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Research Article

Evaluating Stress-Strength Reliability Estimation Technique: A Study on Ranked Set Sampling for the Beta-Lomax Distribution

Mujtaba Zuhair Ali Abuhedary[1,2], Hossein Jabbari Khamnei[1*], Ali Akbar Heydari[1]

[1] Department of Statistics, Faculty of Mathematics, Statistics and Computer Science, University of Tabriz, Tabriz, Iran
[2] Department of Information Technology, Management Technical College, Al-Furat Al-Awsat Technical University, Kufa, Iraq
E-mail: h_jabbari@tabrizu.ac.ir

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Abstract: Reliability, a crucial aspect in engineering and project management, signifies the likelihood of a system or project enduring without malfunction over a specified duration. Typically, the random variable X embodies the lifespan of the system or project. Stress-strength reliability, on the other hand, gauges the assurance that a product or process remains unaffected by stress Y. Extensive literature explores the point estimation and testing of $R(t) = P(X > t)$ (the survival function of the random variable), and $P = P(Y < X)$, delving into methods to enhance reliability assessment. This paper delves into the estimation of $R(t)$ and stress-strength reliability P through ranked set sampling (RSS), assuming independence between stress Y and strength X, both following the Beta-Lomax (BL) distribution. Through rigorous analysis, the maximum likelihood (ML) estimator for $R(t)$ and P is derived, subsequently juxtaposed with its simple random sampling (SRS) equivalent to gauge performance. By applying this methodology to real data from Wheaton River, the study underscores the practicality and efficacy of the proposed approach. By offering a comprehensive analysis of reliability and stress-strength reliability estimation utilizing RSS and the BL distribution, this research furnishes valuable insights for practitioners and researchers in the field. The integration of innovative sampling techniques and statistical methodologies not only enhances the precision of estimations but also underscores the importance of robust reliability assessments in ensuring the longevity and efficiency of systems and projects.

Keywords: Beta-Lomax distribution, maximum likelihood estimator, ranked set sampling, stress-strength reliability

MSC: 65L05, 34K06, 34K28

1. Introduction

The estimation of stress-strength reliability is crucial in engineering, manufacturing, and quality control. One approach to estimating it is through *RSS*, which provides more efficient and precise estimates compared to traditional sampling methods. Researchers have explored the application of *RSS* in stress-strength reliability estimation, aiming to enhance precision and efficiency. Incorporating the *BL* distribution, which combines the flexibility of the Beta distribution and the heavy-tailed characteristics of the Pareto distribution, has shown promise in improving estimation accuracy. In this study, we investigate stress-strength reliability estimation using *RSS* in the *BL* distribution, exploring its theoretical foundations, advantages, and integration into the estimation process. Real-world data analysis demonstrates

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the effectiveness of this approach. In this study, we aim to investigate the estimation of stress-strength reliability using *RSS* in the *BL* distribution. We will explore the theoretical foundations of *RSS*, discuss its advantages over traditional sampling methods, and examine how it can be integrated into the estimation process. Additionally, will analyze real-world data to demonstrate the effectiveness of the proposed approach in estimating stress-strength reliability parameters.

1.1 *The problem of stress and strength*

In the field of statistical quality control, there is a problem known as the stress-strength problem. It involves determining the probability that the strength of a given material exceeds the applied stress on that material. To express this in terms of probability, we need to evaluate $P = P(Y < X)$ where Y and X are random variables representing strength and stress, respectively [1]. The *BL* distribution, also known as the Lomax distribution or the Pareto Type II distribution, is a heavy-tailed distribution commonly used in reliability and survival analysis. It has applications in various domains, particularly when modeling the behavior of stress and strength systems. In stress-strength estimation, it provides several advantages:

1. Flexibility in Modeling

The Lomax beta distribution is flexible in modeling different types of data. It can handle a variety of shapes, including symmetrical and skewed distributions. This flexibility makes it suitable for stress-strength models, where the relationship between stress (applied force) and strength (material durability) may not always be linear or symmetric.

2. Capturing Heavy Tails

Since the Lomax distribution has heavy tails, it is ideal for modeling extreme values or outliers. In many real-world scenarios, particularly in engineering and materials science, extreme events (like sudden stress failures) are significant. The Lomax beta distribution accounts for these cases effectively, providing a more realistic representation of stress and strength characteristics.

3. Parameterization

The Lomax beta distribution can be parameterized easily, typically requiring only two parameters. This simplicity makes it more manageable in practice, allowing researchers and engineers to obtain estimates from limited data effectively.

4. Stress-Strength Model Relevance

In reliability engineering, the Lomax beta distribution can model the distribution of applied stress and the distribution of material strength. The probability that a component will fail can be evaluated as $P(Y < X)$, where Y is the stress and X is the strength. The site for potential failure is where the stress exceeds the strength, which aligns well with the Lomax structure.

5. Real-World Applications

The Lomax beta distribution is useful in diverse fields such as engineering (materials science), finance (modeling income distributions), and epidemiology (modeling time until failure or death), allowing for cross-disciplinary applications in stress-strength estimation.

This problem can be divided into two parts: (i) under certain assumptions about the distribution of the variables, finding an expression for P, and (ii) using a set of data that satisfies these assumptions to estimate P through point estimation and interval estimation [2].

Early research in this area focused on assuming a normal distribution for Y and X. [3] and [4] provided fundamental results in this regard. Nevertheless, subsequent research has explored various other distributions and their implications, leading to a deeper comprehension of the stress-strength problem. [5] offer an introductory account on this topic, while [6] provide a more extensive review.

The aim of this note is to investigate the aforementioned problem when one or both of the underlying probability distributions follow a *BL* distribution. The focus will primarily be on the scenario where only either Y or X has a *BL* distribution. A brief overview of the *BL* distribution is provided below [7]. In summary, the motivation for employing the *BL* distribution in stress-strength estimation lies in its flexibility, capability to model extreme events, ease of parameterization, and its compatibility with modern statistical approaches. By effectively capturing the underlying dynamics of stress and strength distributions, it enhances the reliability and accuracy of assessments in various practical scenarios.

1.2 *Probability results*

The estimation of stress-strength reliability plays a crucial role in various fields, including engineering, manufacturing, and quality control. It involves assessing the probability that a system or component will withstand applied stress levels without failure, considering the strength of the material or structure [8]. Accurate estimation of stress-strength reliability is essential for ensuring the safety and performance of critical systems, optimizing design parameters, and making informed decisions regarding maintenance and operational strategies.

The estimation of stress-strength reliability, $P = P(Y < X)$, is a well-researched problem in the statistical, industrial, and mechanical engineering fields. Y represents stress and X represents strength, making P a measure of system reliability. If stress exceeds strength, the system fails; otherwise, it continues to function. Stress-strength reliability was first proposed by [8], and subsequent studies have explored estimating P under various distributions of Y and X, including Weibull [9], Frechet [10], and Pareto [11].

1.3 *Ranked set ampling (RSS)*

RSS stands as a highly advantageous statistical sampling technique that surpasses traditional random sampling methods, initially proposed by McIntyre in 1952 to enhance the efficiency of population parameter estimation [12]. Rather than directly measuring the values of interest, *RSS* involves ranking the sampled units based on the variable under study. This distinctive approach has gained significant attention and widespread application across diverse fields such as environmental studies, biological research, and quality control [13]. The *RSS* methods provide more flexibility in adjusting the sampling volume, especially when the population is dynamic or when there is a high variability in responses. Researchers can adapt the number of repetitions based on the observed data or variability rather than adhering to a fixed sample size upfront. Also when employing *RSS*, data collection can be more robust, as researchers can gather information across varying conditions or time frames. This approach enriches the dataset, which can be particularly valuable in studies requiring longitudinal data or repeated measures. If the correlation between repeated measurements is high, *RSS* sampling can make better use of available data, leading to increased statistical efficiency. This efficiency can enable smaller sample sizes while still achieving robust power in hypothesis testing. One of the key benefits of *RSS* is its ability to reduce sampling variability and increase estimation precision compared to sampling method *SRS*. By utilizing the relative ordering of observations, *RSS* captures information about the entire distribution of the variable, leading to more efficient estimators. This makes it particularly useful when the variable of interest exhibits high variability or extreme values [14]. To implement *RSS*, a set of ranked units is selected from the population, typically through a two-stage process. In the first stage, a preliminary random sample is drawn from the population. Then, in the second stage, the units in the sample are ranked based on the variable of interest. The final analysis is performed using the ranks rather than the actual measurements, which helps mitigate the impact of outliers and measurement errors [15]. Several studies have demonstrated the effectiveness of *RSS* improving estimation accuracy and reducing costs. For instance, [16] applied *RSS* to estimate the abundance of endangered species in a wildlife reserve, achieving more precise estimates compared to traditional sampling methods. In [17] Fisher's data for generalization Rayleigh distribution has been investigated in ranked set sampling. Additionally, [18] utilized *RSS* in a quality control study to assess the performance of manufacturing processes, resulting in improved process monitoring and defect detection.

