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Research Article

On $\underline{PGK}_2$-algebras and Perfect Extensions

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Abstract: The purpose of this paper is threefold. First, We study some basic features of principal GK_2-algebras with distributive skeletons ($\underline{PGK}_2$-algebras). The S-algebras, PS-algebras and modular S-algebras are determined with many properties. Second, the interplay between $\underline{PGK}_2$-algebras and $\underline{PGK}_2$-triples is revealed. PS-triples and principal Stone triples are used to build PS-algebras and principal Stone algebrasn respectively. We round off with perfect extensions of principal GK_2-algebras.

***Keywords*:** MS-algebras, K_2-algebras, GMS-algebras, GK_2-algebras, PGK_2-algebras, $\underline{PGK}_2$-algebras, d-subalgebras, perfect extensions

MSC: 08A05, 08A30

1. Introduction

In 1983, Blyth and Varlet [1] introduced the class **MS** of MS-algebras and, in [2], they obtained all the subclasses of **MS**. This class is an abstraction of the classes of de Morgan and Stone algebras. Many results on MS-algebras and related structures are established in [3–8].

In 1996, Ševcovic [9] dropped the distributive property of MS-algebras to get a new more general class the so called generalized MS-algebras (GMS-algebras). Badawy [10] introduced and characterized modular GK_2-algebras with distributive skeletons in terms of quadruples. In 2015, Badawy [11] considered a subclass $\mathbf{GK_2}$ (GK_2-algebras) of the class **GMS** (of all generalized MS-algebras) which contains the class $\mathbf{K_2}$. He constructed $\underline{PGK}_2$-algebras from PGK_2-triples and defined the isomorphism between two PGK_2-triples. Also, he proved a full correspondence between PGK_2-algebras and the associated PGK_2-triples. In [12], Badawy et al. studied 2-Permutability, n-Permutability, and strong extensions for PGK_2-algebras by using the congruence pair technique.

The present work build upon the previous as follows: In section 3, we introduce and characterize many special cases of principal GK_2-algebras. We introduce and constructed principal $\underline{GK}_2$-algebras with distributive skeletons ($\underline{PGK}_2$-algebras). Also, $\underline{PGK}_2$-triples are defined and utilised to reveal many properties of $\underline{PGK}_2$-algebras. Also, we determine S-algebras, principal S-algebras (PS-algebras) and modular S-algebras and study their properties. Also, we introduce and characterize S-triple, principal S-triples and principal Stone triples, then we construct principal S-algebras and principal

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Stone algebras via principal S-triples (PS-triples) and principal Stone triples, respectively. Finally, we determine and describe the largest principal S-algebras and principal Stone algebras. In section 4, perfect extensions of $\underline{PGK_2}$-algebras are considered. We proved that a $\underline{PGK_2}$-algebra C is a perfect extension of its d-subalgebra C_1 if and only if $C^{\circ\circ}$ is a perfect extension of $C_1^{\circ\circ}$ and $F(C)$ is a perfect extension of $F(C_1)$.

2. Preliminaries

This section contains the background material which we need in this paper. For details on lattices we refer to [13] and [14]; for details on MS-algebras and GMS-algebras see [1, 2, 9], and [15] and for details on GK_2-algebras and PGK_2-algebras we refer to [10–12].

A generalized De Morgan algebra (GM-algebra) is an algebra $(C;\ \vee,\ \wedge,\ ^{-},\ 0,\ 1)$ of type $(2,\ 2,\ 1,\ 0,\ 0)$ where $(C;\ \vee,\ \wedge,\ 0,\ 1)$ is a bounded lattice and for every $i,\ j \in C$ the unary operation $^{-}$ of involution satisfies:

$$\bar{\bar{i}} = i,$$

$$\overline{(i \vee j)} = \bar{i} \wedge \bar{j},$$

$$\bar{1} = 0.$$

A generalized Kleene algebra (GK-algebra) is a generalized De Morgan algebra with

$$i \wedge i^{\circ} \leq j \vee j^{\circ}, \text{ for every } i,\ j \in C.$$

A universal algebra $(C;\ \vee,\ \wedge,\ ^{\circ},\ 0,\ 1)$ where $(C;\ \vee,\ \wedge,\ 0,\ 1)$ is a bounded lattice is called a generalized MS-algebra (GMS-algebras) if:

$$i \leq i^{\circ\circ},$$

$$(i \wedge j)^{\circ} = i^{\circ} \vee j^{\circ},$$

$$1^{\circ} = 0.$$

Lemma 1 [13] Let C be a GMS-algebra, then for any elements $i,\ j$ of C, we have
(1) $0^{\circ} = 1$,
(2) $i \leq j \Rightarrow i^{\circ} \geq j^{\circ}$,
(3) $i^{\circ} = i^{\circ\circ\circ}$,
(4) $(i \vee j)^{\circ} = i^{\circ} \wedge j^{\circ}$,
(5) $(i \wedge j)^{\circ\circ} = i^{\circ\circ} \wedge j^{\circ\circ}$,
(6) $(i \vee j)^{\circ\circ} = i^{\circ\circ} \vee j^{\circ\circ}$.
Definition 1 [11] A GK_2-algebra C is a GMS-algebra satisfying
(1) $i \wedge j^{\circ} = i^{\circ\circ} \wedge i^{\circ}$,
(2) $i \wedge i^{\circ} \leq j \vee j^{\circ}$.

