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Research Article

Coefficient Estimates for New Subclasses of Bi-Univalent Functions Associated with Jacobi Polynomials

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Abstract: Our research introduces new subclasses of analytical functions that are defined by Jacobi polynomials. We then proceed to estimate the Fekete-Szegö functional problem and the Maclaurin coefficients for this specific subfamily, denoted as $|a_2|$ and $|a_3|$. Furthermore, we demonstrate several new results that emerge when we specialize the parameters used in our main findings.

***Keywords*:** analytic functions, univalent functions, bi-univalent functions, Jacobi polynomials, fekete-szegö problem

MSC: 30C45

1. Preliminaries

Legendre first introduced orthogonal polynomials in 1784 [1]. These polynomials are frequently employed in solving ordinary differential equations with specific model constraints. Additionally, they play a crucial role in approximation theory [2].

Two polynomials Y_n and Y_m of order n and m, respectively, are said to be orthogonal if

$$\int_{\varepsilon}^{I} Y_n(x)Y_m(x)v(x)dx = 0, \text{ for } n \neq m$$

Assuming $v(x)$ is non-negative within the interval (ε, I), all polynomials of finite order $Y_n(x)$ possess a clearly defined integral. Jacobi polynomials belong to the category of orthogonal polynomials.

As a result of the widespread use of Jacobi polynomials in pure mathematics, many scholars have begun to investigate various areas. The present research in geometric function theory mainly focuses on the geometric properties of special functions and their associated counterparts.

Let f be the class of analytic functions b in the unit disk $\Lambda = \{\kappa \in \mathbb{C} : |\kappa| < 1\}$ and normalized by $b(0) = b'(0) - 1 = 0$ of the form:

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$$b(\kappa)=\kappa+\sum_{n=2}^{\infty}c_n\kappa^n,\ (\kappa\in\Lambda). \tag{1}$$

We also let Ψ consisting of functions univalent in Λ.

Every mathematical function $b\in\Psi$ has an inverse b^{-1}, defined by

$$b^{-1}(b(k))=k \text{ and } w=b\left(b^{-1}(w)\right)$$

where

$$b^{-1}(w)=g(w)=w-c_2w^2+\left(2c_2^2-c_3\right)w^3-\left(c_4+5c_2^3-5c_3c_2\right)w^4+\cdots.$$

A function b is said to be bi-univalent in Λ if both b and b^{-1} are univalent in Λ. Let Π denote the class of all bi-univalent functions in Λ given by (1).

Example in the class Π is $h(k)=\dfrac{k}{1-k}$ but $h(k)=\dfrac{k}{1-k^2}$ not members of Π. For interesting function classes in class Π, (see [3]).

Miller and Mocanu [4] introduced the first differential subordination problem, see [5] and [6]. We say that the function b is subordinate to q, written as $p\prec q$, if b and q are analytic in Λ and exists function $w\in F$ in Λ with

$$w(0)=0 \text{ and } |w(k)|<1,$$

such that

$$b(\kappa)=q(w(\kappa)).$$

Also, if q is univalent in Λ, then

$$b(\kappa)<q(\kappa) \text{ if and only if } b(0)=q(0) \text{ and } b(\Lambda)\subset q(\Lambda).$$

Jacobi polynomials play a significant role in geometric function theory due to their rich mathematical structure and versatility in approximating functions, solving boundary value problems, and providing insights into the geometric properties of analytic functions.

Jacobi polynomials are part of a larger family of orthogonal polynomials that include Legendre and Chebyshev polynomials as special cases. These polynomials arise as solutions to the Jacobi differential equation, which is a second-order linear equation. The orthogonality of these polynomials makes them particularly useful in approximating functions and solving boundary value problems in geometric contexts. In geometric function theory, special functions, including Jacobi polynomials, are often used to construct or approximate functions that exhibit specific geometric properties, such as univalence, starlikeness, or convexity.

The aim of this study is to construct a new and comprehensive subclass of bi-univalent functions based on the Jacobi polynomials, a specific special function.

For $n,\ n+\vartheta,\ n+s$ are nonnegative integers, a generating function of Jacobi polynomials is defined by

$$J_n(x,\ z)=2^{\vartheta+\varsigma}R^{-1}(1-x+R)^{-\vartheta}(1+x+R)^{-\varsigma},$$

where $R=R(x,\ z)=\left(1-2zx+x^2\right)^{0.5},\ \vartheta>-1,\ \varsigma>-1,\ x\in[-1,\ 1]$ and $z\in\mathbb{U}$, (see [7]).

For a fixed x, the function $J_n(x,\ z)$ is analytic in $\mathbb{U}$, allowing it to be represented by a Taylor series expansion as follows:

$$J_n(x, z) = \sum_{n=0}^{\infty} P_n^{(\vartheta, \varsigma)}(x) z^n \tag{2}$$

where $P_n^{(\vartheta, \varsigma)}$ is Jacobi polynomial of degree n.