RSS is a powerful statistical technique that leverages the relative ordering of observations enhance estimation precision and reduce sampling variability. Its applications span across diverse fields, offering valuable insights and cost-effective solutions. As demonstrated by the studies conducted by [19] and [20], *RSS* has proven be an effective tool for various research and practical applications. In [21] the author advances the field by using ranked set sampling techniques in the inverse Kumaraswamy distribution in multi-stress resistance reliability estimation. The *RSS* procedure involves two stages. In the first stage, a set of independent and identically distributed items are collected from the population and arranged based on a specific attribute. From this arranged subsample, only one observation is measured and recorded, along with its rank within the subsample. This process is repeated until the resulting sample consists of independent order statistics. If subsamples have a constant size, such as k, and each of the k order statistics are sampled in equal proportion across all subsamples, the ranked set sample is considered to be balanced. In the second stage, a portion of the ranked elements is selected for measurement while retaining their order. The *RSS* is a type of sampling method that leads to more efficient estimators of various population parameters than an sampling method *SRS* of the same size. Let $X_{SRS}=\{X_1,\ X_2,\ \ldots,\ X_n\}$ be an *SRS* of size n drawn from a continuous population with cumulative distribution function (CDF) and probability density function (PDF). To obtain an *RSS* of size n from the same population, we first draw a random sample of size n from the population and arrange them in order without measuring them. Then, we measure only the smallest observation, and the rest are left unmeasured. We repeat this process by drawing another sample of size n and measuring only the second smallest observation, continuing until we measure the largest observation in the n^{th} sample. This process is called a one-cycle *RSS* of size n, and the resulting data are denoted by $X_{RSS}=\{X_{(11)},\ X_{(22)},\ \ldots,\ X_{(nn)}\}$. See the figure below for a visualization of this procedure. Where $X_{(i)n}$, $(i=1,\ \ldots,\ n)$ denote the ith ordered statistics from the Random sample size n_x.

$$\begin{array}{cccccc} \underline{X_{1:1}} & X_{2:1} & \ldots & X_{n:1} & -> & X_{(11)} \\ X_{1:2} & \underline{X_{2:2}} & \ldots & X_{n:2} & -> & X_{(22)} \\ \ldots & \ldots & \ldots & \ldots & -> & \ldots \\ X_{1:n} & X_{2:n} & \ldots & \underline{X_{n:n}} & -> & X_{(nn)} \end{array}$$

Finally, repeat this process r times and obtain the ranked set sample of size $m=nr$. It should be stated that the set size m has a critical role in the *RSS* procedure. We would like to take m as large as possible to obtain more information regarding the variable of interest. However, obviously the possibility of doing error in ranking, called imperfect ranking, increases as m increases. Therefore, the optimal selection of m is very important to avoid the effects of imperfect ranking. To obtain the *RSS* of size $m=nr$, repeat this process r times. It is important to note that the selection of m plays a critical role in the *RSS* procedure. A larger m can provide more information about the variable of interest, but it also increases the risk of imperfect ranking and errors. Therefore, selecting an optimal value for m is crucial to avoid the effects of imperfect ranking.

1.4 *Stress and strength with RSS sampling method*

Stress-strength reliability is a fundamental concept in engineering and statistical analysis, aiming to assess the probability of failure or success in a system under stress. Accurate estimation of stress-strength reliability is crucial for ensuring the safety and performance of various applications, ranging from structural engineering to material science. Traditional methods for estimating stress-strength reliability often rely on sampling method *SRS* techniques, which may not fully capture the underlying variability and correlation structures present in the data [22]. To address these limitations, researchers have turned to alternative sampling strategies, such as *RSS*, which has gained significant attention in recent years. *RSS* is a unique sampling technique that incorporates ranking information into the sampling process, allowing for more efficient and precise estimation of population parameters [23]. By utilizing the order statistics of observations, *RSS* reduces the impact of outliers and improves the estimation accuracy compared to conventional sampling method *SRS* [24].

One specific area where the application of sampling method *RSS* has shown promise is in the estimation stress-strength reliability. The stress-strength model considers two independent random variables: stress (X) and strength (Y). The objective is to estimate the reliability $R=P(Y>X)$, which represents the probability that the strength exceeds the stress level [25]. Traditional approaches based on sampling method *SRS* often assume normality and independence between X and Y, which may not hold in practice. Traditional methods for stress-strength reliability estimation often rely on simple random sampling (*SRS*) and make assumptions about the independence of stress and strength variables. However, these approaches have several limitations: A) Dependency between stress and strength: In many real-world scenarios, stress and strength variables are not independent. This dependency can lead to biased estimates when using traditional methods that assume independence. B) Efficiency: Simple random sampling may not always provide the most efficient estimates, especially when dealing with complex distributions like the beta-lomax. C) Sample size requirements: Accurate estimation using traditional methods often requires larger sample sizes, which can be costly or impractical in some situations. To address these limitations, researchers have explored alternative sampling methods, such as ranked set sampling (*RSS*) and its variations. The motivation for using *RSS* in stress-strength reliability estimation includes: A) Improved efficiency: *RSS* has been shown to provide more efficient estimates compared to sampling method *SRS*, often requiring smaller sample sizes to achieve the same level of precision. B) Flexibility: *RSS* and its variations (e.g., median ranked set sampling, extreme ranked set sampling) can be adapted to different distribution types and estimation scenarios. C) Handling complex distributions: *RSS* methods have demonstrated effectiveness in estimating parameters and reliability for various distributions, including the exponentiated Pareto distribution. D) Practical applicability: In some situations, it may be easier or more cost-effective to rank a small set of units rather than obtain precise measurements for a large sample. Recent research has focused on applying *RSS* and its variations to stress-strength reliability estimation for various distributions. Al-Omari [26] investigated the use of *RSS* and median *RSS* for estimating stress-strength reliability in the exponentiated Pareto distribution. Their results the *MLE* estimation based on *RSS* samples were more efficient than those based on sampling method *SRS*. Furthermore, researchers have explored the use of copula functions to model the dependency between stress and strength variables, addressing one of the key limitations of traditional methods [27]. This approach, combined with advanced sampling techniques like *RSS*, offers promising avenues for improving the accuracy and efficiency of stress-strength reliability estimation. As a result, distributions such as beta-Lomax and beta-Pareto using *RSS* sampling methods show significant improvements in stress-strength reliability estimation. By addressing the limitations of traditional approaches and providing greater flexibility and efficiency, these techniques help improve the accuracy and applicability of reliability analysis in various engineering domains. Several studies have demonstrated the advantages of using *RSS* for stress-strength reliability estimation. For instance, [28] applied *RSS* to estimate the reliability of a bridge structure subjected to varying stress loads. Their results showed that *RSS* provided more accurate estimates compared to sampling method *SRS*, particularly when dealing with non-normal and dependent stress-strength distributions. Additionally, [29] conducted a simulation study comparing different sampling methods for stress-strength reliability estimation, including sampling method *RSS*. Their findings indicated that sampling method *RSS* consistently outperformed *SRS* in terms of bias reduction and confidence interval precision.

In light of these advancements, this study aims to further investigate the application of sampling method *RSS* in estimating stress-strength reliability. We will build upon the existing literature by considering more complex dependence structures, incorporating covariates, and exploring the robustness of *MLE* estimation based on *RSS* samples under various scenarios. The findings from this research will contribute to the growing body of knowledge on efficient and accurate estimation techniques for stress-strength reliability, with potential implications for engineering design, risk assessment, and decision-making process.

1.5 *The BL distribution*

Let $G(x)$ denote the cumulative distribution function CDF and probability density function PDF of a random variable X. The cumulative distribution function for a generalized class of distribution for the random variable X, as defined by [30], is generated by applying the inverse CDF to a beta distributed random variable to obtain

$$F(x) = \frac{1}{B(\alpha,\ \beta)} \int_0^{G(x)} t^{\alpha-1}(1-t)^{\beta-1}dt. \quad \alpha > 0,\ \ \beta > 0 \tag{1}$$

In the present study, we let $G(x)$ be the CDF of the Lomax random variable with parameters $(\lambda,\ \alpha)$ and density function $g(x) = \frac{\alpha}{\lambda}\left[1+\frac{x}{\lambda}\right]^{-(\alpha+1)}$ and CDF $G(x) = 1-\left[1+\frac{x}{\lambda}\right]^{-\alpha}$ for $x \geq 0$.

The corresponding probability density function for $F(x)$ is given by

$$f(x) = \frac{1}{B(\alpha,\ \beta)}[G(X)]^{\alpha-1}[1-G(X)]^{\beta-1}\acute{G}(X). \tag{2}$$

From Equations (1) and (2), the PDF and CDF of the beta-Lomax random variable is given by respectively,

$$f(x)=\frac{\alpha}{\lambda B(a,\, b)}\left[1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}\right]^{a-1}\left(1+\frac{x}{\lambda}\right)^{-(\alpha b+1)}. \tag{3}$$

and

$$F(x)=\frac{\alpha}{B(a,\, b)}\sum_{i=0}^{a-1}\binom{a-1}{i}(-1)^{i}\frac{1}{\alpha(i+b)}\left[1-\left(1+\frac{x}{\lambda}\right)^{-\alpha(i+b)}\right]. \tag{4}$$

The *BL* distribution is a compound distribution characterized by a beta distribution that governs the strength and a Lomax distribution that represents the applied stress. The beta distribution is defined on the interval $(0,\ 1)$, while the Lomax distribution is a heavy-tailed distribution often used in survival analysis. Also the *BL* distribution can exhibit a variety of shapes depending on the parameters of the beta and Lomax components, ranging from unimodal to bimodal distributions. The presence of the Lomax component can introduce heavy-tails in the stress distribution, meaning that there is a higher probability of extreme values compared to lighter-tailed distributions. The *BL* distribution provides a flexible and powerful framework for modeling the stress-strength relationship, especially in contexts where traditional simple models may fall short due to the nature of the data. It helps in understanding the likelihood of failure under varying stress scenarios, making it essential for effective risk management and reliability engineering.

