Contemporary Mathematics
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Research Article

El-Zaki Transform Method for Solving Applications of Some Integral Differential Equations

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Received: 17 July 2024; **Revised:** 6 September 2024; **Accepted:** 10 October 2024

Abstract: Integral equations have close contact with many different science and engineering problems, such as contact problems in the theory of elasticity, potential theory, fluid mechanics, quantum mechanics, healthcare, and others. So, in this presentation, we will study the following models: determining the blood glucose concentration by using Volterra integral equations. Integral equations are used in electrical engineering applications. Abel's integral equation has applications in a variety of scientific domains, including plasma diagnostics, radio astronomy, and optical fibre evaluation. This is an ancient example of an integral equation. The El-Zaki transformation method is used to solve three models and compare the exact and approximate results. The findings are calculated using Maple 18.

***Keywords*:** volterra integral equation, El-Zaki transformation method, blood glucose concentration electrical engineering, Abel's integral equation

MSC: 45D05, 44A15, 92C50, 45E10

1. Introduction

The El-Zaki transformation is a critical tool for solving integral equations, first introduced by El-Zaki in 2011 as a novel integral transform [1]. Over the years, it has been extensively explored, with studies delving into its fundamental characteristics and various applications across different fields of science and engineering. For instance, El-Zaki et al. presented its most important characteristics and applications in 2012 [2]. Additionally, Kim discussed the shifting theorems for the El-Zaki transform in 2014 [3], while Theta et al. provided analytic solutions to the first and second kinds of linear Volterra integral equations using this transform in 2020 [4]. The relationship between El-Zaki and Laplace transforms was further explored by Tariq et al. [5]. In recent years, Al-Rawi et al. [6] focused on solving higher-order integro-differential equations using the El-Zaki decomposition method, while Higazy et al. [7] employed the Sawi decomposition method alongside the El-Zaki transform to solve Volterra integral problems. Furthermore, Al-Bugami [8] investigated numerical methods such as the sixth-kind Chebyshev and polynomial methods for solving nonlinear mixed partial integro-differential equations with continuous kernels. The shifting Legendre and least squares techniques for solving fractional integro-differential equations were discussed by Mahdy in 2024 [9], while Luchko [10] and Gutierrez [11] provided insights into fractional calculus and its applications. Additionally, Ziane et al. [12] combined the El-Zaki transform with the variational iteration method to address fractional-order partial differential equations. The study by

DOI: https://doi.org/10.37256/cm.6220255326

Bouazza et al. [13] maintained the solvability of Urysohn-type quadratic integral equations involving Hadamard variable-order operators, while Wang and Zhou [14] applied the two-gride iterative method based and convergence analysis for systems of Fredholm integral equations, which were further explored by Zhou in 2019 [15]. Additionally, Al-Bugami et al. [16] presented a scientific paper in numerical simulation on existence and uniquness in solving nonlinear mixed partial integro-differential equations with discontinuous kernels. Mahdi et al. [17] discussed the solution of the quadratic mixed integral equation and analysed its convergence and error stability, while Naveen and Parthiban [18] focused on variable-order Caputo derivatives in LC and RC circuit systems. Gomes-Aguilar et al. [19] explored the use of fractional derivatives, such as those of Atangana-Baleanu, in modelling RC, LC, and RL electrical circuits. Despite these significant advancements, further exploration is needed to expand the application of the El-Zaki transformation to more complex and emerging mathematical models, particularly those involving fractional calculus and nonlinear integro-differential equations. Fractional calculus, which deals with non-integre order derivatives and integrals, plays a critical role in accurately modelling complex physics and engineering systems. This study aims to apply the El-Zaki transformation to specific types of fractional calculus problems, such as fractional partial differential equations and nonlinear integro-differential equations. The development of mathematical tools like the El-Zaki transform is crucial for addressing the increasing complexity in fields such as engineering physics and applied science. As systems become more interconnected, especially with the integration of fractional derivatives and nonlinear dynamics, rotors and versatile methods like the El-Zaki transformation are becoming increasingly essential.

This work aims to explore and apply the El-Zaki transformation to new classes of problems, including fractional differential equations and nonlinear integro-differential equations. The study will begin by introducing the fundamental definitions and properties of the transform, followed by its application to scientific problems in applied science, such as the analysis of glucose levels in diabetic patients and the modeling of electrical circuits. The results will be analysed and compared with traditional methods to validate the effectiveness and relevance of the El-Zaki transformations in these new contexts.

2. Basic concepts and definitions

The El-Zaki transformation offers a powerful way to improve the numerical analysis of both fractional differential equations and Volterra integral equations. In this section, we will discuss the fundamental concepts associated with the El-Zaki transformation. It will cover specific definitions of fractional derivatives, such as Riemann-Liouville derivatives.

Definition 1 El-Zaki's transformation on integral equations is defined by the operator E[.] as the following formula (see [1, 3, 4])

$$E[g(t)] = \int_0^\infty \omega g(t) e^{-1/\omega} dt = G(\omega), \ \ t \geq 0 \tag{1}$$

where the set

$$g(t) \in Y = \left\{ g(t) : \ \exists R, \ s_1, \ s_2 > 0, \ |g(t)| < Re^{\frac{t}{s_n}}, \ \ \text{if}\, t \in (-1)^n \times [0, \ \infty) \right\}, \tag{2}$$

where $s_1 \leq \omega \leq s_2$ are finite or infinite and R is a real finite number.