The Jacobi polynomial $P_n^{(\vartheta, \varsigma)}$ satisfies a second-order linear homogeneous differential equation:

$$\left(1-x^2\right) y'' + (\varsigma - \vartheta - (\vartheta + \varsigma + 2)x) y' + n(n + \vartheta + \varsigma + 1) y = 0.$$

Jacobi polynomials can alternatively be characterized by the following recursive relationships:

$$P_n^{(\vartheta, \varsigma)}(x) = \left(a_{n-1} z - b_{n-1}\right) P_{n-1}^{(\vartheta, \varsigma)}(x) - c_{n-1} P_{n-2}^{(\vartheta, \varsigma)}(x), \ n \geqslant 2$$

where

$$a_n = \frac{(2n + \vartheta + \varsigma + 1)(2n + \vartheta + \varsigma + 2)}{2(n+1)(n + \vartheta + \varsigma + 1)}, \ b_n = \frac{(2n + \vartheta + \varsigma + 1)\left(\varsigma^2 - \vartheta^2\right)}{2(n+1)(n + \vartheta + \varsigma + 1)(2n + \vartheta + \varsigma)}$$

$$\text{and } c_n = \frac{(2n + \vartheta + \varsigma + 2)(n + \vartheta)(n + \varsigma)}{(n+1)(n + \vartheta + \varsigma + 1)(2n + \vartheta + \varsigma)},$$

with the initial values

$$P_0^{(\vartheta, \varsigma)}(x) = 1, \ P_1^{(\vartheta, \varsigma)}(x) = (\vartheta + 1) + \frac{1}{2}(\vartheta + \varsigma + 2)(x - 1) \text{ and} \tag{3}$$

$$P_2^{(\vartheta, \varsigma)}(x) = \frac{(\vartheta + 1)(\vartheta + 2)}{2} + \frac{1}{2}(\vartheta + 2)(\vartheta + \varsigma + 3)(x - 1) + \frac{1}{8}(\vartheta + \varsigma + 3)(\vartheta + \varsigma + 4)(x - 1)^2 \tag{4}$$

To begin, we introduce certain special instances of the polynomials $P_n^{(\vartheta, \varsigma)}$:

1. For $\vartheta = \varsigma = 0$, we get the Legendre Polynomials.
2. For $\vartheta = \varsigma = -0.5$, this results in the Chebyshev Polynomials of the first kind.
3. For $\vartheta = \varsigma = 0.5$, this results in the Chebyshev Polynomials of the second kind.
4. For $\vartheta = \varsigma$, we get the Gegenbauer Polynomials and each is replaced by $(\vartheta - 0.5)$.

Ezrohi [8] introduced the class $\mathcal{U}(\varepsilon)$ as follows:

$$\mathcal{U}(\varepsilon) = \left\{\Theta : \Theta \in \mathcal{S} \text{ and } \operatorname{Re}\left\{\Theta'(z)\right\} > \varepsilon, \ (z \in \mathbb{U}; \ 0 \le \varepsilon < 1)\right\}.$$

A lot of studies have looked at the geometric function theory in recent years, including coefficient estimates [9-13].

Several subclasses of the class Π were introduced and non-sharp estimates on the coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1) were obtented in [14-18].

However, when it comes to Jacobi polynomials, to the best of our knowledge, there has been a dearth of research on bi-univalent functions in existing literature [19-23]. The motivation is to create new subclasses of bi-univalent functions using Jacobi polynomials to bridge two areas of interest: geometric function theory and orthogonal polynomials. By introducing Jacobi polynomials into the study of bi-univalent functions, researchers hope to derive new results for coefficient estimates, Fekete-Szegő inequalities, and other function-theoretic properties.

In this study, we define new subclass of Π involving the Jacobi polynomials which are denote by $\mathcal{F}_{\Pi}^{\mu}(\alpha, \varphi)$, and derive bounds for the $|a_2|$ and $|a_3|$ Taylor-Maclaurin coefficients and Fekete-Szegö functional problems. Furthermore, several novel findings are shown to ensue.

2. Definition and examples

At the beginning of this section, we present a definition of the new subclasses $\mathcal{F}_{\Pi}^{\mu}(\alpha,\ \varphi)$ that is associated with Jacobi polynomials.

