Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Certain Weighted Fractional Integral Inequalities Involving Convex Functions

Majid K. Neamah[1,2*], Alawiah Ibrahim[2], Tariq A. Aljaaidi[3], Mohammed S. Abdo[4]

[1] Department of Mathematics, College of Sciences, University of Baghdad, Baghdad, Iraq
[2] Department of Mathematical Sciences, Faculty of Science and Technology, University of Kebangsaan Malaysia, 43600 Bangi, Selangor, Malaysia
[3] Department of Artificial Intelligence, Faculty of Computer Science and Information Technology, Alrazi University, Sana'a, Yemen
[4] Department of Mathematics, Hodeidah University, Al-Hodeidah, Yemen
E-mail: mmathemtic@yahoo.com

Received: 25 July 2024; **Revised:** 10 October 2024; **Accepted:** 10 October 2024

Abstract: A comprehensive examination of applied sciences and their advancement necessitates an expansion of analytical studies. Our objective in this article is to unveil and present a fresh perspective on weighted integral inequalities by introducing the concept of the weighted proportional Hadamard fractional integral operator. To achieve this generalization, we have used positive and continuous functions, while some of the functions used during our generalization of these inequalities must fulfill the condition of being convex over a certain period that represents the range of functions used for the generalization. Additionally, we establish some novel inequalities using this fractional integral operator. We also delve into specific instances of the findings we present. This study significantly contributes to the literature by bridging gaps in the understanding of fractional integrals and their relationship with convex functions, thereby paving the way for future research in this dynamic area.

Keywords: weighted proportional Hadamard fractional integral, fractional integral inequalities, convex functions

MSC: 26D10, 26A33, 26D53, 05A30

1. Introduction

Undoubtedly, fractional calculus has gained significant prominence among calculus researchers due to its myriad and significant applications in various fields of natural sciences and technology. Particularly, it holds great importance in physics, fluid dynamics, biology, image processing, control theory, computer networking, and signal processing.

Fractional calculus represents a generalized version of traditional integrals and derivatives, accommodating non-integer orders. It is a testament to mathematicians' relentless pursuit of developing a more comprehensive and applicable form of mathematics, capable of addressing a wide array of scenarios encountered in the study and analysis of natural phenomena. As a consequence, fractional calculus has garnered significant attention from researchers, leading to numerous expansions and refinements, particularly within classical fractional calculus, such as the renowned Riemann-Liouville and Liouville-Caputo definitions. The Reimann-Liouville derivative is widely regarded as the most

DOI: https://doi.org/10.37256/cm.5420245366

comprehensive, consistent, and intuitive concept among fractional operators. Despite its merits, there are certain drawbacks associated with its use in specific domains, particularly in the modeling of physical phenomena. In fact, various alternative definitions of fractional operators, including but not limited to Hilfer, Katugampola, Riesz, Erdelyi-Kober, and Hadamard. These alternatives are often referred to [1, 2].

Weighted functions are essential tools in the study of inequalities, allowing for the incorporation of varying levels of importance across different values. Their significance lies in their ability to generalize classical inequalities, establish conditions for validity, and apply to various fields, enhancing our understanding of mathematical relationships and their applications. By using weighted functions, researchers can derive more comprehensive and applicable results in the realm of integral inequalities and beyond. A weighted function is a mathematical function that incorporates weights to emphasize certain values or intervals more than others do.

Convex functions, with their distinctive characteristics, hold significant importance among various categories of functions in the fields of Mathematics, Statistics, and several other applied sciences. Their useful definition, which can be geometrically interpreted, further enhances their significance. Convexity plays a vital role in the theory of inequalities and serves as a foundation for establishing a multitude of inequalities, including well-known ones such as Jensen's inequality, Hadamard's inequality, and Steffensen's inequality. One prominent inequality closely related to the convexity of functions is the renowned Hermite-Hadamard inequality, which occupies a prominent position within the realm of inequality theory. Integral inequalities play a crucial role in the analysis of various classes of differential and integral equations, we mention for Petrovic-type, Minkowiski, Jensen and Hermite-Hadamard type inequalities [3–7].

Ngo et al. [8], in 2006 gave the following integral inequalities for $\tau > 0$,

$$\int_0^1 \varsigma^{\tau+1}(\gamma)\, d\gamma \geq \int_0^1 \gamma^{\tau} \varsigma(\gamma)\, d\gamma, \tag{1}$$

and

$$\int_0^1 \varsigma^{\tau+1}(\gamma)\, d\gamma \geq \int_0^1 \gamma \varsigma^{\tau}(\gamma)\, d\gamma, \tag{2}$$

where, ς is a positive and continuous function on $[0,\ 1]$ such that

$$\int_x^1 \varsigma(\gamma)\, d\gamma \geq \int_x^1 \gamma d\gamma, \quad x \in [0,\ 1].$$

Two years later, Liu et al. [9], presented a generalized form of (1) for two parameters as follows

$$\int_{c_1}^{c_2} \varsigma^{\tau+\sigma}(\gamma)\, d\gamma \geq \int_{c_1}^{c_2} (\gamma - c_1)^{\tau} \varsigma^{\sigma}(\gamma)\, d\gamma, \tag{3}$$

which assumes that $\tau > 0,\ \sigma > 0$, and ς is a positive and continuous function on $[c_1,\ c_2]$ such that

$$\int_x^{c_2} \varsigma^{\min[1,\ \sigma]}(\gamma)\, d\gamma \geq \int_x^{c_2} (\gamma - c_1)^{\min[1,\ \sigma]}\, d\gamma, \quad \forall x \in [c_1,\ c_2].$$

Liu et al. also in 2009, using three positive and continuous function $\varsigma,\ \varphi$, and ψ on $[c_1,\ c_2]$ for the assumption that $\varsigma(\gamma) \le \varphi(\gamma)$ for all γ belong to $[c_1,\ c_2]$ satisfying $\dfrac{\varsigma}{\varphi}$ is decreasing and ς is increasing on its defined interval, established the following inequality

$$\frac{\int_{c_1}^{c_2} \varsigma(\gamma)\, d\gamma}{\int_{c_1}^{c_2} \varphi(\gamma)\, d\gamma} \ge \frac{\int_{c_1}^{c_2} f[\varsigma(\gamma)]\, d\gamma}{\int_{c_1}^{c_2} f[\varphi(\gamma)]\, d\gamma}, \tag{4}$$

where, f is a convex function with $f(0) = 0$. They also at same work for same assumptions gave the following inequality

$$\frac{\int_{c_1}^{c_2} \varsigma(\gamma)\, d\gamma}{\int_{c_1}^{c_2} \varphi(\gamma)\, d\gamma} \ge \frac{\int_{c_1}^{c_2} f[\varsigma(\gamma)]\, \psi(\gamma)\, d\gamma}{\int_{c_1}^{c_2} f[\varphi(\gamma)]\, \psi(\gamma)\, d\gamma}. \tag{5}$$

Similar to other branches of mathematics, integral and differential inequalities have been enhanced through the inclusion of fractional calculus. In order to make the differential and fractional inequalities more effective and applicable, it was necessary to extend them to include fractional orders. This led mathematicians to focus on integral and differential inequalities within the framework of fractional calculus. Within these research endeavors, Dahmani [10] introduced the fractional version of the inequalities (4), (5) by utilizing the Riemann-Liouville fractional integral in the following results.

