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Research Article

Generation of Fractal Attractor for Controlled Metric Based Dynamical Systems

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Abstract: Mandelbrot initiated the term "Fractal" in 1975, and it has since gained popularity among mathematicians and physicists alike. The mathematical properties of fractals are available and applied in the chaotic structures of various systems, which are generally experienced in science and technology. The iterated function system (IFS) evolved as a practical application of the theory of discrete dynamical systems and is a valuable tool to generate fractal attractors. In this context, the Hutchinson-Barnsley (HB) theory is generalized to construct fractal sets using an IFS of contractions on a controlled metric space (CMS). The HB theorem of IFS is proved in a Hausdorff controlled metric space (HCMS), and it is also guaranteed that the HB operator has merely a single fixed point in a Hausdorff controlled metric space, known as a controlled fractal. This study also links the extended rectangular b-metric space (ERbMS) and the controlled metric space to create a new metric space, the controlled extended rectangular b-metric space (CERbMS), by incorporating the control factor as a function in the rectangular inequality. In addition, the fixed point theorem is also proved with specific conditions for contractions in the proposed metric space CERbMS and illustrated with an example. Finally, the structure of IFS is defined in CERbMS to construct the HB theory for generating the controlled extended rectangular b-fractals.

***Keywords*:** fractal analysis, controlled metric space, contraction, iterated function system, attractor

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1. Introduction

The fixed point theory is essential in dynamical systems because it allows us to design multiple chaotic attractors with contraction mappings. One of the most well-known applications of fixed point theory is to find an approximate solution for different physical systems represented by a suitable differential equation. Banach [1] developed the well-structured principle known as the Banach contraction principle to create a distinct point in space that is also unique.

Caccioppoli demonstrated the generalization and extensions of Banach's fixed point theorem in the whole metric space. The generalization and extension of Banach's theorem are based on altering the structure and conditions of the

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mapping under examination. New metrics have a variety of forms, including partial metric, quasi metric, *b*-metric, rectangular metric, rectangular *b*-metric, controlled metric, fuzzy metric, intuitionistic fuzzy metric, probabilistic metric, and their extensions in various spaces.

Benoit Mandelbrot coined the term *Fractal* in 1975, derived from the Latin *fractus*, which means shattered or fractured, to describe objects too irregular to fit onto a classical geometric platform [2]. Generally, a fractal object is a less predictable set than the sets considered in standard Euclidean geometry. Hutchinson and Barnsley [3–5] introduced the concept of IFS in 1981 by defining a fractal set as a non-empty compact invariant subset of a complete metric space generated by Banach contractions and its fixed point theorem [6, 7]. In many extended spaces, including generalized metric spaces, multivalued metric spaces, partial metric spaces, quasi metric spaces, *b*-metric spaces, rectangular metric spaces, fuzzy metric spaces, intuitionistic fuzzy spaces, probabilistic spaces, topological spaces, and others, there are numerous generalizations to create a new type of fractal sets as an attractor through HB theory by utilizing the various contraction and the corresponding fixed point theorems [8–20]. Building distinct classes of Fractal Interpolation Functions (FIF) in the general form of metric spaces has been made possible according to the theory of IFS for these generalized spaces. Numerous research problems on dynamical systems, Brownian motion, image compression, and other subjects extensively use Fractal Sets and Curves derived from IFS and FIF [21–26].

Bourbaki introduced the idea of *b*-metric space (bMS), and its partial forms were developed as a powerful generalization for metric space [27, 28]. It extended the metric space by multiplying the triangle equality by a positive constant. By generalizing the constant by a function that depends on the parameters of the triangle inequality's left side, recent researchers successively created and developed an extended *b*-metric space. The researchers determined the various forms of extensions for bMS by altering the positive constant using a two-variable function in the triangle inequality's right-hand side [29, 30]. Afterward, in 2017 Kamran et al. presented the extended *b*-metric and also gave the notable fixed point outcomes over the extended space [31].

Branciari introduced a new metric space called the rectangular metric space by increasing the terms in the triangle inequality of the general metric space in 2000 [32]. The standard form of rectangular metric space, known as rectangular *b*-metric space, was introduced by George et al. in 2015 [33]. Rectangular *b*-metric space (RbMS) is one of the most intriguing metric spaces in the category of generalized metric spaces since it possesses the generic form of distance measure in various contexts. The extended rectangular *b*-metric space (ERbMS) was popularized successively in the line, and interesting results were obtained [34]. The CMS, a new extension of the bMS, is implemented using a two-variable control function in the *b*-triangle inequality. The majority of researchers developed the general spaces with significant fixed point theorems, and a lot of substantial applications of those fixed point theorems have been established [35–44].

The preceding sequence of extensions leads us to construct a new type of metric space and demonstrate the essential fixed point theorems in the context of the ERbMS, in which we serve as the control factor. We also define IFS and talk about how HB theory can be used to develop this novel kind of fractal in the suggested generalized space; these ideas are then visually demonstrated by means of certain instances [45–47]. The above flow of extensions prompts us to study the Banach contraction maps in CMS and CERbMS to construct IFS, explain HB theory, and create a new class of fractal sets in the suggested CMS and CERbMS.

The Hutchinson-Barnsley (HB) theory is generalized in this paper to construct fractal sets using an IFS of contractions in controlled metric spaces. The HB theorem of IFS is proved in a Hausdorff controlled metric space, and it is also assured that the HB operator has only one fixed point in a Hausdorff controlled metric space. Also, the iterated function system (IFS) is defined in controlled extended rectangular *b*-metric space to construct the Hutchinson-Barnsley (HB) Theory for generating the controlled extended rectangular *b*-fractals.

Our work is divided into six parts, the first of which is the Introduction. Section 2 provides some essential definitions and results for bMSs, rectangular metric spaces, controlled metric spaces, and their expansions. In Section 3, we discuss the fractals for single and multivalued contractions. Section 4 discusses an interesting fixed point theorem on a generalized metric space and the concept of CERbMS and its Hausdorff version. In Section 5, we define the IFS alongside a graphical illustration and deduce the HB theorem to produce the novel fractal, namely the controlled extended rectangular *b*-fractal in the proposed space. The last thoughts are discussed briefly in Section 6.

2. Preliminaries

In this section, we define some key terms and discuss some preliminary contents required for this research work.

Definition 1 (b-Metric Space [27]) Given a set $\Lambda \neq \emptyset$. The mapping κ: $\Lambda \times \Lambda \to [0, \infty)$ is known as b-metric on Λ, if $\exists$ a constant $\beta \geq 1$ such that for all $\phi, \varphi, \mu \in \Lambda$, the following conditions are satisfied,

(a). $\kappa(\phi, \upsilon) = 0$ iff $\phi = \upsilon$,

(b). $\kappa(\phi, \upsilon) = \kappa(\phi, \upsilon)$,

(c). $\kappa(\phi, \upsilon) \leq \beta[\kappa(\phi, \mu) + \kappa(\mu, \upsilon)]$.

Then (Λ, κ) is known as b-metric space.

Definition 2 (Extended b-Metric Space [31]) Take $\Lambda \neq \emptyset$ and the mapping β: $\Lambda \times \Lambda \to [1, \infty)$. Then a mapping κ: $\Lambda \times \Lambda \to [0, \infty)$ is said to be an extended b-metric on Λ, for all distinct $\phi, \upsilon, \mu \in \Lambda$, if it fulfills the following conditions:

(a). $\kappa(\phi, \upsilon) = 0$ iff $\phi = \upsilon$,

(b). $\kappa(\phi, \upsilon) = \kappa(\upsilon, \phi)$,

(c). $\kappa(\phi, \upsilon) \leq \beta(\phi, \upsilon)[\kappa(\phi, \mu) + \kappa(\mu, \upsilon)]$.

Then (Λ, κ) is said to be an extended b-metric space.

Definition 3 (Rectangular Metric Space [32]) Let $\Lambda \neq \emptyset$. The function κ: $\Lambda \times \Lambda \to [0, \infty)$ is said to be a rectangular metric on Λ, if $\forall \phi, \upsilon \in \Lambda$ and for every distinct $\mu, \eta \in \Lambda / \{\phi, \upsilon\}$, it satisfies the following conditions:

(a). $\kappa(\phi, \upsilon) = 0$ iff $\phi = \upsilon$,

(b). $\kappa(\phi, \upsilon) = \kappa(\upsilon, \phi)$,

(c). $\kappa(\phi, \upsilon) \leq [\kappa(\phi, \mu) + \kappa(\mu, \eta) + \kappa(\eta, \upsilon)]$.

Then (Λ, κ) is called a rectangular metric space.

Branciari introduced RbMS as a generalization of bMS in 2000 [32]. George also analyzed this space and extensively proved the fixed point theorems in 2015 [33].

Definition 4 (Rectangular b-Metric Space [33]) Let $\Lambda \neq \emptyset$ and the function κ: $\Lambda \times \Lambda \to [0, \infty)$ is called a rectangular b-metric on Λ, if $\forall\ \phi, \upsilon \in \Lambda$ and for every distinct $\mu, \eta \in \Lambda / \{\phi, \upsilon\}$, it fulfills the following conditions:

(a). $\kappa(\phi, \upsilon) = 0$ iff $\phi = \upsilon$,

(b). $\kappa(\phi, \upsilon) = \kappa(\upsilon, \phi)$,

(c). $\kappa(\phi, \upsilon) \leq \beta[\kappa(\phi, \mu) + \kappa(\mu, \eta) + \kappa(\eta, \upsilon)]$, where $\beta \geq 1$ is a real constant.

Then, the pair (Λ, κ) is called a rectangular b-metric space.

Definition 5 (Extended Rectangular b-Metric Space (ERbMS) [34]) Let $\Lambda \neq \emptyset$ and β: $\Lambda \times \Lambda \to [1, \infty)$. A mapping κ: $\Lambda \times \Lambda \to [0, \infty)$ is called an extended rectangular b-metric on Λ, for all distinct $\phi, \upsilon, \mu, \eta \in \Lambda$, if it fulfills the following conditions:

(a). $\kappa(\phi, \upsilon) = 0$ iff $\phi = \upsilon$,

(b). $\kappa(\phi, \upsilon) = \kappa(\upsilon, \phi)$,

(c). $\kappa(\phi, \upsilon) \leq \beta(\phi, \upsilon)[\kappa(\phi, \mu) + \kappa(\mu, \eta) + \kappa(\eta, \upsilon)]$.

Then, (Λ, κ) is called as an extended rectangular b-metric space.