Definition 1 If the following subordinations are met for a function $b \in \Lambda$ given by (1), then $b \in \mathcal{F}_{\Pi}^{\mu}(\alpha,\ \varphi)$:

$$\mu\left[\left(\frac{b(\kappa)}{\kappa}\right)^{\alpha}+\frac{1+e^{i\varphi}}{2}\left(\frac{\kappa b''(\kappa)}{b'(\kappa)}\right)\right]+(1-\mu)\left[\left(b'(\kappa)\right)^{\alpha}+\frac{1+e^{i\varphi}}{2}\left(\kappa b''(\kappa)\right)\right]\prec J_{i}(x,\ \kappa) \tag{5}$$

and

$$\mu\left[\left(\frac{g(w)}{w}\right)^{\alpha}+\frac{1+e^{i\varphi}}{2}\left(\frac{wg''(w)}{g(w)}\right)\right]+(1-\mu)\left[\left(q'(w)\right)^{\alpha}+\frac{1+e^{i\varphi}}{2}\left(wg''(w)\right)\right]\prec J_{i}(x,\ \varpi)$$

where $0 \leq \mu \leq 1,\ \vartheta > -1,\ \varsigma > -1,\ -\pi < \varphi \leq \pi,\ \alpha \geq 1,\ x \in \left(\frac{1}{2},\ 1\right],\ \kappa,\ w \in \Lambda,\ q = b^{-1}$ and $i,\ i+\vartheta,\ i+\varsigma$ are nonnegative integers.

Remark 1 Many subclasses can be found by taking special values for the parameters $\mu,\ \alpha$ and φ in Definition 2.

Example 1 If the following subordinations are met for a function $b \in \Lambda$ given by (1), then $b \in \mathcal{F}_{\Pi}^{\mu}(\alpha,\ \varphi)$:

$$\left(b'(\kappa)\right)^{\alpha}+\frac{1+e^{i\varphi}}{2}\left(\kappa b''(\kappa)\right)\prec J_{i}(x,\ \kappa)$$

and

$$\left(q'(w)\right)^{\alpha}+\frac{1+e^{i\varphi}}{2}\left(wg''(w)\right)\prec J_{i}(x,\ \varpi)$$

where $\vartheta > -1,\ \varsigma > -1,\ \kappa,\ w \in \Lambda$ and $q = b^{-1}$.

Example 2 If the following subordinations are met for a function $b \in \Lambda$ given by (1), then $b \in \mathcal{F}_{\Pi}^{\mu}(\alpha,\ \varphi)$:

$$\left(\frac{b(\kappa)}{\kappa}\right)^{\alpha}+\frac{1+e^{i\varphi}}{2}\left(\frac{\kappa b''(\kappa)}{b'(\kappa)}\right)\prec J_{i}(x,\ \kappa)$$

and

$$\left(\frac{g(w)}{w}\right)^{\alpha}+\frac{1+e^{i\varphi}}{2}\left(\frac{wg''(w)}{g(w)}\right)\prec J_{i}(x,\ \varpi)$$

where $\vartheta > -1,\ \varsigma > -1,\ \kappa,\ w \in \Lambda$ and $q = b^{-1}$.

Example 3 If the following subordinations are met for a function $b \in \Lambda$ given by (1), then $b \in \mathcal{F}_{\Pi}^{1}(1,\ 0)$:

$$\frac{b(\kappa)}{\kappa}+\frac{\kappa b''(\kappa)}{b'(\kappa)}\prec J_{i}(x,\ \kappa)$$

and

$$\frac{g(w)}{w}+\frac{wg''(w)}{g(w)}\prec J_i(x,\ \varpi)$$

where $\vartheta>-1,\ \varsigma>-1,\ \kappa,\ w\in\Lambda$ and $q=b^{-1}$.

Example 4 If the following subordinations are met for a function $b\in\Lambda$ given by (1), then $b\in\mathcal{F}_{\Pi}^{1}(1,\ 0)$:

$$b'(\kappa)+\kappa b''(\kappa)<J_i(x,\ \kappa)$$

and

$$q'(w)+wg''(w)\prec J_i(x,\ \varpi)$$

where $\vartheta>-1,\ \varsigma>-1,\ \kappa,\ w\in\Lambda$ and $q=b^{-1}$.

Lemma 1 [24] If $d\in\mathcal{D}$, then $|m_n|\le 2$ for each n, where $\mathcal{D}$ is the family of all analytic functions in Λ for which

$$\mathrm{Re}(d(\kappa))>0,\ d(\kappa)=1+m_1\kappa+m_2^2\kappa+\cdots(\kappa\in\Lambda)$$

3. Bounds of the class $\mathcal{F}_{\Pi}^{\mu}(\alpha,\ \varphi)$

For a function $b\in\Lambda$, we give the coefficient estimates and solve Fekete-Szegö problem (see [25]) for the class $\mathcal{F}_{\Pi}^{\mu}(\alpha,\ \varphi)$, respectively.