Theorem 1 [10] For the positive and continuous functions ς and φ satisfying $\varsigma(\gamma) \le \varphi(\gamma)$ for all γ belong to its defined interval $[c_1,\ c_2]$. Assume that $\dfrac{\varsigma}{\varphi}$ is decreasing and ς is increasing on the interval $[c_1,\ c_2]$. If f is a convex function such that $f(0) = 0$. Then for all $\tau > 0$, we have

$$\frac{\mathscr{I}^\tau \varsigma(\gamma)}{\mathscr{I}^\tau \varphi(\gamma)} \ge \frac{\mathscr{I}^\tau f(\varsigma(\gamma))}{\mathscr{I}^\tau f(\varphi(\gamma))}. \tag{6}$$

Theorem 2 [10] For the positive and continuous functions ς, φ and ψ satisfying $\varsigma(\gamma) \le \varphi(\gamma)$ for all γ belong to its defined interval $[c_1,\ c_2]$. Assume that $\dfrac{\varsigma}{\varphi}$ is decreasing and ς is increasing on the interval $[c_1,\ c_2]$. If f is a convex function such that $f(0) = 0$. Then for all $\tau > 0$, we have

$$\frac{\mathscr{I}^\tau \varsigma(\gamma)}{\mathscr{I}^\tau \varphi(\gamma)} \ge \frac{\mathscr{I}^\tau f[\varsigma(\gamma)]\, \psi(\gamma)}{\mathscr{I}^\tau f[\varphi(\gamma)]\, \psi(\gamma)}. \tag{7}$$

In the past few decades, numerous researchers have derived a wide range of fractional integral inequalities that cover different fractional differential and integral operators. In 2016, Chinchane and Pachpatte [11] employed Hadamard fractional integral to establish some fractional integral inequalities involving convex functions. In 2016, Aldhaifallah et al. [12] presented some new inequalities using $(k,\ s)$-fractional integrals. Agarwal and Ozbekler [13], in 2017, generated some inequalities of Lyapunov type for mixed nonlinear Riemann-Liouville fractional differential equations concerning a forcing term. In 2019, Rahman et al. [14] have used generalized proportional Hadamard fractional integral operators to generalize certain fractional integral inequalities. Much research and generalizations on these type of inequalities and other using advanced fractional differential and integral operators have been developed in recent decades. We recommend that readers to see [15–25].

The obvious importance and large amount of previous studies on this type of inequality attracted the attention of researchers in the current manuscript to provide a new generalization for these inequalities in a weighted framework that contains convex functions due to its applications in image processing, signal analysis, finance, and physics, where weighted functions play a significant role in system behavior. We have developed and presented the generalization inequalities in a more general way so that mathematicians can use them more generally and thus more effectively. To do this, we use the weighted proportional Hadamard fractional integral operator. In addition, we prove some new relevant inequalities using the existing fractional integral operator. Some special cases of the results presented will be discussed.

This paper is organized as follows: The next section will contain some definitions, properties, and facts of the employed fractional integral operators. In the third section, we present our major results of weighted inequalities.

2. Essential preliminaries

Here, we drop some elementary definitions and properties of some basic fractional integral operators and the present fractional integral operator utilized to discuss and obtain our new results.

Definition 1 [1] For the integrable function ς on $[r, t]$ and $r \geq 0$. We have for all $\tau > 0$

$$\left(\mathscr{I}_{r^+}^{\tau}\varsigma\right)(\gamma) = \frac{1}{\Gamma(\tau)}\int_r^{\gamma}(\gamma-\kappa)^{\tau-1}\varsigma(\kappa)\,d\kappa, \quad \kappa > r, \tag{8}$$

and

$$\left(\mathscr{I}_{t^-}^{\tau}\varsigma\right)(\gamma) = \frac{1}{\Gamma(\tau)}\int_{\gamma}^{t}(\kappa-\gamma)^{\tau-1}\varsigma(\kappa)\,d\kappa, \quad \gamma < t, \tag{9}$$

where, $\Gamma(\tau) = \int_0^{\infty} e^{-x}x^{\tau-1}dx$ is the Gamma function and $\mathscr{I}_{r^+}^{0}\varsigma(\gamma) = \mathscr{I}_{t^-}^{0}\varsigma(\gamma) = \varsigma(\gamma)$. The notations $\left(\mathscr{I}_{r^+}^{\tau}\varsigma\right)(\gamma)$ and $\left(\mathscr{I}_{t^-}^{\tau}\varsigma\right)(\gamma)$ are called the left and right-sided Riemann-Liouville fractional integrals respectively.

Definition 2 [26] For the differentiable strictly increasing function η, and the mapping ω both defined on the interval $[c_1, c_2]$, the notation $\chi_{\omega}^{q}(c_1, c_2)$, where $1 \leq q < \infty$ is called the space of all Lebesgue measurable functions ς defined on the interval $[c_1, c_2]$ for which $\|\varsigma\|_{\chi_{\omega}^{q}}$, where

$$\|\varsigma\|_{\chi_{\omega}^{q}} = \left(\int_{c_1}^{c_2}|\omega(\kappa)\varsigma(\kappa)|^{q}\eta'(\kappa)\,d\kappa\right)^{\frac{1}{q}}, \quad 1 < q < \infty,$$

and

$$\|\varsigma\|_{\chi_{\omega}^{q}} = ess \sup_{c_1 \leq \kappa \leq c_2}|\omega(\kappa)\varsigma(\kappa)| < \infty.$$

Remark 1 Regarding the above definition, it should be noted that

$$\varsigma \in \chi_{\omega}^{q}(c_1, c_2) \implies \left(\omega(\kappa)\varsigma(\kappa)\eta'(\kappa)\right)^{1/q} \in L_q(c_1, c_2), \quad 1 \leq q < \infty,$$

and

$$\varsigma \in \chi_{\omega}^{q}(c_1, c_2) \implies \omega(\kappa)\varsigma(\kappa) \in L_{\infty}(c_1, c_2).$$

Definition 3 [27] For the function ς defined on $\chi_{\omega}^{q}[1, \infty)$, and the function $(\omega \neq 0)$ defined on $[1, \infty)$, we have for all $\tau \in \mathbb{C}$, $Re(\tau) > 0$,

$$\left({}_{\omega}^{H}\mathscr{I}_{c_1}^{\tau,\, z}\varsigma\right)(\gamma) = \frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)} \int_{c_1}^{\gamma} \frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\kappa}\right)\right]}{\left(\ln\frac{\gamma}{\kappa}\right)^{1-\tau}} \frac{\varsigma(\kappa)\,\omega(\kappa)}{\kappa} d\kappa, \quad c_1 < \gamma, \tag{10}$$

and

$$\left({}_{\omega}^{H}\mathscr{I}_{c_2}^{\tau,\, z}\varsigma\right)(\gamma) = \frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)} \int_{\gamma}^{c_2} \frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\kappa}{\gamma}\right)\right]}{\left(\ln\frac{\kappa}{\gamma}\right)^{1-\tau}} \frac{\varsigma(\kappa)\,\omega(\kappa)}{\kappa} d\kappa, \quad \gamma < c_2, \tag{11}$$

the notations $\left({}_{\omega}^{H}\mathscr{I}_{c_1}^{\tau,\, z}\varsigma\right)(\gamma)$ and $\left({}_{\omega}^{H}\mathscr{I}_{c_2}^{\tau,\, z}\varsigma\right)(\gamma)$ are called respectively the left and right-sided weighted generalized proportional Hadamard fractional integrals of the function ς for the order τ, where $\tau \in (0, 1]$ is the proportionality index and $\Gamma(\tau)$ is the gamma function.

Now, we define the one-sided weighted generalized proportional Hadamard fractional integral as follows.

Definition 4 For the function ς defined on $\chi_{\omega}^{q}[1, \infty)$, and the function $(\omega \neq 0)$ defined on $[1, \infty)$, we have for all $\tau \in \mathbb{C}$, $Re(\tau) > 0$,

$$\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\, z}\varsigma\right)(\gamma) = \frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)} \int_{1}^{\gamma} \frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\kappa}\right)\right]}{\left(\ln\frac{\gamma}{\kappa}\right)^{1-\tau}} \frac{\varsigma(\kappa)\,\omega(\kappa)}{\kappa} d\kappa, \quad \gamma > 1, \tag{12}$$

the notations $\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\, z}\varsigma\right)(\gamma)$ is called the one-sided weighted generalized proportional Hadamard fractional integral of the function ς for the order τ, where $\tau \in (0, 1]$ is the proportionality index and $\Gamma(\tau)$ is the gamma function.