For any elements $i,\ j$ of C.

Definition 2 [12] An algebra $(C;\ \vee,\ \wedge,\ ^*,\ 0,\ 1)$ is called an S-algebra if $(C;\ \vee,\ \wedge,\ 0,\ 1)$ is a bounded lattice and a unary operation * satisfying

(1) $i \wedge i^* = 0$,

(2) $(i \vee j)^* = i^* \wedge j^*$,

(3) $1^* = 0$,

(4) $i \vee i^{**} = 1$.

It is known that an S-algebra $(C;\ ^*)$ is pseudo-complemented lattice (p-algebra) satisfying the Stone identity $i^* \vee i^{**} = 1$, where * is called the pseudo-complementation and $i^* = \max\{j \in C : i \wedge j = 0\}$.

Lemma 2 [11] Let C be a GK_2-algebra. Then

(1) $C^{\circ\circ} = \{i \in C : i = i^{\circ\circ}\}$ is a GK-algebra,

(2) $F(C) = \{i \in C : i^{\circ} = 0\}$ is a filter of C.

The algebra $C^{\circ\circ}$ is called the skeleton of C and $F(C)$ is called the filter of dense elements of L.

Definition 3 [11] A GK_2-algebra $(C;\ \vee,\ \wedge,\ ^{\circ},\ 0,\ 1)$ is said to be a PGK_2-algebra if:

(1) $F(C) = [d)$ for some $d \in C$, that is, $F(C)$ is a principal filter of C,

(2) The generator d is a distributive element of C, that is, $d \vee (i \wedge j) = (d \vee i) \wedge (d \vee j)$ for any $i,\ j \in C$,

(3) $i = i^{\circ\circ} \wedge (i \vee d)$ for any $i \in C$.

Definition 4 [11] A PGK_2-triple is $(N,\ F,\ \vartheta)$, where

(1) N is a GK-algebra,

(2) F is a bounded lattice,

(3) $\vartheta : N \longrightarrow F$ is a (0, 1)-lattice homomorphism from N into F and $\vartheta(n) = 0_F$ for any $n \in K^{\wedge}$.

Theorem 1 [11] Let $(N,\ F,\ \vartheta)$ be a PGK_2-triple.Then

$$I = \{(q,\ m) : q \in N,\ m \in F,\ m \leq \vartheta(q)\};$$

is a PGK_2-algebra with $F(I) = [(1_N,\ 0_F))$ if we define

$$(q,\ m) \vee (w,\ n) = (q \vee w,\ m \vee n)$$

$$(q,\ m) \wedge (w,\ n) = (q \wedge w,\ m \wedge n)$$

$$(q,\ m)^{\circ} = (q^{\circ},\ \vartheta(m^{\circ}))$$

$$1_I = (1_N,\ 1_F)$$

$$0_I = (0_N,\ 0_F).$$

Moreover, $I^{\circ\circ} \cong N$ and $F(I) \cong F$.

Theorem 2 [11] Let C be a principal GK_2-algebra with a smallest dense element d. Then any congruence relation θ of C determines a congruence pair $(\theta_{C^{\circ\circ}},\ \theta_{F(C)})$. Conversely, every congruences pair $(\theta_1,\ \theta_2)$ uniquely determines a congruence relation θ on C satisfies $\theta_{C^{\circ\circ}} = \theta_1$ and $\theta_{F(C)} = \theta_2$, by the rule $i \equiv j(\theta) \Leftrightarrow i^{\circ\circ} \equiv j^{\circ\circ}(\theta_1)$ *and* $i \vee d \equiv j \vee d(\theta_2)$.

Lemma 3 [11] Let C be a principal GK_2-algebra and $A(C)$ be the set of all congruence pairs of C. Then the following statements hold:

(1) $(\forall \beta \in \mathrm{Con}(F(C)))(\triangle_{C^{\circ\circ}},\ \beta) \in A(C)$,

(2) $(\forall \alpha \in \text{Con}(C^{\circ\circ}))(\alpha, \nabla_{F(C)}) \in A(C)$.