Theorem 1 Let $b\in\Pi$ given by (1) belongs to the class $\mathcal{F}_{\Pi}^{\mu}(\alpha,\ \varphi)$ where $0\le\mu\le 1,\ \vartheta>-1,\ \varsigma>-1,\ -\pi<\varphi\le\pi$, $\alpha\ge 1,\ \kappa,\ w\in\Lambda$ and $q=b^{-1}$. Then

$$|c_2|\le\sqrt{\Upsilon(\mu,\ \alpha)}$$

$$|c_3|\le\frac{\left|2\alpha(\alpha(2-\mu)+1)+\mu\alpha(3-\alpha)+2\left(e^{i\varphi}+1\right)(3-2\mu)\right|\left((\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right)^2}{2\left|3e^{i\varphi}+\alpha(3-2\mu)+3\right|\left(e^{i\varphi}+\alpha(2-\mu)+1\right)^2}$$

$$+\left|\frac{\left((\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right)}{\left(3e^{i\varphi}+\alpha(3-2\mu)+3\right)}\right|$$

and

$$\left|c_3 - \varkappa c_2^2\right| \le \begin{cases} 0 & \le \quad \mathcal{F}^{\varphi}(\alpha,\ \mu) \\ \dfrac{2(\vartheta+1)+(\vartheta+\varsigma+2)(x-1)}{\left(3e^{i\varphi}+\alpha(3-2\mu)+3\right)} & < \quad \dfrac{(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)}{\left(3e^{i\varphi}+\alpha(3-2\mu)+3\right)}, \\ 2\mathcal{F}^{\varphi}(\alpha,\ \mu)\ \mathcal{F}^{\varphi}(\alpha,\ \mu) & \ge \quad \dfrac{(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)}{\left(3e^{i\varphi}+\alpha(3-2\mu)+3\right)} \end{cases}$$

Where

$$\Upsilon(\mu,\ \alpha)\frac{\left((\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right)^3}{\left(\begin{array}{c} 3\left(e^{i\varphi}+1\right)+\alpha(3-2\mu)\left[(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right]^2-\left(e^{i\varphi}+\alpha(2-\mu)+1\right)^2 \\ \left(\frac{(\vartheta+1)(\vartheta+2)}{2}+\frac{1}{2}(\vartheta+2)(\vartheta+\varsigma+3)(x-1)+\frac{1}{8}(\vartheta+\varsigma+3)(\vartheta+\varsigma+4)(x-1)^2\right) \end{array}\right)}$$

and

$$\mathcal{F}^{\varphi}(\alpha,\ \mu)=\left|\frac{\left[2\alpha(\alpha(2-\mu)+1)+\mu\alpha(3-\alpha)+2\left(e^{i\varphi}+1\right)(3-2\mu)\right]}{2\left(3e^{i\varphi}+\alpha(3-2\mu)+3\right)}-\chi\right|\Upsilon(\mu,\ \alpha)$$

Proof. Since $b(\kappa)=\kappa+\sum_{i=2}^{\infty}c_i\kappa^i \in \mathcal{F}_{\Pi}^{\mu}(\alpha,\ \varphi)$, so from Definition 1 we can write

$$\mu\left[\left(\frac{b(\kappa)}{\kappa}\right)^{\alpha}+\frac{1+e^{i\varphi}}{2}\left(\frac{\kappa b''(\kappa)}{b'(\kappa)}\right)\right]+(1-\mu)\left[\left(b'(\kappa)\right)^{\alpha}+\frac{1+e^{i\varphi}}{2}\left(\kappa b''(\kappa)\right)\right]\prec J_i(x,\ \kappa) \tag{6}$$

and

$$\mu\left[\left(\frac{g(w)}{w}\right)^{\alpha}+\frac{1+e^{i\varphi}}{2}\left(\frac{wg''(w)}{g(w)}\right)\right]+(1-\mu)\left[\left(q'(w)\right)^{\alpha}+\frac{1+e^{i\varphi}}{2}\left(wg''(w)\right)\right]\prec J_i(x,\ \varpi) \tag{7}$$

We can consider two functions $r,\ s:\Lambda\to\Lambda$, with $r(0)=s(0)=0$ and $|r(\kappa)|<1,\ |s(w)|<1$ for all $\kappa,\ w\in\Lambda$. So we can define $b,\ d\in\mathcal{D}$ as following:

$$J_n(x,\ r(\kappa))=1+P_1^{(\vartheta,\ \varsigma)}(x)b_1\kappa+\left[P_1^{(\vartheta,\ \varsigma)}(x)b_2+P_2^{(\vartheta,\ \varsigma)}(x)b_1^2\right]\kappa^2+\cdots \tag{8}$$

and

$$J_n(x,\ s(w))=1+P_1^{(\vartheta,\ \varsigma)}(x)d_1\varpi+\left[P_1^{(\vartheta,\ \varsigma)}(x)d_2+P_2^{(\vartheta,\ \varsigma)}(x)d_1^2\right]\varpi^2+\cdots \tag{9}$$

then

$$\left|b_j\right| \leq 1 \text{ and } \left|d_j\right| \leq 1 \text{ for all } j \in \mathbb{N} \tag{10}$$