For convenience, throughout this article, we will denote the following function as

$$A(\kappa, \tau) = \frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)} \frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\kappa}\right)\right]}{\left(\ln\frac{\gamma}{\kappa}\right)^{1-\tau}} \frac{\omega(\kappa)}{\kappa}. \tag{13}$$

3. Main results

Within this section, we give our main contributions. Here, we generalize some weighted inequalities involving convex function for weighted proportional Hadamard fractional integral operators.

Theorem 3 For the positive and continuous functions ς and φ satisfying $\varsigma(\gamma) \leq \varphi(\gamma)$ for all γ belong to its defined interval $[1, \infty)$. Assume that $\frac{\varsigma}{\varphi}$ is decreasing and ς is increasing on the interval $[1, \infty)$, which containing the weight function ω with $(\omega \neq 0)$. If f is a convex function such that $f(0) = 0$. Then for all $\tau \in \mathbb{C}$, $Re(\tau) > 0$, and $z \in (0, 1]$, we have

$$\frac{\left({}^{H}_{\omega}\mathscr{I}_{1}^{\tau,\, z}\varsigma\right)(\gamma)}{\left({}^{H}_{\omega}\mathscr{I}_{1}^{\tau,\, z}\varphi\right)(\gamma)} \geq \frac{{}^{H}_{\omega}\mathscr{I}_{1}^{\tau,\, z} f(\varsigma(\gamma))}{{}^{H}_{\omega}\mathscr{I}_{1}^{\tau,\, z} f(\varphi(\gamma))}. \tag{14}$$

Proof. According to the convexity of f with $f(0) = 0$, and convexity properties, the function $\frac{f(\gamma)}{\gamma}$ is increasing. Again, as ς is increasing, the function $\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}$ is also increasing. Since $\frac{\varsigma(\gamma)}{\varphi(\gamma)}$ is decreasing, and considering all the hypotheses and facts mentioned above, we can conclude for all cases of any $\kappa,\ \mu \in [1,\ \infty)$ that

$$\left(\frac{f(\varsigma(\kappa))}{\varsigma(\kappa)} - \frac{f(\varsigma(\mu))}{\varsigma(\mu)}\right)\left(\frac{\varsigma(\mu)}{\varphi(\mu)} - \frac{\varsigma(\kappa)}{\varphi(\kappa)}\right) \geq 0.$$

It follows that

$$\frac{f(\varsigma(\kappa))}{\varsigma(\kappa)}\frac{\varsigma(\mu)}{\varphi(\mu)} + \frac{f(\varsigma(\mu))}{\varsigma(\mu)}\frac{\varsigma(\kappa)}{\varphi(\kappa)} - \frac{f(\varsigma(\kappa))}{\varsigma(\kappa)}\frac{\varsigma(\kappa)}{\varphi(\kappa)} - \frac{f(\varsigma(\mu))}{\varsigma(\mu)}\frac{\varsigma(\mu)}{\varphi(\mu)} \geq 0. \tag{15}$$

Multiplying (15) by the positive function $\varphi(\kappa)\varphi(\mu)$, we get

$$\frac{f(\varsigma(\kappa))}{\varsigma(\kappa)}\varsigma(\mu)\varphi(\kappa) + \frac{f(\varsigma(\mu))}{\varsigma(\mu)}\varsigma(\kappa)\varphi(\mu) - \frac{f(\varsigma(\kappa))}{\varsigma(\kappa)}\varsigma(\kappa)\varphi(\mu) - \frac{f(\varsigma(\mu))}{\varsigma(\mu)}\varsigma(\mu)\varphi(\kappa) \geq 0. \tag{16}$$

On both sides of (16), taking product by $A(\kappa,\ \tau)$, which is positive as $\kappa \in (1,\ \gamma)$, $\gamma > 1$, then integrating with respect to κ over $(1,\ \gamma)$, we have

$$\varsigma(\mu)\frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)}\int_{1}^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\kappa}\right)\right]}{\left(\ln\frac{\gamma}{\kappa}\right)^{1-\tau}}\frac{f(\varsigma(\kappa))}{\varsigma(\kappa)}\frac{\varphi(\kappa)\omega(\kappa)}{\kappa}d\kappa$$

$$+\varphi(\mu)\frac{f(\varsigma(\mu))}{\varsigma(\mu)}\frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)}\int_{1}^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\kappa}\right)\right]}{\left(\ln\frac{\gamma}{\kappa}\right)^{1-\tau}}\frac{\varsigma(\kappa)\omega(\kappa)}{\kappa}d\kappa$$

$$-\varphi(\mu)\frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)}\int_{1}^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\kappa}\right)\right]}{\left(\ln\frac{\gamma}{\kappa}\right)^{1-\tau}}\frac{f(\varsigma(\kappa))}{\varsigma(\kappa)}\frac{\varsigma(\kappa)\omega(\kappa)}{\kappa}d\kappa \tag{17}$$

$$-\varsigma(\mu)\frac{f(\varsigma(\mu))}{\varsigma(\mu)}\frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)}\int_1^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\kappa}\right)\right]}{\left(\ln\frac{\gamma}{\kappa}\right)^{1-\tau}}\frac{\varphi(\kappa)\omega(\kappa)}{\kappa}d\kappa \geq 0,$$

which leads to

$$\varsigma(\mu)\,{}^{H}_{\omega}\mathscr{I}_1^{\tau,\,z}\left(\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\varphi(\gamma)\right)+\left(\varphi(\mu)\frac{f(\varsigma(\mu))}{\varsigma(\mu)}\right)\left({}^{H}_{\omega}\mathscr{I}_1^{\tau,\,z}\varsigma\right)(\gamma)$$

$$-\varphi(\mu)\,{}^{H}_{\omega}\mathscr{I}_1^{\tau,\,z}f(\varsigma(\gamma))-\left(\varsigma(\mu)\frac{f(\varsigma(\mu))}{\varsigma(\mu)}\right)\left({}^{H}_{\omega}\mathscr{I}_1^{\tau,\,z}\varphi\right)(\gamma)\geq 0. \tag{18}$$

Again, on both sides of (18), taking product by $A(\mu,\ \tau)$, which is positive as $\mu \in (1,\ \gamma),\ \gamma > 1$, then integrating the resulting inequality with respect to μ over $(1,\ \gamma)$, we get

$${}^{H}_{\omega}\mathscr{I}_1^{\tau,\,z}\left(\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\varphi(\gamma)\right)\left[\frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)}\int_1^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\mu}\right)\right]}{\left(\ln\frac{\gamma}{\mu}\right)^{1-\tau}}\frac{\varsigma(\mu)\omega(\mu)}{\mu}d\mu\right]$$

$$+\left({}^{H}_{\omega}\mathscr{I}_1^{\tau,\,z}\varsigma\right)(\gamma)\left[\frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)}\int_1^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\mu}\right)\right]}{\left(\ln\frac{\gamma}{\mu}\right)^{1-\tau}}\left(\varphi(\mu)\frac{f(\varsigma(\mu))}{\varsigma(\mu)}\right)\frac{\omega(\mu)}{\mu}d\mu\right]$$

$$-\,{}^{H}_{\omega}\mathscr{I}_1^{\tau,\,z}f(\varsigma(\gamma))\left[\frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)}\int_1^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\mu}\right)\right]}{\left(\ln\frac{\gamma}{\mu}\right)^{1-\tau}}\frac{\varphi(\mu)\omega(\mu)}{\mu}d\mu\right]$$

$$-\left({}^{H}_{\omega}\mathscr{I}_1^{\tau,\,z}\varphi\right)(\gamma)\left[\frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)}\int_1^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\mu}\right)\right]}{\left(\ln\frac{\gamma}{\mu}\right)^{1-\tau}}\left(\varsigma(\mu)\frac{f(\varsigma(\mu))}{\varsigma(\mu)}\right)\frac{\omega(\mu)}{\mu}d\mu\right]\geq 0,$$

which leads to

$$\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varsigma\right)(\gamma)\,{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\left(\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\varphi(\gamma)\right)+{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\left(\varphi(\gamma)\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\right)\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varsigma\right)(\gamma)$$

$$-\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varphi\right)(\gamma)\,{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}f(\varsigma(\gamma))-{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}f(\varsigma(\gamma))\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varphi\right)(\gamma)\geq 0.$$

