts the soft discreteness of (Z, ϖ). □

Theorem 1 Given a soft topological space $(\mathcal{X}, \nu, \varpi)$. The following conditions are equivalent:

(1) $(\mathcal{X}, \nu, \varpi)$ is soft submaximal;

(2) $cl(Z, \varpi) - (Z, \varpi)$ is soft closed for each soft subset $(Z, \varpi) \,\widetilde{\subseteq}\, (\mathcal{X}, \nu, \varpi)$;

(3) For each $(Z, \varpi) \,\widetilde{\subseteq}\, (\mathcal{X}, \nu, \varpi)$, if (Z, ϖ) is a soft codense set, then (Z, ϖ) is a soft discrete and soft closed set;

(4) $cl(Z, \varpi) - (Z, \varpi)$ is a soft discrete and soft closed set for each soft set (Z, ϖ) in $(\mathcal{X}, \nu, \varpi)$.

Proof. $(1 \Longrightarrow 2)$ Let (Z, ϖ) be a soft set in a soft submaximal space $(\mathcal{X}, \nu, \varpi)$. By [36, Theorem 4.4, part 2], $b(Z, \varpi)$ is soft discrete. Hence, since $(cl(Z, \varpi) - (Z, \varpi)) \,\widetilde{\subseteq}\, b(Z, \varpi)$, we deduce that $cl(Z, \varpi) - (Z, \varpi)$ is both soft closed in $b(Z, \varpi)$ and soft discrete. Since $b(Z, \varpi)$ is soft closed in $(\mathcal{X}, \nu, \varpi)$, we have that $cl(Z, \varpi) - (Z, \varpi)$ is soft closed in $(\mathcal{X}, \nu, \varpi)$.

$(2 \Longrightarrow 3)$ Suppose $int(Z, \varpi) = \Phi_\varpi$ and $x_\rho \in SP(\mathcal{X})_\varpi$. Then

$$\begin{aligned}(Z, \varpi)^c &= (Z, \varpi)^c \,\widetilde{\cup}\, int(Z, \varpi)\\ &= (Z, \varpi)^c \,\widetilde{\cup}\, (cl((Z, \varpi)^c))^c\\ &= ((Z, \varpi) \,\widetilde{\cap}\, cl((Z, \varpi)^c))^c\\ &= (cl((Z, \varpi)^c) - (Z, \varpi)^c)^c\end{aligned}$$

is soft open since $cl((Z, \varpi)^c) - (Z, \varpi)^c$ is soft closed. So (Z, ϖ) is soft closed. Thus, $\mathscr{D}(Z, \varpi) \,\widetilde{\subseteq}\, (Z, \varpi)$. We show that $\mathscr{D}(Z, \varpi) = \Phi_\varpi$. Let $x_\rho \in (Z, \varpi)$. Since $int(Z, \varpi) = \Phi_\varpi$, we also have that $int((Z, \varpi) - \{x_\rho\}) = \Phi_\varpi$. Therefore, $(Z, \varpi) - \{x_\rho\} = (Z, \varpi) \,\widetilde{\cap}\, \{x_\rho\}^c$ is soft closed, and so $\{x_\rho\} \,\widetilde{\cup}\, (Z, \varpi)^c$ is soft open. But then there is a soft open neighborhood $(L, \varpi) = \{x_\rho\} \,\widetilde{\cup}\, (Z, \varpi)^c$ of x_ρ such that $(L, \varpi) \,\widetilde{\cap}\, ((Z, \varpi) - \{x_\rho\}) = \Phi_\varpi$. Thus, $\{x_\rho\} \notin \mathscr{D}(Z, \varpi)$. It follows that $\mathscr{D}(Z, \varpi) = \Phi_\varpi$. Therefore, by Lemma 5, (Z, ϖ) is soft closed and soft discrete.

$(3 \Longrightarrow 4)$ Since

$$int(cl(Z, \varpi) - (Z, \varpi)) = int(cl(Z, \varpi)) \widetilde{\cap}\, int((Z, \varpi)^c)$$

$$= int(cl(Z, \varpi)) \widetilde{\cap}\, (cl(Z, \varpi))^c$$

$$= \Phi_{\varpi},$$

we have that $cl(Z, \varpi) - (Z, \varpi)$ is soft closed and soft discrete.

($4 \Longrightarrow 1$) Let (Z, ϖ) be a soft dense subset of $(\mathcal{X}, \nu, \varpi)$. Then $cl(Z, \varpi) - (Z, \varpi)$ is soft closed and soft discrete. By soft density of (Z, ϖ) in $(\mathcal{X}, \nu, \varpi)$, we have $cl(Z, \varpi) - (Z, \varpi) = \mathcal{X}_{\varpi} - (Z, \varpi)$ is soft closed and soft discrete, thus (Z, ϖ) is soft open. □

Theorem 2 Given a soft topological space $(\mathcal{X}, \nu, \varpi)$. The following conditions are equivalent:

(1) $(\mathcal{X}, \nu, \varpi)$ is soft submaximal;

(2) $b(Z, \varpi)$ has no soft limit points for each soft set (Z, ϖ) in $(\mathcal{X}, \nu, \varpi)$;

(3) Each residual soft set has no soft limit points.