Theorem 1 (Linearity of El-Zaki. see [6]) For two functions, $f(t)$ and $g(t)$, with constants a_1 and a_2, we may express the El-Zaki transform's linearity feature as:

$$E\left[a_1 f(t)+a_2 g(t)\right]=a_1 E[f(t)]+a_2 E[g(t)]$$
$$=a_1 F(\omega)+a_2 G(\omega), \tag{3}$$

where represents the El-Zaki transform operator.

Proof. Let's assume that $f(t)$ and $g(t)$ are two functions, and a and b are constants. Then:

$$E\left[a_1 f(t)+\ a_2 g(t)\right]=\omega \int_0^{\infty}\left[a_1 f(t)+\ a_2 g(t)\right] e^{-1/\omega} dt$$
$$=a_1\left[\omega \int_0^{\infty} f(t)\ e^{-1/\omega} dt\right]+a_2\left[\omega \int_0^{\infty} g(t)\ e^{-1/\omega} dt\right]$$
$$=a_1 E\left[f(t)\right]+a_2 E\left[g(t)\right]$$
$$=a_1 F(\omega)+a_2 G(\omega).$$

linearity is thus achieved by the El-Zaki transform. □

Theorem 2 (Convolution theorem. see [4, 6]) If $f(t)$ and $g(t)$ represent both functions, then the convolution of f and g

$$E[f * g](t)=\frac{1}{\omega}[F(\omega) G(\omega)] \tag{4}$$

See [1–6] for further information on the characteristics of the El-Zaki transformation technique.

Definition 2 If $f^{(m)} \in C_\delta$ is a real function and $\delta \geq -1$ then the Riemann-Liouville fractional integral ${}_{RL}I^{\xi}$ of order ξ is defined as:

$${}_{RL}I^{\xi} f(t)=\frac{1}{\Gamma(\xi)} \int_0^t (t-x)^{\xi-1} f(x) dx \tag{5}$$

where $\xi,\ t>0$ and if $\xi=0$ then ${}_{RL}I^{0} f(x)=f(t)$.

The Riemann-Liouville fractional integral in (5) has the following properties:

$${}_{RL}I^{\xi}\,{}_{RL}I^{\eta} f(t)=RLI^{\xi+\eta} f(t)$$

$${}_{RL}I^{\xi}\,{}_{RL}I^{\eta} f(t)={}_{RL}I^{\xi}\,{}_{RL}I^{\eta} f(t)$$

$${}_{RL}I^{\xi} t^{\alpha}=\frac{\Gamma(\alpha+1)}{\Gamma(\xi+\alpha+1)} t^{\xi+\alpha}$$

where $\xi,\ \eta \geq 0$.

To get additional information on the fundamental characteristics of fractional calculus, refer to the following sources: [20, 11, 10].

3. Methods

The El-Zaki transformation is an effective technique for solving diverse integral equations, as emphasized in the introduction of this study. This method offers a highly efficient approach for dealing with various integral equation models, such as Abel equations and second-order Volterra integral equations. The following comprehensive solution strategy demonstrates how to solve these equations using the El-Zaki transformation.

Consider an Volterra integral equation of the form:

$$u(t) = f(t) + \lambda \int_0^t k(t,\ x)u(x)dx \tag{6}$$

here $f(t)$ is a known function, $k(t,\ x)$ is the kernel, $u(t)$ is the unknown function, and λ is a constant. On both sides of the equation, apply the El-Zaki transformation.

$$E[u(t)] = E[f(t)] + \lambda E\left[\int_0^t k(t,\ x)u(x)dx\right] \tag{7}$$

To simplify the integral term, apply the El-Zaki transformation's properties, which can be represented as follows:

$$E[u(t)] = \frac{E[f(t)]}{1 - \dfrac{\lambda}{\omega}E[k(t)]} \tag{8}$$

the inverse El-Zaki transformation to obtain

$$u(t) = E^{-1}\left[\frac{E[f(t)]}{1 - \dfrac{\lambda}{\omega}E[k(t)]}\right]. \tag{9}$$

To review the inverse of the El-Zaki transform, see the source referred to as a reference [4].

3.1 *Application 1 (blood glucose concentration)*

Representing this model in the form of a Volterra integral equation as, (see [21])

$$C_i = -\frac{\alpha}{V}t + C(t) + \int_0^t kC(x)dx \tag{10}$$

the blood glucose concentration at a certain time t is shown by $C(t)$. Each minute is the unit of measurement for the fixed elimination velocity, k. α represents the ratio of the infusion amount in milligrams per minute. The variable V represents the amount in which glucose is spread, measured in decilitres dL. Measured in milligrams/dL, the variable C_i denotes the initial blood glucose concentration.

• The exact solution for this model (10) by using Laplace transforms as,

$$\mathscr{L}\{C(t)\}+k\mathscr{L}\left\{\int_0^t C(x)dx\right\}-\frac{\alpha}{V}\mathscr{L}\{t\}=\mathscr{L}\{C_i\} \tag{11}$$

$$C(s)+k\frac{1}{s}C(s)-\frac{\alpha}{V}\frac{1}{s^2}=\frac{C_i}{s} \tag{12}$$

$$C(s)\left[\frac{s+k}{s}\right]=\frac{C_i}{s}+\frac{\alpha}{Vs^2} \tag{13}$$

$$C(s)=\frac{C_i}{s+k}+\frac{\alpha}{Vs(s+k)} \tag{14}$$

from partial fractions

$$C(s)=C_i\left[\frac{1}{s+k}\right]+\frac{\alpha}{V}\left[\frac{A}{s}+\frac{B}{s+k}\right] \tag{15}$$

we have $A=\frac{1}{k},\ B=-\frac{1}{k}$. Take inverse of Laplace transform

$$\mathscr{L}^{-1}\{C(s)\}=C_i\mathscr{L}^{-1}\left\{\frac{1}{s+k}\right\}+\mathscr{L}^{-1}\left\{\frac{a}{V}\left[\frac{A}{s}+\frac{B}{s+k}\right]\right\} \tag{16}$$

finally, we get the exact solution as formula,

$$C(t)=C_ie^{-kt}+\frac{\alpha}{Vk}\left(1-e^{-kt}\right). \tag{17}$$

• The approximate solution is obtained by taking the transformation of El-Zaki for both sides in the Equation (10) as in [1, 2, 5].