It follows that

$$\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varsigma\right)(\gamma)\,{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\left(\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\varphi(\gamma)\right)\geq{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}f(\varsigma(\gamma))\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varphi\right)(\gamma). \tag{19}$$

Since $\left(\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\varphi(\gamma)\right)$, $\varphi(\gamma)$ are all positive functions, then the inequality (19) leads to

$$\frac{\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varsigma\right)(\gamma)}{\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varphi\right)(\gamma)}\geq\frac{{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}f(\varsigma(\gamma))}{{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\left(\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\varphi(\gamma)\right)}. \tag{20}$$

Now, considering the increase of the function $\frac{f(\gamma)}{\gamma}$ and that $\varsigma(\gamma)\leq\varphi(\gamma)$ for all $\gamma\in[1,\,\infty)$, we have for all $\mu\in[1,\,\infty)$

$$\frac{f(\varsigma(\mu))}{\varsigma(\mu)}\leq\frac{f(\varphi(\mu))}{\varphi(\mu)}. \tag{21}$$

On both sides of (21), taking product by $A(\mu,\,\tau)\,\varphi(\mu)$, $\mu\in(1,\,\gamma)$, $\gamma>1$, then integrating with respect to μ over $(1,\,\gamma)$, we have

$$\frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)}\int_{1}^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\mu}\right)\right]}{\left(\ln\frac{\gamma}{\mu}\right)^{1-\tau}}\frac{f(\varsigma(\mu))\,\varphi(\mu)}{\varsigma(\mu)}\frac{\omega(\mu)}{\mu}d\mu$$

$$\leq\frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)}\int_{1}^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\mu}\right)\right]}{\left(\ln\frac{\gamma}{\mu}\right)^{1-\tau}}\frac{f(\varphi(\mu))\,\varphi(\mu)}{\varphi(\mu)}\frac{\omega(\mu)}{\mu}d\mu,$$

which yields

$${}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\left(\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\varphi(\gamma)\right)\leq{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}f(\varphi(\gamma)). \tag{22}$$

Hence, in the view of (20) and (22), we can complete the proof. □

Example 1 By defining the required functions γ, φ, convex function f, and weight function $\omega(z)$, we shall prove the correctness of Theorem 3. Additionally, we will compute and numerically evaluate the necessary integrals. Consider the functions $\gamma(z)$, $\varphi(z)$, f, and $\omega(z)$ as follows

$$\gamma(z) = z, \quad \varphi(z) = z^2, \quad f(x) = x^2, \quad f(0) = 0, \quad \omega(z) = 1.$$

Clearly, $\gamma(z) \leq \varphi(z)$ for $z \geq 1$, and $\frac{\gamma(z)}{\varphi(z)} = \frac{z}{z^2} = \frac{1}{z}$, which is decreasing. Also, the function $\gamma(z)$ is increasing because $\gamma'(z) = 1 > 0$ and $\omega(z)$ is positive and continuous on $[1, \infty)$.

Now, we calculate $\left({}^{H}_{\omega}\mathscr{I}_{1}^{\tau,\,\delta}\gamma\right)(z)$, $\left({}^{H}_{\omega}\mathscr{I}_{1}^{\tau,\,\delta}\varphi\right)(z)$, $\left({}^{H}_{\omega}\mathscr{I}_{1}^{\tau,\,\delta}f(\gamma)\right)(z)$, $\left({}^{H}_{\omega}\mathscr{I}_{1}^{\tau,\,\delta}f(\varphi)\right)(z)$. For $\tau = 0.5$ and $\delta = 1$ and using the Definition 4, we obtain

$$\left({}^{H}_{1}\mathscr{I}_{1}^{0.5,\,1}\gamma\right)(z) = \frac{1}{\sqrt{\pi}}\int_{1}^{z}\frac{1}{(\ln(z/\kappa))^{0.5}}d\kappa.$$

$$\left({}^{H}_{1}\mathscr{I}_{1}^{0.5,\,1}\varphi\right)(z) = \frac{1}{\sqrt{\pi}}\int_{1}^{z}\frac{\kappa}{(\ln(z/\kappa))^{0.5}}d\kappa.$$

$$\left({}^{H}_{1}\mathscr{I}_{1}^{0.5,\,1}f(\gamma)\right)(z) = \left({}^{H}_{1}\mathscr{I}_{1}^{0.5,\,1}(z^2)\right)(z) = \frac{1}{\sqrt{\pi}}\int_{1}^{z}\frac{\kappa}{(\ln(z/\kappa))^{0.5}}d\kappa,$$

$$\left({}^{H}_{1}\mathscr{I}_{1}^{0.5,\,1}f(\varphi)\right)(z) = \left({}^{H}_{1}\mathscr{I}_{1}^{0.5,\,1}(z^4)\right)(z) = \frac{1}{\sqrt{\pi}}\int_{1}^{z}\frac{\kappa^3}{(\ln(z/\kappa))^{0.5}}d\kappa.$$

To evaluate these integrals numerically, we can use Python with the 'scipy' library. For $z = 2$, we get

$$\left({}^{H}_{1}\mathscr{I}_{1}^{\tau,\,\delta}\gamma\right)(2) = 1.5219362171002615; \quad \left({}^{H}_{1}\mathscr{I}_{1}^{0.5,\,1}\varphi\right)(2) = 2.5572065122560286;$$

$$\left({}^{H}_{1}\mathscr{I}_{1}^{0.5,\,1}f(\gamma\right)(2) = 2.5572065122560286; \quad \left({}^{H}_{1}\mathscr{I}_{1}^{0.5,\,1}f(\varphi\right)(2) = 7.094816890863363.$$

The inequality from Theorem 3 states:

$$\frac{\left({}^{H}_{\omega}\mathscr{I}_{1}^{\tau,\,\delta}\gamma\right)(z)}{\left({}^{H}_{\omega}\mathscr{I}_{1}^{\tau,\,\delta}\varphi\right)(z)} \geq \frac{\left({}^{H}_{\omega}\mathscr{I}_{1}^{\tau,\,\delta}f(\gamma)\right)(z)}{\left({}^{H}_{\omega}\mathscr{I}_{1}^{\tau,\,\delta}f(\varphi)\right)(z)}.$$

For $z = 2$, $\gamma(z) = z$ and $\varphi(z) = z^2$, $f(\gamma(z)) = z^2$, $f(\varphi(z)) = z^4$ and $\omega(z) = 1$ over the interval $[1,\ 2]$, with $\tau = 0.5$ and $\delta = 0.9$, we obtain

$$\frac{\left({}_{1}^{H}\mathscr{I}_{1}^{0.5,\,0.9}\gamma\right)(2)}{\left({}_{1}^{H}\mathscr{I}_{1}^{0.5,\,0.9}\varphi\right)(2)}=\frac{1.5219362171002615}{2.5572065122560286}=0.59516,$$

$$\frac{\left({}_{1}^{H}\mathscr{I}_{1}^{0.5,\,0.9}f(\gamma)\right)(2)}{\left({}_{1}^{H}\mathscr{I}_{1}^{0.5,\,0.9}f(\varphi)\right)(2)}=\frac{2.5572065122560286}{7.094816890863363}=0.36043.$$

Since $0.59516>0.36043$, the inequality from Theorem 3 is verified for the selected functions $\gamma(z)=z$, $\varphi(z)=z^2$, $f(x)=x^2$ and $\omega(z)=1$, for $z=2.0$.

In the following, we present a generalized version of the previous result, accounting for different parameters.