2. Inferential aspects

In the subsequent sections, our focus will be on examining the inference for *MLE* based on assumptions (3) and (4). Specifically, we assume that there is a set of independent samples of size n, denoted as $X\sim BL(a,\ b,\ \lambda,\ \alpha)$, at our disposal. Then we estimate the *MLE* of the parameters using two sampling methods, *SRS* and *RSS*.

2.1 *MLE of the parameters with SRS*

Maximum Likelihood Estimation (MLE) is a method of estimating the parameters of a probability distribution by maximizing a likelihood function. This function expresses the likelihood of observing the given data as a function of the parameters. The core idea is to choose parameter values that make the observed data "most likely" to have occurred. As sample size increases, the estimate converges to the true parameter value and for large samples, the distribution of the *MLE* is approximately normal. In general, *MLE*s achieve the Kramer-Rao lower bound asymptotically, which makes them asymptotically efficient. *MLE* can be applied to a wide range of statistical models and distributions. It's particularly useful for complex distributions often encountered in reliability analysis, such as the inverted Kumaraswamy or beta-Pareto and beta-lomax distributions.

The log-likelihood function of $BL(a,\ b,\ \lambda,\ \alpha)$ distribution may be expressed as

$$\ln L(x;\ a,\ b,\ \lambda,\ \alpha)=n\left[\ln\frac{\alpha}{\lambda}+\ln\frac{\Gamma(a+b)}{(\Gamma(a)+\Gamma(b))}\right]+(a-1)\sum_{i=1}^{n}\ln\left[1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}\right]+(\alpha b+1)\sum_{i=1}^{n}\ln\left(1+\frac{x}{\lambda}\right). \tag{5}$$

Differentiating Equation (5) with respect to k, α and β, respectively, and setting the results equal to zero, we have

$$\frac{\partial \ln L(x)}{\partial \alpha}=\frac{n}{\alpha}+n\alpha(a-1)\sum_{i=1}^{n}\frac{\left(1+\frac{x_i}{\lambda}\right)^{-(\alpha+1)}}{\left[1-\left(1+\frac{x_i}{\lambda}\right)^{-\alpha}\right]}+b\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\lambda}\right)=0 \tag{6}$$

$$\frac{\partial \ln L(x)}{\partial \lambda}=\frac{n\alpha(a-1)}{\lambda^2}\sum_{i=1}^{n}\frac{\left(1+\frac{x_i}{\lambda}\right)^{-(\alpha+1)}}{\left[1-\left(1+\frac{x_i}{\lambda}\right)^{-\alpha}\right]}-\frac{n(\alpha b+1)}{\lambda^2}\sum_{i=1}^{n}\frac{1}{\left(1+\frac{x_i}{\lambda}\right)}-\frac{n}{\lambda}=0 \tag{7}$$

$$\frac{\partial \ln L(x)}{\partial a}=n\{\Psi(a+b)-\Psi(a)\}+\sum_{i=1}^{n}\ln\left[1+\left(1+\frac{x_i}{\lambda}\right)^{-\alpha}\right]=0 \tag{8}$$

$$\frac{\partial \ln L(x)}{\partial b}=n\{\Psi(a+b)-\Psi(b)\}+\alpha\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\lambda}\right)=0. \tag{9}$$

The *ML* estimates of the parameters α, λ, a and b (represented respectively by $\widehat{\alpha}$, $\widehat{\lambda}$, $\hat{a}$ and $\hat{b}$) can be obtained by solving Equations (6)-(9) alternatively. By applying the same procedure to a second sample Y_j, $(j=1, \ldots, n)$, analogous results to those derived for the first sample are obtained. Expectations can be obtained using a numerical method.

2.2 *MLE of the parameters with RSS*

Ranked set sampling *RSS* is an effective technique for acquiring data when measuring units in a population is costly, but ranking them according to the variable of interest is relatively easy. *MLE* estimation based on *RSS* samples, including *MLE*, have been shown to outperform their Simple Random Sampling (*SRS*) counterparts significantly, providing more accurate parameter estimates. Studies have demonstrated that *RSS*-based estimators are more efficient than *SRS*-based methods, especially when using the same number of measured units. *MLE* with sampling method *RSS* has been successfully applied to estimate parameters of various complex distributions used in reliability studies and life testing, such as the Inverted Kumaraswamy distribution [31]. The effectiveness of *RSS*-based *MLE* has been demonstrated through applications to real-world datasets, such as waiting times between consecutive eruptions of natural phenomena [31]. *RSS* can be used not only with *MLE* but also with other estimation techniques like maximum product of spacings, least squares, and various goodness-of-fit based methods, allowing for comprehensive comparisons. These motivations highlight that *MLE* with *RSS* offers a powerful and flexible approach to parameter estimation, particularly valuable in scenarios where data collection is challenging or expensive, and when dealing with complex or bounded distributions common in reliability and life testing applications. When applying *MLE* within the framework of *RSS*, here's how the two concepts interact:

1) *MLE* with Ranked Data: By using the ranked data from *RSS*, the likelihood function can be constructed to reflect the rankings. This allows for potentially more informative data to be used in the estimation process.

2) Asymptotic Properties: Similar to standard *MLE*, the estimates derived from *RSS* will converge to the true underlying parameter values as the sample size increases.

3) Efficiency Gains: The combination of ranked data and *MLE* often leads to estimators that not only approximate the Cramer-Rao lower bound more effectively but also maintain desirable asymptotic properties, making them robust for a variety of applications in reliability and other fields.

Let $X_{(i)is}$, $(i=1, \ldots, n_x, s=1, \ldots, r_x)$ denote the ith ordered statistics from the ith set of size n_x in the sth cycle of size r_x, where $m=n_x r_y$ sample of size for X and $Y_{(j)jl}$, $(j=1, \ldots n_y, l=1, \ldots, r_y)$ denote the jth ordered statistics from the jth set of size m_y in the lth cycle of size r_y, where $m=n_y r_y$ sample of size for Y. Here, X and Y have $BL(a, b, \alpha, \lambda)$ and $BL(c, d, \beta, \theta)$ densities, respectively. For the sake of simplicity, we use the notations X_{is} and Y_{jl} instead of the notations $X_{(i)is}$ and $Y_{(j)jl}$, respectively. It should be mentioned that if the judgment ranking is perfect, the pdf of the ith ordered statistics X_{is} is given below

$$f_i(x_{is}) = \frac{1}{B(i,\ n_x - i + 1)} [F(x_{is})]^{i-1} [1 - F(x_{is})]^{n_x - i} f(x_{is}), \tag{10}$$

In addition, the *PDF* of Y_{jl} has a form similar to (10). First, we need to calculate the relationship (10) for the *BL* distribution with $(a,\ b,\ \alpha,\ \lambda)$.

$$f_{i:n}(x) = \frac{1}{B(i,\ n-i+1)} [F(x)]^{i-1} [1 - F(x)]^{n-i} f(x)$$

$$= \frac{1}{B(i,\ n-i+1)} [F(x)]^{i-1} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} [F(x)]^j f(x)$$

$$= \frac{1}{B(i,\ n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} [F(x)]^{i+j-1} f(x).$$

Then we can write:

$$F_{i:n}(x) = \int_0^\infty f_{i:n}(x) dx$$

$$= \int_0^\infty \frac{1}{B(i,\ n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} [F(x)]^{i+j-1} f(x) dx$$

$$= \sum_{j=0}^{n-i} \frac{(-1)^j \binom{n-i}{j}}{B(i,\ n-i+1)} \int_0^\infty [F(x)]^{i+j-1} f(x) dx$$

$$= \sum_{j=0}^{n-i} \frac{(-1)^j \binom{n-i}{j}}{B(i,\ n-i+1)(i+j)} [F(x)]^{i+j}.$$

In this part, we use the relationships found in the [32] to calculate

$$(I)\ \ F(x) = \sum_{t=0}^{\infty} b_t G(x)^t \tag{11}$$

$$(II)\ \ F(x)^n = \sum_{t=0}^{\infty} d_{n,\ t} G(x)^t \tag{12}$$

where

$$b_t = \sum_{j=0}^{\infty}\sum_{l=t}^{\infty} p_j (-1)^{l+t} \binom{a+j}{l}\binom{l}{t} \quad \& \quad p_j = \frac{(-1)^j \Gamma(a+b)}{\Gamma(a)\Gamma(b-j)\Gamma(j+1)(a+j)}$$

and $d_{n,\,t};\ t = 1,\ 2,\ ...$ are easily determined from the recurrence equation

$$d_{n,\,t} = (tb_0)^{-1} \sum_{m=1}^{t} [m(n+1)-t] b_m d_{n,\,t-m},$$

and $d_{n,\,0} = b_0^n$. Hence, $d_{n,\,t}$ comes directly from $d_{n,\,0},\ \ldots,\ d_{n,\,t-1}$ and, therefore, from $b_0,\ \ldots,\ b_t$. With this description, the proposition (II) (12) is confirmed. And finally, according to the presented relations, we have

$$F_{i:n}(x) = \sum_{t=0}^{\infty}\sum_{j=0}^{n-i} \frac{(-1)^j \binom{n-i}{j} d_{i+j,\,t}}{B(i,\ n-i+1)(i+j)} G(x)^t \tag{13}$$

and

$$f_{i:n}(x) = \sum_{t=0}^{\infty}\sum_{j=0}^{n-i} \frac{t(-1)^j \binom{n-i}{j} d_{i+j,\,t}}{B(i,\ n-i+1)(i+j)} G(x)^{t-1} g(x). \tag{14}$$