Definition 6 (Controlled Metric Space [35]) Let $\Lambda \neq \emptyset$ and β: $\Lambda \times \Lambda \to [1, \infty)$. A mapping κ: $\Lambda \times \Lambda \to [0, \infty)$ is known as controlled metric on Λ, if it fulfills the following conditions:

(a). $\kappa(\phi, \upsilon) = 0$ iff $\phi = \upsilon$,

(b). $\kappa(\phi, \upsilon) = \kappa(\upsilon, \phi)$,

(c). $\kappa(\phi, \upsilon) \leq \beta(\phi, \mu)\kappa(\phi, \mu) + \beta(\mu, \upsilon)\kappa(\mu, \upsilon)\ \ \forall\ \phi, \upsilon, \mu \in \Lambda$.

Then (Λ, κ) is called a controlled metric space.

Theorem 1 [31] If every Cauchy Sequence (CS) is convergent, then the CMS (Λ, κ) is said to be a complete controlled metric space (CCMS).

Definition 7 [31] Let (Λ, κ) be a CMS and let $\phi \in \Lambda$, there exist a positive ε such that

(a). The open ball $B(\phi, \varepsilon)$ is defined as

$$B(a, \varepsilon) = \{x \in \Lambda, \ \kappa(\phi, x) < \varepsilon\}.$$

(b). The function $\mathscr{T}\colon \Lambda \to \Lambda$ is denoted as continuous at $\phi \in \Lambda$ if $\forall\ \varepsilon > 0$, $\exists$ positive $\delta \ni \mathscr{T}(B(\phi, \delta)) \subseteq B(\mathscr{T}\phi, \varepsilon)$. Obviously, in CMS $\mathscr{T}$ is continuous at ϕ, then $\phi_n \to \phi$ and $\mathscr{T}(\phi_n)$ converges to $\mathscr{T}(\phi)$ as $n \to \infty$.

Definition 8 [31] If the function $\mathscr{T}\colon \Lambda \longrightarrow \Lambda$ is said to be contraction on a CMS (Λ, κ), if $\exists\ \zeta \in [0, 1)$ such that

$$\kappa(\mathscr{T}(\phi), \mathscr{T}(\upsilon)) \leq \zeta \kappa(\phi, \upsilon), \quad \forall\ \phi, \upsilon \in \Lambda.$$

Here ζ is the contractivity ratio of $\mathscr{T}$.

Theorem 2 (Contraction Theorem on Controlled Metric Space) [31] Let us consider (Λ, κ) as a CMS. If $\mathscr{T}\colon \Lambda \to \Lambda$ is a function with

$$\kappa(\mathscr{T}(\phi), \mathscr{T}(\upsilon)) \leq \zeta \kappa(\phi, \upsilon),$$

for every $\phi, \upsilon \in \Lambda$ and $\zeta \in (0, 1)$. For $\phi_0 \in \Lambda$, take $\phi_n = \mathscr{T}^n \phi_0$, where $\mathscr{T}^n \phi_0 = \underbrace{\mathscr{T} \circ \mathscr{T} \circ \mathscr{T} \circ \cdots \circ \mathscr{T}}_{n\ times}$. Suppose that

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\beta(\phi_{i+1}, \phi_{i+2})}{\beta(\phi_i, \phi_{i+1})} \beta(\phi_{i+1}, \phi_m) < \frac{1}{\zeta}.$$

Moreover, assume that, $\forall\ \phi \in \Lambda$, we have

$$\lim_{n \to \infty} \beta(\phi_n, \phi) \text{ and } \lim_{n \to \infty} \beta(\phi, \phi_n)$$

exist and have a limit. Then "$\mathscr{T}$ has a unique fixed point".

3. Fractals for single and multivalued contractions

If $(\Lambda, \mathscr{H}_\kappa)$ is complete, $\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_n$ are Banach contractions and $F\colon \mathscr{K}_o(\Lambda) \to \mathscr{K}_o(\Lambda)$, then

$$F = F(\mathscr{A}) = \mathscr{T}_1(\mathscr{A}) \cup \mathscr{T}_2(\mathscr{A}) \cup \ldots \cup \mathscr{T}_n(\mathscr{A}).$$

The HB theorem shows that F has a unique fixed point in $\mathscr{K}_o(\Lambda)$, referred to as a fractal.

Definition 9 If (Λ, κ) is a CMS, then the function $\mathscr{H}_\kappa\colon CLD(\Lambda) \times CLD(\Lambda) \to [0, \infty)$ is defined as

$$\mathscr{H}_\kappa(\mathscr{A}, \mathscr{B}) = \begin{cases} \max\left\{ \sup\limits_{\phi \in \mathscr{A}} \kappa(\phi, \mathscr{B}), \ \sup\limits_{\upsilon \in \mathscr{B}} \kappa(\upsilon, \mathscr{A}) \right\}, & \text{if the maximum exists;} \\ \infty, & \text{otherwise.} \end{cases}$$

Theorem 3 [36] Let (Λ, κ) be a CCMS with $\lim_{n, m\to\infty} \beta(\phi_n, \phi_m)\zeta \leq 1\ \forall\ \phi_n, \phi_m \in \Lambda$, where $\zeta \geq 1$. Then $(CLD(\Lambda), \mathscr{H}_\kappa)$ is complete.

Theorem 4 [36] Let (Λ, κ) be a nonempty set CCMS and the mapping $\mathscr{T}\colon \Lambda \to CLD(\Lambda)$ is satisfies the contraction condition $\mathscr{H}_\kappa(\mathscr{T}_\phi, \mathscr{T}_\phi) \leq \zeta\kappa(\phi, \upsilon),\ \forall\ \phi, \upsilon \in \Lambda,$ where $\zeta \in [0, 1)$ such that $\lim_{n, m\to\infty} \beta(\phi_n, \phi_m)\zeta < 1, \forall\ \phi_n, \phi_m \in H$. Then "$\mathscr{T}$ has a unique fixed point".

Theorem 5 [36] If (Λ, κ) is a CCMS, then $(\mathscr{K}_o(\Lambda), \mathscr{H}_\kappa)$ is also a CCMS.

Definition 10 Let (Λ, κ) be a CMS and $\mathscr{T}_\omega\colon \Lambda \longrightarrow \Lambda,\ \omega = 1, 2, 3, ..., \mathbb{K}$ ($\mathbb{K} \in \mathbb{N}$) be $\mathbb{K}$-contraction functions with the associated contraction ratios ζ_ω, $\omega = 1, 2, 3, ..., \mathbb{K}$. Then $\{\Lambda; \mathscr{T}_\omega, \omega = 1, 2, 3, ..., \mathbb{K}\}$ is said to be Controlled IFS (C-IFS) or Controlled Hyperbolic IFS of contractions with the ratio $\zeta = \max_{\omega=1}^{\mathbb{K}} \zeta_\omega$.

Definition 11 If (Λ, κ) is a CMS and the system

$$\{\Lambda; \mathscr{T}_\omega,\ \omega = 1, 2, 3, ..., \mathbb{K}; \mathbb{K} \in \mathbb{N}\}$$

is C-IFS of contractions. Then the Controlled HB operator of C-IFS of contractions is a mapping $F\colon \mathscr{K}_o(\Lambda) \longrightarrow \mathscr{K}_o(\Lambda)$ defined by

$$F(\mathscr{B}) = \bigcup_{\omega=1}^{\mathbb{K}} \mathscr{T}_\omega(\mathscr{B}), \quad \text{for all } \mathscr{B} \in \mathscr{K}_o(\Lambda).$$

Theorem 6 If (Λ, κ) is a CMS and

$$\{\mathscr{T}_\omega\colon \Lambda \to \Lambda, \omega = 1, 2, ..., \mathbb{K}; \mathbb{K} \in \mathbb{N}\} \tag{1}$$

is a system of contractions, *i.e*,

$$\kappa(\mathscr{T}_\omega(\phi), \mathscr{T}_\omega(\upsilon)) \leq \zeta_\omega \kappa(\phi, \upsilon) \tag{2}$$

$\forall\ \phi, \upsilon \in \Lambda$ and $\zeta_\omega \in [0, 1)$, $\omega = 1, 2, ..., \mathbb{K}$.

Then there is the attractor, the only invariant element $\mathscr{A}^* \in \mathscr{K}_o(\Lambda)$ of the HB operator

$$F(\mathscr{B}) = \bigcup_{\omega=1}^{\mathbb{K}} \mathscr{T}_\omega(\mathscr{B}), \quad \phi \in \Lambda. \tag{3}$$

Here, $\mathscr{A}^*$ is called as a Controlled Metric Fractal for the IFS on CMS and we can also represent $\mathscr{A}^*$ alternatively as follows

$$\mathscr{A}^* = F(\mathscr{B}) = \overline{\bigcup_{\upsilon\in\mathscr{B}} F(\upsilon)} \left(= \bigcup_{\upsilon\in\mathscr{B}} F(\upsilon) \right), \quad B \in \mathscr{K}_o(\Lambda). \tag{4}$$

Moreover,

$$\lim_{\omega \to \infty} \mathscr{H}_\kappa(F^m(\mathscr{B}, \mathscr{A}^*)) = 0, \quad \forall\ \mathscr{B} \in \mathscr{K}_o(\Lambda), \tag{5}$$

and

$$\mathscr{H}_\kappa(\mathscr{B}, \mathscr{A}^*) \leq \frac{1}{1-\zeta} \mathscr{H}_\kappa(\mathscr{B}, F(\mathscr{A}^*)), \tag{6}$$

where $\zeta = \max_{\omega=1,\,2,\,...,\,\mathbb{K}} (\zeta_\omega)$.

Theorem 7 Let $H \neq \emptyset$ and $\mathscr{T}_1,\ \mathscr{T}_2,\ ...,\ \mathscr{T}_\omega$ be self-maps of Λ and

$$F(\mathscr{B}) = \bigcup_{\omega=1}^{\mathbb{K}} \mathscr{T}_\omega(\mathscr{B}), \quad \mathscr{B} \subseteq \Lambda. \tag{7}$$

If all fibers of the function $\mathscr{T}_\omega$ are finite for $\omega = 1,\ 2,\ ...,\ \mathbb{K},$ then $F(\mathscr{B}) \subseteq \mathscr{B}$ and $\bigcap_{\omega \in \mathbb{N}} F^\omega(\mathscr{B})$ are invariant with respect to C-IFS $\{\mathscr{T}_1,\ \mathscr{T}_2,\ ...,\ \mathscr{T}_\omega\},$ for any $\mathscr{B} \subseteq \Lambda$ such that $F(\mathscr{B}) \subseteq \mathscr{B}$. The set $\bigcap_{\omega \in \mathbb{N}} F^\omega(\mathscr{B})$, in the context of this C-IFS, is the largest invariant set. As a result, if the set $\bigcap_{\omega \in \mathbb{N}} F^\omega(\mathscr{B})$ is nonempty, then the system $\{\mathscr{T}_1,\ \mathscr{T}_2,\ ...,\ \mathscr{T}_\omega\}$ has a nonempty invariant set.