Theorem 4 For the positive and continuous functions ς and φ satisfying $\varsigma(\gamma)\leq\varphi(\gamma)$ for all γ belong to its defined interval $[1,\,\infty)$. Assume that $\frac{\varsigma}{\varphi}$ is decreasing and ς is increasing on the interval $[1,\,\infty)$, which containing the weight function ω with $(\omega\neq 0)$. If f is a convex function such that $f(0)=0$. Then for all $\tau,\ \sigma\in\mathbb{C}$, $Re(\tau)$, $Re(\sigma)>0$, and $z\in(0,\ 1]$, the following inequality holds

$$\frac{\left({}_{\omega}^{H}\mathscr{I}_{1}^{\sigma,\,z}\varsigma\right)(\gamma)\,{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}f(\varsigma(\gamma))+\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varsigma\right)(\gamma)\,{}_{\omega}^{H}\mathscr{I}_{1}^{\sigma,\,z}f(\varsigma(\gamma))}{\left({}_{\omega}^{H}\mathscr{I}_{1}^{\sigma,\,z}\varphi\right)(\gamma)\,{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}f(\varphi(\gamma))+\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varphi\right)(\gamma)\,{}_{\omega}^{H}\mathscr{I}_{1}^{\sigma,\,z}f(\varphi(\gamma))}\geq 1. \tag{23}$$

Proof. According to the convexity of f with $f(0)=0$, and convexity properties, the function $\frac{f(\gamma)}{\gamma}$ is increasing. Again, as ς is increasing, the function $\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}$ is also increasing. Since $\frac{\varsigma(\gamma)}{\varphi(\gamma)}$ is decreasing, and considering all the hypotheses and facts mentioned above, and by the same arguments as in the proof of Theorem 3, we can have for all $\mu\in[1,\,\infty)$, $\gamma>1$ that

$$\varsigma(\mu)\,{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\left(\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\varphi(\gamma)\right)+\left(\varphi(\mu)\frac{f(\varsigma(\mu))}{\varsigma(\mu)}\right)\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varsigma\right)(\gamma)$$

$$-\varphi(\mu)\,{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}f(\varsigma(\gamma))-f(\varsigma(\mu))\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varphi\right)(\gamma)\geq 0,$$

so, multiplying both sides by $A(\mu,\,\sigma)$, which is positive as $\mu\in(1,\,\gamma)$, $\gamma>1$, then integrating the resulting inequality with respect to μ over $(1,\,\gamma)$, we get

$$
{}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\left(\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\varphi(\gamma)\right)\left[\frac{\omega^{-1}(\gamma)}{z^{\sigma}\Gamma(\sigma)}\int_1^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\mu}\right)\right]}{\left(\ln\frac{\gamma}{\mu}\right)^{1-\sigma}}\frac{\varsigma(\mu)\,\omega(\mu)}{\mu}d\mu\right]
$$

$$
+\left({}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\varsigma\right)(\gamma)\left[\frac{\omega^{-1}(\gamma)}{z^{\sigma}\Gamma(\sigma)}\int_1^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\mu}\right)\right]}{\left(\ln\frac{\gamma}{\mu}\right)^{1-\sigma}}\left(\varphi(\mu)\frac{f(\varsigma(\mu))}{\varsigma(\mu)}\right)\frac{\omega(\mu)}{\mu}d\mu\right]
$$

$$
-\,{}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}f(\varsigma(\gamma))\left[\frac{\omega^{-1}(\gamma)}{z^{\sigma}\Gamma(\sigma)}\int_1^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\mu}\right)\right]}{\left(\ln\frac{\gamma}{\mu}\right)^{1-\sigma}}\frac{\varphi(\mu)\,\omega(\mu)}{\mu}d\mu\right]
$$

$$
-\left({}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\varphi\right)(\gamma)\left[\frac{\omega^{-1}(\gamma)}{z^{\sigma}\Gamma(\sigma)}\int_1^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\mu}\right)\right]}{\left(\ln\frac{\gamma}{\mu}\right)^{1-\sigma}}\left(\varsigma(\mu)\frac{f(\varsigma(\mu))}{\varsigma(\mu)}\right)\frac{\omega(\mu)}{\mu}d\mu\right]\geq 0.
$$

So, we have

$$
\left({}^{H}_{\omega}\mathscr{I}_1^{\sigma,\, z}\varsigma\right)(\gamma)\,{}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\left(\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\varphi(\gamma)\right)+{}^{H}_{\omega}\mathscr{I}_1^{\sigma,\, z}\left(\varphi(\gamma)\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\right)\left({}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\varsigma\right)(\gamma)
$$

$$
\geq\left({}^{H}_{\omega}\mathscr{I}_1^{\sigma,\, z}\varphi\right)(\gamma)\,{}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}f(\varsigma(\gamma))+{}^{H}_{\omega}\mathscr{I}_1^{\sigma,\, z}f(\varsigma(\gamma))\left({}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\varphi\right)(\gamma). \tag{24}
$$

Considering the increase of the function $\frac{f(\gamma)}{\gamma}$ and $\varsigma(\gamma)\leq\varphi(\gamma)$ for all $\gamma\in[1,\,\infty)$, we have for all $\mu\in[1,\,\infty)$

$$
\frac{f(\varsigma(\mu))}{\varsigma(\mu)}\leq\frac{f(\varphi(\mu))}{\varphi(\mu)}. \tag{25}
$$

Taking product by $A(\mu,\,\tau)\,\varphi(\mu)$, $\mu\in(1,\,\gamma),\ \gamma>1$ on both sides of (25), then integrating with respect to μ over $(1,\,\gamma)$, we have

$$\frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)}\int_{1}^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\mu}\right)\right]}{\left(\ln\frac{\gamma}{\mu}\right)^{1-\tau}}\frac{f(\varsigma(\mu))\varphi(\mu)}{\varsigma(\mu)}\frac{\omega(\mu)}{\mu}d\mu$$

$$\leq\frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)}\int_{1}^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\mu}\right)\right]}{\left(\ln\frac{\gamma}{\mu}\right)^{1-\tau}}\frac{f(\varphi(\mu))\varphi(\mu)}{\varphi(\mu)}\frac{\omega(\mu)}{\mu}d\mu,$$

which yielas

$$ {}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\left(\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\varphi(\gamma)\right)\leq {}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}f(\varphi(\gamma)). \tag{26}$$

In view of (24) and (26), we get

$$\left({}_{\omega}^{H}\mathscr{I}_{1}^{\sigma,\,z}\varsigma\right)(\gamma)\,{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}f(\varsigma(\gamma))+\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varsigma\right)(\gamma)\,{}_{\omega}^{H}\mathscr{I}_{1}^{\sigma,\,z}f(\varsigma(\gamma))$$

$$\geq\left({}_{\omega}^{H}\mathscr{I}_{1}^{\sigma,\,z}\varphi\right)(\gamma)\,{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}f(\varsigma(\gamma))+\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varphi\right)(\gamma)\,{}_{\omega}^{H}\mathscr{I}_{1}^{\sigma,\,z}f(\varsigma(\gamma)). \tag{27}$$

Since $f(\varsigma(\gamma))$, $\varphi(\gamma)$ are all positive functions, then, using inequality (27) we can conclude the inequality (23). □

Remark 2 It is worth noting the following:

• Applying Theorem 4 for $\tau=\sigma$, we get Theorem 3.

• Putting $\omega=1$ in Theorem 3 and Theorem 4, we get thier unweighted proportional Hadamard fractional integral version obtained in [14].

The following result is another type of inequality that represents a quotient of fractional integrals of three positive and continuous functions over a certain interval.