Proof. ($1 \Longrightarrow 2$) By [36, Theorem 4.4. part 2 & 3], $b(Z, \varpi)$ is soft discrete and soft closed. Thus, by Lemma 5, it has no soft limit points.

($2 \Longrightarrow 3$) Let (Z, ϖ) be a residual soft set in $(\mathcal{X}, \nu, \varpi)$. If $int(Z, \varpi) = \Phi_{\varpi}$, then

$$(Z, \varpi) \widetilde{\subseteq} cl(Z, \varpi)$$

$$= cl(Z, \varpi) - int(Z, \varpi)$$

$$= b(Z, \varpi).$$

By (2), $b(Z, \varpi)$ has no soft limit points; therefore, also (Z, ϖ) has no soft limit points.

($3 \Longrightarrow 1$) Since each residual soft set is soft closed, hence every soft dense set in $(\mathcal{X}, \nu, \varpi)$ belongs ν. Therefore, $(\mathcal{X}, \nu, \varpi)$ is soft submaximal. □

Theorem 3 Let $(\mathcal{X}, \nu, \varpi)$ be a soft space that has a finite number of soft limit points. If each soft limit point of $(\mathcal{X}, \nu, \varpi)$ is soft closed, then $(\mathcal{X}, \nu, \varpi)$ is a soft submaximal space.

Proof. Let (Z, ϖ) be a soft dense subset of $(\mathcal{X}, \nu, \varpi)$ and $\mathscr{D}(\mathcal{X}_{\varpi})$ be the set of soft limit points of $(\mathcal{X}, \nu, \varpi)$. Then, clearly, $cl(Z, \varpi) - (Z, \varpi) \widetilde{\subseteq} \mathscr{D}(\mathcal{X}_{\varpi})$, which is a finite union of soft closed soft point sets, hence (Z, ϖ) is soft closed and soft discrete; so by Theorem 1, $(\mathcal{X}, \nu, \varpi)$ is a soft submaximal space. □

Theorem 4 If $(\mathcal{X}, \nu, \varpi)$ is soft submaximal space and $(Z, \varpi) \widetilde{\subseteq} (\mathcal{X}, \nu, \varpi)$, then (Z, ϖ) is soft open if and only if it is the soft intersection of a soft dense and a soft regular open set.

Proof. Let (Z, ϖ) be a soft open subset of $(\mathcal{X}, \nu, \varpi)$. Clearly $(Z, \varpi) \widetilde{\subseteq} int(cl(Z, \varpi))$. Thus

$$(Z,\,\varpi) = cl(Z,\,\varpi) - (cl(Z,\,\varpi) - (Z,\,\varpi))$$

$$= cl(Z,\,\varpi)\,\widetilde{\cap}\,[\mathcal{X}_{\varpi} - (cl(Z,\,\varpi) - (Z,\,\varpi))]$$

$$= int(cl(Z,\,\varpi))\,\widetilde{\cap}\,[(Z,\,\varpi)\,\widetilde{\cup}\,(\mathcal{X}_{\varpi} - cl(Z,\,\varpi))],$$

where $int(cl(Z,\,\varpi))$ is soft regular open and $[(Z,\,\varpi)\,\widetilde{\cup}\,(\mathcal{X}_{\varpi} - cl(Z,\,\varpi))]$ is soft dense. The reverse is trivial. □

Theorem 5 If $(\mathcal{X},\,\nu,\,\varpi)$ is a soft submaximal space, $(\mathcal{X},\,\nu,\,\varpi)$ is soft T_0.

Proof. Suppose $(\mathcal{X},\,\nu,\,\varpi)$ is a soft submaximal space and $x_\rho,\,y_\varsigma \in SP(\mathcal{X})_{\varpi}$ such that $x_\rho \neq y_\varsigma$. If $(\mathcal{X}_{\varpi} - \{x_\rho\})\,\widetilde{\subseteq}\,\mathcal{X}_{\varpi}$ is soft dense in $(\mathcal{X},\,\nu,\,\varpi)$, then it is soft open containing y_ς, but not x_ρ. Suppose that $\mathcal{X}_{\varpi} - \{x_\rho\}$ is not soft dense, then $cl(\mathcal{X}_{\varpi} - \{x_\rho\}) \neq \mathcal{X}_{\varpi}$, hence $\mathcal{X}_{\varpi} - \{x_\rho\}$ is soft closed in $(\mathcal{X},\,\nu,\,\varpi)$; so $\{x_\rho\}$ is soft open set containing x_ρ, but not y_ς. □

Let $(\mathcal{X},\,\nu,\,\varpi)$ be a soft T_0-space, $x_\rho,\,y_\varsigma \in SP(\mathcal{X})_{\varpi}$, and $\leq_\nu$ be an ordering defined on $SP(\mathcal{X})_{\varpi}$ by

$$x_\rho \leq_\nu y_\varsigma \Longleftrightarrow y_\varsigma \in cl(\{x_\rho\}). \tag{1}$$

The ν-ordering generated by ν will be referred to as the order $\leq_\nu$. The "Krull dimension" of a soft topological space $(\mathcal{X},\,\nu,\,\varpi)$ is defined to be the supremum of the lengths of chains of soft points of the given soft space. The length of the chain $x^1_\rho \leq_\nu x^2_\varsigma \leq_\nu x^3_\zeta \leq_\nu \dots \leq_\nu x^n_\eta$ is the positive integer n. We symbolize the Krull dimension of $(\mathcal{X},\,\nu,\,\varpi)$ by κ-$dim(\mathcal{X},\,\nu,\,\varpi)$. A soft point $x_\rho \in SP(\mathcal{X})_{\varpi}$ is said to have the height n, denoted by $H(x_\rho) = n$, if the supremum of the lengths of all the chains that arrive to x_ρ is n. Let $X^{(n)}_{\varpi} := \{x_\rho \in SP(\mathcal{X})_{\varpi} : H(x_\rho) = n\}$, where $n \in \mathbb{N}$.