Theorem 8 If Λ denotes a CCMS, and $\mathscr{T}_1,\ \mathscr{T}_2,\ ...,\ \mathscr{T}_\omega$ denote continuous self-maps on Λ and let us consider Eqn. (7) and

$$\mathscr{A}^* = \bigcap_{\omega \in \mathbb{N}} F^\omega(\mathscr{B})$$

is the definition of F respectively. Then $\mathscr{A}^* = F(\mathscr{B}),$ and $\mathscr{A}^*$ is the largest invariant set with respect to the C-IFS $\{\mathscr{T}_1,\ \mathscr{T}_2,\ ...,\ \mathscr{T}_\omega\}$.

Theorem 9 (C-IFS Collage Theorem in Controlled Metric Space) If $(\mathscr{K}_o(\Lambda),\ \mathscr{H}_\kappa)$ is a CHMS, where $(\Lambda,\ \kappa)$ is a CMS, then

$$\mathscr{H}_\kappa(\mathscr{A}_\infty, \mathscr{B}) \leq \left(\frac{\mathscr{H}_\kappa(\mathscr{B}, \mathscr{T}(\mathscr{B}))}{(1-\zeta)} \right),\ \forall\ \mathscr{B} \in \mathscr{K}_o(\Lambda),$$

where $\mathscr{T}$ is the HB operator of the given IFS $\{\Lambda;\ \mathscr{T}_\omega,\ \omega = 1,\ 2,\ ...,\ \mathbb{K}\},\ \mathbb{K} \in \mathbb{N}$, with the contraction ratio ζ and the controlled metric fractal $\mathscr{A}_\infty$.

Proof. This proof is evident [4, 6]. □

Theorem 10 Let $(\Lambda,\ \kappa)$ be a CCMS and

$$\{\mathscr{T}_\omega\colon \Lambda \to CLD(H),\ \omega = 1,\ 2,\ ...,\ \mathbb{K};\ \mathbb{K} \in \mathbb{N}\} \tag{8}$$

be a system of multivalued contractions, *i*, *e*.,

$$\mathscr{H}_\kappa(\mathscr{T}_\omega(\phi),\, \mathscr{T}_\omega(\upsilon)) \leq \zeta_\omega\, \kappa(\phi,\, \upsilon), \tag{9}$$

$\forall\, \phi,\, \upsilon \in \Lambda$ and $\zeta_\omega \in [0,\, 1)$, $\omega = 1,\, 2,\, ...,\, \mathbb{K}$.
Then there is a unique invariant element $\mathscr{A}^* \in CLD(\Lambda)$ of the HB operator

$$F(\Lambda) = \bigcup_{\omega=1}^{N_o} \mathscr{T}_\omega(\phi), \quad \phi \in \Lambda \tag{10}$$

called the Controlled Metric Fractal of the given IFS of multivalued contractions.

Theorem 11 If $(\Lambda,\, \kappa)$ is a CCMS and the mapping $\mathscr{T}_\omega\colon \Lambda \to CLD(\Lambda)$ such that

$$\Lambda_\kappa(\mathscr{T}_\omega(\phi),\, \mathscr{T}_\omega(\upsilon)) \leq \zeta_\omega\, \kappa(\phi,\, \upsilon) + \eta_\omega\, [\kappa(\phi,\, \mathscr{T}_\omega(\phi)) + \kappa(\upsilon,\, \mathscr{T}_\omega(\upsilon)]$$

$\forall\, \phi,\, \upsilon \in \Lambda$, where $\zeta_\omega,\, \eta_\omega \geq 0$, $\zeta_\omega + 2\eta_\omega < 1$, $\omega = 1,\, 2,\, ...,\, \mathbb{K}$. Then there exists a closed set $\mathscr{B} \neq \emptyset \subseteq H$ that satisfies

$$\mathscr{B} \subseteq \mathscr{T}_\omega(\mathscr{B}),\ \forall\, \omega = 1,\, 2,\, ...,\, \mathbb{K}.$$

Lemma 1 Suppose we take $\eta_\omega = 0$, then Theorem 11 implies Theorem 10. This shows that Theorem 11 is the generalization of Theorem 10.

4. Controlled extended rectangular *b*-metric space

Here, we develop a novel metric, the controlled extended rectangular *b*-metric, and thoroughly analyze the associated fixed point theorem.

Definition 12 (Controlled Extended Rectangular *b*-Metric Space (CERbMS)) Consider $\Lambda \neq \emptyset$ and a mapping $\beta\colon \Lambda \times \Lambda \to [1,\, \infty)$. Then a mapping $\kappa\colon \Lambda \times \Lambda \to [0,\, \infty)$ is called a controlled extended rectangular *b*-metric, if it satisfies the following conditions:

(a). $\kappa(\phi,\, \upsilon) = 0$ iff $\phi = \upsilon$,

(b). $\kappa(\phi,\, \upsilon) = \kappa(\upsilon,\, \phi)$,

(c). $\kappa(\phi,\, \upsilon) \leq [\beta(\phi,\, \mu)\kappa(\phi,\, \mu) + \beta(\mu,\, \eta)\kappa(\mu,\, \eta) + \beta(\eta,\, \upsilon)\kappa(\eta,\, \upsilon)]\ \forall$ distinct $\phi,\, \upsilon,\, \mu,\, \eta \in \Lambda$.

Then the pair $(\Lambda,\, \kappa)$ is said to be CERbMS.

Example 1 Let $\Lambda = Y \cup Z^+$, $Y = \left\{\dfrac{1}{m};\, m \in N\right\}$, $Z^+ = \{1,\, 2,\, 3\, ...\}$, $\beta\colon \Lambda \times \Lambda \to [1,\, \infty)$

$$\beta(\phi,\, \upsilon) = \begin{cases} \phi; & \text{if} \quad \phi \quad \text{is even,} \quad \upsilon \quad \text{is odd,} \\ \upsilon; & \text{if} \quad \phi \quad \text{is odd,} \quad \upsilon \quad \text{is even,} \\ 1; & \text{otherwise.} \end{cases}$$

$\kappa\colon \Lambda \times \Lambda \to [0,\, \infty)$

$$\kappa(\phi, \upsilon) = \begin{cases} 0; & \text{if} \quad \phi = \upsilon, \\ 2\beta; & \text{if} \quad \phi, \upsilon \in Y, \\ \dfrac{\beta}{2}; & \text{otherwise.} \end{cases}$$

where β is a positive real constant and κ is a controlled extended rectangular b-metric on Λ.

On the other hand,

$$\kappa\left(\frac{1}{a}, \frac{1}{a+1}\right) = 2\beta > \beta\left(\frac{1}{a}, \frac{1}{a+1}\right)\left[\kappa\left(\frac{1}{a}, a\right) + \kappa(a, a+1) + \kappa\left(a+1, \frac{1}{a+1}\right)\right]$$

$$\Longrightarrow 2\beta > \frac{3\beta}{2}$$

Therefore, the example shows that every CERbMS is not necessarily an ERbMS.

The notions of convergent sequence, Cauchy sequence, completeness, and compactness are defined in CERbMS below to proceed further to derive related results.

Definition 13 Let (Λ, κ) be a CERbMS and the sequence $\{\phi_n\}_{n\geq 0}$ in Λ.

(a). For $\phi \in \Lambda$ and $\varepsilon > 0$, the open ball about ϕ of radius ε is defined as $B_\varepsilon(\phi) = \{x \in \Lambda\colon \kappa(x, \phi) < \varepsilon\}$.

(b). The sequence ϕ_n is convergent to $\phi \in \Lambda$, if $\forall\, \varepsilon > 0$, $\exists N \in \mathbb{N} \ni \kappa(\phi_n, \phi) < \varepsilon$, $\forall\, n \geq N$, *i.e.*, $\lim_{n\to\infty} \phi_n = \phi$.

(c). The sequence $\{\phi_n\}$ is Cauchy, if $\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N} \ni \kappa(\phi_n, \phi_m) < \varepsilon$, $\forall\, n, m \geq N$.

(d). If every CS converges in Λ, then (Λ, κ) is complete.

(e). The space (Λ, κ) is known to be a compact CERbMS if every sequence in Λ has a convergent subsequence converging to a point in Λ.

As a consequence, if $\mathscr{T}$ is continuous at ϕ in (Λ, κ), then $\phi_n \to \phi \Longrightarrow \mathscr{T}\phi_n \to \mathscr{T}\phi$, as $n \to \infty$.

Theorem 12 If (Λ, κ) is a complete CERbMS and if a function $\mathscr{T}\colon \Lambda \to \Lambda$ such that

$$\kappa(\mathscr{T}(\phi), \mathscr{T}(\upsilon)) \leq \zeta\kappa(\phi, \upsilon) \tag{11}$$

$\forall\, \phi, \upsilon \in \Lambda$ where $\zeta \in (0, 1)$. For $\phi_0 \in \Lambda$, take $\phi_n = \mathscr{T}^n\phi_0$. Suppose that

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\beta(\phi_{i+1}, \phi_{i+2})}{\beta(\phi_i, \phi_{i+1})} \beta(\phi_{i+1}, \phi_m) < \frac{1}{\zeta}. \tag{12}$$

In addition that, $\forall \phi \in \Lambda$,

$$\lim_{n\to\infty} \beta(\phi_n, \phi) \quad \textit{and} \quad \lim_{n\to\infty} \beta(\phi, \phi_n) \tag{13}$$

exist and finite. Then $\mathscr{T}$ has a unique fixed point.