To obtain the *ML* estimator of *R* we first derive the *ML* estimators of parameters. Therefore, the likelihood function based on *RSS* is written as shown below

$$\begin{aligned} L(x;\ a,\ b,\ \alpha,\ \lambda) &= \prod_{i=1}^{n_x}\prod_{s=1}^{r_x} f_i(x_{is}) \\ &= \prod_{i=1}^{n_x}\prod_{s=1}^{r_x}\sum_{l=0}^{\infty}\sum_{t=0}^{\infty}\sum_{j=0}^{m-i} \frac{\alpha t(-1)^{j+l}\binom{t-1}{l}\binom{m-i}{j} d_{i+j,\,t}}{\lambda B(i,\ m-i+1)(i+j)} \left[1+\frac{x_{is}}{\lambda}\right]^{-\alpha(l+1)} \\ &= \left(\sum_{l=0}^{\infty}\sum_{t=0}^{\infty}\sum_{j=0}^{m-i} \frac{\alpha t(-1)^{j+l}\binom{t-1}{l}\binom{m-i}{j} d_{i+j,\,t}}{\lambda B(i,\ m-i+1)(i+j)}\right)^{n_x r_x} \left(\prod_{i=1}^{n_x}\prod_{s=1}^{r_x}\sum_{l=0}^{\infty}\left[1+\frac{x_{is}}{\lambda}\right]^{-\alpha(l+1)}\right). \end{aligned} \tag{15}$$

Then, the log-likelihood function is

$$\ln L(x;\ a,\ b,\ \alpha,\ \lambda) = Q + n_x r_x(\ln\alpha) - n_x r_x(\ln\lambda) + r_x \ln\left(\sum_{i=1}^{n_x}\sum_{t=0}^{\infty}\sum_{j=0}^{m-i} d_{i+j,\,t}\right) + \sum_{i=1}^{n_x}\sum_{s=1}^{r_x} \ln\left(\sum_{l=0}^{\infty}\left[1+\frac{x_{is}}{\lambda}\right]^{-\alpha(l+1)}\right). \tag{16}$$

Differentiating Equation (16) with respect to a, b, α and λ, respectively, and setting the results equal to zero, we have

$$\frac{\partial \ln L(x)}{\partial \alpha} = \frac{n_x r_x}{\alpha} - \sum_{i=1}^{n_x} \sum_{s=1}^{r_x} \sum_{l=0}^{\infty} (l+1) \ln\left[1 + \frac{x_{is}}{\lambda}\right] = 0 \tag{17}$$

$$\frac{\partial \ln L(x)}{\partial \lambda} = \frac{n_x r_x \alpha}{\lambda^2} \sum_{i=1}^{n_x} \sum_{s=1}^{r_x} \sum_{l=0}^{\infty} \frac{(l+1)}{\left[1 + \frac{x_{is}}{\lambda}\right]} - \frac{n_x r_x}{\lambda} = 0 \tag{18}$$

$$\frac{\partial \ln L(x)}{\partial a} = r_x \sum_{i=1}^{n_x} \sum_{t=0}^{\infty} \sum_{j=0}^{m-i} \frac{\frac{\partial l}{\partial a} d_{i+j,\, t}}{d_{i+j,\, t}} = 0 \tag{19}$$

$$\frac{\partial \ln L(x)}{\partial b} = r_x \sum_{i=1}^{n_x} \sum_{t=0}^{\infty} \sum_{j=0}^{m-i} \frac{\frac{\partial l}{\partial b} d_{i+j,\, t}}{d_{i+j,\, t}} = 0 \tag{20}$$

The maximum likelihood estimates $\hat{\alpha}$, $\hat{\lambda}$, $\hat{a}$ and $\hat{b}$ for the parameters α, λ, a and b, respectively, are obtained by solving alternatively Equations (17)-(20). By applying the same procedure for the second sample $Y_{(j)jl}$, $(j = 1, \ldots, n_y,\ l = 1, \ldots, r_y)$, our results are similar to those of the higher process. The expectations are not in close form. we resort to iterative methods for the *ML* estimators.

2.3 *MLE of R(t) = P(X > t)*

Suppose X follows the BL distribution with parameters $a,\ b,\ \alpha,\ \lambda$. Using the Beta distribution function, it can be easily shown that,

$$R = P(X > t) = F_B(b,\ a;\ W).$$

Where $F_B(b,\ a;\ W) = I_W(b,\ a)$ is the distribution function of the beta distribution with $W = \left(1 + \frac{t}{\lambda}\right)^{-\alpha}$.
It can be easily seen from (3)

$$R = P(X > t) = \int_t^{\infty} f_X(x)dx = \int_t^{\infty} \frac{\alpha}{\lambda B(a,\ b)} \left[1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}\right]^{a-1} \left(1 + \frac{x}{\lambda}\right)^{-(\alpha b+1)} dx$$

$$= \int_0^W \frac{1}{B(a,\ b)} [1 - z]^{a-1} z^{b-1} dz = F_B(b,\ a;\ W). \tag{21}$$

By setting $z = \left(1 + \frac{x}{\lambda}\right)^{-\alpha}$ and $W = \left(1 + \frac{t}{\lambda}\right)^{-\alpha}$ the above integration becomes. Now to compute the *MLE* of $R(t)$, we use the estimates of parameters that were calculated in the previous section. Therefore, according to the reliability property of *MLE*, we can write,

$$\hat{R(t)} = R(\hat{\alpha},\ \hat{\beta},\ \hat{\theta},\ \hat{k}). \tag{22}$$

2.4 *MLE of P = P(Y < X)*

Suppose that $X \sim BL(a,\ b,\ \alpha,\ \lambda)$ and $Y \sim BP(c,\ d,\ \beta,\ \theta)$ are independent. That it can be shown that,

$$R = P(Y < X) = E[P(Y < X|X)] = E\left[\int_0^X f_Y(y)dy\right] = E[F_y(X)].$$

As the same method reported in [32] for BL distribution we have,

$$F_y(X) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} W_{kj}X^j - C_1. \tag{23}$$

Where

$$W_{kj} = \frac{\binom{\beta-1}{k}\binom{-\beta(k+d)}{j}(-1)^{k+1}}{(k+d)B(c,\ d)\theta^j}. \tag{24}$$

And

$$C_1 = \sum_{0}^{\infty}\frac{\binom{\beta-1}{k}(-1)^{k+1}}{(k+d)B(c,\ d)}.$$

Therefore we can write,

$$R = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} W_{kj}E[X^j] - C_1. \tag{25}$$

In Equation (25), $X \sim BL(a,\ b,\ \alpha,\ \lambda)$ with PDF (3). Then it can be easily for the $X \sim BL(a,\ b,\ \alpha,\ \lambda)$ that,

$$f_X(x) = \sum_{r=0}^{\infty}\sum_{p=0}^{\infty} Q_{rp}x^p$$

Where

$$Q_{rp} = \frac{\alpha\binom{\alpha-1}{r}\binom{-\alpha(r+b)-1}{p}(-1)^r}{\theta^{p+1}B(a,\ b)}.$$

Therefore we can write,

$$E(X^k) = \sum_{r=0}^{\infty}\sum_{p=0}^{\infty}\frac{Q_{rp}}{(p+k+1)}.$$

And finally, using the above relations, we have

$$R(a,\ b,\ \alpha,\ \lambda,\ c,\ d,\ \beta,\ \theta) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\sum_{p=0}^{\infty}\frac{W_{kj}Q_{rp}}{(p+k+1)} - C_1. \tag{26}$$

Now to compute the *MLE* of *R*, we use the estimates of parameters that calculated in the previous section. Therefore, according to the reliability property of *MLE*, we can write,

$$\hat{R} = R(\hat{a},\ \hat{b},\ \hat{\alpha},\ \hat{\lambda},\ \hat{c},\ \hat{d},\ \hat{\beta},\ \hat{\theta}). \tag{27}$$

3. Numerical experiments and discussions

The *BL* distribution is particularly useful in engineering fields, financial risk assessment, and any area where fatigue or failures due to stress are of concern. It can help in quality control, assessing acceptable levels of stress in materials and components. The model can be used for reliability function analysis, allowing for the calculation of the reliability $R(t)$, which gives the probability that the system will perform successfully up to time t. In this section, based on the relationships obtained in the previous sections, we perform simulation studies to compare the performance of $\hat{R(t)}$ for different sample sizes. We generate 10,000 samples each of size n from the Beta-Lomax distrobution and repeat this procedure for several values of $R(t)$. Figures 1 and 2 show the *MSE* of the *MLE* of $R(t)$ for different sample sizes n and parameters. From these figures we note that the *MSE* of the *MLE* of $R(t)$ is always greate when n is less than 20, however for large sample sizes (more than 20) this estimator of $R(t)$ is better and almost efficient. In Table 1 the estimation of $P = P(Y < X)$, when $X \sim BL(3,\ 2,\ 3,\ 4)$ and $Y \sim BL(1,\ 2,\ 2,\ 1.5)$ are independent random variables from Beta-Lomax distribution using ranked set sampling and simple random sampling has been compared. The plots in Figure 3 shows the process of change *Bias* and *MSE* in P when $X \sim BL(3,\ 2,\ 3,\ 4)$ and $Y \sim BL(1,\ 2,\ 2,\ 1.5)$ are independent random variables from Beta-Lomax distribution. We can see from Figure 3 that the *RSS* produces smaller absolute *MSEs* compared to sampling method *SRS* for sample size 40 or less. In large samples, many statistical properties hold (e.g., normality due to the Central Limit Theorem), making *MSE* a robust metric for comparing model performance. Beyond a certain sample size, improvements in *MSE* may become marginal. Larger datasets can help refine the model, but the marginal gain might not justify the added complexity or cost of data collection.