Let $(\mathcal{X},\,\nu,\,\varpi)$ be an Alexandroff soft topological space and $\mathscr{R}$ be a soft preorder relation on $SP(\mathcal{X})_{\varpi}$ (i.e., $\mathscr{R}$ is reflexive and transitive in $SP(\mathcal{X})_{\varpi}$, see [37]).

For each $x_\rho \in SP(\mathcal{X})_{\varpi}$, we define:

- $(\downarrow x_\rho)_{\mathscr{R}} := \{y_\varsigma \in SP(\mathcal{X})_{\varpi} : y_\varsigma \mathscr{R} x_\rho\}$;
- $(x_\rho \uparrow)_{\mathscr{R}} := \{y_\varsigma \in SP(\mathcal{X})_{\varpi} : x_\rho \mathscr{R} y_\varsigma\}$.

Then the family $\mathscr{B} := \{(\downarrow x_\rho)_{\mathscr{R}} : x_\rho \in SP(\mathcal{X})_{\varpi}\}$ is a soft base of a soft topology over $\mathcal{X}$, referred to as the Alexandroff soft topology related with $\mathscr{R}$ and symbolized by $\nu(\mathscr{B})_{\mathscr{R}}$.

On the other hand, each Alexandroff soft topology over $\mathcal{X}$ can be identified with $\nu(\mathscr{B})_{\mathscr{R}}$, where $\mathscr{R}$ is a soft preorder given by

$$\mathcal{X}_{\varpi} \mathscr{R} y_\varsigma \text{ if and only if } cl(\{x_\rho\})\,\widetilde{\subseteq}\,cl(\{y_\varsigma\}). \tag{2}$$

By Equation (1), one can easily conclude that

$$y_\varsigma \mathscr{R} x_\rho \text{ if and only if } x_\rho \leq_\nu y_\varsigma. \tag{3}$$

Theorem 6 Let $(\mathcal{X},\,\nu,\,\varpi)$ be a soft T_0-space. The following conditions are equivalent:

(1) If $(\mathcal{X},\,\nu,\,\varpi)$ is a soft submaximal space, then κ-$dim(\mathcal{X},\,\nu,\,\varpi) \leq 1$;

(2) If, additionally, ν is an Alexandroff soft topology, then $(\mathcal{X},\,\nu,\,\varpi)$ is a soft submaximal space, whenever κ-$dim(\mathcal{X},\,\nu,\,\varpi) \leq 1$.

Proof. (1) Let $x_\rho, y_\varsigma \in SP(\mathcal{X})_\varpi$. Then, by part (4) in Theorem 1, $cl\{x_\rho\} - \{x_\rho\}$ is a soft closed and soft discrete set in $(\mathcal{X}, \nu, \varpi)$. Therefore, if $y_\varsigma \in (cl\{x_\rho\} - \{x_\rho\})$, then $\{y_\varsigma\}$ is soft closed. Hence, $\{y_\varsigma\}$ is a maximal soft point in the preorder $\leq_\nu$ of $(\mathcal{X}, \nu, \varpi)$. Thus $\kappa\text{-}dim(\mathcal{X}, \nu, \varpi) \leq 1$.

(2) Let $(Z, \varpi) \widetilde{\subseteq} (\mathcal{X}, \nu, \varpi)$. Assume (Z, ϖ) is not soft closed and $x_\rho \in (cl(Z, \varpi) - (Z, \varpi))$, where $x_\rho \in SP(\mathcal{X})_\varpi$. Since ν is an Alexandroff soft topology, $cl(Z, \varpi) = \bigcup\{(y_\varsigma \uparrow) \colon y_\varsigma \in (Z, \varpi)\}$ and, therefore, there exists $y_\varsigma \in (Z, \varpi)$ such that $y_\varsigma \leq_\nu x_\rho$. Hence, $(x_\rho \uparrow) = \{x_\rho\}$, since $\kappa\text{-}dim(\mathcal{X}, \nu, \varpi) \leq 1$. Thus, $cl(Z, \varpi) - (Z, \varpi) = \bigcup\{(x_\rho \uparrow) \colon x_\rho \in (cl(Z, \varpi) - (Z, \varpi))\}$ implies $cl(Z, \varpi) - (Z, \varpi)$ is soft closed. Therefore, by part (4) in Theorem 1, $(\mathcal{X}, \nu, \varpi)$ is soft submaximal. □

Here, we remark that Theorem 6 was proved for classical submaximal spaces in Proposition 2.2 in [38].

Theorem 7 Let $(\mathcal{X}, \nu, \varpi)$ be a soft submaximal soft compact space. Then $cl(Z, \varpi) - (Z, \varpi)$ is finite for each $(Z, \varpi) \widetilde{\subseteq} (\mathcal{X}, \nu, \varpi)$.