$$E[C(t)]+kE\left[\int_0^t C(x)dx\right]-E\left[\frac{\alpha}{V}t\right]=E\left[C_i\right] \tag{18}$$

$$E[C(t)]=C_iE[1]+\left(\frac{\alpha}{V}\right)E[t]-kE\left[\int_0^t C(x)dx\right] \tag{19}$$

from Theorem 2

$$E[C(t)] = C_i\omega^2 + \left(\frac{\alpha}{V}\right)\omega^3 - k\left(\frac{1}{\omega}\right)E[1]E[C(t)] \tag{20}$$

$$E[C(t)] = C_i\omega^2 + \left(\frac{\alpha}{V}\right)\omega^3 - k\left(\frac{1}{\omega}\right)\omega^2 E[C(t)] \tag{21}$$

using the inverse of the El-Zaki transform

$$C(t) = C_i E^{-1}\left[\omega^2\right] + \left(\frac{\alpha}{V}\right)E^{-1}\left[\omega^3\right] - k\omega E[C(t)] \tag{22}$$

$$C(t) = C_i + \left(\frac{\alpha}{V}\right)t - kE^{-1}[\omega E[C(t)]] \tag{23}$$

assume the solution of $C(t)$ can be expressed as an infinite series

$$C(t) = \sum_{m=0}^{\infty} C_m(t) \tag{24}$$

substituting into (23)

$$\sum_{m=0}^{\infty} C_m(t) = C(t) = C_i + \left(\frac{\alpha}{V}\right)t - kE^{-1}\left[\omega E\left[\sum_{m=0}^{\infty} C_m(t)\right]\right]. \tag{25}$$

Using the recursive relation, we can derive the general formulas as

$$C_0(t) = C_i + \left(\frac{\alpha}{V}\right)t \tag{26}$$

$$C_{m+1}(t) = -kE^{-1}\left[\omega E\left[\sum_{m=0}^{\infty} C_m(t)\right]\right] \tag{27}$$

at $m = 0$

$$\begin{aligned} C_1(t) &= -kE^{-1}\left[\omega E\left[C_0(t)\right]\right] = -kE^{-1}\left[\omega E\left[C_i + \left(\frac{\alpha}{V}\right)t\right]\right] \\ &= -kE^{-1}\left[\omega\left[C_i\omega^2 + \left(\frac{\alpha}{V}\right)\omega^3\right]\right] = -k\left[C_i t + \left(\frac{\alpha}{V}\right)\frac{t^2}{2!}\right] \end{aligned} \tag{28}$$

at $m = 1$

$$C_2(t) = -kE^{-1}\left[\omega E\left[C_1(t)\right]\right] = -kE^{-1}\left[\omega E\left[-kC_it - k\left(\frac{\alpha}{V}\right)\frac{t^2}{2!}\right]\right]$$

$$= -kE^{-1}\left[\omega\left[-kC_i\omega^3 - k\left(\frac{\alpha}{V}\right)\frac{2!\omega^4}{2!}\right]\right] = k^2\left[C_i\frac{t^2}{2!} + \left(\frac{\alpha}{V}\right)\frac{t^3}{3!}\right]. \tag{29}$$

The solution will be as follows:

$$C(t) = \sum_{m=0}^{\infty} C_m(t) = C_0(t) + C_1(t) + C_2(t) + \ldots$$

$$= \left[C_1 + \left(\frac{\alpha}{V}\right)t\right] - k\left[C_it + \left(\frac{\alpha}{V}\right)\frac{t^2}{2!}\right] + k^2\left[C_i\frac{t^2}{2!} + \left(\frac{\alpha}{V}\right)\frac{t^3}{3!}\right]\cdots \tag{30}$$

$$= C_i\left[1 - kt - k^2\frac{t^2}{2!}\cdots\right] + \left(\frac{\alpha}{Vk}\right)\left[1 - 1 + kt - k^{t^2}\frac{t^2}{2!} + k^3\frac{t^3}{3!}\cdots\right] \tag{31}$$

finally, we obtain

$$C(t) = C_ie^{-kt} + \frac{\alpha}{Vk}\left(1 - e^{-kt}\right). \tag{32}$$

The results of the solution concentration of blood glucose using the El-Zaki transform technique between the exact solution (17) and the El-Zaki transform method (32) are obtained in Tables 1-5 and Figures 1-5.