Theorem 5 For the positive and continuous functions ς, φ, and ψ satisfying $\varsigma(\gamma)\leq\varphi(\gamma)$ for all γ belong to its defined interval $[1,\,\infty)$. Assume that $\frac{\varsigma}{\varphi}$ is decreasing and ς and ψ is increasing on the interval $[1,\,\infty)$, which containing the weight function ω with $(\omega\neq 0)$. If f is a convex function such that $f(0)=0$. Then for all $\tau\in\mathbb{C}$, $Re(\tau)>0$, and $z\in(0,\,1]$, we have

$$\frac{\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varsigma\right)(\gamma)}{\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varphi\right)(\gamma)}\geq\frac{{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}f[\varsigma(\gamma)]\,\psi(\gamma)}{{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}f[\varphi(\gamma)]\,\psi(\gamma)}. \tag{28}$$

Proof. Considering the increase of the function $\frac{f(\gamma)}{\gamma}$ and that $\varsigma(\gamma)\leq\varphi(\gamma)$ for all $\gamma\in[1,\,\infty)$, we have for all $\kappa\in[1,\,\infty)$ that

$$\frac{f(\varsigma(\kappa))}{\varsigma(\kappa)} \leq \frac{f(\varphi(\kappa))}{\varphi(\kappa)}. \tag{29}$$

Taking product by $A(\kappa,\ \tau)\,\varphi(\kappa)\,\psi(\kappa)$, $\kappa \in (1,\ \gamma),\ \gamma > 1$ on both sides of (29), then integrating with respect to κ over $(1,\ \gamma)$, we have

$$\frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)}\int_{1}^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\kappa}\right)\right]}{\left(\ln\frac{\gamma}{\kappa}\right)^{1-\tau}}\frac{f(\varsigma(\kappa))\,\varphi(\kappa)\,\psi(\kappa)}{\varsigma(\kappa)}\frac{\omega(\kappa)}{\kappa}d\kappa$$

$$\leq \frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)}\int_{1}^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\kappa}\right)\right]}{\left(\ln\frac{\gamma}{\kappa}\right)^{1-\tau}}\frac{f(\varphi(\kappa))\,\varphi(\kappa)\,\psi(\kappa)}{\varphi(\kappa)}\frac{\omega(\kappa)}{\kappa}d\kappa,$$

which yielas

$${}^{H}_{\omega}\mathscr{I}_{1}^{\tau,\,z}\left(\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\varphi(\gamma)\,\psi(\gamma)\right) \leq {}^{H}_{\omega}\mathscr{I}_{1}^{\tau,\,z}\, f[\varphi(\gamma)]\,\psi(\gamma). \tag{30}$$

Now, in the veiw of the convexity of f with $f(0) = 0$, and convexity properties, the function $\frac{f(\gamma)}{\gamma}$ is increasing. As ς is increasing, the function $\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}$ is also increasing. Since $\frac{\varsigma(\gamma)}{\varphi(\gamma)}$ is decreasing, and considering all the hypotheses and facts mentioned above, we can conclude for all cases of $\kappa,\ \mu \in [1,\ \infty)$ that

$$\left(\frac{f(\varsigma(\kappa))}{\varsigma(\kappa)} - \frac{f(\varsigma(\mu))}{\varsigma(\mu)}\right)\left(\frac{\varsigma(\mu)}{\varphi(\mu)} - \frac{\varsigma(\kappa)}{\varphi(\kappa)}\right) \geq 0. \tag{31}$$

Since $\psi(\gamma)$ is increasing and positive for all $\gamma \in [1,\ \infty)$, we can rewrite (31) as

$$\left(\frac{f(\varsigma(\kappa))}{\varsigma(\kappa)}\psi(\kappa) - \frac{f(\varsigma(\mu))}{\varsigma(\mu)}\psi(\mu)\right)\left(\frac{\varsigma(\mu)}{\varphi(\mu)} - \frac{\varsigma(\kappa)}{\varphi(\kappa)}\right) \geq 0. \tag{32}$$

Multiplying (32) by the positive function $\varphi(\kappa)\,\varphi(\mu)$, we get

$$\left(\frac{f(\varsigma(\kappa))}{\varsigma(\kappa)}\psi(\kappa) - \frac{f(\varsigma(\mu))}{\varsigma(\mu)}\psi(\mu)\right)(\varsigma(\mu)\,\varphi(\kappa) - \varsigma(\kappa)\,\varphi(\mu)) \geq 0. \tag{33}$$

It follows that

$$\frac{f(\varsigma(\kappa))}{\varsigma(\kappa)}\psi(\kappa)\varsigma(\mu)\varphi(\kappa)+\frac{f(\varsigma(\mu))}{\varsigma(\mu)}\psi(\mu)\varsigma(\kappa)\varphi(\mu)$$

$$-\frac{f(\varsigma(\kappa))}{\varsigma(\kappa)}\psi(\kappa)\varsigma(\kappa)\varphi(\mu)-\frac{f(\varsigma(\mu))}{\varsigma(\mu)}\psi(\mu)\varsigma(\mu)\varphi(\kappa)\geq 0. \tag{34}$$

On both sides of (34), taking product by $A(\kappa,\ \tau)$, which is positive as $\kappa\in(1,\ \gamma)$, $\gamma>1$, then integrating with respect to κ over $(1,\ \gamma)$, we have

$$\varsigma(\mu)\frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)}\int_{1}^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\kappa}\right)\right]}{\left(\ln\frac{\gamma}{\kappa}\right)^{1-\tau}}\frac{f(\varsigma(\kappa))}{\varsigma(\kappa)}\frac{\psi(\kappa)\varphi(\kappa)}{\kappa}d\kappa$$

$$+\frac{f(\varsigma(\mu))}{\varsigma(\mu)}\psi(\mu)\varphi(\mu)\frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)}\int_{1}^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\kappa}\right)\right]}{\left(\ln\frac{\gamma}{\kappa}\right)^{1-\tau}}\frac{\varsigma(\kappa)}{\kappa}d\kappa$$

$$-\varphi(\mu)\frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)}\int_{1}^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\kappa}\right)\right]}{\left(\ln\frac{\gamma}{\kappa}\right)^{1-\tau}}\frac{f(\varsigma(\kappa))}{\varsigma(\kappa)}\frac{\psi(\kappa)\varsigma(\kappa)}{\kappa}d\kappa$$

$$-\frac{f(\varsigma(\mu))}{\varsigma(\mu)}\psi(\mu)\varsigma(\mu)\frac{\omega^{-1}(\gamma)}{z^{\tau}\Gamma(\tau)}\int_{1}^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\kappa}\right)\right]}{\left(\ln\frac{\gamma}{\kappa}\right)^{1-\tau}}\frac{\varphi(\kappa)}{\kappa}d\kappa\geq 0.$$

So, we get

$$\varsigma(\mu)\,{}^{H}_{\omega}\mathscr{I}_{1}^{\tau,\,z}\left(\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\psi(\gamma)\varphi(\gamma)\right)+\frac{f(\varsigma(\mu))}{\varsigma(\mu)}\psi(\mu)\varphi(\mu)\left({}^{H}_{\omega}\mathscr{I}_{1}^{\tau,\,z}\varsigma\right)(\gamma)$$

$$-\varphi(\mu)\,{}^{H}_{\omega}\mathscr{I}_{1}^{\tau,\,z}\left(\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\psi(\gamma)\varsigma(\gamma)\right)-\frac{f(\varsigma(\mu))}{\varsigma(\mu)}\psi(\mu)\varsigma(\mu)\left({}^{H}_{\omega}\mathscr{I}_{1}^{\tau,\,z}\varphi\right)(\gamma)\geq 0. \tag{35}$$

Again, taking product by $A(\mu,\ \tau)$ on both sides of (35), then integrating the resulting inequality with respect to μ over $(1,\ \gamma)$, we get

$$\left({}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\varsigma\right)(\gamma)\, {}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\left(\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\psi(\gamma)\varphi(\gamma)\right) + {}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\left(\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\psi(\gamma)\varphi(\gamma)\right)\left({}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\varsigma\right)(\gamma)$$

$$-\left({}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\varphi\right)(\gamma)\, {}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\left(f[\varsigma(\gamma)]\psi(\gamma)\right) - {}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\left(f[\varsigma(\gamma)]\psi(\gamma)\right)\left({}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\varphi\right)(\gamma) \geq 0, \tag{36}$$

which yields

$$\frac{\left({}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\varsigma\right)(\gamma)}{\left({}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\varphi\right)(\gamma)} \geq \frac{{}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z} f[\varsigma(\gamma)]\psi(\gamma)}{{}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\left(\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\psi(\gamma)\varphi(\gamma)\right)}. \tag{37}$$

Hence, by comparing each of the inequality (30) and inequality (37), we can deduce (28). □