3. Basic properties of $\underline{PGK}_2$-algebras

In this section, we construct certain $\underline{PGK}_2$-algebras via certain $\underline{PGK}_2$-triples and study their related properties. We determine S-algebras, PS-algebras and modular S-algebras and study their properties. Also, we introduce and characterize S-triples, PS-triples and principal Stone triples. Then we construct PS-algebras and principal Stone algebras via PS-triples and principal Stone triples, respectively. Finally, we determine and describe the d-S-subalgebra and the largest d-Stone subalgebra of a modular $\underline{PGK}_2$-algebra.

Definition 5 If a PGK_2-algebra C has a distributive skeleton, that is, $C^{\circ\circ}$ is a Kleene algebra, we call it a $\underline{PGK}_2$-algebra.

Definition 6 A PGK_2-triple (N, F, ϑ) is called a $\underline{PGK}_2$-triple if N is a Kleene algebra.

Example 1 Figure 1 represents $\underline{PGK}_2$-algebra C with $F(C) = [d)$.

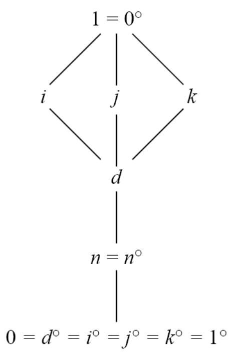


Figure 1. C is a $\underline{PGK}_2$-algebra

It is clear that $F(C) = [d)$ is a modular lattice and $C^{\circ\circ} = \{0, n, 1\}$ is a Kleene algebra which is isomorphic to K.

Theorem 3 Let (N, F, ϑ) be a $\underline{PGK}_2$-triple. Then

$$C = \{(n, i) : n \in N, \in F, i \leq \vartheta(n)\}$$

is a $\underline{PGK}_2$-algebra.

Proof. We know that C is a PGK_2-algebra from Theorem 1 such that $C^{\circ\circ} \cong N$. Since N is a Kleene algebra, then $C^{\circ\circ}$ is distributive. Thus $C^{\circ\circ}$ is a Kleene algebra. Hence, C is a $\underline{PGK}_2$-algebra. □

Let $(C; \vee, \wedge, {}^{\circ}, 0, 1)$ be a Kleene algebra. An element $j \in C$ is called a central element of C if $j \vee j^{\circ} = 1$. Then the set $T(C) = \{j \in C : j \vee j^{\circ} = 1\}$ is the greatest Boolean subalgebra of C and $T(C)$ is called the center of C.

Definition 7 A $\underline{PGK}_2$-triple (N, F, ϑ) is called a modular $\underline{PGK}_2$-triple if F is a bounded modular lattice. Theorem 4 describes the greatest d-S-subalgebra of a modular $\underline{PGK}_2$-algebra which is constructed from a modular $\underline{PGK}_2$-triple (N, F, ϑ).

Theorem 4 Let (N, F, ϑ) be a modular $\underline{PGK}_2$-triple, $T(N)$ the center of N and $F_1 = F$. Then the $\underline{PGK}_2$-algebra which associated with (N, F, ϑ) is a modular $\underline{GK}_2$-algebra and the following statements hold:

(1) $T(C^{\circ\circ}) = \{(n, \vartheta(n)) : n \in T(N)\} \cong T(N)$,

(2) $C_1 = \{(n, i) \in C : n \in T(N)\}$ is the largest d-S-subalgebra of C,

(3) $F(C_1) \cong F(C)$ and $T(C^{\circ\circ}) \cong C_1^{\circ\circ}$.

Proof. We have to prove that the PGK_2-algebra

$$C = \{(n,\ i) : n \in N,\ i \in F,\ i \leq \vartheta(n)\}$$

which is found from the $\underline{PGK_2}$-triple $(N,\ F,\ \vartheta)$ is a modular $\underline{GK_2}$-algebra. Let $(n,\ i),\ (m,\ j),\ (s,\ k) \in C$ and $(n,\ i) \geq (s,\ k)$.

Then, we have

$$\begin{aligned}(n,\ i) \wedge ((m,\ j) \vee (s,\ k)) &= (n,\ i) \wedge (m \vee s,\ j \vee k)\\ &= (n \wedge (m \vee s),\ i \wedge (m \vee s)) \text{ by the modularity of } N \text{ and } F\\ &= ((n \wedge m) \vee s,\ (i \wedge j) \vee k) \text{ as } n \geq s,\ i \geq k\\ &= (n \wedge m,\ i \wedge j) \vee (s,\ k).\end{aligned}$$

Therefore, C is a modular lattice. Thus, C is a modular $\underline{GK_2}$-algebra.