Table 1. The initial concentration of glucose $C_i = 320$ mg/dL, $k = 0.058$ per minute, $V = 45$ dL, along with $\alpha = 280$ mg/min

t	$C_i = 320$		Absolute error
	Exact solution	Approximate solution	
0	320	320	0.000000000
10	226.3814456	226.3814456	0.000000000
20	173.9645700	173.9645700	0.000000000
30	144.6164469	144.6164469	0.000000000
40	128.1844808	128.1844808	0.000000000
50	118.9842497	118.9842497	0.000000000
60	113.8330554	113.8330554	0.000000000
70	110.9489101	110.9489101	0.000000000
80	109.3340819	109.3340819	0.000000000
90	108.4299422	108.4299422	0.000000000

Table 2. The initial concentration of glucose $C_i = 325$ mg/dL, $k = 0.058$ per minute, $V = 45$ dL, along with $\alpha = 280$ mg/min

t	$C_i = 325$		Absolute error
	Exact solution	Approximate solution	
0	325	325	0.000000000
10	229.1809374	229.1809374	0.000000000
20	175.5320009	175.5320009	0.000000000
30	145.4940489	145.4940489	0.000000000
40	128.6758487	128.6758487	0.000000000
50	119.2593658	119.2593658	0.000000000
60	113.9870925	113.9870925	0.000000000
70	111.0351552	111.0351552	0.000000000
80	109.3823704	109.3823704	0.000000000
90	108.4569789	108.4569789	0.000000000

Table 3. The initial concentration of glucose $C_i = 330$ mg/dL, $k = 0.058$ per minute, $V = 45$ dL, along with $\alpha = 280$ mg/min

t	$C_i = 330$		Absolute error
	Exact solution	Approximate solution	
0	330	330	0.000000000
10	231.9804293	231.9804293	0.000000000
20	177.0994318	177.0994318	0.000000000
30	146.3716509	146.3716509	0.000000000
40	129.1672166	129.1672166	0.000000000
50	119.5344819	119.5344819	0.000000000
60	114.1411295	114.1411295	0.000000000
70	111.1214003	111.1214003	0.000000000
80	109.4306589	109.4306589	0.000000000
90	108.4840155	108.4840155	0.000000000

Table 4. The initial concentration of glucose $C_i = 335$ mg/dL, $k = 0.058$ per minute, $V = 45$ dL, along with $\alpha = 280$ mg/min

t	$C_i = 335$		Absolute error
	Exact solution	Approximate solution	
0	335	335	0.000000000
10	234.7799211	234.7799211	0.000000000
20	178.6668627	178.6668627	0.000000000
30	147.2492529	147.2492529	0.000000000
40	129.6585845	129.6585845	0.000000000
50	119.8095980	119.8095980	0.000000000
60	114.2951666	114.2951666	0.000000000
70	111.2076454	111.2076454	0.000000000
80	109.4789474	109.4789474	0.000000000
90	108.5110522	108.5110522	0.000000000

Table 5. The initial concentration of glucose $C_i = 340$ mg/dL, $k = 0.058$ per minute, $V = 45$ dL, along with $\alpha = 280$ mg/min

t	$C_i = 340$		Absolute error
	Exact solution	Approximate solution	
0	340	340	0.000000000
10	237.5794129	237.5794129	0.000000000
20	180.2342936	180.2342936	0.000000000
30	148.1268549	148.1268549	0.000000000
40	130.1499525	130.1499525	0.000000000
50	120.0847141	120.0847141	0.000000000
60	114.4492036	114.4492036	0.000000000
70	111.2938905	111.2938905	0.000000000
80	109.5272359	109.5272359	0.000000000
90	108.5380888	108.5380888	0.000000000