Proof. Since $cl(Z, \varpi) - (Z, \varpi)$ is a soft closed set with a discrete relative soft topology (see part (4) in Theorem 1). Since $(\mathcal{X}, \nu, \varpi)$ is soft compact, hence by Lemma 4, hence $cl(Z, \varpi) - (Z, \varpi)$ is a soft compact set. However, $cl(Z, \varpi) - (Z, \varpi)$ is a soft discrete set, therefore it is finite. □

Theorem 8 Let $(\mathcal{X}, \nu, \varpi)$ be a soft infinite soft discrete topological space. Then $(\mathcal{X}^*, \nu^*, \varpi)$ is soft submaximal.

Proof. (Z, ϖ) be a soft dense subset of $(\mathcal{X}^*, \nu^*, \varpi)$ and let $x_\rho \in SP(\mathcal{X})_\varpi$. Since each $\{x_\rho\} \widetilde{\subseteq} \mathcal{X}_\varpi$ is soft open in $(\mathcal{X}^*, \nu^*, \varpi)$, therefore $\{x_\rho\} \widetilde{\cap} (Z, \varpi) \neq \Phi_\varpi$. Thus, $\mathcal{X}_\varpi \widetilde{\subseteq} (Z, \varpi)$. Since, on the other hand, $cl(\mathcal{X}_\varpi) = \mathcal{X}^*_\varpi$, then the space $(\mathcal{X}^*, \nu^*, \varpi)$ has exactly two soft dense subsets; namely $\mathcal{X}_\varpi$ and $\mathcal{X}^*_\varpi$, and both of them are soft open, thus $(\mathcal{X}^*, \nu^*, \varpi)$ is soft submaximal. □

Theorem 9 Given a soft topological space $(\mathcal{X}, \nu, \varpi)$. The following conditions are equivalent:

(1) $(\mathcal{X}^*, \nu^*, \varpi)$ is soft submaximal;

(2) For every soft dense set (Z, ϖ) in $(\mathcal{X}, \nu, \varpi)$, the following conditions hold:

(i) (Z, ϖ) is cofinite in $(\mathcal{X}^*, \nu^*, \varpi)$;

(ii) For every $x_\rho \in ((\mathcal{X}^*, \varpi) - (Z, \varpi))$, $\{x_\rho\}$ is soft closed, where $x_\rho \in SP(\mathcal{X}^*_\varpi)$.

Proof. $(1 \Longrightarrow 2)$ Let (Z, ϖ) be a soft dense subset of $(\mathcal{X}, \nu, \varpi)$. Then (Z, ϖ) is a soft dense subset of $(\mathcal{X}^*, \nu^*, \varpi)$. Hence, by part (4) in Theorem 1, $cl_{\mathcal{X}^*}(Z, \varpi) - (Z, \varpi) = (\mathcal{X}^*, \varpi) - (Z, \varpi)$ is a soft closed and soft discrete subset of $(\mathcal{X}^*, \nu^*, \varpi)$. Thus, $\{x_\rho\}$ is soft closed, for every $x_\rho \in ((\mathcal{X}^*, \varpi) - (Z, \varpi))$. Since $(\mathcal{X}^*, \nu^*, \varpi)$ is soft compact, $(\mathcal{X}^*, \varpi) - (Z, \varpi)$ is finite; so that (Z, ϖ) is cofinite in $(\mathcal{X}^*, \nu^*, \varpi)$.

$(2 \Longrightarrow 1)$ Let (Z, ϖ) be a soft dense subset of $(\mathcal{X}^*, \nu^*, \varpi)$. Then $(Z, \varpi) \widetilde{\cap} \mathcal{X}_\varpi$ is a soft dense subset of $\mathcal{X}_\varpi$. By hypothesis, $(\mathcal{X}^*, \varpi) - ((Z, \varpi) \widetilde{\cap} \mathcal{X}_\varpi)$ is finite and for each $x_\rho \in ((\mathcal{X}^*, \varpi) - ((Z, \varpi) \widetilde{\cap} \mathcal{X}_\varpi))$, $\{x_\rho\}$ is a soft closed set, where $x_\rho \in SP(\mathcal{X}^*_\varpi)$. Consequently, $(\mathcal{X}^*, \varpi) - (Z, \varpi)$ is soft closed, hence (Z, ϖ) is a soft open set in $(\mathcal{X}^*, \nu^*, \varpi)$. □

Corollary 1 Given a soft topological space $(\mathcal{X}, \nu, \varpi)$. The following conditions are equivalent:

(1) $(\mathcal{X}^*, \nu^*, \varpi)$ is a soft submaximal soft T_1-space;

(2) Each soft dense subset of $(\mathcal{X}, \nu, \varpi)$, (Z, ϖ) is cofinite in $(\mathcal{X}^*, \nu^*, \varpi)$.

We end this section by highlighting that:

Remark 2 The family of soft submaximal spaces along with the same nonempty set of parameters is hereditary, additive, but not productive, see [36].

4. On soft door spaces

This section begins with an introduction of the definition of soft door spaces, which is followed by some additional findings.