(i) We have

$$\begin{aligned}T(C^{\circ\circ}) &= \{(n,\ \vartheta(n)) \in C^{\circ\circ} : (n,\ \vartheta(n))^{\circ} \vee (n,\ \vartheta(n))^{\circ\circ} = (1,\ 1)\}\\ &= \{(n,\ \vartheta(n)) \in C^{\circ\circ} : (n^{\circ},\ \vartheta(n^{\circ})) \vee (n,\ \vartheta(n)) = (1,\ 1)\}\\ &= \{(n,\ \vartheta(n)) \in C^{\circ\circ} : (n \vee n^{\circ},\ \vartheta(n \vee n^{\circ})) = (1,\ 1)\}\\ &= \{(n,\ \vartheta(n)) \in C^{\circ\circ} : n \vee n^{\circ} = 1\}\\ &= \{(n,\ \vartheta(n)) \in C^{\circ\circ} : n \in T(N)\}\\ &\cong T(N).\end{aligned}$$

(ii) We need to show that the Stone identity $j^{\circ} \vee j^{\circ\circ} = 1$ holds for any $j = (n,\ i) \in C_1$.

$$\begin{aligned}(n,\ i)^{\circ} \vee (n,\ i)^{\circ\circ} &= (n^{\circ},\ \vartheta(n^{\circ})) \vee (n,\ \vartheta(n))\\ &= (n \vee n^{\circ},\ \vartheta(n \vee n^{\circ}))\\ &= (1,\ 1)\ as\ n \in T(N).\end{aligned}$$

Thus C_1 is an S-subalgebra of a $\underline{PGK_2}$-algebra. Since $(0,\ d) \in C$, then C_1 is an d-S-subalgebra of C. Let W be any d-S-subalgebra of C. Let $(n,\ i) \in W$. Then, $(n,\ i)^{\circ} \vee (n,\ i)^{\circ\circ} = (1,\ 1)$. Then $n \vee n^{\circ} = 1$ and so $n \in T(N)$. Then $(n,\ i) \in C_1$. Therefore, $W \subseteq C_1$.

(iii) We notice that

$$\begin{aligned}
F(C_1) &= \{(n,\ i) \in C_1 : (n,\ i)^{\circ} = (0_N,\ 0_F)\} \\
&= \{(n,\ i) \in C_1 : (n,\ \vartheta(n))^{\circ} = (0_N,\ 0_F)\} \\
&= \{(n,\ i) \in C_1 : n = 1_N,\ i \in F,\ i \leq \vartheta(n)\} \\
&= \{(1,\ i) : i \in F\} \\
&\cong F(C),
\end{aligned}$$

and

$$\begin{aligned}
C_1^{\circ\circ} &= \{(n,\ i)^{\circ\circ} : (n,\ i) \in C_1\} \\
&= \{(n,\ \vartheta(n)) : n \in T(N)\} \\
&\cong T(C^{\circ\circ}).
\end{aligned}$$

□

Remark 1 A GK_2-algebra C satisfying the Stone identity

$$i^{\circ} \vee i^{\circ\circ} = 1,$$

for all $i \in C$ need not to be an S-algebra as explained in the next example.

Example 2 In the following algebra, we see that C satisfies the Stone identity but C is not a pseudo-complemented lattice (p-algebra) as each of the elements $n,\ m,\ s,\ q,\ \alpha,\ \beta,\ \gamma$ and μ does not have a pseudo-complement on C.

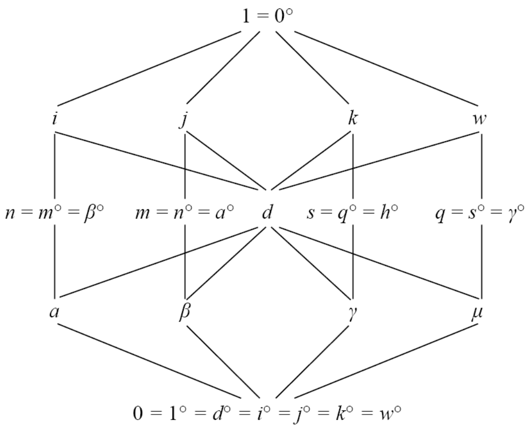


Figure 2. C is not a pseudo-complemented lattice

Remark 2 If C is a GK_2-algebra, then the identity $i \wedge i^\circ = i^\circ \wedge i^{\circ\circ}$ is not equivalent to the identity $i = i^{\circ\circ} \wedge (i \vee i^\circ)$, for all $i \in C$. For example, consider the GK_2-algebra C as in Figure 3.

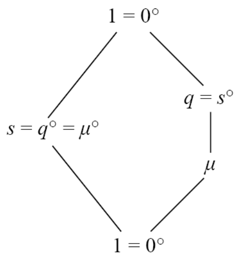


Figure 3. C is a GK_2-algebra

We observe that $\gamma^{\circ\circ} \wedge (\gamma \vee \gamma^\circ) = s \neq \gamma$ but C satisfies the identity $i \wedge i^\circ = i^{\circ\circ} \wedge i^\circ$ for all $i \in C$.