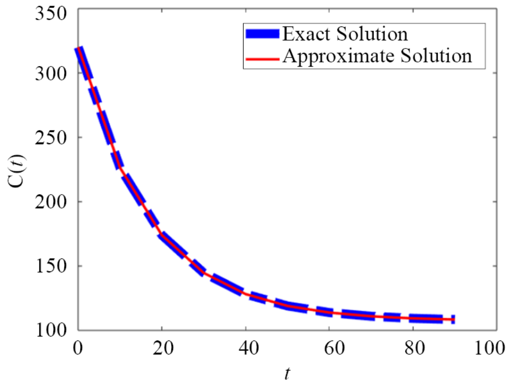

Figure 1. $C_i = 340$ mg/dL

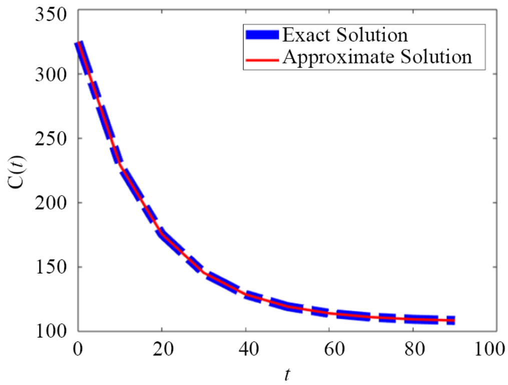

Figure 2. $C_i = 340$ mg/dL

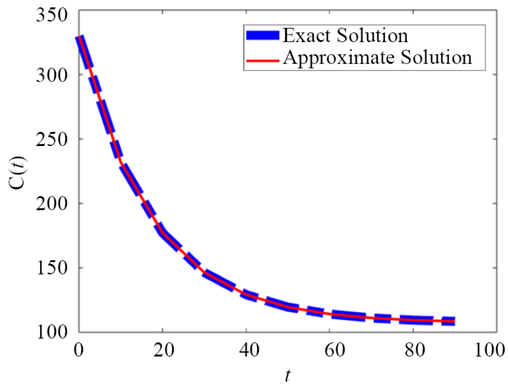

Figure 3. $C_i = 330$ mg/dL

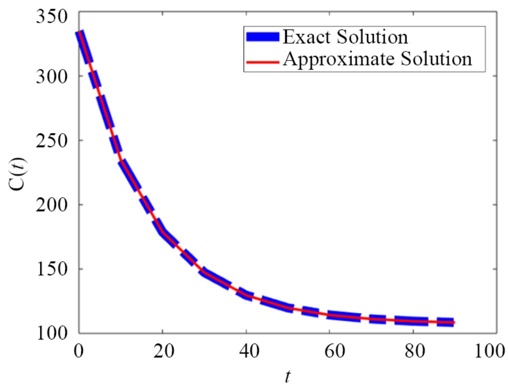

Figure 4. $C_i = 335$ mg/dL

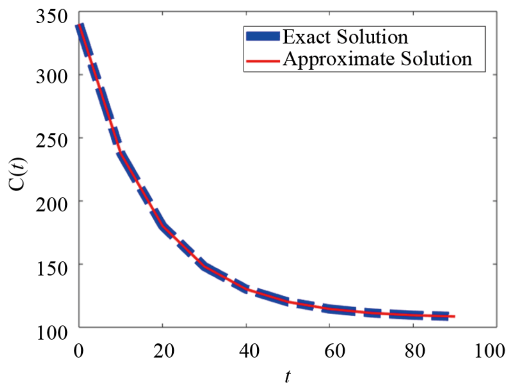

Figure 5. $C_i = 340$ mg/dL

3.2 *Application 2 (electrical engineering)*

In this section, we explore the practical application of the El-Zaki transformation to solve specific problems in electrical engineering. The El-Zaki transform has been employed to analyze and model complex electrical circuits, particularly in systems involving dynamics such as RC and RCL circuits. By applying the El-Zaki transform, we can convert complex differential equations that describe circuit behavior into more manageable algebraic equations, facilitating the analysis and design of electrical systems.

3.2.1 *RC electrical circuit*

For the RC circuit's in Figure 6 capacitor volume as a function of electrical charge q, a capacitance $C(farads)$, a resistance $R(ohms)$, and no inductance L are connected to an electromotive force $E(volts)$, which is a differential equation that uses Kirchhoff's Laws (33) to determine the current $I = \dot{q}$ after a close time (see [22]) as follows:

$$\dot{q} + \frac{1}{RC}q = \frac{E}{R}. \tag{33}$$

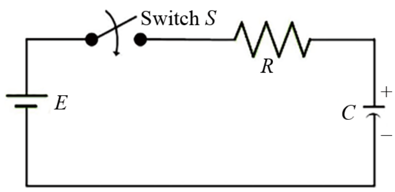

Figure 6. RC electrical circuit

To convert Kirchhoff's voltage laws (33) from a differential equation to a Volterra integral equation with the initial conditions for q being

$$q(0) = q_0. \tag{34}$$

Let $\dot{q} = \varphi(t)$ and

$$q = \int_0^t \varphi(x)\, dx + q_0 \tag{35}$$

substitute from (35) and $\dot{q} = \varphi(t)$ into (33), we obtain

$$\phi(t) + \frac{1}{RC}\left[\int_0^t \phi(x)dx + q_0\right] = \frac{E}{R} \tag{36}$$

this formula (36) as the Volterra integral equations in second kind.

Example 1 Find the current in an RC circuit

$$\dot{q} + 20q = 20 \tag{37}$$

at $t \in [0,\ 1]$, subject to initial conditions $q(0) = 0$, and specific values $R = 5\ ohms,\ C = 0.01\ farads$, and $E = 100\ volts$.

Solution:

• The first is to discover the exact solution

$$\mu(t) = e^{\int 20dt} = e^{20t} \tag{38}$$

and

$$q(t) = \frac{1}{e^{20t}}\left[\int 20e^{20}dt + c\right] \tag{39}$$

then by using the initial condition, the general solution of RC circuit is

$$q(t) = 1 - 20\ e^{20t}. \tag{40}$$

• The second is to discover the approximate solution by converting the (37) from a differential equation to an Volterra integral equation for the second kind $\dot{q} = \varphi(t)$.

Then

$$q = \int_0^t \varphi(x)\ dx \tag{41}$$

substitute from (41) and $\dot{q} = \varphi(t)$ into (37), we obtain

$$\phi(t) + 20\left[\int_0^t \phi(x)dx\right] = 20 \tag{42}$$

take an El-Zaki transform; see ([1, 2, 5]) for both sides

$$E[\phi(t)] + 20E\left[\int_0^t \phi(x)dx\right] = 20E[1] \tag{43}$$

$$E[\phi(t)] + \frac{20}{\omega}E\left[[1]E[\phi(t)]\right] = 20E[1] \tag{44}$$

$$E[\phi(t)] = 20\omega^2 - 20\omega E[\phi(t)] \tag{45}$$

take the inverse

$$\phi(t) = 20 - 20E^{-1}[\omega E[\phi(t)]] \tag{46}$$

let the solution of $\phi(t)$ can be expressed as a infinite series

$$\phi(t) = \sum_{m=0}^{\infty} \phi_m(t) \tag{47}$$

using the recursive relation, we can derive the general formulas as

$$\begin{aligned} &\phi_0(t) = 20 \\ &\phi_{m+1}(t) = -20E^{-1}\left[\omega E\left[\phi_m(t)\right]\right] \end{aligned} \tag{48}$$

at $m = 0$

$$\begin{aligned} \phi_1(t) &= -20E^{-1}\left[\omega E\left[\phi_0(t)\right]\right] = -20E^{-1}[\omega E[20]] \\ &= -20^2E^{-1}\left[\omega^3\right] = -20^2t \end{aligned} \tag{49}$$