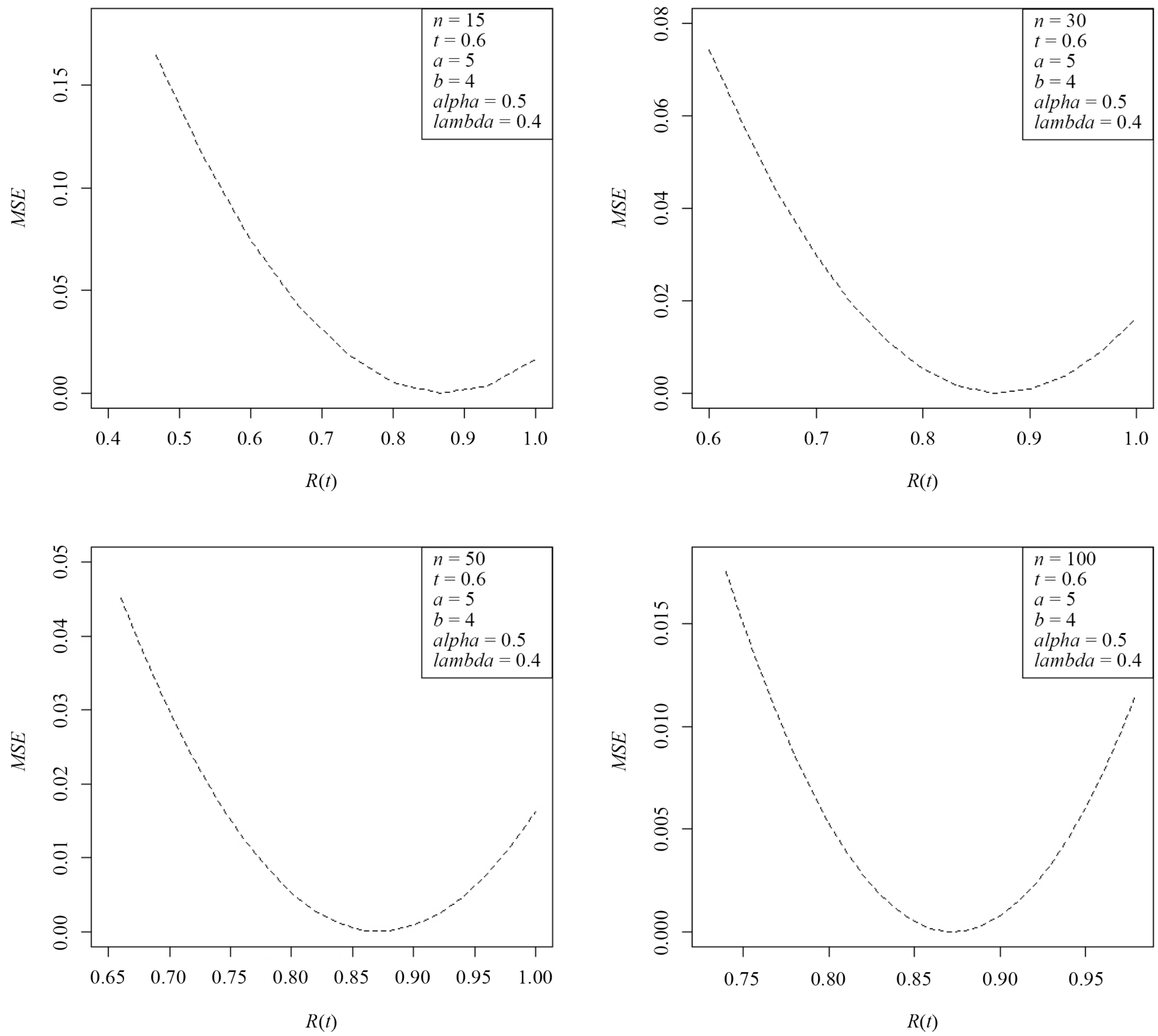

Figure 1. The performance of MSE of $R(t)$ by the MLE for different sample sizes n

Table 1. Estimation of $P = P(Y < X)$

Sampling method	results	$n = 9$	$n = 15$	$n = 25$	$n = 40$	$n = 50$	$n = 100$
MLE (SRS)	P	0.8887	0.8887	0.8887	0.8887	0.8887	0.8887
	$\hat{P}$	0.8231	0.8347	0.8362	0.8408	0.8441	0.8447
	$Bias$ (P)	0.0656	0.0540	0.0525	0.0480	0.0446	0.0440
	MSE (P)	0.0087	0.0055	0.0045	0.0034	0.0029	0.0024
MLE (RSS)	P	0.8887	0.8887	0.8887	0.8887	0.8887	0.8887
	$\hat{P}$	0.8283	0.8358	0.8412	0.8414	0.8444	0.8444
	$Bias$ (P)	0.0604	0.0529	0.0475	0.0573	0.0443	0.0443
	MSE (P)	0.0065	0.0042	0.0032	0.0037	0.0023	0.0022

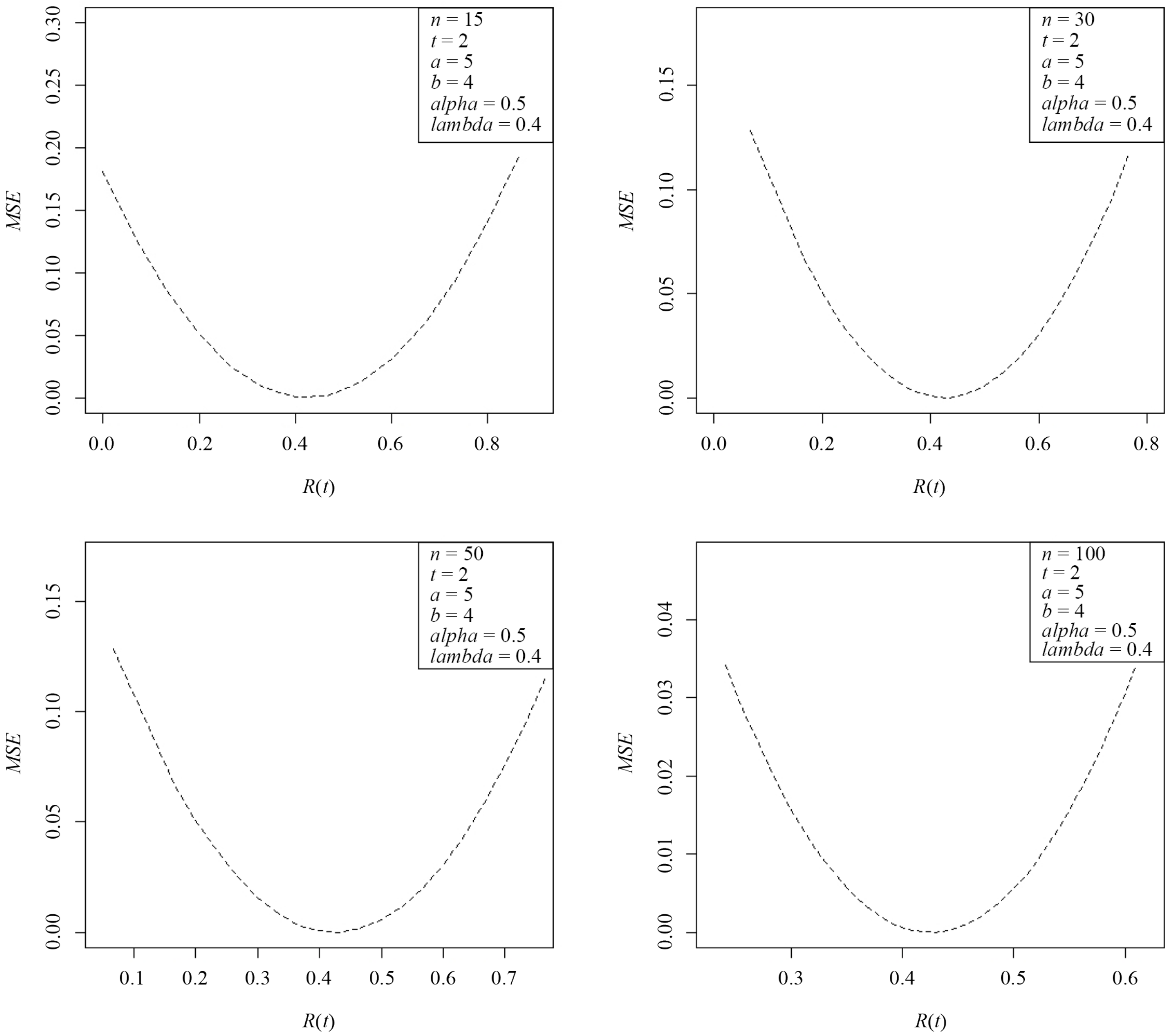

Figure 2. The performance of MSE of $R(t)$ by the MLE for different sample sizes n