Definition 8 A PS-algebra is a $\underline{PGK_2}$-algebra C with $i \wedge i^\circ = 0$, for every $i \in C$.

Example 3 Figure 4 represents a modular PS-algebra C with $F(C) = [d)$.

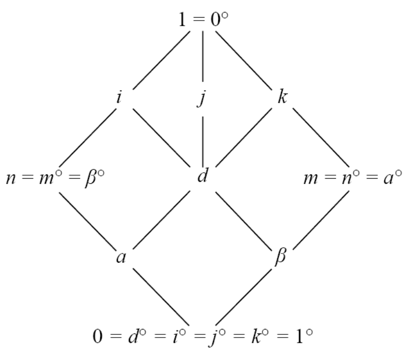


Figure 4. C is a modular PS-algebra

Definition 9 A principal S-triple (briefly PS-triple) is a $\underline{PGK_2}$-triple $(N,\ F,\ \vartheta)$, whenever N is a Boolean algebra.

Example 4 Figure 5 describes a PS-algebra. Also, we notice that $C^{\circ\circ} = \{0,\ a,\ b,\ 1\}$ is a Boolean subalgebra of C.

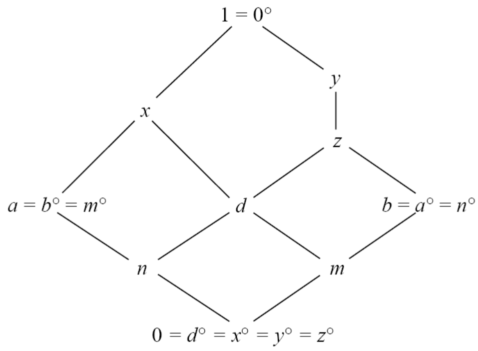


Figure 5. C is a non-modular PS-algebra

Now, we describe the ideal $C^{\wedge}$ and the filter $C^{\vee}$ of a $\underline{PGK_2}$-algebra from the $\underline{PGK_2}$-triple $(N,\ F,\ \vartheta)$ as follows.

Lemma 4 Let C be a PGK_2-algebra found from the $\underline{PGK_2}$-triple $(N,\ F,\ \vartheta)$. Then

(1) $C^{\wedge} = \{(n,\ 0_F) \in C : n \in N^{\wedge}\}$,

(2) $C^{\vee} = \{(n,\ i) \in C : n \in N^{\vee}\}$. Moreover $F(C) \subseteq C^{\vee}$,

(3) $C^{\circ\circ\vee} = C^{\circ\circ} \cap C^{\vee}$, where $C^{\circ\circ\vee}$ is a filter of $C^{\circ\circ}$.

Proof. (1) Let $(N,\ F,\ \vartheta)$ be a $\underline{PGK_2}$-triple. Using Theorem 1, we obtain the $\underline{PGK_2}$-algebra $C = \{(n,\ i) : n \in N,\ i \in F,\ i \leq \vartheta(n)\}$. Now,

$$C^{\wedge} = \{(n \wedge n^{\circ},\ i \wedge \vartheta(n^{\circ})) : n \in N,\ i \in F,\ i \leq \vartheta(n)\}$$

$$= \{(n \wedge n^{\circ},\ i \wedge \vartheta(n) \wedge \vartheta(n^{\circ})) : n \in N,\ i \in F,\ i \leq \vartheta(n)\}$$

$$= \{(n \wedge n^{\circ},\ i \wedge \vartheta(n \wedge n^{\circ})) : n \wedge n^{\circ} \in N^{\wedge},\ i \leq \vartheta(n)\}$$

$$= \{(n \wedge n^{\circ},\ i \wedge 0_F)\} \text{ as } \vartheta(n \wedge n^{\circ}) = 0_F \text{ by definition 3.5(3) of [1]}$$

$$= \{(m,\ 0_F) \in C : m = n \wedge n^{\circ} \in N^{\wedge}\}.$$

(2) We have

$$C^{\vee} = \{(n \vee n^{\circ},\ i \vee \vartheta(n^{\circ})) : n \in N,\ i \in F,\ i \leq \vartheta(n)\}$$

$$= \{(n \vee n^{\circ},\ \acute{i}) : n \in N,\ \acute{i} \in F,\ \acute{i} \leq \vartheta(n \vee n^{\circ})\}$$

$$= \{(\acute{n},\ \acute{i}) \in C : \acute{n} = n \vee n^{\circ} \in N^{\vee}\}.$$

Now, let $(1,\ i) \in F(C)$. Then $(1,\ i) = (1,\ i) \vee (1,\ i)^{\circ} \in C^{\vee}$, as $(1,\ i)^{\circ} = (0_N,\ 0_F)$. Therefore, $F(C) \subseteq C^{\vee}$.