at $m = 1$

$$\begin{aligned} \phi_2(t) &= -20E^{-1}\left[\omega E\left[\phi_1(t)\right]\right] = -20E^{-1}\left[\omega E\left[-20^2t\right]\right] \\ &= 20^3E^{-1}\left[\omega^4\right] = 20^3\frac{t^2}{2!} \end{aligned} \tag{50}$$

the solution will be as this following:

$$\begin{aligned} \phi(t) &= \sum_{m=0}^{\infty} \phi_m(t) = \phi_0(t) + \phi_1(t) + \phi_2(t) + \cdots \\ &= 20 - 20^2t + 20^3\frac{t^2}{2!}\cdots \end{aligned} \tag{51}$$

finally, we obtain

$$\phi(t) = 20e^{-20t} \tag{52}$$

to find the current, substituting (52) into $\dot{q} = \varphi(t)$ and (41) and then in (37), to obtain

$$q(t) = 1 - e^{-20t}. \tag{53}$$

For results for both the exact and approximate solution, see Table 6 as illustrated in Figure 7.

Table 6. Comparison between the exact solution and the approximate solution for diverse values of time $t \in [0, 1]$

t	Exact solution	Approximate solution	Absolute error
0	0.000000000	0.000000000	0.000000000
0.1	0.8646647168	0.8646647168	0.000000000
0.2	0.9816843611	0.9816843611	0.000000000
0.3	0.9975212478	0.9975212478	0.000000000
0.4	0.9996645374	0.9996645374	0.000000000
0.5	0.9999546001	0.9999546001	0.000000000
0.6	0.9999938558	0.9999938558	0.000000000
0.7	0.9999991685	0.9999991685	0.000000000
0.8	0.9999998875	0.9999998875	0.000000000
0.9	0.9999999848	0.9999999848	0.000000000
1	0.9999999979	0.9999999979	0.000000000

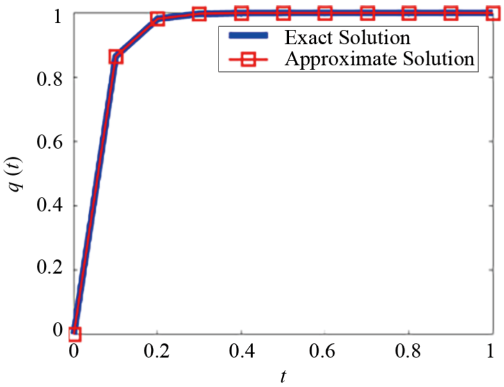

Figure 7. The RC circuit's current at $R = 5$ *ohms*, $C = 0.01$ *farads*, and $E_0 = 100$ *volts*, as well as a comparison of the approximate and exact solutions for a range of time $t \in [0, 1]$

3.2.2 *RCL electrical circuit*

In Figure 8 capacitor with a capacity of C, a resistance of R, and induction from a battery of L is connected to an electrical circuit. E.M.F. equals $E(t)$, and the following integro-differential equation uses Kirchhoff's Laws (54) to determine the circuit's voltage $V(t)$ and current I after a close time.

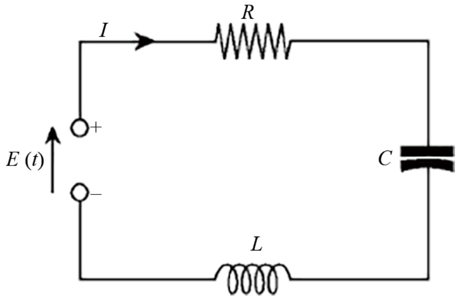

Figure 8. RCL electrical circuit. E.M.F

From Kirchhoff loop law Equation (see [23])

$$RI + L\frac{dI}{dt} + \frac{1}{C} - E(t) = 0 \tag{54}$$

the relationship between q and I is

$$I = \frac{dq}{dt}, \; \frac{dI}{dt} = \frac{d^2q}{dt^2} \tag{55}$$

substituting this value into (54), we have

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = \frac{1}{L}E(t) \tag{56}$$

the initial conditions for q are

$$\dot{q}(0) = I_0 = I(0), \qquad q(0) = q_0 \tag{57}$$

let $\frac{d^2q}{dt^2} = \varphi(t)$ and

$$\frac{dq}{dt} = \int_0^t \phi(x)dx + c_1 \tag{58}$$

$$q = \int_0^t (t-x)\phi(x)dx + c_1 t + c_2 \tag{59}$$

substitute from (58) and (59) into (56), we obtain

$$\phi(t)+\frac{R}{L}\left[\int_0^t \phi(x)dx+c_1\right]+\frac{1}{LC}\left[\int_0^t (t-x)\phi(x)dx+c_1t+c_2\right]=\frac{1}{L}E(t) \tag{60}$$

$$\phi(t)+\left[\int_0^t \left[\frac{R}{L}+\frac{1}{LC}(t-x)\right]\phi(x)dx\right]+\frac{1}{LC}\left[c_1t+c_2\right]=\frac{1}{L}E(t). \tag{61}$$

This formula (61) is a general solution of Volterra integral equations in the second kind, which is applied in Kirchhoff's Law. The El-Za ki transform method will then be used; for further information, see [1, 2, 5].

Example 2 Let the Kirchhoff's loop law equation

$$\ddot{q}+20\dot{q}+200q=24 \tag{62}$$

subject to the initial conditions $q(0)=0$, $\dot{q}(0)=0$ represents particular values. $R=10$ *ohm*, $C=10^{-2}farad$, $L=\frac{1}{2}henries$, $E(t)=12$ *volts*.