From (6), (7) and the previous two equations, we have

$$\left(e^{i\varphi} + \alpha(2-\mu) + 1\right)c_2 = P_1^{(\vartheta,\, \varsigma)}(x)b_1 \tag{11}$$

$$\left(3e^{i\varphi} + \alpha(3-2\mu) + 3\right)c_3 - \left[2\alpha(\alpha-1) - \mu\left(\frac{3\alpha(\alpha-1)}{2} + 2\left(e^{i\varphi}+1\right)\right)\right]c_2^2 = P_1^{(\vartheta,\, S)}(x)b_2 + P_2^{(\vartheta,\, \zeta)}(x)b_1^2 \tag{12}$$

$$-\left(e^{i\varphi} + \alpha(2-\mu) + 1\right)c_2 = P_1^{(\vartheta,\, \varsigma)}(x)d_1 \tag{13}$$

and

$$\left[2\left(3\left(e^{i\varphi}+1\right) + \alpha(\alpha+2)\right) - \mu\left(2\left(e^{i\varphi}+1\right) - \frac{\alpha(\alpha+3)}{2} + 2\alpha(\alpha+2)\right)\right]c_2^2$$

$$-\left[3e^{i\varphi} + \alpha(3-2\mu) + 3\right]c_3 = P_1^{(\vartheta,\, \varsigma)}(x)d_2 + P_2^{(\vartheta,\, \varsigma)}(x)d_1^2 \tag{14}$$

Adding equations (11) and (13) and some simplification, we get

$$b_1 = -d_1 \text{ and } b_1^2 = d_1^2 \tag{15}$$

and

$$2\left(e^{i\varphi} + \alpha(2-\mu) + 1\right)^2 c_2^2 = \left[P_1^{(\vartheta,\, \varsigma)}(x)\right]^2 \left(b_1^2 + d_1^2\right) \tag{16}$$

$$\Rightarrow c_2^2 = \frac{\left[P_1^{(\vartheta,\, \varsigma)}(x)\right]^2 \left(b_1^2 + d_1^2\right)}{2\left(e^{i\varphi} + \alpha(2-\mu) + 1\right)^2} \tag{17}$$

Adding (12) to (14) gives

$$2\left(3\left(e^{i\varphi}+1\right) + \alpha(3-2\mu)\right)c_2^2 = P_1^{(\vartheta,\, \varsigma)}(x)\left(b_2 + d_2\right) + P_2^{(\vartheta,\, \varsigma)}(x)\left(b_1^2 + d_1^2\right)$$

By (15), we have

$$2\left(3\left(e^{i\varphi}+1\right) + \alpha(3-2\mu)\right)c_2^2 = P_1^{(\vartheta,\, \varsigma)}(x)\left(b_2 + d_2\right) + 2b_1^2 P_2^{(\vartheta,\, \varsigma)}(x) \tag{18}$$

Also, appling (15) in (16)

$$b_1^2 = \frac{\left(e^{i\varphi} + \alpha(2-\mu) + 1\right)^2 c_2^2}{\left[P_1^{(\vartheta,\, \varsigma)}(x)\right]^2} \tag{19}$$

Replacing b_1^2 in (18)

$$c_2^2 = \frac{\left[P_1^{(\vartheta,\,\varsigma)}(x)\right]^3\left(b_2+d_2\right)}{2\left(3\left(e^{i\varphi}+1\right)+\alpha(3-2\mu)\right)\left[P_1^{(\vartheta,\,\varsigma)}(x)\right]^2-2\left(e^{i\varphi}+\alpha(2-\mu)+1\right)^2 P_2^{(\vartheta,\,\varsigma)}(x)}$$

$$\Rightarrow |c_2|^2 = \frac{\left[P_1^{(\vartheta,\,\varsigma)}(x)\right]^3\left(|b_2|+|d_2|\right)}{\left|2\left(3\left(e^{i\varphi}+1\right)+\alpha(3-2\mu)\right)\left[P_1^{(\vartheta,\,\varsigma)}(x)\right]^2-2\left(e^{i\varphi}+\alpha(2-\mu)+1\right)^2 P_2^{(\vartheta,\,\varsigma)}(x)\right|} \tag{20}$$

Applying Lemma 3 and (20), we have:

$$= \sqrt{\Upsilon(\mu,\ \alpha)}$$

Subtracting (14) from (12), then view (15) and with some computations, we obtain

$$2\left(3e^{i\varphi}+\alpha(3-2\mu)+3\right)c_3-\left[2\alpha(\alpha(2-\mu)+1)+\mu\alpha(3-\alpha)+2\left(e^{i\varphi}+1\right)(3-2\mu)\right]c_2^2 = P_1^{(\vartheta,\,\varsigma)}(x)\left(b_2-d_2\right)$$