Proof. Let us consider $\{\phi_n \in \Lambda\colon \phi_n = \mathscr{T}^n a_0\}$. By using Eqn. (11),

$$\kappa(\phi_{n+1}, \phi_n) \leq \zeta^n \kappa(\phi_0, \phi_1), \quad \forall\, n \geq 0.$$

$\forall\, n, m \in \mathbb{N} \ni n < m$, we have

$$\begin{aligned}
\kappa(\phi_n, \phi_m) &\leq \beta(\phi_n, \phi_{n+1})\kappa(\phi_n, \phi_{n+1}) + \beta(\phi_{n+1}, \phi_{n+2})\kappa(\phi_{n+1}, \phi_{n+2}) \\
&\quad + \beta(\phi_{n+2}, \phi_m)\kappa(\phi_{n+2}, \phi_m) \\
&\leq \beta(\phi_n, \phi_{n+1})\kappa(\phi_n, \phi_{n+1}) + \beta(\phi_{n+1}, \phi_{n+2})\kappa(\phi_{n+1}, \phi_{n+2}) \\
&\quad + \beta(\phi_{n+2}, \phi_m)\beta(\phi_{n+2}, \phi_{n+3})\kappa(\phi_{n+2}, \phi_{n+3}) \\
&\quad + \beta(\phi_{n+2}, \phi_m)\beta(\phi_{n+3}, \phi_{n+4})\kappa(\phi_{n+3}, \phi_{n+4}) \\
&\quad + \beta(\phi_{n+2}, \phi_m)\beta(\phi_{n+4}, \phi_m)\kappa(\phi_{n+4}, \phi_m) \\
&\leq \cdots \\
&\leq \beta(\phi_n, \phi_{n+1})\kappa(\phi_n, \phi_{n+1}) + \beta(\phi_{n+1}, \phi_{n+2})\kappa(\phi_{n+1}, \phi_{n+2}) \\
&\quad + \sum_{i=n+2}^{m-2} \left[\prod_{j=n+2}^{i} \beta(\phi_j, \phi_m)\right] \beta(\phi_i, \phi_{i+1})\kappa(\phi_i, \phi_{i+1}) \\
&\quad + \prod_{\zeta=n+2}^{m-1} \beta(\phi_\zeta, \phi_m)\kappa(\phi_{m-1}, \phi_m) \\
&\leq \beta(\phi_n, \phi_{n+1})\zeta^n d(\phi_0, \phi_1) + \beta(\phi_{n+1}, \phi_{n+2})\zeta^{n+1}\kappa(\phi_0, \phi_1) \\
&\quad + \sum_{i=n+2}^{m-2} \left[\prod_{j=n+2}^{i} \beta(\phi_j, \phi_m)\right] \beta(\phi_i, \phi_{i+1})\zeta^i \kappa(\phi_0, \phi_1) \\
&\quad + \sum_{i=n+2}^{m-1} \beta(\phi_i, \phi_m)\zeta^{m-1}\kappa(\phi_0, \phi_1) \\
&\leq \beta(\phi_n, \phi_{n+1})\zeta^n \kappa(\phi_0, \phi_1) + \beta(\phi_{n+1}, \phi_{n+2})\zeta^{n+1}\kappa(\phi_0, \phi_1)
\end{aligned}$$

$$+\sum_{i=n+2}^{m-2}\left[\prod_{j=n+2}^{i}\beta(\phi_j,\,\phi_m)\right]\beta(\phi_i,\,\phi_{i+1})\zeta^i\kappa(\phi_0,\,\phi_1)$$

$$+\prod_{i=n+2}^{m-1}\beta(\phi_i,\,\phi_m)\zeta^{m-1}\beta(\phi_{m-1},\,\phi_m)\kappa(\phi_0,\,\phi_1)$$

$$=\beta(\phi_n,\,\phi_{n+1})\zeta^n\kappa(\phi_0,\,\phi_1)+\beta(\phi_{n+1},\,\phi_{n+2})\zeta^{n+1}\kappa(\phi_0,\,\phi_1)$$

$$+\sum_{i=n+2}^{m-1}\left[\prod_{j=n+2}^{i}\beta(\phi_j,\,\phi_m)\right]\beta(\phi_i,\,\phi_{i+1})\zeta^i\kappa(\phi_0,\,\phi_1)$$

$$\leq\beta(\phi_n,\,\phi_{n+1})\zeta^n\kappa(\phi_0,\,\phi_1)+\beta(\phi_{n+1},\,\phi_{n+2})\zeta^{n+1}\kappa(\phi_0,\,\phi_1)$$

$$+\sum_{i=n+2}^{m-1}\left[\prod_{j=0}^{i}\beta(\phi_j,\,\phi_m)\right]\beta(\phi_i,\,\phi_{i+1})\zeta^i\kappa(\phi_0,\,\phi_1)$$

It is noted that $\beta(x,\,y)\geq 1$. Let $s_p=\sum_{i=1}^{p}\left[\prod_{j=1}^{i}\beta(\phi_j,\,\phi_m)\right]\beta(\phi_i,\,\phi_{i+1})\zeta^i$.
Hence, we have

$$\kappa(\phi_n,\,\phi_m)\leq\kappa(\phi_0,\,\phi_1)\left[\zeta^n\beta(\phi_n,\,\phi_{n+1})+\zeta^{n+1}\beta(\phi_{n+1},\,\phi_{n+2})+(s_{m-1}-s_n)\right] \tag{14}$$

By using the Ratio Test, Eqn. (12) confirms that $\lim_{n\to\infty}s_n$ exists. So, $\{s_n\}$ is a CS. By using the inequality (14), as $n,\,m\to\infty$

$$\lim_{n,\,m\to\infty}\kappa(\phi_n,\,\phi_m)=0 \tag{15}$$

i.e., $\{\phi_n\}$ is a CS in CERbMS $(\Lambda,\,\kappa)$. So $\{\phi_n\}$ converges to $\phi\in\Lambda$.
It is required to prove that ϕ is a fixed point of T. By triangle inequality,

$$\kappa(\phi_n,\,\phi_{n+2})\leq\beta(\phi,\,\phi_n)\kappa(\phi,\,\phi_n)+\beta(\phi_n,\,\phi_{n+1})\kappa(\phi_n,\,\phi_{n+1})+\beta(\phi_{n+1},\,\phi_{n+2}).$$

It is concluded from Eqns. (12), (13) & (15) that,

$$\lim_{n\to\infty}\kappa(\phi_n,\,\phi_{n+1})=0 \tag{16}$$

By using the triangle inequality,

$$\kappa(\phi, T\phi) \leq \beta(\phi, \phi_{n+1})\kappa(\phi, \phi_{n+1}) + \beta(\phi_{n+1}, \phi_{n+2})\kappa(\phi_{n+1}, \phi_{n+2}) + \beta(\phi_{n+2}, \phi_{n+3})\kappa(\phi_{n+2}, \phi_{n+3})$$

$$\leq \beta(\phi, \phi_{n+1})\kappa(\phi, \phi_{n+1}) + \zeta\beta(\phi_{n+1}, \phi_{n+2})\kappa(\phi_{n+1}, \phi_{n+2}) + \zeta\beta(\phi_{n+2}, \phi_{n+3})\kappa(\phi_{n+2}, \phi_{n+3})$$

As $n \to \infty$ and from Eqns. (13) & (16), it is obtained as $\kappa(\phi, \mathscr{T}\phi) = 0$ and hence $\phi = \mathscr{T}\phi$.

To prove the uniqueness of the obtained fixed point, the following argument has to be proceeded.

Suppose ϕ, υ are two fixed points of $\mathscr{T}$. Then $\kappa(\phi, \upsilon) \leq \zeta\kappa(\phi, \upsilon)$, which holds for $\kappa(\phi, \upsilon) = 0$. This implies $\phi = \upsilon$. As a result, "$\mathscr{T}$ has a unique fixed point". □

Example 2 Let $\Lambda = \{0, 1, 2, 3\}$. Let κ be a symmetric function on Λ and given with $\kappa(\phi, \phi) = 0$ for all $\phi \in \Lambda$ and also

$$\kappa(0, 1) = 1, \quad \kappa(0, 2) = \frac{11}{10}, \quad \kappa(0, 3) = \frac{12}{10}, \quad \kappa(1, 2) = \frac{2}{5}, \quad \kappa(1, 3) = \frac{4}{5}, \quad \kappa(2, 3) = \frac{1}{3}.$$

and the function β: $\Lambda \times \Lambda \to [1, \infty)$ is defined as

$$\beta(0, 1) = 1, \quad \beta(0, 2) = \frac{11}{10}, \quad \beta(0, 3) = 1, \quad \beta(1, 2) = \frac{2}{5}, \quad \beta(1, 3) = \frac{4}{5}, \quad \beta(2, 3) = \frac{1}{3}.$$

Clearly (Λ, κ) is a CERbMS.

Consider the mapping $\mathscr{T}$ from Λ to Λ such that $\mathscr{T}(\phi) = \begin{cases} 2; & \text{if} \quad \phi = 0 \\ 1; & \text{if} \quad \phi \in \{1, 2, 3\} \end{cases}$.

Choose $\zeta = \dfrac{12}{25}$. Then, $\mathscr{T}$ satisfies all conditions of Theorem 4 and hence "$\mathscr{T}$ has a unique fixed point", which is $\phi = 1$.

4.1 *Hausdorff controlled extended rectangular b-metric space*

The Hausdorff version of "proposed metric space" is defined in this section as an analogue of classical Hausdorff metric space.

We consider $\beta(\phi, \mathscr{A}) = \inf_{x \in A} \beta(\phi, x)$ and $\kappa(\phi, \mathscr{A}) = \inf_{x \in A} \kappa(\phi, x)$, where $\mathscr{A} \subset \Lambda$.

Lemma 1 Let (Λ, κ) be a CERbMS. Then $\kappa(\phi_1, \mathscr{A}) \leq \beta(\phi_1, \phi_2)\kappa(\phi_1, \phi_2) + \beta(\phi_2, \phi_3)\kappa(\phi_2, \phi_3) + \beta(\phi_3, x)\kappa(\phi_3, x)$, for all $\phi_1, \phi_2, \phi_3 \in \Lambda$ and $x \in \mathscr{A} \subset \Lambda$, where $\beta(\phi_3, \mathscr{A}) = \inf_{x \in \mathscr{A}} \beta(\phi_3, x)$.

Proof. From the definition of the CERbMS, we have $\kappa(\phi_1, x) \leq \beta(\phi_1, \phi_2)\kappa(\phi_1, \phi_2) + \beta(\phi_2, \phi_3)\kappa(\phi_2, \phi_3) + \beta(\phi_3, x)\kappa(\phi_3, x)$ and taking infimum on both side over $\mathscr{A}$, then we get $\inf\limits_{x \in \mathscr{A}} \kappa(\phi_1, x) \leq \beta(\phi_1, \phi_2)\kappa(\phi_1, \phi_2) + \beta(\phi_2, \phi_3)\kappa(\phi_2, \phi_3) + \inf\limits_{x \in \mathscr{A}} \beta(\phi_3, x)\kappa(\phi_3, x)$.