(3) We have

$$C^{\circ\circ\vee} = \{(n,\ i) \in C : (n,\ i) \in C^{\circ\circ},\ (n,\ i) = (m,\ \vartheta(m)) \vee (m,\ \vartheta(m))^{\circ} \text{ for some } (m,\ \vartheta(m)) \in C^{\circ\circ}\}$$

$$= \{(n,\ i) \in C : (n,\ i) \in C^{\circ\circ},\ (n,\ i) = (m \vee m^{\circ},\ \vartheta(m \vee m^{\circ})),\ m \vee m^{\circ} \in N^{\vee}\}$$

$$= \{(n,\ i) : (n,\ i) \in C^{\circ\circ}\} \cap \{(n,\ i) : n = m \vee m^{\circ} \in N^{\vee}\}$$

$$= C^{\circ\circ} \cap C^{\vee}.$$

□

Theorem 5 Let $(N,\ F,\ \vartheta)$ be a PS-triple. Then

$$C = \{(n,\ i) : n \in N,\ i \in F,\ i \leq \vartheta(n)\}$$

is a PS-algebra such that $T(C) = C^{\circ\circ}$ and $C^{\vee} = F(C) = [(0,\ 1))$.

Proof. Let $(n,\ i) \in C$. Then

$$(n,\ i) \wedge (n,\ i)^{\circ} = (n,\ i) \wedge (n^{\circ},\ \vartheta(n^{\circ}))$$

$$= (n \wedge n^{\circ},\ \vartheta(n \wedge n^{\circ}))$$

$$= (0_N,\ 0_F) \text{ as } i \wedge i^{\circ} = 0.$$

Hence C is a PS-algebra. Now,

$$C^{\circ\circ} = \{(n,\ i) \in C : (n,\ i)^{\circ\circ} \vee (n,\ i)^{\circ} = (n,\ i) \vee (n,\ i)^{\circ}\}$$

$$= \{(n,\ i) \in C : ((n,\ i)^{\circ} \wedge (n,\ i))^{\circ} = (n,\ i) \vee (n,\ i)^{\circ}\}$$

$$= \{(n,\ i) \in C : (0,\ 0)^{\circ} = (n,\ i) \vee (n,\ i)^{\circ}\}$$

$$= \{(n,\ i) \in C : (1,\ 1) = (n,\ i) \vee (n,\ i)^{\circ}\}$$

$$= T(C),$$

where $(n,\ i) \wedge (n,\ i)^{\circ} = (0,\ 0)$. By lemma 4, we obtain

$$C^{\vee} = \{(n,\ i) \in C : n \in N^{\vee}\}$$

$$= \{(1_N,\ i) \in C : i \in F\}$$

$$= F(C).$$

□

Definition 10 An S-algebra C is called a modular S-algebra (Stone algebra) if C is a modular (distributive) lattice.

Definition 11 A PS-triple $(N,\ F,\ \vartheta)$ is called a modular PS-triple (principal Stone triple) if F is a bounded modular (distributive) lattice.

Corollary 1 Let $(N,\ F,\ \vartheta)$ be a modular PS-triple (principal Stone triple). Then

$$C = \{(n,\ i) : n \in N,\ i \in F,\ i \leq \vartheta(n)\}$$

is a modular PS-algebra (principal Stone algebra) such that $T(C) = C^{\circ\circ}$ and $F(C) = C^{\vee}$.

Proof. By Theorem 5, C is a PS-algebra. Now, we need only to show that C is modular. Indeed, let $(n,\ i),\ (m,\ j),\ (s,\ k) \in C$ and $(n,\ i) \geq (s,\ k)$. Then we have

$$(n,\ i) \wedge ((m,\ j) \vee (s,\ k)) = (n,\ i) \wedge (m \vee s,\ j \vee k)$$

$$= (n \wedge (m \vee s),\ i \wedge (j \vee k)) \text{ by the modularity of } N \text{ and } F$$

$$= ((n \wedge m) \vee s,\ (i \wedge j) \vee k) \text{ as } n \geq s,\ i \geq k$$

$$= (n \wedge m,\ i \wedge j) \vee (s,\ k).$$

Thus, C is a modular PS-algebra. On the other hand, if $(N,\ F,\ \vartheta)$ is a principal Stone triple, then we have to prove that a PS-algebra which constructed by Theorem 5 is a Stone algebra, that is, C is distributive. Let $(n,\ i),\ (m,\ j),\ (s,\ k) \in C$. Then, we have

$$(n,\ i) \wedge ((m,\ j) \vee (s,\ k)) = (n \wedge (m \vee s),\ i \wedge (j \vee k))$$

$$= ((n \wedge m),\ (i \wedge j)) \vee ((n \wedge s),\ (i \wedge k))$$

$$= ((n,\ i) \wedge (m,\ j)) \vee ((n,\ i) \wedge (s,\ k)).$$

Thus, C is a principal Stone algebra such that $T(C) = C^{\circ\circ}$ and $F(C) = C^{\vee} = [(0,\ 1)).$ □

4. Perfect extensions of $\underline{PGK}_2$-algebras

In this section, a full description of perfect extensions of $\underline{PGK}_2$-algebras is given in Theorem 6. Also, examples are provided to illustrate all concepts and results.