By (17) we obtain

$$c_3 = \frac{\left[2\alpha(\alpha(2-\mu)+1)+\mu\alpha(3-\alpha)+2\left(e^{i\varphi}+1\right)(3-2\mu)\right]\left[P_1^{(\vartheta,\,\varsigma)}(x)\right]^2\left(b_1^2+d_1^2\right)}{4\left(3e^{i\varphi}+\alpha(3-2\mu)+3\right)\left(e^{i\varphi}+\alpha(2-\mu)+1\right)^2}$$

$$+\frac{P_1^{(\vartheta,\,\varsigma)}(x)\left(b_2-d_2\right)}{2\left(3e^{i\varphi}+\alpha(3-2\mu)+3\right)} \tag{21}$$

By (21) and (15)

$$c_3 = \frac{\left[2\alpha(\alpha(2-\mu)+1)+\mu\alpha(3-\alpha)+2\left(e^{i\varphi}+1\right)(3-2\mu)\right]\left((\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right)^2}{2\left(3e^{i\varphi}+\alpha(3-2\mu)+3\right)\left(e^{i\varphi}+\alpha(2-\mu)+1\right)^2}$$

$$+\frac{\left((\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right)}{\left(3e^{i\varphi}+\alpha(3-2\mu)+3\right)} \tag{22}$$

Applying Lemma 1 and (15), we have:

$$\left|c_3\right| \leq \frac{\left|2\alpha(\alpha(2-\mu)+1)+\mu\alpha(3-\alpha)+2\left(e^{i\varphi}+1\right)(3-2\mu)\right|\left((\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right)^2}{2\left|3e^{i\varphi}+\alpha(3-2\mu)+3\right|\left(e^{i\varphi}+\alpha(2-\mu)+1\right)^2}$$

$$+\left|\frac{(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)}{\left(3e^{i\varphi}+\alpha(3-2\mu)+3\right)}\right|$$

From (21), we obtain

$$c_3-xc_2^2=\frac{P_1^{(\vartheta,\,\varsigma)}(x)\left(b_2-d_2\right)}{2\left(3e^{i\varphi}+\alpha(3-2\mu)+3\right)}+\left[\frac{\left[2\alpha(\alpha(2-\mu)+1)+\mu\alpha(3-\alpha)+2\left(e^{i\varphi}+1\right)(3-2\mu)\right]}{2\left(3e^{i\varphi}+\alpha(3-2\mu)+3\right)-\varkappa}\right]c_2^2$$

Applying the triangular inequality with assist (15), we obtain:

$$\left|c_3-\varkappa c_2^2\right| \leq \frac{(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)}{\left(3e^{i\varphi}+\alpha(3-2\mu)+3\right)}+\mathcal{F}^{\varphi}(\alpha,\ \mu)$$

If

$$\mathcal{F}^{\varphi}(\alpha,\ \mu) \leq \frac{(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)}{\left(3e^{i\varphi}+\alpha(3-2\mu)+3\right)}$$

we obtain

$$\left|c_3-\varkappa c_2^2\right| \leq \frac{2(\vartheta+1)+(\vartheta+\varsigma+2)(x-1)}{\left(3e^{i\varphi}+\alpha(3-2\mu)+3\right)}$$

and if:

$$\mathcal{F}^{\varphi}(\alpha,\ \mu) \geq \frac{(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)}{\left(3e^{i\varphi}+\alpha(3-2\mu)+3\right)}$$

we obtain

$$\left|c_3-\varkappa c_2^2\right| \leq 2\mathcal{F}^{\varphi}(\alpha,\ \mu)$$

Which are asserted by the Theorem 1.

4. Some corollaries

Each new corollary and implication presented here is based on the key findings from this section.

If we set $\mu = 1$ in Theorems 3 we get the next corollary.

Corollary 1 Let $b \in \Pi$ given by (1) belongs to the class $b \in \mathcal{F}_{\Pi}^{1}(\alpha, \varphi)$ where $-\pi < \varphi \leq \pi$, $\alpha \geq 1$, κ, $w \in \Lambda$ and $q = b^{-1}$. Then

$$|c_2| \leq \sqrt{\Upsilon(0, \alpha)}$$

$$|c_3| \leq \frac{\left|2\alpha(2\alpha+1)+6\left(e^{i\varphi}+1\right)\right|\left((\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right)^2}{2\left|3e^{i\varphi}+3\alpha+3\right|\left(e^{i\varphi}+2\alpha+1\right)^2}+\left|\frac{\left((\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right)}{\left(3e^{i\varphi}+3\alpha+3\right)}\right|$$