Therefore, $\kappa(\phi_1, \mathscr{A}) \leq \beta(\phi_1, \phi_2)\kappa(\phi_1, \phi_2) + \beta(\phi_2, \phi_3)\kappa(\phi_2, \phi_3) + \beta(\phi_3, x)\kappa(\phi_3, x)$ for all $\phi_1, \phi_2, \phi_3 \in \Lambda$. □

Here we can define the Hausdorff CERbMS. Let us consider $CLD(\Lambda)$ as a set of non-empty closed subsets of Λ and $\mathscr{K}_o(\Lambda)$ as a set of non-empty compact subsets of Λ.

Definition 14 (Hausdorff Controlled Extended Rectangular b-Metric Space (HCERbMS)) If (Λ, κ) is a CERbMS, then the Hausdorff Controlled Extended Rectangular b-Metric is defined as a mapping $\mathscr{H}_\kappa$: $\mathscr{K}_o(\Lambda) \times \mathscr{K}_o(\Lambda) \to [0, \infty)$ such that

$$\mathscr{H}_\kappa(\mathscr{A}, \mathscr{B}) = \begin{cases} \max\left\{ \sup_{\phi \in \mathscr{A}} \kappa(\phi, \mathscr{B}), \sup_{\upsilon \in \mathscr{B}} \kappa(\upsilon, \mathscr{A}) \right\}, & \text{if the maximum exists;} \\ \infty, & \text{otherwise.} \end{cases}$$

Then, $(\mathscr{K}_o(\Lambda), \mathscr{H}_\kappa)$ is called Hausdorff Controlled Extended Rectangular b-Metric Space.
Theorem 13 Let $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D} \subset \mathscr{K}_o(\Lambda)$. Then

$$\mathscr{H}_\kappa(\mathscr{A}, \mathscr{B}) \le \max\left\{ \sup_{\beta \in \mathscr{A}} \beta(\phi, \upsilon), \beta(\upsilon, \mathscr{A}) \right\} \mathscr{H}_\kappa(\mathscr{A}, \mathscr{B})$$

$$+ \max\left\{ \beta(\upsilon, \mathscr{C}), \sup_{\mu \in \mathscr{C}} \beta(\upsilon, \mu) \right\} \mathscr{H}_\kappa(\mathscr{B}, \mathscr{C})$$

$$+ \max\left\{ \beta(\mu, \mathscr{D}), \sup_{\eta \in \mathscr{D}} \beta(\mu, \eta) \right\} \mathscr{H}_\kappa(\mathscr{C}, \mathscr{D}).$$

Proof. Consider $\mathscr{H}_\kappa(\mathscr{A}, \mathscr{B})$, $\mathscr{H}_\kappa(\mathscr{B}, \mathscr{C})$ and $\mathscr{H}_\kappa(\mathscr{C}, \mathscr{D})$ are finite. From Lemma 1, for $\phi \in \mathscr{A}$, $\upsilon \in \mathscr{B}$, we have

$$\kappa(\phi, \mathscr{D}) \le \beta(\phi, \upsilon)\kappa(\phi, \upsilon) + \beta(\upsilon, \mu)\kappa(\upsilon, \mu) + \beta(\mu, \mathscr{D})\kappa(\mu, \mathscr{D}).$$

As $\kappa(\mu, \mathscr{D}) \le \mathscr{H}_\kappa(\mathscr{C}, \mathscr{D})$, $\kappa(\upsilon, \mathscr{C}) \le \mathscr{H}_\kappa(\mathscr{B}, \mathscr{C})$. Therefore

$$\kappa(\phi, \mathscr{D}) \le \beta(\phi, \upsilon)\kappa(\phi, \upsilon) + \beta(\upsilon, \mu)\kappa(\upsilon, \mu) + \beta(\mu, \mathscr{D})\kappa(\mu, \mathscr{D}).$$

$$\kappa(\phi, \mathscr{D}) \le \beta(\phi, \upsilon)\kappa(\phi, \mathscr{B}) + \beta(\upsilon, \mathscr{C})\kappa(\upsilon, \mathscr{C}) + \beta(\mu, \mathscr{C})\kappa(\mathscr{C}, \mathscr{D}).$$

Taking supremum on both side over A, then

$$\sup_{\phi \in \mathscr{A}} \kappa(\phi, D\mathscr{D}) \le \sup_{\phi \in \mathscr{A}} \beta(\phi, \upsilon)\mathscr{H}_\kappa(\mathscr{A}, \mathscr{B})$$

$$+ \beta(\upsilon, \mu)\mathscr{H}_\kappa(\mathscr{B}, \mathscr{C})$$

$$+ \beta(\mu, \eta)\mathscr{H}_\kappa(\mathscr{C}, \mathscr{D}).$$

Similarly,

$$\sup_{\eta\in\mathscr{D}} \kappa(\eta, \mathscr{A}) \leq \beta(\upsilon, \mathscr{A})\mathscr{H}_\kappa(\mathscr{A}, \mathscr{B})$$

$$+\beta(\mu, \upsilon)\mathscr{H}_\kappa(\mathscr{B}, \mathscr{C})$$

$$+\sup_{\eta\in\mathscr{D}} \beta(\mu, \eta)\mathscr{H}_\kappa(\mathscr{C}, \mathscr{D}).$$

Hence,

$$\max\left\{\sup_{\phi\in A}\kappa(\phi, D), \sup_{\eta\in D}\kappa(\eta, A)\right\} \leq \left\{\sup_{\phi\in A}\beta(\phi, \upsilon), \beta(\upsilon, A)\right\}\mathscr{H}_\kappa(A, B)$$

$$+\left\{\beta(\upsilon, C), \sup_{\mu\in C}\beta(\mu, \upsilon)\right\}\mathscr{H}_\kappa(B, C)$$

$$+\left\{\beta(\mu, D), \sup_{\eta\in D}\beta(\mu, \eta)\right\}\mathscr{H}_\kappa(C, D).$$

Therefore, by definition, we get

$$\mathscr{H}_\kappa(\mathscr{A}, \mathscr{D}) \leq \max\left\{\sup_{\phi\in\mathscr{A}}\beta(\phi, \upsilon), \beta(\upsilon, \mathscr{A})\right\}\mathscr{H}_\kappa(\mathscr{A}, \mathscr{B})$$

$$+\max\left\{\beta(\upsilon, \mathscr{C}), \sup_{\mu\in\mathscr{C}}\beta(\upsilon, \mu)\right\}\mathscr{H}_\kappa(\mathscr{B}, \mathscr{C})$$

$$+\max\left\{\beta(\mu, \mathscr{D}), \sup_{\eta\in\mathscr{D}}\beta(\mu, \eta)\right\}\mathscr{H}_\kappa(\mathscr{C}, \mathscr{D}).$$

□

Definition 15 If $x \in \overline{\mathscr{A}}$, where $\overline{\mathscr{A}}$ is the closure of a set $\mathscr{A} \subset \Lambda$, iff $\exists$ a sequence $\{x_n\}_{n=0}^{\infty}$ in $\mathscr{A}$ such that $\lim\limits_{n\to\infty} x_n = x$. Denote for $\varepsilon > 0$ and $\mathscr{A} \subset \Lambda$, $\mathscr{A}_\varepsilon = \{\phi \in \Lambda\colon \kappa(\phi, \mathscr{A}) \leq \varepsilon\}$.

Theorem 14 If $\phi \in \overline{\mathscr{A}}_\varepsilon$, then $\kappa(\phi, \mathscr{A}) \leq \lim\limits_{n\to\infty}\beta(\phi_n, \mathscr{A})\varepsilon$, where $\beta(\phi_n, \mathscr{A}) = \inf_{\phi\in\mathscr{A}}\beta(\phi_n, \phi)$.

Proof. Let $\phi \in \overline{\mathscr{A}}_\varepsilon$. Then, there exist a sequence $\{\phi_n\}$ in $\mathscr{A}_\varepsilon$ such that $\lim_{n\to\infty}\phi_n = \phi$. By Lemma 1, we have

$$\kappa(\phi, \mathscr{A}) \leq \beta(\phi, \phi_n)\kappa(\phi, \phi_n) + \beta(\phi_n, \phi_{n+1})\kappa(\phi_n, \phi_{n+1}) + \beta(\phi_{n+1}, \mathscr{A})\kappa(\phi_{n+1}, \mathscr{A})$$

Take $n \to \infty$ then, $\kappa(\phi, \mathscr{A}) \leq \lim_{n\to\infty} \beta(\phi_n, \mathscr{A})\varepsilon$. $\square$

Definition 16 The upper topological limit of a sequence $\{\mathscr{A}_l\}_{l=1}^{\infty}$ in CERbMS Λ is denoted by $\overline{Lt}\mathscr{A}_l$ and determined by $x \in \overline{Lt}\mathscr{A}_l$ iff $\lim_{l\to\infty} \inf \kappa(x, \mathscr{A}_l) = 0$.

Theorem 15 If a subsequence $\{x_{n_l}\}$ in $\mathscr{A}$ is convergent to x and $x_{n_l} \in \mathscr{A}_{n_l}$ for $l = 1, 2, 3, ...$ if and only if the point $x \in \overline{Lt}\mathscr{A}_l$.

Theorem 16 $L = \overline{Lt} \quad \overline{\mathscr{A}_l}$ is closed.

Corollary 1 $\overline{Lt}\mathscr{A}_l = \cap_{l=1}^{\infty} \cup_{n=0}^{\infty} \mathscr{A}_{l+n}$.

Corollary 2 $\lim_{l\to\infty} \mathscr{A}_l = \overline{\overline{Lt}\mathscr{A}_l} = \overline{Lt\mathscr{A}_l}$.

Theorem 17 If (Λ, κ) is a complete CERbMS with

$$\lim_{n, m\to\infty} \beta(\phi_n, \phi_m)\zeta < 1$$

for all $\phi_n, \phi_m \in \Lambda$, where $\zeta \geq 1$. Then $(CLD(\Lambda), \mathscr{H}_\kappa)$ is complete.