• The first is to discover the exact solution, the general solution is

$$q=\frac{3}{25}-e^{-10t}\left[\frac{3}{25}cos(10t)+\frac{3}{25}sin(10t)\right] \tag{63}$$

$$I=\frac{dq}{dt}=\frac{12}{5}e^{-10t}sin(10t). \tag{64}$$

• The first is to second is to discover the approximate solution by applying the El-Zaki transform method to Kirchoff's loop law equation

$$\ddot{q}+20\dot{q}+200q=24 \tag{65}$$

let $\ddot{q}+\varphi(t)$, then

$$\dot{q}(t)=\int_0^t \phi(x)dx \tag{66}$$

$$q(t)=\int_0^t (t-x)\phi(x)dx \tag{67}$$

substituting (67) and (66) into (63)

$$\phi(t)+20\int_0^t \phi(x)dx+200\int_0^t (t-x)\phi(x)dx=24 \tag{68}$$

take an El-Zaki transform for both sides

$$E[\phi(t)] + 20E\left[\int_0^t \phi(x)dx\right] + 200E\left[\int_0^t (t-x)\phi(x)dx\right] = 24E[1] \tag{69}$$

$$E[\phi(t)] + 20\omega E[\phi(t)] + 200\omega^2 E[\phi(t)] = 24\omega^2 \tag{70}$$

the inverse

$$\begin{aligned} E^{-1}[\phi(t)] = E^{-1}\left[\frac{24\omega^2}{1+20\omega+200\omega^2}\right] \\ = E^{-1}\left[\frac{24\omega^2}{\frac{2(10)^2}{\omega}\left[\frac{\omega}{2(10)^2} + \frac{10\omega^2}{2(10)^2}(1+10\omega)\right]}\right], \\ = E^{-1}\left[\frac{24\omega^3}{2(10)^2\left[\frac{\omega}{2(10)^2} + \frac{10\omega^2}{2(10)^2}(1+10\omega)\right]}\right], \end{aligned} \tag{71}$$

therefore

$$\phi(t) = \frac{12}{5}e^{-10t} sin(10t) \tag{72}$$

$$q(t) = -e^{-10t}\left[\frac{3}{25}cos(10t) + \frac{3}{25}sin(10t)\right] + \frac{3}{25}. \tag{73}$$

The results of the El-Zaki transform method are obtained in Table 7 and Figure 9 between the El-Zaki transform method and the exact solution.

Table 7. Comparison between the exact solution and the approximate solution for diverse values of time t

t	Exact solution	Approximate solution	Absolute error
0	0.000000000	0.000000000	0.000000000
0.1	0.05900088167	0.05900088167	0.000000000
0.2	0.1119911190	0.1119911190	0.000000000
0.3	0.1250715447	0.1250715447	0.000000000
0.4	0.1230999866	0.1230999866	0.000000000
0.5	0.1205459856	0.1205459856	0.000000000
0.6	0.1197975093	0.1197975093	0.000000000
0.7	0.1198456123	0.1198456123	0.000000000
0.8	0.1199660301	0.1199660301	0.000000000
0.9	0.1200073900	0.1200073900	0.000000000
1	0.1200075351	0.1200075351	0.000000000

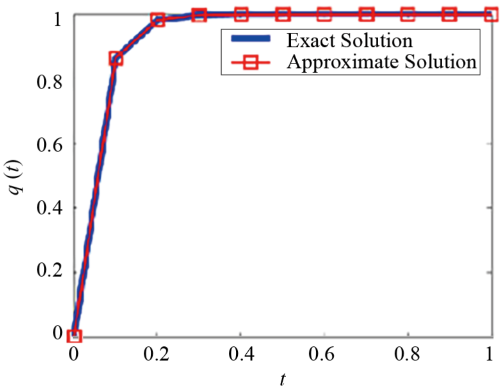

Figure 9. The RCL circuit's current at $R = 10\ ohms$, $C = 10^{-2} farads$, $L = \frac{1}{2}\ henries$, and $E_0 = \frac{1}{2} volts$, as well as a comparison of the approximate and exact solutions for a range of time $t \in [0,\ 1]$

3.3 *Application 3 (abel integral equation)*

In this section, we focus on how analysing Abel's integral equation can enrich our understanding of the techniques used in solving fractional differential equations and their applications. We will discuss how the analytical methods applied to Abel's equation can lead to improved solutions and analyses for fractional differential equations. See defined in [24, 25]:

$$\int_0^x \frac{u(t)}{(x-t)^{\beta}} dt = f(x) \tag{74}$$

which $u(t)$ is unknown, $f(x)$ is known, and $0 < \beta < 1$.

The Laplace transformation method is used in this section and converts it from the Abel integration equation in general to the fractional differential equation as a formula:

$$u(x) = \frac{sin(\pi\beta)}{\pi} \frac{d}{dx} \left[\int_0^x \frac{f(t)}{(x-t)^{1-\beta}} dt \right]. \tag{75}$$

Example 1 In the special at $\beta = \frac{1}{3}$ and $f(x) = \frac{2\pi}{3\sqrt{3}}x$ the general formula of Abel's integral as

$$\frac{2\pi}{3\sqrt{3}}x = \int_0^x \frac{u(t)}{(x-t)^{1/3}}dt. \tag{76}$$

• The exact solution obtained by applying the Laplace transform from (75) is

$$u(x) = \frac{\sin(\pi\frac{1}{3})}{\pi}\frac{d}{dx}\left[\int_0^x \frac{2\pi}{3\sqrt{3}}\frac{t}{(x-t)^{1-\beta}}dt\right]. \tag{77}$$