and

$$\left|c_3-\varkappa c_2^2\right| \leq \begin{cases} 0 \leq \mathcal{F}\varphi(\alpha, 1) \\ \dfrac{2(\vartheta+1)+(\vartheta+\varsigma+2)(x-1)}{\left(3e^{i\varphi}+\alpha+3\right)2\mathcal{F}^{\varphi}(\alpha, 1)} < \dfrac{(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)}{\left(3e^{\varphi\varphi}+\alpha+3\right)\mathcal{F}^{\varphi}(\alpha, 1)} \geq \dfrac{(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)}{\left(3e^{i\varphi}+\alpha+3\right)} \end{cases}$$

where

$$\Upsilon(1, \alpha)=\frac{\left((\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right)^3}{\left|\begin{array}{l}\left(3\left(e^{i\varphi}+1\right)+\alpha\right)\left[(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right]^2-\left(e^{i\varphi}+\alpha+1\right)^2\frac{(\vartheta+1)(\vartheta+2)}{2} \\ +\frac{1}{2}(\vartheta+2)(\vartheta+\varsigma+3)(x-1)+\frac{1}{8}(\vartheta+\varsigma+3)(\vartheta+\varsigma+4)(x-1)^2\end{array}\right|}$$

and

$$\mathcal{F}^{\varphi}(\alpha, 1)=\left|\frac{\left[2\alpha(\alpha+1)+\alpha(3-\alpha)+2\left(e^{i\varphi}+1\right)\right]}{2\left(3e^{i\varphi}+\alpha+3\right)}-\chi\right|\Upsilon(1, \alpha)$$

If we set $\mu = 0$ in Theorems 1 we get the next corollary.

Corollary 2 Let $b \in \Pi$ given by (2) belongs to the class $\mathcal{F}_{\Pi}^{0}(\alpha, \varphi)$ where $-\pi < \varphi \leq \pi$, $\alpha \geq 1$, κ, $w \in \Lambda$ and $q = b^{-1}$. Then

$$|c_2| \leq \sqrt{\Upsilon(0, \alpha)}$$

$$|c_3| \leq \frac{\left|2\alpha(2\alpha+1)+6\left(e^{i\varphi}+1\right)\right|\left((\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right)^2}{2\left|3e^{i\varphi}+3\alpha+3\right|\left(e^{i\varphi}+2\alpha+1\right)^2}+\left|\frac{\left((\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right)}{\left(3e^{i\varphi}+3\alpha+3\right)}\right|$$

and

$$\left|c_3-\varkappa c_2^2\right| \le \begin{cases} 0 \le \mathcal{F}^{\varphi}(\alpha,\ 0) \\ \dfrac{2(\vartheta+1)+(\vartheta+\varsigma+2)(x-1)}{\left(3e^{i\varphi}+3\alpha+3\right)2\mathcal{F}^{\varphi}(\alpha,\ 0)} < \dfrac{(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)}{\left(3e^{i\varphi}+3\alpha+3\right)\mathcal{F}^{\varphi}(\alpha,\ 0)} \ge \dfrac{(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)}{\left(3e^{i\varphi}+3\alpha+3\right)} \end{cases}$$

where

$$\Upsilon(0,\ \alpha)=\frac{\left((\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right)^3}{\left|\begin{array}{c}\left(3\left(e^{i\varphi}+1\right)+3\alpha\right)\left[(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right]^2-\left(e^{i\varphi}+2\alpha+1\right)^2 \\ \left(\frac{(\vartheta+1)(\vartheta+2)}{2}+\frac{1}{2}(\vartheta+2)(\vartheta+\varsigma+3)(x-1)+\frac{1}{8}(\vartheta+\varsigma+3)(\vartheta+\varsigma+4)(x-1)^2\right)\end{array}\right|}$$

and

$$\mathcal{F}^{\varphi}(\alpha,\ 0)=\left|\frac{\left[2\alpha(\alpha(2-\mu)+1)+\mu\alpha(3-\alpha)+6\left(e^{i\varphi}+1\right)\right]}{2\left(3e^{i\varphi}+3\alpha+3\right)}-\varkappa\right|\Upsilon(0,\ \alpha)$$