Definition 12 A subalgebra C_1, with $F(C_1) = [d_1)$, of a $\underline{PGK}_2$-algebra C with $F(C) = [d)$ is a d-subalgebra if $d_1 = d$, that is, $F(C_1)$ is a bounded sublattice of $F(C)$.

Definition 13 A $\underline{PGK}_2$-algebra C_1 is an extension of a $\underline{PGK}_2$-algebras if C is a d-subalgebra of C_1. If every congruence on C has exactly one extension to C_1, we say that C_1 is a perfect extension of C.

Example 5 Figure 6 represents a $\underline{PGK}_2$-algebra C with $F(C) = [d)$.

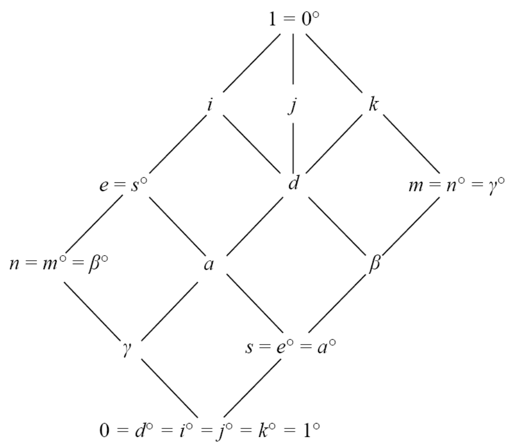


Figure 6. C is a $\underline{PGK}_2$-algebra

Consider the d-subalgebra C_1 of a $\underline{PGK}_2$-algebra C (see Figure 7: C_1).

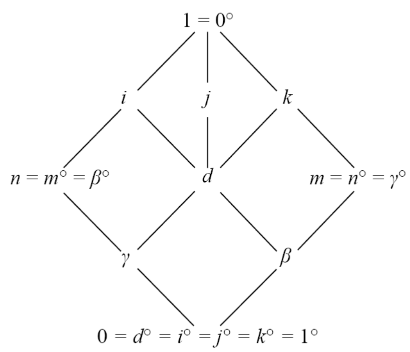


Figure 7. C_1 is a $\underline{PGK}_2$-algebra

Now, we describe $\text{Con}(C)$ and $\text{Con}(C_1)$ as in follows:

$\bigtriangledown_C = C \times C$,

$\triangle_C = \{(i, i) : i \in C\}$,

$\Phi_C = \{\{0\}, \{n, \gamma\}, \{e, \alpha\}, \{m, \beta\}, \{s\}, \{i, j, k, d, 1\}\}$,

$\theta_C = \{\{0, n, \gamma\}, \{e, \alpha, s\}, \{i, j, k, d, m, \beta, 1\}\}$,

$\vartheta_C = \{\{0, s, m, \beta\}, \{i, j, k, d, n, e, \alpha, \gamma, 1\}\}$,

and

$\bigtriangledown_{C_1} = C_1 \times C_1$,
$\triangle_{C_1} = \{(i,\ i) : i \in C_1\}$,
$\Phi_{C_1} = \{\{0\},\ \{n,\ \gamma\},\ \{m,\ \beta\},\ \{d,\ i,\ j,\ k,\ 1\}\}$,
$\theta_{C_1} = \{\{0,\ n,\ \gamma\},\ \{m,\ \beta,\ d,\ i,\ j,\ k,\ 1\}\}$,
$\vartheta_{C_1} = \{\{0,\ m,\ \beta\},\ \{i,\ j,\ k,\ d,\ n,\ \gamma,\ 1\}\}$.
The description of restrictions is summarized in Table 1.

Table 1. Description of restrictions

$\mid$	C_1
$\bigtriangledown_C$	$\bigtriangledown_{C_1}$
$\triangle_C$	$\triangle_{C_1}$
Φ_C	Φ_{C_1}
θ_C	θ_{C_1}
ϑ_C	ϑ_{C_1}

We observe that
$\bigtriangledown_C$ is a unique extension of $\bigtriangledown_{C_1}$,
$\triangle_C$ is a unique extension of $\triangle_{C_1}$,
Φ_C is a unique extension of Φ_{C_1},
θ_C is a unique extension of θ_{C_1},
ϑ_C is a unique extension of ϑ_{C_1}.
Therefore every congruence on C_1 has exactly one extension to C. Thus C is a perfect extension of C_1.