Table 2. Estimation of $P = P(Y < X)$

Sampling method	results	$n = 9$	$n = 15$	$n = 25$	$n = 40$	$n = 50$	$n = 100$
$MLE\ (SRS)$	P	0.5750	0.5750	0.5750	0.5750	0.5750	0.5750
	$\hat{P}$	0.5634	0.5590	0.5644	0.5658	0.5653	0.5639
	$Bias\ (P)$	0.0116	0.0081	0.0106	0.093	0.0097	0.0112
	$MSE\ (P)$	0.0107	0.0084	0.0051	0.0009	0.0027	0.0014
$MLE\ (RSS)$	P	0.5750	0.5750	0.5750	0.5750	0.5750	0.5750
	$\hat{P}$	0.5621	0.5655	0.5660	0.5655	0.5674	0.5643
	$Bias\ (P)$	0.0129	0.0095	0.090	0.0096	0.0077	0.0109
	$MSE\ (P)$	0.0069	0.0041	0.0018	0.0009	0.0006	0.0004

In Table 2 the estimation of $P = P(Y < X)$, when $X \sim BL(3, 2, 5, 4)$ and $Y \sim BL(1, 2, 1, 1.5)$ are independent random variables from Beta-Lomax distribution using ranked set sampling and simple random sampling has been compared. The plots in Figure 4 shows the process of change *Bias* and *MSE* in *P* when $X \sim BL(3, 2, 5, 4)$ and $Y \sim BL(1, 2, 1, 1.5)$ are independent random variables from Beta-Lomax distribution. According to the graphs, it can be seen that in n less than 50, the performance of the *MLE* estimation based on *RSS* samples is better than the *MLE* estimation based on *SRS* samples. With increasing *n*, there is no significant difference between the performance of estimators in these two methods.

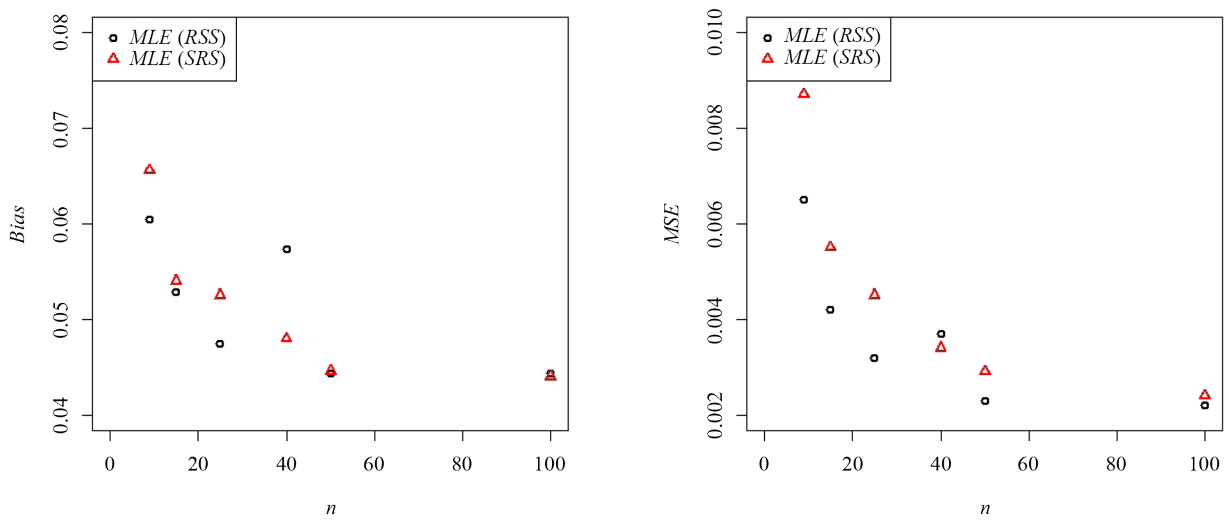

Figure 3. Performance of *Bias* and *MSE* in *P*

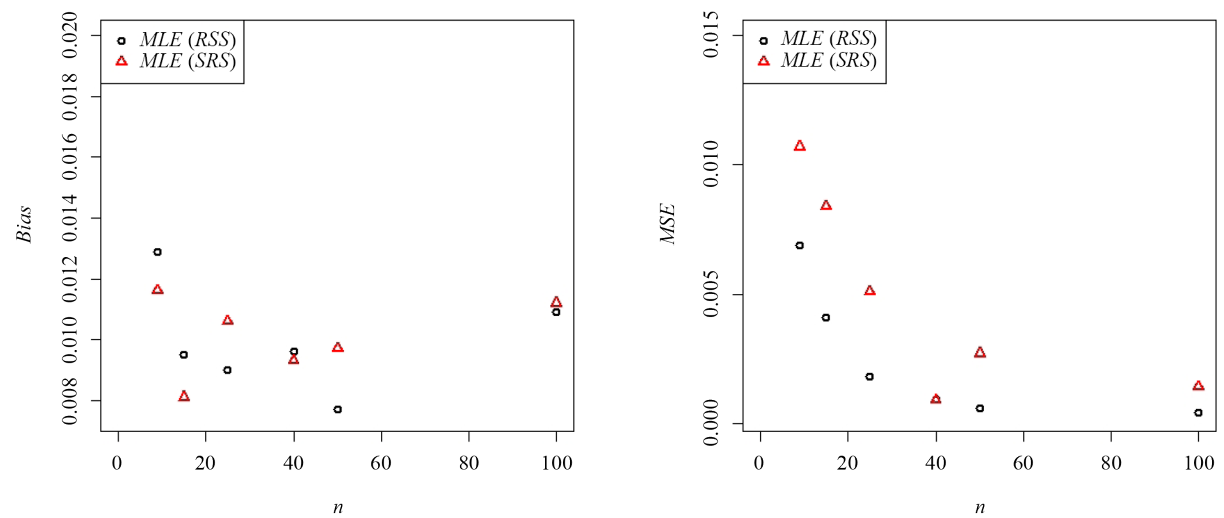

Figure 4. Performance of *Bias* and *MSE* in *P*

In Table 3 the estimation of $P = P(Y < X)$, when $X \sim BL(2, 4, 3, 2)$ and $Y \sim BL(1, 2.5, 3, 4)$ are independent random variables from Beta-Lomax distribution using ranked set sampling and simple random sampling has been compared.

Table 3. Estimation of $P = P(Y < X)$

Sampling method	results	$n = 9$	$n = 15$	$n = 25$	$n = 40$	$n = 50$	$n = 100$
MLE (SRS)	P	0.9324	0.9324	0.9324	0.9324	0.9324	0.9324
	$\hat{P}$	0.8879	0.8935	0.8966	0.8979	0.8987	0.8987
	$Bias$ (P)	0.0445	0.0389	0.0358	0.0345	0.0337	0.0337
	MSE (P)	0.0047	0.0030	0.0021	0.0017	0.0016	0.0013
MLE (RSS)	P	0.9324	0.9324	0.9324	0.9324	0.9324	0.9324
	$\hat{P}$	0.8927	0.8951	0.8989	0.9002	0.9013	0.9008
	$Bias$ (P)	0.0397	0.0373	0.0335	0.0322	0.0311	0.0316
	MSE (P)	0.0047	0.0021	0.0013	0.0011	0.0010	0.0010

The plots in Figure 5 shows the process of change $Bias$ and MSE in P when $X \sim BL(2,\ 4,\ 3,\ 2)$ and $Y \sim BL(1,\ 2.5,\ 3,\ 4)$ are independent random variables from Beta-Lomax distributin. According to the graphs, it can be seen that in n less than 50, the performance of the MLE estimator based on RSS is better than the MLE estimation based on SRS samples. With increasing n, there is no significant difference between the performance of estimators in these two methods. Considering that in this research, the comparison between the MLE estimation based on RSS and SRS samples was done considering different parameters and no restrictions were applied in the selection of parameters, it can be concluded that in general, for n less than 50, the error of the MLE estimation based on RSS samples is less and therefore more suitable. Given that large sample sizes can impose large costs on researchers, it is worthwhile to use estimators that provide better results in small sample sizes. RSS sampling can improve the precision of estimates by allowing the same sampling unit to be observed multiple times. By accumulating data across several repetitions, researchers can derive more accurate estimates and reduce variability, thereby requiring a smaller sample size compared to SRS for the same level of precision. While both SRS and RSS sampling methods have distinct advantages, RSS sampling can provide significant benefits related to sampling volume, particularly in improving precision, enhancing data collection, managing rare events, and reducing bias. These strengths make RSS especially useful in contexts requiring detailed understanding or frequent measurements.