Corollary 3 Let $b\in\Pi$ given by (1) belongs to the class $b\in\mathcal{F}_{\Pi}^{1}(1,\ \varphi)$ where $\kappa,\ w\in\Lambda$ and $q=b^{-1}$. Then

$$|c_2|\le\sqrt{\Upsilon(1,\ 1)}$$

$$|c_3|\le\frac{\left|6+2\left(e^{i\varphi}+1\right)\right|\left((\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right)^2}{2\left|3e^{i\varphi}+1+3\right|\left(e^{i\varphi}+2\right)^2}+\left|\frac{\left((\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right)}{\left(3e^{i\varphi}+4\right)}\right|$$

and

$$\left|c_3-\varkappa c_2^2\right| \le \begin{cases} 0 \le \mathcal{F}^{\varphi}(1,\ 1) \\ \dfrac{(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)}{\left(3e^{i\varphi}+4\right)2\mathcal{F}^{\varphi}(1,\ 1)} < \dfrac{2(\vartheta+1)+(\vartheta+\varsigma+2)(x-1)}{\left(3e^{i\varphi}+4\right)\mathcal{F}^{\varphi}(1,\ 1)} \ge \dfrac{(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)}{\left(3e^{i\varphi}+4\right)} \end{cases}$$

where

$$\Upsilon(1,\ 1)=\frac{\left((\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right)^{3}}{\left|\begin{array}{l}\left(3\left(e^{i\varphi}+1\right)+1\right)\left[(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right]^{2}-\left(e^{i\varphi}+2\right)^{2}\\ \left(\frac{(\vartheta+1)(\vartheta+2)}{2}+\frac{1}{2}(\vartheta+2)(\vartheta+\varsigma+3)(x-1)+\frac{1}{8}(\vartheta+\varsigma+3)(\vartheta+\varsigma+4)(x-1)^{2}\right)\end{array}\right|}$$

and

$$\mathcal{F}^{\varphi}(1,\ 1)=\left|\frac{\left[6+2\left(e^{i\varphi}+1\right)\right]}{2\left(3e^{i\varphi}+4\right)}-\varkappa\right|\Upsilon(1,\ 1)$$

Corollary 4 Let $b\in\Pi$ given by (1) belongs to the class $\mathcal{F}_{\Pi}^{0}(1,\ \varphi)$ where $\kappa,\ w\in\Lambda$ and $q=b^{-1}$. Then

$$|c_2|\leq\sqrt{\Upsilon(0,\ 1)}$$

$$|c_3|\leq\frac{\left|6+6\left(e^{i\varphi}+1\right)\right|\left((\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right)^{2}}{2\left|3e^{i\varphi}+6\right|\left(e^{i\varphi}+3\right)^{2}}+\left|\frac{\left((\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right)}{\left(3e^{i\varphi}+6\right)}\right|$$

and

$$\left|c_3-\varkappa c_2^{2}\right|\leq\begin{cases}0\leq\mathcal{F}^{\varphi}(\alpha,\ 0)\\ \frac{2(\vartheta+1)+(\vartheta+\varsigma+2)(x-1)}{\left(3e^{i\varphi}+6\right)2\mathcal{F}^{\varphi}(1,\ 0)}<\frac{(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)}{\left(3e^{\varphi}+3\alpha+3\right)\mathcal{F}^{\varphi}(1,\ 0)}\geq\frac{(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)}{\left(3e^{i\varphi}+6\right)}\end{cases}$$

where

$$\Upsilon(0,\ 1)=\frac{\left((\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right)^{3}}{\left|\begin{array}{l}\left(3\left(e^{i\varphi}+1\right)+3\right)\left[(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right]^{2}-\left(e^{i\varphi}+3\right)^{2}\\ \left(\frac{(\vartheta+1)(\vartheta+2)}{2}+\frac{1}{2}(\vartheta+2)(\vartheta+\varsigma+3)(x-1)+\frac{1}{8}(\vartheta+\varsigma+3)(\vartheta+\varsigma+4)(x-1)^{2}\right)\end{array}\right|}$$

and

$$\mathcal{F}^{\varphi}(1,\ 1)=\left|\frac{\left[8+2\left(e^{i\varphi}+1\right)\right]}{2\left(3e^{i\varphi}+4\right)}-\varkappa\right|\Upsilon(1,\ 1)$$

5. Conclusions

Recently, there has been a surge of interest among prominent mathematicians in studying polynomials and special functions due to their applications in various mathematical and scientific fields. The objective of this paper is to introduce new subclasses of analytical and univalent functions, utilizing Jacobi polynomials. For functions belonging to these classes $\mathcal{F}_{\Pi}^{\mu}(\alpha, \varphi, \ell)$, $\mathcal{F}_{\Pi}^{1}(\alpha, \varphi, \ell)$ and $\mathcal{F}_{\Pi}^{0}(\alpha, \varphi, \ell)$, we have established an upper bound estimate for the coefficients and successfully solved the Fekete-Szeg problem. The sharp upper bounds for $|c_2|$, $|c_3|$ and $|c_3 - xc_2^2|$ are still an interesting challenge to discover, as well as the open problem regarding $|c_i|$, $i \geq 3$. This investigation can utilize bi-univalent functions that employ the modified Caputo's derivative operator. In the future, it may be worthwhile to explore Hankel determinants for this distribution. The Caputo derivative operator is anticipated to be significant in various fields of mathematics, science, and technology.

Conflict of interest

The authors declare no competing financial interest.

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