Proof. Let $\{\mathscr{A}_n\}_{n=1}^{\infty}$ be a CS in $CLD(\Lambda)$. Then by definition, $\forall\, \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\mathscr{H}_\kappa(\mathscr{A}_n, \mathscr{A}_m) < \varepsilon, \ \forall\, m, n \geq N \ \text{ and } \ \mathscr{H}_\kappa(\mathscr{A}_m, \mathscr{A}_p) < \varepsilon \ \forall\, m, p \geq N. \tag{17}$$

Let $\mathscr{A} = \overline{Lt}\mathscr{A}_n$. It is required to prove that, $\mathscr{A} \in CLD(\Lambda)$ and $\mathscr{A}_n \to \mathscr{A}$. From Theorem 16, $L = \overline{Lt}\mathscr{A}_l$ is closed and then we get $\mathscr{A} \in CLD(\Lambda)$. It is enough to claim that, $\{\mathscr{A}_n\}$ is convergent to $\mathscr{A}$. We have to show that, there exists a positive integer N such that $\mathscr{H}_\kappa(\mathscr{A}_n, \mathscr{A}) < \varepsilon$, $\forall\, n \geq N$. By rectangle inequality, $\forall\, n, m \geq N$,

$$\mathscr{H}_\kappa(\mathscr{A}_n, \mathscr{A}) \leq \max\left\{\sup_{\phi_n\in\mathscr{A}_n} \beta(\phi_n, \phi_m), \beta(\phi_n, \mathscr{A}_m)\right\}\mathscr{H}_\kappa(\mathscr{A}_n, \mathscr{A}_m)$$

$$+ \max\left\{\sup_{\phi_m\in\mathscr{A}_m} \beta(\phi_m, \phi_p), \beta(\phi_m, \mathscr{A}_p)\right\}\mathscr{H}_\kappa(\mathscr{A}_m, \mathscr{A}_p)$$

$$+ \max\left\{\sup_{\phi_p\in\mathscr{A}_p} \beta(\phi_p, \phi), \beta(\phi, \mathscr{A}_p)\right\}\mathscr{H}_\kappa(\mathscr{A}_p, \mathscr{A}).$$

For $n, m \geq N$, we have

$$\mathscr{H}_\kappa(A_n, \mathscr{A}) \leq \max\left\{\sup_{\phi_n\in\mathscr{A}_n} \beta(\phi_n, \phi_m), \beta(\phi_n, \mathscr{A}_m)\right\}\varepsilon$$

$$+ \max\left\{\sup_{\phi_m\in\mathscr{A}_m} \beta(\phi_m, \phi_p), \beta(\phi_m, \mathscr{A}_p)\right\}\varepsilon$$

$$+ \max\left\{\sup_{\phi_p\in\mathscr{A}_p} \beta(\phi_p, \phi), \beta(\phi, \mathscr{A}_p)\right\}\mathscr{H}_\kappa(\mathscr{A}_p, \mathscr{A}).$$

To prove that,

$$\mathscr{H}_\kappa(\mathscr{A}_p,\, \mathscr{A}) \leq \max\left\{ \sup_{\phi_p \in \mathscr{A}_p} \beta(\phi_p,\, \phi_{p_r}),\, \beta(\phi_{p_r},\, \mathscr{A}_p) \right\} \varepsilon.$$

For this reason, we claim the following inequalities for $\phi^* \in \mathscr{A}$,

$$\kappa(\phi_p,\, \phi^*) \leq \beta(\phi_p,\, \phi_{n_r})\varepsilon,\ \forall\ \phi_p \in \mathscr{A}_p. \tag{18}$$

$$\kappa(\phi*,\, \mathscr{A}_p) \leq \beta(\phi_{n_r},\, \mathscr{A}_p)\varepsilon. \tag{19}$$

From Eqn. (17), we get $\mathscr{A}_n \subset \mathscr{A}_{p_\varepsilon}$, for all $m > p \geq N$. By Corollary 2, we have $\mathscr{A} \subset \overline{\mathscr{A}_n \cup \mathscr{A}_{n+1} \cup \ldots} \subset \overline{\mathscr{A}_{p_\varepsilon}}$. From Theorem 14, for $\phi^* \in \mathscr{A}$, we get

$$\kappa(\phi^*,\, \mathscr{A}_p) \leq \beta(\phi_{n_r},\, \mathscr{A}_p)\varepsilon.$$

Hence Eqn. (19) is proved.

Now, it is required to prove Eqn. (18). Since $\{\mathscr{A}_n\}$ is a CS in $CLD(\Lambda)$, $\{n_r\}_{r=1}^{\infty} = \{\varepsilon l^{-r}\}_{r=1}^{\infty}$ such that $n_r > N$, where $N \in \mathbb{N}$ and $\mathscr{H}_\kappa(\mathscr{A}_n,\, \mathscr{A}_p) < \varepsilon l^{-r}$ for all $n,\, m \geq n_r$.

Take arbitrary $\phi_p \in \mathscr{A}_p$, where $\phi_p = \phi_{n_0}$. Since $\mathscr{H}_\kappa(\mathscr{A}_n,\, \mathscr{A}_{n_0}) < \varepsilon$ for $n > n_0$, there exists $\phi_{n_1} \in \mathscr{A}_n$, such that $\kappa(\phi_{n_0},\, \phi_{n_1}) < \varepsilon$ for $n = n_1 > n_0$.

Similarly, $\mathscr{H}_\kappa(\mathscr{A}_n,\, \mathscr{A}_{n_1}) < \dfrac{\varepsilon}{l}$, so there exists $\phi_{n_2} \in \mathscr{A}_{n_2}$ such that $\kappa(\phi_{n_1},\, \phi_{n_2}) < \dfrac{\varepsilon}{l^2}$ for $n = n_2 > n_1$.

Continuing the same, we can construct a sequence $\{\phi_{n_r}\}$ with $\phi_{n_r} \in \mathscr{A}_{n_r}$, for $r = 0,\, 1,\, 2,\, \ldots$ and

$$\kappa(\phi_{n_r},\, \phi_{n_{r+l}}) < \frac{\varepsilon}{l^r}, \quad \phi_{n_0} = \phi. \tag{20}$$

Next, we need to claim that $\{\phi_{n_r}\}$ is CS, from the rectangle inequality, we have

$$
\begin{aligned}
\kappa(\phi_{n_r}, \phi_{n_{r+l}}) &\leq \beta(\phi_{n_r}, \phi_{n_{r+1}})\kappa(\phi_{n_r}, \phi_{n_{r+1}}) \\
&\quad + \beta(\phi_{n_{r+1}}, \phi_{n_{r+2}})\kappa(\phi_{n_{r+1}}, \phi_{n_{r+2}}) \\
&\quad + \beta(\phi_{n_{r+2}}, \phi_{n_{r+l}})\kappa(\phi_{n_{r+2}}, \phi_{n_{r+l}}) \\
&\leq \beta(\phi_{n_r}, \phi_{n_{r+1}})\kappa(\phi_{n_r}, \phi_{n_{r+1}}) \\
&\quad + \beta(\phi_{n_{r+1}}, \phi_{n_{r+2}})\kappa(\phi_{n_{r+1}}, \phi_{n_{r+2}}) \\
&\quad + \beta(\phi_{n_{r+2}}, \phi_{n_{r+l}})\beta(\phi_{n_{r+2}}, \phi_{n_{r+3}})\kappa(\phi_{n_{r+2}}, \phi_{n_{r+3}}) \\
&\quad + \beta(\phi_{n_{r+2}}, \phi_{n_{r+l}})\beta(\phi_{n_{r+3}}, \phi_{n_{r+4}})\kappa(\phi_{n_{r+3}}, \phi_{n_{r+4}}) \\
&\quad + \beta(\phi_{n_{r+2}}, \phi_{n_{r+l}})\beta(\phi_{n_{r+4}}, \phi_{n_{r+l}})\kappa(\phi_{n_{r+4}}, \phi_{n_{r+l}}) \\
&\leq \cdots \\
&\leq \beta(\phi_{n_r}, \phi_{n_{r+1}})\kappa(\phi_{n_r}, \phi_{n_{r+1}}) \\
&\quad + \beta(\phi_{n_{r+1}}, \phi_{n_{r+2}})\kappa(\phi_{n_{r+1}}, \phi_{n_{r+2}}) \\
&\quad + \sum_{i=r+2}^{r+l-2}\left(\prod_{j=r+2}^{i}\beta(\phi_{n_j}, \phi_{n_{r+l}})\right)\beta(\phi_{n_i}, \phi_{n_{i+1}})\kappa(\phi_{n_i}, \phi_{n_{i+1}}) \\
&\quad + \prod_{j=r+2}^{r+l-1}\beta(\phi_{n_j}, \phi_{n_{r+l}})\kappa(\phi_{n_{r+l-1}}, \phi_{n_{r+l}}) \\
&\leq \beta(\phi_{n_r}, \phi_{n_{r+1}})\kappa(\phi_{n_r}, \phi_{n_{r+1}}) \\
&\quad + \beta(\phi_{n_{r+1}}, \phi_{n_{r+2}})\kappa(\phi_{n_{r+1}}, \phi_{n_{r+2}}) \\
&\quad + \sum_{i=r+2}^{r+l-1}\left(\prod_{j=r+2}^{i}\beta(\phi_{n_j}, \phi_{n_{r+l}})\right)\beta(\phi_{n_i}, \phi_{n_{i+1}})\kappa(\phi_{n_i}, \phi_{n_{i+1}}).
\end{aligned}
$$

From Eqn. (20), we get

$$\kappa(\phi_{n_r}, \phi_{n_{r+l}}) \leq \beta(\phi_{n_r}, \phi_{n_{r+l}})\frac{\varepsilon}{l^r} + \beta(\phi_{n_{r+1}}, \phi_{n_{r+2}})\frac{\varepsilon}{l^{r+1}}$$

$$+\sum_{i=r+2}^{r+l-1}\left(\prod_{j=r+2}^{i}\beta(\phi_{n_j}, \phi_{n_{r+l}})\right)\beta(\phi_{n_i}, \phi_{n_{i+1}})\frac{\varepsilon}{l^i}. \tag{21}$$