• The approximate solution by taking the El-Zaki transform for each side of (76) and using Theorem 2, we have

$$\frac{2\pi}{3\sqrt{3}}E[x] = \frac{1}{\omega}E[u(x)]E\left[x^{-\frac{1}{3}}\right] \tag{78}$$

$$\frac{2\pi}{3\sqrt{3}}\omega^3 = \omega^{2/3}\Gamma\left(\frac{2}{3}\right)E[u(x)] \tag{79}$$

$$\frac{2\pi}{3\sqrt{3}\Gamma\left(\frac{2}{3}\right)}\omega^{7/3} = E[u(x)] \tag{80}$$

using the inverse of the El-Zaki transform to find the approximate solution

$$u(x) = \frac{2\pi}{3\sqrt{3}\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right)}x^{1/3}. \tag{81}$$

The results of the El-Zaki transform method are obtained in Table 8 and Figure 10 between El-Zaki transform method and the exact solution.

Table 8. A comparison between the exact and approximate solutions

x	Exact solution	Approximate solution	Absolute error
0	0.000000000	0.000000000	0.000000000
0.1	0.4641588834	0.4641588834	0.000000000
0.2	0.5848035476	0.5848035476	0.000000000
0.3	0.6694329501	0.6694329501	0.000000000
0.4	0.7368062997	0.7368062997	0.000000000
0.5	0.7937005260	0.7937005260	0.000000000
0.6	0.8434326653	0.8434326653	0.000000000
0.7	0.8879040017	0.8879040017	0.000000000
0.8	0.9283177667	0.9283177667	0.000000000
0.9	0.9654893846	0.9654893846	0.000000000
1	1	1	0.000000000

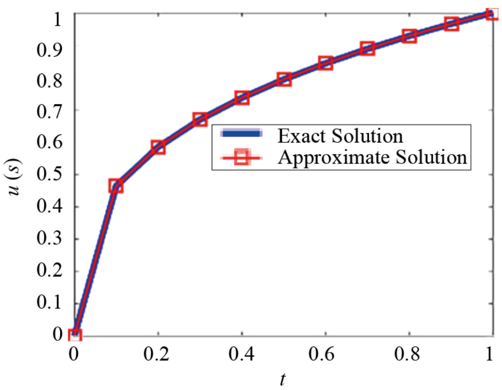

Figure 10. Using Abel's integral equations' exact and numerical solutions

4. Discussion

We compared the exact solution and the approximate solution for diverse values of time t from 0 to 90. In Table 1, at C_i = 320 mg/dL; for Table 2, C_i = 325 mg/dL; at about C_i = 330 mg/dL in Table 3, C_i = 335 mg/dL for Table 4, and for Table 5, C_i = 340 mg/dL. Great results were obtained, basically identical to the exact solution, and the absolute error was acquired. That is, demonstrate an increase in blood glucose concentration over time at different initial concentrations, with the values of k, V, and α constant. These are 0.058, 45, and 280, in that order. The results are supported in Figures 1-5, as is the no-gap between the exact solution and the informational technique.

Based on the application of the El-Zaki transform technique to a RC circuit and its comparison with the exact solution, which includes an amount of time values as seen in Figure 7 and documented in Table 6, it is obvious that the results are characteristic results.

By comparing the exact and approximate solution of the El-Zaki conversion on the circuit

RCL using the loop law, when $R = 10\ ohm$, $C = 10-2\ farad$, $L = 12\ henries$, and $E(t) = 12\ volts$, we got the results shown in Table 7 and Chart 9. The amount of absolute error was calculated to be almost zero for all the different values of time, showing that the method used had been successful.

In the Abel integral problem, if the exponent of β is bound between 0 and 1, such as a $\beta = \frac{1}{3}$, it may be solved using the El-Zaki method. The result obtained using this method is then compared to the exact solution, as seen in Figure 10. The absolute error and its related results are presented in Table 8.

5. Conclusions

The conclusion gives the reader a clear and concise understanding of your final idea. It reviews key points from the paper and summarizes the data in simple terms. It can also be used as a platform and an opportunity to call for further action.

In this research, we explored the application of the El-Zaki transformation technique to approximate solutions for Volterra integral equations. This method was effectively applied to three distant, distinct models:

• Blood glucose concentration detection:

To analyse and approximate the dynamic of blood concentration in the diabetic patients, we used the El-Zaki transformation. The result demonstrated that the El-Zaki transformation provides a course and rely on estimate, which can be instrumental in predicting breathing and managing blood cause levels.

• Electrical engineering applications:

The technique was applied to electrical engineering problems, specifically analyzing and solving circuits described by integral equations. The El-Zaki transformation demonstrated strong performance in simplifying complex circuit equations and providing free solutions, which is valuable for both thermal analysis and practical engineering applications.

• Abel's integral equation:

The transformation was also used to solve Abel's integral equation, a type of fractional differential equation. The results confirmed the El-Zaki transformation's effectiveness in handling fractional calculus problems and provided solutions that aligned with the expectation.

Overall, the application of the El-Zaki transformation technique yielded an excellent result across all models. The approximate solution obtained was in close argument with the exact solution, demonstrating the method's robustness and accuracy. The El-Zaki transformations ability to handle complex integral and differential equations effectively demonstrates their potential for broader applications in various fields of science and engineering.

Acknowledgments

The authors would like to acknowledge Deanship of Graduate Students and Scientific Research, Taif University for funding this work.

Conflict of interest

There is no conflict of interest for this study.

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