Theorem 6 For the positive and continuous functions ς, φ, and ψ satisfying $\varsigma(\gamma) \leq \varphi(\gamma)$ for all γ belong to its defined interval $[1, \infty)$. Assume that $\frac{\varsigma}{\varphi}$ is decreasing and ς and ψ is increasing on the interval $[1, \infty)$, which containing the weight function ω with $(\omega \neq 0)$. If f is a convex function such that $f(0) = 0$. Then for all $\tau,\ \sigma \in \mathbb{C}$, $Re(\tau), Re(\sigma) > 0$, and $z \in (0,\ 1]$, we have

$$\frac{\left({}^{H}_{\omega}\mathscr{I}_1^{\sigma,\, z}\varsigma\right)(\gamma)\, {}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z} f[\varphi(\gamma)]\psi(\gamma) + \left({}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\varsigma\right)(\gamma)\, {}^{H}_{\omega}\mathscr{I}_1^{\sigma,\, z} f[\varphi(\gamma)]\psi(\gamma)}{\left({}^{H}_{\omega}\mathscr{I}_1^{\sigma,\, z}\varphi\right)(\gamma)\, {}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z} f[\varsigma(\gamma)]\psi(\gamma) + \left({}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\varphi\right)(\gamma)\, {}^{H}_{\omega}\mathscr{I}_1^{\sigma,\, z} f[\varsigma(\mu)]\psi(\mu)} \geq 1. \tag{38}$$

Proof. Following the same arguments as in the proof of Theorem 5, and according to the convexity of f with $f(0) = 0$, and convexity properties, the function $\frac{f(\gamma)}{\gamma}$ is increasing. As ς is increasing, the function $\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}$ is also increasing. Since $\frac{\varsigma(\gamma)}{\varphi(\gamma)}$ is decreasing, and considering all the hypotheses and facts mentioned above, we can deduce for all $\mu \in [1, \infty)$, $\gamma > 1$ the inequality (35). Now, taking product by $A(\mu,\ \sigma)$ on both sides of (35), then integrating the resulting inequality with respect to μ over $(1,\ \gamma)$, we get

$${}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\left(\frac{f(\varsigma(\gamma))}{\varsigma(\gamma)}\psi(\gamma)\varphi(\gamma)\right)\left[\frac{\omega^{-1}(\gamma)}{z^{\sigma}\Gamma(\sigma)}\int_1^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\mu}\right)\right]}{\left(\ln\frac{\gamma}{\mu}\right)^{1-\sigma}}\frac{\varsigma(\mu)\,\omega(\mu)}{\mu}d\mu\right]$$

$$+\left({}^{H}_{\omega}\mathscr{I}_1^{\tau,\, z}\varsigma\right)(\gamma)\left[\frac{\omega^{-1}(\gamma)}{z^{\sigma}\Gamma(\sigma)}\int_1^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\mu}\right)\right]}{\left(\ln\frac{\gamma}{\mu}\right)^{1-\sigma}}\left(\frac{f(\varsigma(\mu))}{\varsigma(\mu)}\psi(\mu)\varphi(\mu)\right)\frac{\omega(\mu)}{\mu}d\mu\right]$$

$$-{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\, f\left[\varsigma\left(\gamma\right)\right]\psi\left(\gamma\right)\left[\frac{\omega^{-1}\left(\gamma\right)}{z^{\sigma}\Gamma\left(\sigma\right)}\int_{1}^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\mu}\right)\right]}{\left(\ln\frac{\gamma}{\mu}\right)^{1-\sigma}}\frac{\varphi\left(\mu\right)\omega\left(\mu\right)}{\mu}d\mu\right]$$

$$-\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varphi\right)\left(\gamma\right)\left[\frac{\omega^{-1}\left(\gamma\right)}{z^{\sigma}\Gamma\left(\sigma\right)}\int_{1}^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\mu}\right)\right]}{\left(\ln\frac{\gamma}{\mu}\right)^{1-\sigma}}\left(\frac{f\left(\varsigma\left(\mu\right)\right)}{\varsigma\left(\mu\right)}\psi\left(\mu\right)\varsigma\left(\mu\right)\right)\frac{\omega\left(\mu\right)}{\mu}d\mu\right]\geq 0,$$

which leads to

$$\left({}_{\omega}^{H}\mathscr{I}_{1}^{\sigma,\,z}\varsigma\right)\left(\gamma\right){}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\left(\frac{f\left(\varsigma\left(\gamma\right)\right)}{\varsigma\left(\gamma\right)}\psi\left(\gamma\right)\varphi\left(\gamma\right)\right)+\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varsigma\right)\left(\gamma\right){}_{\omega}^{H}\mathscr{I}_{1}^{\sigma,\,z}\left(\frac{f\left(\varsigma\left(\gamma\right)\right)}{\varsigma\left(\gamma\right)}\psi\left(\gamma\right)\varphi\left(\gamma\right)\right)$$

$$\geq\left({}_{\omega}^{H}\mathscr{I}_{1}^{\sigma,\,z}\varphi\right)\left(\gamma\right){}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\, f\left[\varsigma\left(\gamma\right)\right]\psi\left(\gamma\right)+\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varphi\right)\left(\gamma\right){}_{\omega}^{H}\mathscr{I}_{1}^{\sigma,\,z}\, f\left[\varsigma\left(\mu\right)\right]\psi\left(\mu\right).$$

So, we have

$$\frac{\left({}_{\omega}^{H}\mathscr{I}_{1}^{\sigma,\,z}\varsigma\right)\left(\gamma\right){}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\left(\frac{f\left(\varsigma\left(\gamma\right)\right)}{\varsigma\left(\gamma\right)}\psi\left(\gamma\right)\varphi\left(\gamma\right)\right)+\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varsigma\right)\left(\gamma\right){}_{\omega}^{H}\mathscr{I}_{1}^{\sigma,\,z}\left(\frac{f\left(\varsigma\left(\gamma\right)\right)}{\varsigma\left(\gamma\right)}\psi\left(\gamma\right)\varphi\left(\gamma\right)\right)}{\left({}_{\omega}^{H}\mathscr{I}_{1}^{\sigma,\,z}\varphi\right)\left(\gamma\right){}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\, f\left[\varsigma\left(\gamma\right)\right]\psi\left(\gamma\right)+\left({}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\varphi\right)\left(\gamma\right){}_{\omega}^{H}\mathscr{I}_{1}^{\sigma,\,z}\, f\left[\varsigma\left(\mu\right)\right]\psi\left(\mu\right)}\geq 1. \tag{39}$$

In the view of the increase of the function $\frac{f\left(\gamma\right)}{\gamma}$ and that $\varsigma\left(\gamma\right)\leq\varphi\left(\gamma\right)$ for all $\gamma\in[1,\,\infty)$, we have for all $\kappa\in[1,\,\infty)$ that

$$\frac{f\left(\varsigma\left(\kappa\right)\right)}{\varsigma\left(\kappa\right)}\leq\frac{f\left(\varphi\left(\kappa\right)\right)}{\varphi\left(\kappa\right)}. \tag{40}$$

Taking product by $A\left(\kappa,\,\tau\right)\psi\left(\kappa\right)\varphi\left(\kappa\right)$, $\kappa\in\left(1,\,\gamma\right)$, $\gamma>1$ on both sides of (40), then integrating with respect to κ over $\left(1,\,\gamma\right)$, we have

$$\frac{\omega^{-1}\left(\gamma\right)}{z^{\tau}\Gamma\left(\tau\right)}\int_{1}^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\kappa}\right)\right]}{\left(\ln\frac{\gamma}{\kappa}\right)^{1-\tau}}\frac{f\left(\varsigma\left(\kappa\right)\right)\psi\left(\kappa\right)\varphi\left(\kappa\right)}{\varsigma\left(\kappa\right)}\frac{\omega\left(\kappa\right)}{\kappa}d\kappa$$

$$\leq\frac{\omega^{-1}\left(\gamma\right)}{z^{\tau}\Gamma\left(\tau\right)}\int_{1}^{\gamma}\frac{\exp\left[\frac{z-1}{z}\left(\ln\frac{\gamma}{\kappa}\right)\right]}{\left(\ln\frac{\gamma}{\kappa}\right)^{1-\tau}}\frac{f\left(\varphi\left(\kappa\right)\right)\psi\left(\kappa\right)\varphi\left(\kappa\right)}{\varphi\left(\kappa\right)}\frac{\omega\left(\kappa\right)}{\kappa}d\kappa,$$

which yielas

$$
{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\left(\frac{f\left(\varsigma\left(\gamma\right)\right)}{\varsigma\left(\gamma\right)}\varphi\left(\gamma\right)\psi\left(\gamma\right)\right)\leq{}_{\omega}^{H}\mathscr{I}_{1}^{\tau,\,z}\,f\left[\varphi\left(\gamma\right)\right]\psi\left(\gamma\right). \tag{41}
$$