Definition 24 [39] Let $(\mathcal{X}, \nu, \varpi)$ be a soft topological space. Then $(\mathcal{X}, \nu, \varpi)$ is said to be a soft door space if each soft set in $(\mathcal{X}, \nu, \varpi)$ is either soft open or soft closed.

Lemma 3 Every soft discrete topological space is soft door.

Theorem 10 If a soft topological space $(\mathcal{X}, \nu, \varpi)$ is soft door, then $(\mathcal{X}, \nu, \varpi)$ is soft submaximal.

Proof. Let $(\mathcal{X}, \nu, \varpi)$ be a soft door space. Let (Z, ϖ) be a soft dense set in $(\mathcal{X}, \nu, \varpi)$. If (Z, ϖ) is soft open, there is nothing to prove. If (Z, ϖ) is not soft open, hence by soft doorness of $(\mathcal{X}, \nu, \varpi)$, (Z, ϖ) is soft closed, therefore

$cl(Z, \varpi) = (Z, \varpi)$. Thus, contradicting the fact that (Z, ϖ) is soft dense in $(\mathcal{X}, \nu, \varpi)$. Hence, (Z, ϖ) is soft open. Therefore, $(\mathcal{X}, \nu, \varpi)$ is soft submaximal. $\square$

The next illustration demonstrates that the reverse of Theorem 10 is not always valid:

Example 1 Let ϖ be a nonempty set of parameters. Let ν be the soft topology over the set of integers $\mathbb{Z}$ generated by $\{(\rho, K(\rho))\colon K(\rho) = \{2n-1, 2n, 2n+1\}; n \in \mathbb{Z}, \rho \in \varpi\}$. Then $(\mathbb{Z}, \nu, \varpi)$ is an Alexandroff soft topological T_0-space with $\kappa\text{-}dim(\mathbb{Z}, \nu, \varpi) = 1$. By Theorem 6, $(\mathbb{Z}, \nu, \varpi)$ is a submaximal soft topological space. However, the soft set $(L, \varpi) = \{(\rho, \{2n-1, 2n+2\})\}$, where $n \in \mathbb{Z}$ and $\rho \in \varpi$ is neither soft open nor soft closed. Thus, $(\mathbb{Z}, \nu, \varpi)$ cannot be soft door.

Theorem 11 Soft doorness is hereditary.

Proof. Suppose that $(\Upsilon, \nu_{(\Upsilon, \varpi)}, \varpi)$ is a soft subspace of a soft door space $(\mathcal{X}, \nu, \varpi)$. Let (Z, ϖ) be in $(\Upsilon, \nu_{(\Upsilon, \varpi)}, \varpi)$, then (Z, ϖ) is in $(\mathcal{X}, \nu, \varpi)$, hence (Z, ϖ) is soft open or soft closed in $(\mathcal{X}, \nu, \varpi)$. Thus, (Z, ϖ) is soft open or soft closed in $(\Upsilon, \nu_{(\Upsilon, \varpi)}, \varpi)$. $\square$

Theorem 12 A soft Hausdorff space is soft door if and only if it has at most one soft limit point.

Proof. Given a soft Hausdorff space $(\mathcal{X}, \nu, \varpi)$. Let $(\mathcal{X}, \nu, \varpi)$ be a soft door space. The first part is clear, if $(\mathcal{X}, \nu, \varpi)$ is soft discrete, therefore, $(\mathcal{X}, \nu, \varpi)$ has no soft limit points. On the other hand, suppose $(\mathcal{X}, \nu, \varpi)$ has two distinct soft limit points; namely, x_ρ, y_ς. Since $(\mathcal{X}, \nu, \varpi)$ a soft Hausdorff space, therefore x_ρ, y_ς can be separated by two disjoint soft open sets (Z, ϖ), (W, ϖ) such that $x_\rho \in (Z, \varpi)$ and $y_\varsigma \in (W, \varpi)$. Then the soft set $((Z, \varpi) - \{x_\rho\}) \,\widetilde{\bigcup}\, \{y_\varsigma\}$ is not soft open as it includes y_ς but no other points of (W, ϖ), and it cannot be soft closed, since its soft closure includes x_ρ. This violates the assumption that $(\mathcal{X}, \nu, \varpi)$ is a soft door space.

Conversely, if $(\mathcal{X}, \nu, \varpi)$ has no soft limit point, therefore $(\mathcal{X}, \nu, \varpi)$ is soft discrete and so it soft door. Suppose $(\mathcal{X}, \nu, \varpi)$ has exactly one soft limit point, say, y_ς. Since, for each $x_\rho \in SP(\mathcal{X})_\varpi$ with $x_\rho \neq y_\varsigma$, $\{x_\rho\}$ is a soft open set, otherwise it is soft closed. Thus, $(\mathcal{X}, \nu, \varpi)$ is a soft door space. $\square$

Theorem 13 If $(\bigoplus_{i\in I} \mathcal{X}_i, \nu, \varpi)$ is a soft door space, then $(\mathcal{X}_i, \nu_i, \varpi)$ is soft door for every i.

Proof. Follows from Theorem 11. $\square$

Remark 4 Soft doorness is neither additive nor productive, (*c.f.*, Remark 2).