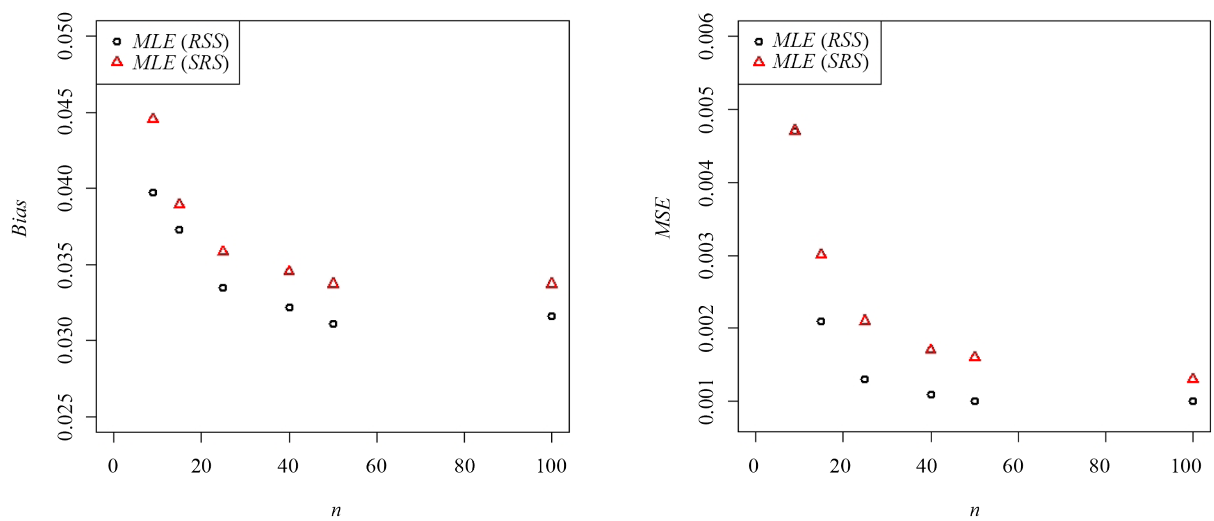

Figure 5. Performance of $Bias$ and MSE in P

4. Real data analysis

Studying the failure times of materials such as Kevlar 49/epoxy composites is very important in various engineering and materials science applications, especially in fields such as aerospace, automotive, and civil engineering. In this section, we review actual data collected on 51 observations of the failure time of a Kevlar 49/epoxy foundation under 90% pressure. Kevlar 49 is a high-strength synthetic fiber known for its excellent tensile strength and durability. When combined with epoxy resin, it forms a composite material that is widely used in high-performance applications. The failure times can be collected through controlled experiments where Kevlar 49/epoxy samples are subjected to increasing pressure until failure occurs. The time until failure is recorded for each sample. The data comes from the studies in [33] based on the recorded failure times in hours. In this research, each unit of the society has an equal chance to be selected. This method reduces bias and helps achieve a representative sample. If the materials were selected without any bias from a larger batch of Kevlar 49/epoxy composites, this could represent simple random sampling (SRS). MLE can be used to estimate the parameters of the underlying distribution of failure times, which can be modeled using various distributions including the beta-Lomax. Data for these sample are provided in Tables 4 and 5. The data are fitted by using the BLD. The Kolmogorov-Smirnov (K-S) goodness-of-fit statistic is used for the comparison of the fits. The parameters are estimated by the maximum likelihood technique. The maximum likelihood estimates and the P-values based on the (K-S) goodness-of-fit statistics are given and presented in Table 6. Let us assign the random variable $X \sim f_X(x)$ to Data set (I) and random variable $Y \sim f_Y(y)$ to Data set (II) that have been reproduced in the following tables. According to the Figures 6 and 7, and Tables 6 and 7, it is clear that our distributions have a good fit on these data. In the following, to show the applicability of the relations obtained in the article, we will calculate $R(t)$. Considering that the real data was collected by simple random sampling method (SRS), we also use parameters estimated by MLE and sampling method (SRS) to estimate $R(t)$. In Table 8 and 9 the observed $R(t)$ values for data set (I) and data set (II), and their predicted values are calculated based on the parameters estimated in Table 6 and 7 for different t. Also, the values of bias and MSE have been calculated and included. The results show that the estimator $R(t)$ is well able to estimate the real probability value.

Table 4. Data set (I)

I												
0.01	0.24	0.80	1.45	0.01	0.24	0.80	1.50	0.02	0.35	0.90	1.53	0.03
0.36	0.92	1.54	0.05	0.42	1.00	1.58	0.06	0.43	1.01	1.60	0.08	0.56
1.05	1.80	0.09	0.60	1.10	1.80	0.10	0.65	1.15	2.05	0.11	0.67	1.18
2.14	0.13	0.72	1.31	3.03	0.18	0.72	1.33	3.03	0.23	1.43	7.89	

Table 5. Data set (II)

II												
0.02	0.29	0.83	1.51	0.02	0.03	0.85	1.52	0.03	0.38	0.95	1.54	0.04
0.40	0.99	1.55	0.07	0.52	1.02	1.63	0.07	0.54	1.03	1.64	0.09	0.60
1.10	1.81	0.10	0.63	1.11	2.02	0.11	0.68	1.20	2.17	0.12	0.72	1.29
2.33	0.19	0.73	1.34	3.34	0.20	0.79	1.40	4.20	0.23	4.69		

Table 6. The parameters estimates and goodness of fit criteria for data set (I)

Distribution	$MLE\ (SRS)$	(K-S) statistics	p-value
Beta-Lomax	$\hat{a} = 2.25$	0.9754	0.0694
	$\hat{b} = 2.38$		
	$\hat{\lambda} = 0.156$		
	$\hat{\alpha} = 5.49$		

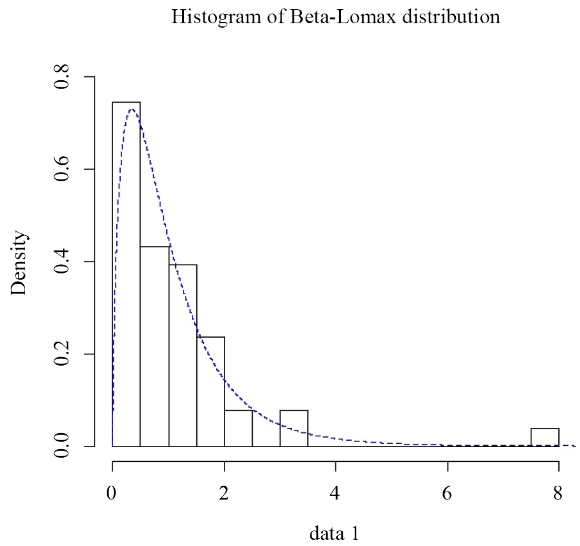

Figure 6. Plot of the PDF for Beta-Lomax based on data set (I)

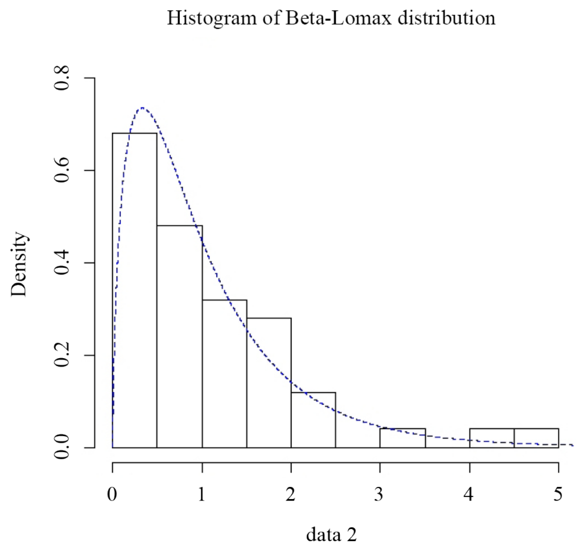

Figure 7. Plot of the PDF for Beta-Lomax based on data set (II)

Table 7. The parameters estimates and goodness of fit criteria for data set (II)

Distribution	*MLE* (*SRS*)	(K-S) statistics	p-value
BL	$\hat{a} = 1.61$	1.1434	0.0714
	$\hat{b} = 1.97$		
	$\hat{\lambda} = 0.14$		
	$\hat{\alpha} = 5.49$		

Table 8. Observed $R(t)$ and their predicted values for data set (I)

Results	$t = 0.05$	$t = 0.1$	$t = 0.5$	$t = 1$	$t = 2$	$t = 3$	$t = 5$
Observed $R(t)$	0.901	0.823	0.627	0.411	0.098	0.059	0.019
Predicted $R(t)$	0.897	0.798	0.574	0.333	0.059	0.035	0.018
Bias	0.004	0.025	0.053	0.078	0.039	0.024	0.001
MSE	0.0000	0.0000	0.0002	0.0006	0.0001	0.0005	0.0000

Table 9. Observed $R(t)$ and their predicted values for data set (II)

Results	$t = 0.05$	$t = 0.1$	$t = 0.5$	$t = 1$	$t = 2$	$t = 3$	$t = 5$
Observed $R(t)$	0.916	0.827	0.662	0.427	0.124	0.062	0.000
Predicted $R(t)$	0.909	0.794	0.587	0.337	0.104	0.043	0.008
Bias	0.007	0.033	0.075	0.090	0.020	0.019	0.008
MSE	0.0000	0.0010	0.0056	0.0081	0.0004	0.0003	0.0000