Similarly, taking product by $A(\kappa,\ \sigma)\,\psi(\mu)\,\varphi(\mu)$, $\mu\in(1,\ \gamma)$, $\gamma>1$ on both sides of (40), then integrating with respect to μ over $(1,\ \gamma)$, we can deduce that

$$
{}_{\omega}^{H}\mathscr{I}_{1}^{\sigma,\,z}\left(\frac{f\left(\varsigma\left(\gamma\right)\right)}{\varsigma\left(\gamma\right)}\varphi\left(\gamma\right)\psi\left(\gamma\right)\right)\leq{}_{\omega}^{H}\mathscr{I}_{1}^{\sigma,\,z}\,f\left[\varphi\left(\gamma\right)\right]\psi\left(\gamma\right). \tag{42}
$$

Hence, from each of the inequalities (39), (41), and (42), we obtain the desired inequality (38). □

Remark 3 Applying Theorem 6 for $\tau=\sigma$, we get Theorem 5.

4. Concluding remarks

Undoubtedly, fractional calculus has garnered significant attention from numerous authors in recent years. In the context of our manuscript, we have uncovered and introduced a fresh perspective on weighted inequalities, relying on the notion of convex functions. To accomplish this, we have utilized the newly developed proportional Hadamard fractional integral operator. Moreover, we have established a series of novel inequalities using this particular fractional integral operator. Additionally, we have discussed specific instances of the results presented. Open problem: While the method is theoretically sound, its applicability may be restricted to specific types of functions or conditions. For instance, the inequalities derived may only hold under certain assumptions about the convexity or continuity of the functions involved, which may not be met in all real-world applications.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Kilbas AA, Srivastava HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations*. Amsterdam: Elsevier; 2006.

[2] Stefan SG, Kilbas AA, Marichev OI. *Fractional Integrals and Derivatives: Theory and Applications*. Yverdon: Gordon and Breach; 1993.

[3] Baloch AI, Chu YM. Petrović-type inequalities for harmonic *h*-convex functions, control, and the performance of firms in Indian industry. *Journal of Function Spaces*. 2020; 2020: 075390.

[4] Bougoffa L. On Minkowski and Hardy integral inequalities, control, and the performance of firms in Indian industry. *Journal of Inequalities in Pure and Applied Mathematics*. 2006; 7(2): 60.

[5] Khan S, Adil Khan M, Chu YM. Converses of the Jensen inequality derived from the Green functions with applications in information theory. *Mathematical Methods in the Applied Sciences*. 2020; 43(5): 2577-2587.

[6] Latif MA, Rashid S, Dragomir SS, Chu YM. Hermite-Hadamard type inequalities for coordinated convex and quasi-convex functions and their applications. *Journal of Inequalities and Applications*. 2019; 2019: 317.

[7] Aljaaidi TA, Pachpatte DB. The Hermite-Hadamard-Mercer type inequalities via generalized proportional fractional integral concerning another function. *International Journal of Mathematics and Mathematical Sciences*. 2022; 2022: 6716830.

[8] Ngo QA, Thang DD, Dat TT, Tuan DA. Notes on an integral inequality. *Journal of Inequalities in Pure and Applied Mathematics*. 2006; 2006: 120.
[9] Liu WJ, Ngo QA, Huy VN. Several interesting integral inequalities. *Journal of Mathematical Inequalities*. 2009; 3(2): 201-212.
[10] Dahmani Z. A note on some new fractional results involving convex functions. *Acta Mathematica Universitatis Comenianae*. 2012; LXXXI(2): 241-246.
[11] Chinchane VL, Pachpatte DB. On new fractional integral inequalities involving convex functions using Hadamard fractional integral. *Bulletin of the Allahabad Mathematical Society*. 2016; 31(2): 183-192.
[12] ldhaifallah M, Tomar M, Nisar KS, Purohit SD. Some new inequalities for (k, s)-fractional integrals. *Journal of Nonlinear Science and Applications*. 2016; 9(9): 5374-5381.
[13] Agarwal RP, Ozbekler A. Lyapunov type inequalities for mixed nonlinear Riemann-Liouville fractional differential equations with a forcing term. *Journal of Computational and Applied Mathematics*. 2017; 314: 69-78.
[14] Rahman G, Abdeljawad T, Jarad F, Khan A, Nisar KS. Certain inequalities via generalized proportional Hadamard fractional integral operators. *Advances in Difference Equations*. 2019; 2019: 454. Available from: https://doi.org/10.1186/s13662-019-2381-0.
[15] Alzabut J, Abdeljawad T, Jarad F, Sudsutad W. Gronwall inequality via the generalized proportional fractional derivative with applications. *Journal of Inequalities and Applications*. 2019; 2019: 101.
[16] Aljaaidi TA, Pachpatte DB, Shatanawi W, Abdo MS, Abodayeh K. Generalized proportional fractional integral functional bounds in Minkowski's inequalities. *Advances in Difference Equations*. 2021; 2021: 419.
[17] Aljaaidi TA, Pachpatte DB. Some Gruss-type inequalities using generalized Katugampola fractional integral. *AIMS Mathematics*. 2020; 5(2): 1011-1024.
[18] Aljaaidi TA, Pachpatte DB. Some Gruss-type Inequalities via ψ-Riemann-Liouville fractional integral. *Indian Journal of Mathematics*. 2020; 62(2): 249-268.
[19] Agarwal P, Tariboon J, Ntouyas SK. Some generalized Riemann-Liouville k-fractional integral inequalities. *Journal of Inequalities and Applications*. 2016; 2016: 122.
[20] Anber A, Dahmani Z. New integral results using Polya-Szegö inequality. *Acta et Commentationes Universitatis Tartuensis de Mathematic*. 2013; 17(2): 171-178.
[21] Dahmani Z, Tabharit L, Taf S. New generalisations of Gruss inequality using Riemann-Liouville fractional integrals. *Bulletin of Mathematical Analysis and Applications*. 2010; 2(3): 93-99.
[22] Dragomir SS, Diamond NT. Integral inequalities of Gruss type via Polya-Szegö and Shisha-Mond results. *East Asian Mathematical Journal*. 2003; 19(1): 27-39.
[23] Dragomir SS. Some integral inequalities of Gruss type. *Indian Journal of Pure and Applied Mathematics*. 2000; 31: 397-415.
[24] Dragomir SS. A generalization of Gruss's inequality in inner product spaces and applications. *Journal of Mathematical Analysis and Applications*. 1999; 237: 74-82.
[25] Aljaaidi TA, Pachpatte DB, Abdo MS, Botmart T, Ahmad H. (k, ψ)-Proportional fractional integral Polya-Szegö and Gruss-type inequalities. *Fractal and Fractional*. 2021; 5: 172.
[26] Jarad F, Abdeljawad T, Shah K. On the weighted fractional operators of a function with respect to another function. *Fractals*. 2020; 28(8): 2040011.
[27] Zhou SS, Rashid S, Rauf A, Jarad F, Hamed YS, Abualnaja KM. Ecient computations for weighted generalized proportional fractional operators with respect to a monotone function. *AIMS Mathematics*. 2021; 6: 8001-8029.