Example 2 Given $\varpi = \{\rho\}$ and let $(\mathbb{N}^*, \nu_D^*, \varpi)$ be the soft one-point compactification of the soft discrete topological space $(\mathbb{N}, \nu_D, \varpi)$, where $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$. Notice that $(\mathbb{N}^*, \nu_D^*, \varpi)$ has the unique soft limit point ∞_ρ. The sum $\left(\mathbb{N}^*_\varpi \times (\{0\}, \varpi)\right) \bigoplus \left(\mathbb{N}^*_\varpi \times (\{1\}, \varpi)\right)$ is a soft T_2-space of two soft limit points. According to Theorem 12, it is not a soft door space (but it is a soft submaximal space). Moreover, the product of the soft topological space $(\mathbb{N}^*, \nu_D^*, \varpi)$ with itself is a soft T_2-space with infinitely many soft limit points, so it is not a door space.

This example also shows that a soft submaximal space need not be soft door.

Theorem 14 Let $(\mathcal{X}, \nu, \varpi)$, $(\mathcal{X}, \nu', \varpi)$ be soft topological spaces and $\nu \,\widetilde{\subseteq}\, \nu'$. If $(\mathcal{X}, \nu, \varpi)$ is a soft door space, then $(\mathcal{X}, \nu', \varpi)$ is a soft door space.

Proof. Let (Z, ϖ) be in $(\mathcal{X}, \nu', \varpi)$. Then by soft doorness of $(\mathcal{X}, \nu, \varpi)$, (Z, ϖ) is either soft open or soft closed with respect to ν. Since $\nu \,\widetilde{\subseteq}\, \nu'$, then (Z, ϖ) is either soft open or soft closed of $(\mathcal{X}, \nu', \varpi)$, hence $(\mathcal{X}, \nu', \varpi)$ is a soft door space. $\square$

Theorem 15 The image of a soft door space under a soft open surjection is soft door.

Proof. Given a soft open surjective function $h\colon (\mathcal{X}, \nu, \varpi) \to (\Upsilon, \nu', \varpi')$. Suppose $(\mathcal{X}, \nu, \varpi)$ is a soft door space. Let (Z, ϖ) be in $(\Upsilon, \nu', \varpi')$, then $h^{-1}(Z, \varpi) \,\widetilde{\subseteq}\, (\mathcal{X}, \nu, \varpi)$. By soft doorness of $(\mathcal{X}, \nu, \varpi)$, $h^{-1}(Z, \varpi)$ is soft open or soft closed. If $h^{-1}(Z, \varpi)$ is soft open, by assumption we have $h(h^{-1}(Z, \varpi)) = (Z, \varpi)$ is soft open in $(\Upsilon, \nu', \varpi')$. Suppose that (Z, ϖ) is soft closed, hence $\mathcal{X}_\varpi - h^{-1}(Z, \varpi)$ is soft open in $(\mathcal{X}, \nu, \varpi)$ and, again by assumption, we have $h(\mathcal{X}_\varpi - h^{-1}(Z, \varpi)) = \Upsilon_{\varpi'} - (Z, \varpi)$ is soft open in $(\Upsilon, \nu', \varpi')$, thus (Z, ϖ) is a soft closed set in $(\Upsilon, \nu', \varpi')$. Therefore, $(\Upsilon, \nu', \varpi')$ is a soft door space. $\square$

Generally, the inverse image of a soft door space under a soft continuous surjection need not be a soft door space can be shown as follows. Take soft topological spaces $(\mathcal{X}, \nu, \varpi)$ and $(\Upsilon, \nu', \varpi')$, such that $(\mathcal{X}, \nu, \varpi)$ is not soft door and Υ is a singleton. Then the constant mapping is a soft continuous surjection from $(\mathcal{X}, \nu, \varpi)$ onto $(\Upsilon, \nu', \varpi')$.

Now, we provide some fundamental information concerning soft door spaces, soft limit points, and soft door soft compact spaces.

Theorem 16 Let $(\mathcal{X}, \nu, \varpi)$ be a soft topological space and $x_\rho, y_\varsigma \in SP(\mathcal{X})_\varpi$. The following conditions are satisfied:

(1) If $(\mathcal{X}, \nu, \varpi)$ is a soft door space, we have that κ-$dim(\mathcal{X}, \nu, \varpi) \leq 1$;

(2) If $(\mathcal{X}, \nu, \varpi)$ is a soft T_0-space, so any soft point of the ordered soft set $(\mathcal{X}_\varpi, \leq_\nu)$ that is not minimal is a soft limit point of $(\mathcal{X}, \nu, \varpi)$;

(3) If $(\mathcal{X}, \nu, \varpi)$ is a soft door space and x_ρ, y_ς are different soft points in $\mathcal{X}_\varpi^{(1)}$, then $(\downarrow x_\rho) - \{x_\rho\} = (\downarrow y_\varsigma) - \{y_\varsigma\}$;

(4) If $(\mathcal{X}, \nu, \varpi)$ is a soft compact soft door space, then, for every $x_\rho \in SP(\mathcal{X})_\varpi$, $(x_\rho \uparrow)$ is finite.

Proof. (1) Since every soft door space is soft submaximal and every soft submaximal space is soft T_0, hence, by Theorem 6 part (1), κ-$dim(\mathcal{X}, \nu, \varpi) \leq 1$.

(2) If x_ρ is not minimum soft point of $(\mathcal{X}_\varpi, \leq_\nu)$, then there exists y_ς such that $x_\rho \in cl\{y_\varsigma\} - \{y_\varsigma\}$. This implies that x_ρ is a soft limit point.