As $\lim_{n, m\to\infty}\beta(\phi_n, \phi_m)\zeta < 1$ for all $\phi_n, \phi_m \in H$.
By using Ratio test, the series $\sum_{i=r+2}^{r+l-1}\left(\prod_{j=r+2}^{i}\beta(\phi_{n_j}, \phi_{n_{r+l}})\right)\beta(\phi_{n_i}, \phi_{n_{i+1}})\frac{\varepsilon}{l^i}$ is convergent.
By taking the limit as $r \to \infty$ in Eqn. (21), we derive $\lim_{r\to\infty}\kappa(\phi_{n_r}, \phi_{n_{r+l}}) = 0$.
Hence, we claim that $\{\phi_{n_r}\}$ is a CS. Since (Λ, κ) is complete then $\exists\phi \in \Lambda$ such that $\phi_{n_r} \to \phi \in \Lambda$ and clearly, $\phi \in \mathscr{A}$.
Again by rectangle inequality, we have,

$$\begin{aligned}
\kappa(\phi_{n_0}, \phi_{n_r}) &\leq \beta(\phi_{n_0}, \phi_{n_1})\kappa(\phi_{n_0}, \phi_{n_1}) + \beta(\phi_{n_1}, \phi_{n_2})\kappa(\phi_{n_1}, \phi_{n_2}) \\
&\quad + \beta(\phi_{n_2}, \phi_{n_r})\kappa(\phi_{n_2}, \phi_{n_r}) \\
&\leq \beta(\phi_{n_0}, \phi_{n_1})\kappa(\phi_{n_0}, \phi_{n_1}) + \beta(\phi_{n_1}, \phi_{n_2})\kappa(\phi_{n_1}, \phi_{n_2}) \\
&\quad + \beta(\phi_{n_2}, \phi_{n_r})\beta(\phi_{n_2}, \phi_{n_3})\kappa(\phi_{n_2}, \phi_{n_3}) \\
&\quad + \beta(\phi_{n_2}, \phi_{n_r})\beta(\phi_{n_3}, \phi_{n_4})\kappa(\phi_{n_3}, \phi_{n_4}) \\
&\quad + \beta(\phi_{n_2}, \phi_{n_r})\beta(\phi_{n_4}, \phi_{n_r})\kappa(\phi_{n_4}, \phi_{n_r}) \\
&\leq \cdots \\
&\leq \beta(\phi_{n_0}, \phi_{n_1})\kappa(\phi_{n_0}, \phi_{n_1}) + \beta(\phi_{n_1}, \phi_{n_2})\kappa(\phi_{n_1}, \phi_{n_2}) \\
&\quad + \sum_{i=2}^{r-2}\left(\prod_{j=2}^{i}\beta(\phi_{n_j}, \phi_{n_r})\right)\beta(\phi_{n_i}, \phi_{n_{i+1}})\kappa(\phi_{n_i}, \phi_{n_{i+1}}) \\
&\quad + \prod_{j=2}^{r-1}\beta(\phi_{n_j}, \phi_{n_r})\beta(\phi_{n_{r-1}}, \phi_{n_r})\kappa(\phi_{n_{r-1}}, \phi_{n_r})
\end{aligned}$$

$$\leq \beta(\phi_{n_0}, \phi_{n_1})\kappa(\phi_{n_0}, \phi_{n_1}) + \beta(\phi_{n_1}, \phi_{n_2})\kappa(\phi_{n_1}, \phi_{n_2})$$

$$+\sum_{i=2}^{r-1}\left(\prod_{j=2}^{i}\beta(\phi_{n_j}, \phi_{n_r})\right)\beta(\phi_{n_i}, \phi_{n_{i+1}})\kappa(\phi_{n_i}, \phi_{n_{i+1}}).$$

From Eqn. (20), we have

$$\kappa(\phi_{n_0}, \phi_{n_r}) \leq \beta(\phi_{n_0}, \phi_{n_1})\varepsilon + \beta(\phi_{n_1}, \phi_{n_2})\frac{\varepsilon}{l} + \sum_{i=2}^{r-1}\left(\prod_{j=2}^{i}\beta(\phi_{n_j}, \phi_{n_r})\right)\beta(\phi_{n_i}, \phi_{n_{i+1}})\frac{\varepsilon}{l^i}. \tag{22}$$

As $\lim_{n, m\to\infty}\beta(\phi_n, \phi_m)\zeta < 1$ for all $\phi_n, \phi_m \in \Lambda$. By Ratio test, the series $\sum_{i=2}^{r-1}\left(\prod_{j=2}^{i}\beta(\phi_{n_j}, \phi_{n_r})\right)\beta(\phi_{n_i}, \phi_{n_{i+1}})\frac{\varepsilon}{l^i}$ is convergent. By taking the limit as $r \to \infty$ in Eqn. (22), we get

$$\lim_{r\to\infty}\kappa(\phi_{n_0}, \phi_{n_r}) \leq \frac{1}{\zeta}\varepsilon < \varepsilon.$$

From the rectangle inequality, we have

$$\kappa(\phi, \phi_p) \leq \beta(\phi, \phi_{n_r})\kappa(\phi, \phi_{n_r}) + \beta(\phi_{n_r}, \phi_{n_{r+1}})\kappa(\phi_{n_r}, \phi_{n_{r+1}}) + \beta(\phi_{n_{r+1}}, \phi_p)\kappa(\phi_{n_{r+1}}, \phi_p).$$

Hence $\kappa(\phi, \phi_p) \leq \beta(\phi_{n_{r+1}}, \phi_p)\varepsilon$, as $r \to \infty$. This implies $\kappa(\phi_p, \phi) \leq \beta(\phi_p, \phi_{n_{r+1}})\varepsilon$. Hence we derive the following expression.

$$\mathscr{H}_\kappa(\mathscr{A}_n, \mathscr{A}) \leq \max\left\{\sup_{\phi_n\in\mathscr{A}_n}\beta(\phi_n, \phi_m), \beta(\phi_n, \mathscr{A}_m)\right\}\mathscr{H}_\kappa(\mathscr{A}_n, \mathscr{A}_m)$$

$$+\max\left\{\sup_{\phi_m\in\mathscr{A}_m}\beta(\phi_m, \phi_p), \beta(\phi_m, \mathscr{A}_p)\right\}\mathscr{H}_\kappa(\mathscr{A}_m, \mathscr{A}_p)$$

$$+\max\left\{\sup_{\phi_p\in\mathscr{A}_p}\beta(\phi_p, \phi), \beta(\phi, \mathscr{A}_p)\right\}\mathscr{H}_\kappa(\mathscr{A}_p, \mathscr{A}).$$

Since $\lim_{n, m\to\infty}\beta(x_n, x_m)\zeta < 1$, for all $x_n, x_m \in \Lambda$ and $n, m \longrightarrow \infty$ in the preceding inequality, we get a positive real number. Therefore, $\mathscr{A}_n$ approaches to A, which completes this proof. □

Proposition 1 If $\mathscr{A}, \mathscr{B} \in \mathscr{K}_o(\Lambda)$ for all $\varepsilon > 0$ and $\upsilon \in \mathscr{B}$, $\exists\ \phi \in \mathscr{A}$ such that

$$\kappa(\phi, \upsilon) \leq \mathscr{H}_\kappa(\mathscr{A}, \mathscr{B}) + \varepsilon$$

Proof. The proposition is proved by using Theorem 13. □

Theorem 18 If (Λ, κ) is a complete CERbMS with the contraction condition

$$\lim_{n, m \to \infty} \beta(\phi_n, \phi_m)\zeta < 1$$

for all $\phi_n, \phi_m \in \Lambda$ where $\zeta \geq 1$. Then $(\mathscr{K}_o(\Lambda), \mathscr{H}_\kappa)$ is complete.

Proof. Let $\{\mathscr{A}_n\}_{n=1}^{\infty}$ be a CS in $\mathscr{K}_o(\Lambda)$. By definition, $\forall\, \varepsilon > 0\, \exists\, N > 0 \in \mathbb{N}$ such that $\mathscr{H}_\kappa(\mathscr{A}_n, \mathscr{A}_m) < \varepsilon, \forall\, n, m \geq N$. It is proved that for $\mathscr{A} \in \mathscr{K}_o(\Lambda)$, $\mathscr{A}_n \to \mathscr{A}$. Theorem 2.4 in [36] confirms that $\{\mathscr{A}_n\}$ converges to $\mathscr{A}$. Also $\{\mathscr{A}_n\}$ is a CS in $\mathscr{K}_o(\Lambda)$. Therefore $(\mathscr{K}_o(\Lambda), \mathscr{H}_\kappa)$ is complete. □

Theorem 19 Let $\mathscr{T}: \Lambda \to \mathscr{K}_o(\Lambda)$ be a mapping on a complete CERbMS (Λ, κ). If $\mathscr{T}$ fullfills the inequality $\mathscr{H}_\kappa(\mathscr{T}(\phi), \mathscr{T}(\upsilon)) \leq \zeta(\phi, \upsilon), \forall\, \phi, \upsilon \in \Lambda$, where $\zeta \in (0, 1)$ is an real constant such that $\lim_{n, m \to \infty} \beta(\phi_n, \phi_m)\zeta < 1$ $\forall\ \phi_n, \phi_m \in \Lambda$. Then "$\mathscr{T}$ has a unique fixed point".

Proof. The proof is evident when using Theorem 18 and Theorem 2.4 in [36] to prove the theorem. □

5. Controlled extended rectangular b-fractals

As a consequence of the previous section, the new version of IFS and Fractal in the CERbMS are initiated and discussed in this section.

Definition 17 (Controlled Extended Rectangular b-Iterated Function System (CERb-IFS)) Let (Λ, κ) be a CERbMS and $\mathscr{T}_n: \Lambda \longrightarrow \Lambda,\ \omega = 1, 2, 3, ..., \mathbb{K}\ (\mathbb{K} \in \mathbb{N})$ be contraction functions on CERbMS with the associated contraction ratios $\zeta_\omega, \omega = 1, 2, 3, ..., \mathbb{K}$. Then $\{\Lambda; \mathscr{T}_\omega,\ \omega = 1, 2, 3, ..., \mathbb{K}\}$ is said to be a controlled extended rectangular b-iterated function system (CERb-IFS) of contractions with the contraction factor $\zeta = \max_{\omega=1}^{\mathbb{K}} \zeta_\omega$.

Example 3 Let $\Lambda = \{0, 1, 2, 3\}$ and κ be a "symmetric function" on Λ. κ is a CERbMS on Λ from Example 2. Next, consider the self mappings T_i from Λ to Λ for $i = 1, 2$ as below.

$$T_1(\phi) = \begin{cases} 1; & \text{if} \quad \phi = 0 \\ 2; & \text{if} \quad \phi \in \{1, 2, 3\} \end{cases}$$

$$T_2(\phi) = \begin{cases} 2; & \text{if} \quad \phi = 0 \\ 1; & \text{if} \quad \phi \in \{1, 2, 3\} \end{cases}$$

Then $\{\Lambda; T_1, T_2\}$ is the system of contraction functions is a CERbIFS.