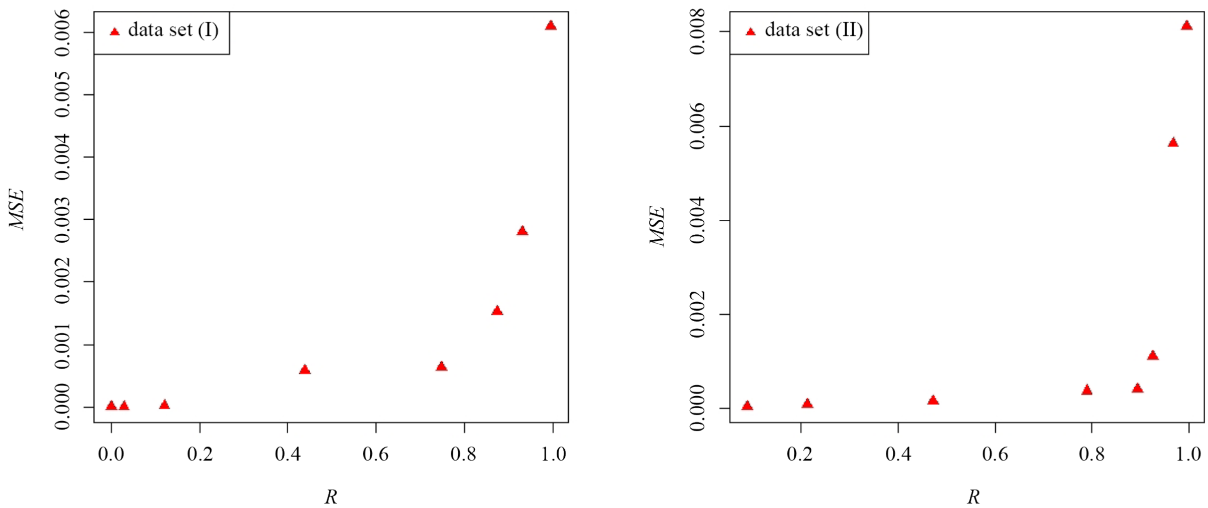

Figure 8. The performance of *MSE* of $R(t)$ for data set (I) and data set (II)

Figures 8 and 9 show the *MSE* and *Bias* of Predicted $R(t)$ for data set (I) and data set (II).

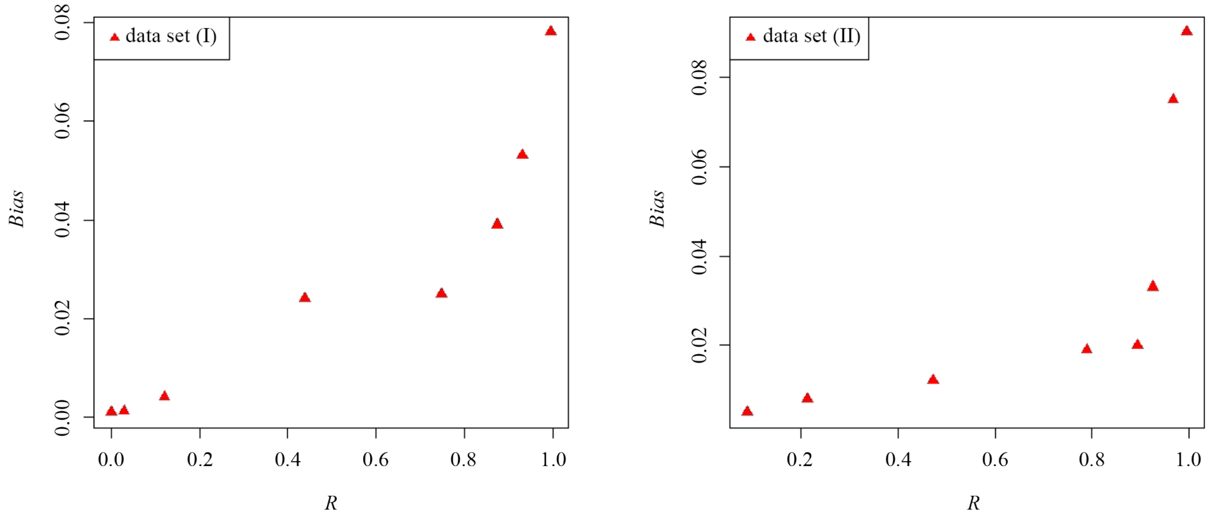

Figure 9. The performance of *Bias* of $R(t)$ for data set (I) and data set (II)

Figures 10 shows the *MSE* and *Bias* of Predicted $R(t)$ for data set (I) and data set (II).

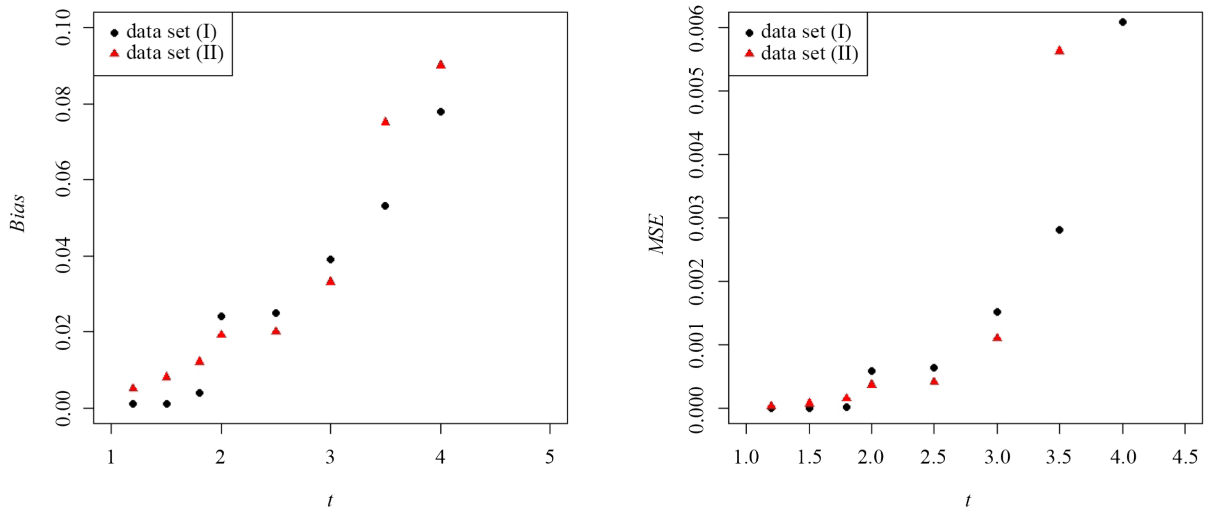

Figure 10. The performance of *MSE* and *Bias* of $R(t)$ for different value t for data set (I) and data set (II)

Now, for the above two data sets, we obtain estimators of $P = P(Y < X)$ for *BL* distribution, and the results are presented in Table 10.

Table 10. The *MLE* and of $P = P(Y < X)$

Distribution	$\hat{P}$
Beta-Lomax	0.327

5. Conclusions

Based on the analysis of the provided data sets (Data set I and Data set II) using the *BL* distribution, the following conclusions can be drawn:

1. Goodness-of-fit: The *BL* distribution provides a good fit for both data sets, as indicated by the K-S goodness-of-fit statistics and p-values presented in Tables 6 and 7.

2. Parameter estimates: The *MLE* technique was used to estimate the parameters of the *BL* distribution for each data set. The estimated parameter values are presented in Tables 6 and 7.

3. Predicted $R(t)$: The observed and predicted values of the reliability function $R(t)$ for different time points (t) were calculated based on the estimated parameters. Tables 8 and 9 show the observed and predicted $R(t)$ values for Data set I and Data set II, respectively. The bias and *MSE* of the predictions were also calculated.

Performance evaluation: Figures 6, 7, 8, 9, and 10 provide visual representations of the performance of the predicted $R(t)$ values in terms *MSE* and bias for both data sets.

6. Suggestion

Based on the results obtained from the analysis, the following suggestions can be made:

1. Further validation: While the *BL* distribution appears to provide a good fit for the given data sets, it is recommended to validate the results using additional statistical tests or alternative distributions to ensure the robustness of the findings.

2. Model comparison: It would be beneficial to compare the performance of the *BL* distribution with other competing distributions commonly used for modeling failure times, such as Weibull or log-normal distributions. This can help determine if the *BL* distribution is the most appropriate choice for these data sets.

3. Sample size consideration: The data sets used in this analysis consist of 51 and 50 observations, respectively. It is worth exploring the impact of sample size on the estimation results and assessing whether larger sample sizes would lead to more reliable parameter estimates and predictions.

4. Sensitivity analysis: Conducting a sensitivity analysis by varying the assumptions or parameters of the *BL* distribution can provide insights into the robustness of the results and help identify potential sources of uncertainty.

5. External validation: available, it would be valuable to compare the estimated parameters and predictions from the *BL* distribution with independent data sets or real-world observations to assess the generalizability and applicability of the findings.

6. Practical implications: Consider the practical implications of the estimated reliability function $R(t)$ and the estimated probability $P(Y < X)$. How can these results be used in decision-making processes or applications related to the failure time of Kevlar 49/epoxy strands under pressure?

Acknowledgement

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Conflict of interest

The authors declare no conflict of interest.

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