(3) Let $x_\rho, y_\varsigma \in \mathcal{X}_\varpi^{(1)}$ such that $x_\rho \neq y_\varsigma$. Suppose that there exists $z_\zeta \in (\downarrow x_\rho) \tilde{\cap} \mathcal{X}_\varpi^{(0)}$ such that $z_\zeta \notin (\downarrow y_\varsigma)$. Then the soft subset $\{y_\varsigma, z_\zeta\}$ is not soft open and not soft closed in $(\mathcal{X}, \nu, \varpi)$, which contradicts that $(\mathcal{X}, \nu, \varpi)$ is soft door. Thus, $(\downarrow x_\rho) - \{x_\rho\} = (\downarrow y_\varsigma) - \{y_\varsigma\}$.

(4) This directly follows from Theorem 7. □

In the following result, by $|(Z, \varpi)|$ we mean the cardinality of $(Z, \varpi) \in SS(\mathcal{X})_\varpi$.

Theorem 17 Let $(\mathcal{X}, \nu, \varpi)$ be a soft topological space and let $x_\rho \in SP(\mathcal{X})_\varpi$. If $(\mathcal{X}, \nu, \varpi)$ is both soft compact soft door having the property that κ-$dim(\mathcal{X}, \nu, \varpi) = 1$. Then, only one of the following conditions is true:

(1) There is a unique soft point x_ρ of $SP(\mathcal{X})_\varpi$ such that $|(\downarrow x_\rho)| \geq 2$;

(2) There is a unique soft point x_ρ of $SP(\mathcal{X})_\varpi$ such that $|(x_\rho \uparrow)| \geq 3$.

Proof. Suppose that neither (1) nor (2) is satisfied. Since κ-$dim(\mathcal{X}, \nu, \varpi) = 1,$ there exist $x_\rho, y_\varsigma \in SP(\mathcal{X})_\varpi$ such that $|(\downarrow x_\rho)| \geq 2$ and $|(\downarrow y_\varsigma)| \geq 2$. By Theorem 16, part (3), $(\downarrow x_\rho) - \{x_\rho\} = (\downarrow y_\varsigma) - \{y_\varsigma\}$. Let z_ζ be a soft point of $(\downarrow x_\rho) - \{x_\rho\}$. Then $|(z_\zeta \uparrow)| \geq 3$. Since (2) is not satisfied, there is $w_k \neq z_\zeta$ such that $|(w_k \uparrow)| \geq 3$. Let $v_l \in (w_k \uparrow) - \{w_k\}$. Again, by Theorem 16, part (3), $(\downarrow x_\rho) - \{x_\rho\} = (\downarrow y_\varsigma) - \{y_\varsigma\} = (\downarrow v_l) - \{v_l\}$.

This gives the following inequalities:

$$z_\zeta \leq x_\rho,\ z_\zeta \leq y_\varsigma,\ w_k \leq x_\rho, \text{ and } w_k \leq y_\varsigma.$$

Consequently, the set $\{z_\zeta, y_\varsigma\}$ cannot be soft open or soft closed, a contradiction to fact that soft doorness of $(\mathcal{X}, \nu, \varpi)$. This implies that one of the requirements meets, at least. If (2) is satisfied, heretofore there are pairwise distinct soft points $z_\zeta, x_\rho, y_\varsigma$ in $\mathcal{X}_\varpi$ such that $z_\zeta \leq x_\rho$ and $z_\zeta \leq y_\varsigma$. Then the two soft points x_ρ, y_ς satisfy the following, $|(\downarrow x_\rho)| \geq 2$ and $|(\downarrow y_\varsigma)| \geq 2$ demonstrating that (1) is not met. Therefore, only (1) or (2) is true. □

Theorem 18 Given a soft compact topological space $(\mathcal{X}, \nu, \varpi)$, which is also soft door. Suppose that there exists a unique soft point x_ρ of $SP(\mathcal{X})_\varpi$ with the property $(\downarrow x_\rho) \neq \{x_\rho\}$. Then x_ρ is the unique soft limit point of $(\mathcal{X}, \nu, \varpi)$.

Proof. Assume that there exists a soft limit point y_ς such that $y_\varsigma \neq x_\rho$. Since $\{y_\varsigma\}$ is soft closed, $y_\varsigma \notin (\downarrow x_\rho)$. Let $z_\zeta \in ((\downarrow x_\rho) - \{x_\rho\})$. Since $x_\rho \in cl\{y_\varsigma, z_\zeta\},$ the collection $\{y_\varsigma, z_\zeta\}$ cannot be soft closed; however, it is a soft open set in $(\mathcal{X}, \nu, \varpi)$. But, since x_ρ is the unique soft point of $(\mathcal{X}, \nu, \varpi)$ that satisfies $(\downarrow x_\rho) \neq \{x_\rho\}$. This gives us $y_\varsigma \notin cl\{z_\zeta\}$. Thus, $\{y_\varsigma\} = \{y_\varsigma, z_\zeta\} \tilde{\cap} (\mathcal{X}_\varpi - cl\{z_\zeta\})$. Consequently, $\{y_\varsigma\}$ is a soft open set, rejecting the truth that y_ς is a soft limit point. □

Theorem 19 Let $(\mathcal{X}, \nu, \varpi)$ be a soft submaximal space in which the soft superset of every non-null soft open set is soft open, then $(\mathcal{X}, \nu, \varpi)$ is a soft door space.