The fixed point of the contractions T_1 and T_2 in CERb-IFS are illustrated graphically in Figure 1.

The point at which the graph of T_1 and T_2 intersects the graph of the identity function represents the fixed points graphically for the given contractions respectively.

Based on the above definitions and results, we can define the HB Operator and HB Theorem on Hausdorff CERbMS as follows.

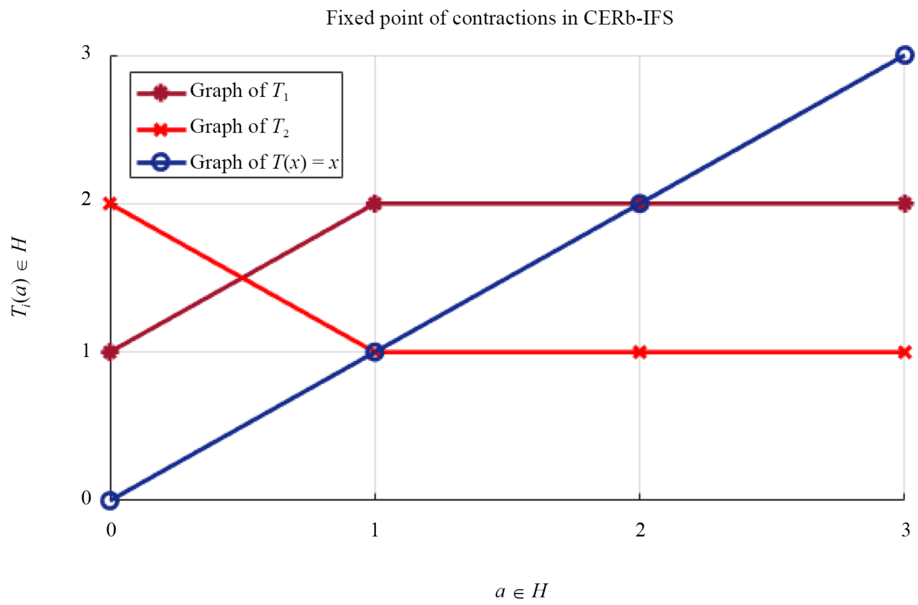

Figure 1. Graphical representation for fixed point of contractions in CER*b*-IFS

Definition 18 (HB Operator on HCERbMS) Let (Λ, κ) be a CERbMS. Let $\{\Lambda; \mathscr{T}_\omega, \omega = 1, 2, 3, ..., \mathbb{K}; \mathbb{K} \in \mathbb{N}\}$ be a CERbIFS system consists of the finite number of contractions on Λ. Then the HB operator on HCERbMS is a function $F: \mathscr{K}_o(\Lambda) \longrightarrow \mathscr{K}_o(\Lambda)$ as

$$F(\mathscr{B}) = \bigcup_{\omega=1}^{\mathbb{K}} \mathscr{T}_\omega(\mathscr{B}), \quad \text{for all } \mathscr{B} \in \mathscr{K}_o(\Lambda).$$

Theorem 20 Let (H, κ) be a CERbMS and let $(\mathscr{K}_o(\Lambda), \mathscr{H}_\kappa)$ be the corresponding Hausdorff space. Let $\mathscr{T}: \Lambda \longrightarrow \Lambda$ be continuous and contraction on (Λ, κ) with the contractivity factor ζ. Then $\mathscr{T}: \mathscr{K}_o(\Lambda) \longrightarrow \mathscr{K}_o(\Lambda)$ defined by $\mathscr{T}(B) = \{\mathscr{T}(\phi): \phi \in \mathscr{B}\}\ \forall\ \mathscr{B} \in \mathscr{K}_o(\Lambda)$ is a contraction on $(\mathscr{K}_o(\Lambda), \mathscr{H}_\kappa)$ with the contractivity factor ζ.

Proof. Since $\mathscr{T}$ is a continuous function on Λ, $\mathscr{T}$ maps elements of $\mathscr{K}_o(\Lambda)$ into itself.

Let $\mathscr{B}, \mathscr{C} \in \mathscr{K}_o(\Lambda)$. Then

$$\mathscr{H}_\kappa\Big(\mathscr{T}(\mathscr{B}), \mathscr{T}(\mathscr{C})\Big) = \max\Big\{\kappa(\mathscr{T}(\mathscr{B}), \mathscr{T}(\mathscr{C})), \kappa(\mathscr{T}(\mathscr{C}), \mathscr{T}(\mathscr{B}))\Big\}$$

$$\leq \max\Big\{\zeta\Big[\kappa(\mathscr{B}, \mathscr{C}), \kappa(\mathscr{C}, \mathscr{B})\Big]\Big\}$$

$$\leq \zeta \max\Big\{\Big[\kappa(\mathscr{B}, \mathscr{C}), \kappa(\mathscr{C}, \mathscr{B})\Big]\Big\}$$

$$\leq \mathscr{H}(\mathscr{B}, \mathscr{C}).$$

This completes our assertion. $\square$

Theorem 21 Let (Λ, κ) be a CERbMS and let $(\mathscr{K}_o(\Lambda), \mathscr{H}_\kappa)$ be the corresponding Hausdorff space. Let $\mathscr{T}_\omega \colon \Lambda \longrightarrow \Lambda$, $\omega = 1, 2, 3, ..., \mathbb{K}$ ($\mathbb{K} \in \mathbb{N}$) be continuous and contraction mappings on (Λ, κ) with the contractivity factors ζ_ω, $\omega = 1, 2, 3, ..., \mathbb{K}$. Then the HB operator $F \colon \mathscr{K}_o(\Lambda) \longrightarrow \mathscr{K}_o(\Lambda)$ on on HCERbMS is also a contraction on $(\mathscr{K}_o(\Lambda), \mathscr{H}_\kappa)$ with contractivity factor $\zeta = \max_{\omega=1}^{N} \zeta_n$.

Proof. It is enough to prove the theorem for the case $N = 2$.

For $N = 2$, by Theorem 20, the following is proceeded.

$$\mathscr{H}_\kappa\Big(F(\mathscr{B}), F(\mathscr{C})\Big) = \mathscr{H}_\kappa\Bigg(\mathscr{T}_1(\mathscr{B}) \cup \mathscr{T}_2(\mathscr{B}),\ \mathscr{T}_1(\mathscr{C}) \cup \mathscr{T}_2(\mathscr{C})\Bigg)$$

$$\leq \max\Bigg\{\mathscr{H}_\kappa(\mathscr{T}_1(\mathscr{B}),\ \mathscr{T}_1(\mathscr{C})),\ \mathscr{H}_\kappa(\mathscr{T}_2(\mathscr{B}),\ \mathscr{T}_2(\mathscr{C}))\Bigg\}$$

$$\leq \max\Bigg\{\zeta_1 \mathscr{H}_\kappa(\mathscr{B}, \mathscr{C}),\ \zeta_2 \mathscr{H}_\kappa(\mathscr{B}, \mathscr{C})\Bigg\}$$

$$\leq \max\{\zeta_1, \zeta_2\}\Bigg\{\mathscr{H}_\kappa(\mathscr{B}, \mathscr{C}),\ \mathscr{H}_\kappa(\mathscr{B}, \mathscr{C})\Bigg\}$$

$$\leq \zeta \max\Bigg\{\mathscr{H}_\kappa(\mathscr{B}, \mathscr{C})\Bigg\}$$

This completes the proof. $\square$

Theorem 22 (HB Theorem for CERb-IFS) Let (Λ, κ) be a Complete CERbMS and $\{\Lambda; \mathscr{T}_\omega, \omega = 1, 2, 3, ..., \mathbb{K}; \mathbb{K} \in \mathbb{N}\}$ be CERb-IFS of continuous contractions with contraction ratios ζ. Then, there exists only one attractor $\mathscr{A}_\infty \in \mathscr{K}_o(\Lambda)$ of the HB operator (F) on HCERbMS or equivalently, "F has a unique fixed point" namely $\mathscr{A}_\infty \in \mathscr{K}_o(\Lambda)$ such that

$$\mathscr{A}_\infty = F(\mathscr{A}_\infty) = \bigcup_{\omega=1}^{\mathbb{K}} \mathscr{T}_\omega(\mathscr{A}_\infty),$$

and is given by $\mathscr{A}_\infty = \lim_{\omega\to\infty} F^{o(\omega)}(\mathscr{B})$ for any $\mathscr{B} \in \mathscr{K}_o(\Lambda)$.

Proof. Since (Λ, κ) complete in the CERbMS, it implies that the Theorem 18 shows that $(\mathscr{K}_o(\Lambda), \mathscr{H}_\kappa)$ is also complete HCERbMS. Also Theorem 21 clears that the HB Operator, F is a contraction mapping on HCERbMS. By using Theorem 19, we conclude that "F has a unique fixed point". This completes our assertion. $\square$

Definition 19 (Controlled Extended Rectangular b-Fractals (CERb-Fractals)) The fixed point $\mathscr{A}_\infty \in \mathscr{K}_o(\Lambda)$ of the HB operator F for CERb-IFS described in Theorem 22 is the controlled extended rectangular b-attractor or controlled extended rectangular b-fractal in CERbMS. So, $\mathscr{A}_\infty \in \mathscr{K}_o(\Lambda)$ is known as a fractal generated by the CERb-IFS on CERbMS.

The derived fractal is constructed in the CERbMS as in the above HB theory; hence, it will be extended to develop the concepts of fractal interpolation theory and other related results in the extended space proposed in this paper. Since the control factor is included in the basic metric space, the fractal developed over the controlled space may be applicable in the fractal and control theories to model specific physical systems and analyze the experimental system's stability and controlability.

6. Conclusion

The contractions over CMSs have been used in this study to create a new type of IFS known as C-IFS, and an IFS of contractions has been built in a CMS to generate controlled fractals. The subsequent results have been demonstrated intriguingly using the CIFS and controlled fractals. Furthermore, the CERbMS has been established and investigated using the fixed point theorem on the proposed metric space, called the CERbMS. Additionally, the IFS has been defined on CERbMS to generate a new fractal attractor known as controlled extended rectangular b-fractals.

The controlled metric can be studied in the fractal interpolation and multifractal analysis. A new type of fractals in the controlled extended rectangular b-metric space with Kannan, Fisher-type and some other contractions can be addressed generically. It can be a new path to further describe the fractal interpolation function and multifractal analysis in the proposed general space.

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Conflict of interest

The authors declare that they have no conflict of interest.

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