Proof. Assume (Z, ϖ) is not a soft closed subset of $(\mathcal{X}, \nu, \varpi)$. Since every soft subset of $(\mathcal{X}, \nu, \varpi)$ which contains a non-null soft open subset is soft open, then (Z, ϖ) is soft dense and, by soft submaximilty of $(\mathcal{X}, \nu, \varpi)$, (Z, ϖ) is soft open. Thus, every soft subset of $(\mathcal{X}, \nu, \varpi)$ is either soft closed or soft open and, hence $(\mathcal{X}, \nu, \varpi)$ is a soft door space. □

Next, our goal is to find a soft topological space $(\mathcal{X}, \nu, \varpi)$ such that its soft one-point compactification $(\mathcal{X}^*, \nu^*, \varpi)$ is soft door.

Theorem 20 Given a soft non-compact space $(\mathcal{X}, \nu, \varpi)$. The following conditions are equivalent:

(1) $(\mathcal{X}^*, \nu^*, \varpi)$ is a soft door space;

(2) Each soft set in $(\mathcal{X}, \nu, \varpi)$ is either a soft open set, or a soft compact and soft closed set in $(\mathcal{X}, \nu, \varpi)$.

Proof. $(1 \Longrightarrow 2)$ Let (Z, ϖ) be a soft subset of $(\mathcal{X}, \nu, \varpi)$. Then $(Z, \varpi) \widetilde{\subseteq} (\mathcal{X}^*, \nu^*, \varpi)$. Since $(\mathcal{X}^*, \nu^*, \varpi)$ is a soft door space, (Z, ϖ) is either soft open or soft closed in $(\mathcal{X}^*, \nu^*, \varpi)$. Therefore, by Lemma 4, (Z, ϖ) is either a soft open or soft compact soft closed in $(\mathcal{X}, \nu, \varpi)$.

$(2 \Longrightarrow 1)$ Let (Z, ϖ) be a soft subset of $(\mathcal{X}^*, \nu^*, \varpi)$, $x \notin \mathcal{X}$, and $\mathcal{X}^* = \mathcal{X} \cup \{x\}$. We consider two cases.

Case 1: If $\{x_\rho\} \notin (Z, \varpi)$, then (Z, ϖ) is a soft subset of $(\mathcal{X}, \nu, \varpi)$ such that (Z, ϖ) is either a soft open or soft compact soft closed subset of $(\mathcal{X}, \nu, \varpi)$, in which case, (Z, ϖ) is either soft open or soft closed in $(\mathcal{X}^*, \nu^*, \varpi)$.

Case 2: If $\{x_\rho\} \in (Z, \varpi)$, then $(\mathcal{X}^*, \varpi) - (Z, \varpi) = \mathcal{X}_\varpi - (Z, \varpi)$. Therefore, $\mathcal{X}_\varpi - (Z, \varpi)$ is either soft open or soft compact soft closed in $(\mathcal{X}, \nu, \varpi)$ and $\mathcal{X}_\varpi - (Z, \varpi)$ is either soft open or soft closed in $(\mathcal{X}^*, \nu^*, \varpi)$. So $(\mathcal{X}^*, \varpi) - (Z, \varpi)$ is either soft open or soft closed in $(\mathcal{X}^*, \nu^*, \varpi)$ implies (Z, ϖ) is either soft open or soft closed in $(\mathcal{X}^*, \nu^*, \varpi)$. Hence, $(\mathcal{X}^*, \nu^*, \varpi)$ is soft door. $\square$

Corollary 2 If $(\mathcal{X}, \nu, \varpi)$ is soft discrete space that is soft non-compact, then $(\mathcal{X}^*, \nu^*, \varpi)$ is a soft door space.

Proof. It follows immediately from Theorem 20. $\square$

5. Concluding remarks

In 2011, the concept of soft topology was introduced in [2, 20]. Following that, numerous types of topological spaces were generalized to soft settings, including "soft compact", "soft one-point compactification", "soft paracompact", "soft connected", and "soft separable spaces", among others.

In 2022, some researchers touched on presenting some fundamental results about soft submaximal spaces; see [36]. In this work, we have demonstrated some other new results on soft submaximal spaces with several illuminating examples. Among others, we have shown that every soft submaximal space is soft T_0 with the Krull dimension less than or equal to one. The reverse is also possible whenever, in addition, the underlying soft topology is an Alexandroff soft topology. We have proven that the soft one-point compactification of a soft topological space is soft submaximal when the soft dense subsets of the soft topological space satisfy certain properties.

In 2017, the concept of soft door spaces was defined by Sabiha in [39] without giving any details. From this standpoint, we have discussed basic properties of soft door spaces, such as hereditary, additive, productive, and image preservations under certain soft functions, with the aid of suitable counter-examples. Further, we have given some connections between soft limit points, soft door spaces, and soft compact spaces. Also, we have characterized soft door spaces by means of a soft preorder defined on soft T_0-topological spaces. Lastly, we have discussed the soft doorness property of a soft one-point compactification of some soft topological space.

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Conflict of interest

The authors declare no competing financial interest.

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