Theorem 6 Let C be a $\underline{PGK}_2$-algebra and let C_1 be a d-subalgebra of C. Then C is a perfect extension of C_1 if and only if

(1) $F(C)$ is a perfect extension of $F(C_1)$,

(2) $C^{\circ\circ}$ is a perfect extension of ${C_1}^{\circ\circ}$.

Proof. Let C be a perfect extension of C_1. Let $\beta_2 \in \mathrm{Con}(F(C_1))$. Now we assume that $\acute{\beta}_2,\ \bar{\beta}_2 \in \mathrm{Con}(F(C))$ such that $\acute{\beta}_{2_{F(C_1)}} = \bar{\beta}_{2_{F(C_1)}} = \beta_2$. Then, by Lemma 3, we get $(\triangle_{C^{\circ\circ}},\ \acute{\beta}_2),\ (\triangle_{C^{\circ\circ}},\ \bar{\beta}_2) \in A(C)$ and $(\triangle_{C_1^{\circ\circ}},\ \beta_2) \in A(C_1)$. According to Theorem 3, there exist $\acute{\beta},\ \bar{\beta} \in \mathrm{Con}(C)$ and $\beta \in \mathrm{Con}(C_1)$ corresponding to $(\triangle_{C^{\circ\circ}},\ \acute{\beta}_2),\ (\triangle_{C^{\circ\circ}},\ \bar{\beta}_2)$ and $\beta = (\triangle_{C_1^{\circ\circ}},\ \beta)$, respectively. We have $\acute{\beta}_{C_1} = \bar{\beta}_{C_1} = \beta$. Since C is a perfect extension of C_1, then $\acute{\beta} = \bar{\beta}$. Hence, $\acute{\beta}_2 = \bar{\beta}_2$, proving (1). On the other hand, we need to show that $C^{\circ\circ}$ is a perfect extension of $C_1^{\circ\circ}$, let $\beta_1 \in \mathrm{Con}({C_1}^{\circ\circ})$ and β_1 has an extension to a congruence of $C^{\circ\circ}$.

To show this extension is unique. Let $\acute{\beta}_1,\ \bar{\beta}_1 \in \mathrm{Con}(C^{\circ\circ})$ with $\acute{\beta}_{1_{C_1^{\circ\circ}}} = \bar{\beta}_{1_{C_1^{\circ\circ}}} = \beta_1$. Then, we have

$$(\acute{\beta}_1,\ \bigtriangledown_{F(C)}),\ (\bar{\beta}_1,\ \bigtriangledown_{F(C)}) \in A(C)$$

and

$$(\beta,\ \bigtriangledown_{F(C_1)}) \in A(C_1).$$

Again, we see that there exist $\acute{\beta},\ \bar{\beta} \in \mathrm{Con}(C)$ and $\beta \in \mathrm{Con}(C)$ corresponding to $(\acute{\beta}_1,\ \bigtriangledown_{F(C)}),\ (\bar{\beta}_1,\ \bigtriangledown_{F(C)})$ and $\beta = (\beta,\ \bigtriangledown_{F(C_1)})$, respectively. We see that

$$\acute{\beta}_{C_1} = \bar{\beta}_{C_1} = \beta.$$

As C is a perfect extension of C_1, then

$$\acute{\beta} = \bar{\beta}.$$

Therefore

$$\acute{\beta}_1 = \bar{\beta}_1.$$

This proves (2). Conversely, let $\beta \in \mathrm{Con}(C_1)$. If β is extended to C, then we will prove that this extension is unique. Let $\acute{\beta}$, $\bar{\beta}$ are extensions of β in $\mathrm{Con}(C)$. Then, the congruences $\acute{\beta}$, $\bar{\beta}$ and β can be represented by the congruence pairs $(\acute{\beta}_1, \acute{\beta}_2)$, $(\bar{\beta}_1, \bar{\beta}_2)$ and (β_1, β_2), respectively. Where

$$\acute{\beta}_{1_{{C_1}^{\circ\circ}}} = \bar{\beta}_{1_{{C_1}^{\circ\circ}}} = \beta_1$$

and

$$\acute{\beta}_{2_{F(C_1)}} = \bar{\beta}_{2_{F(C_1)}} = \beta_2.$$

By (1) and (2), we get

$$\acute{\beta}_1 = \bar{\beta}_1 \text{ and } \acute{\beta}_2 = \bar{\beta}_2.$$

Therefore, $\acute{\beta} = \bar{\beta}$. □

The next examples show how Theorem 6 can be applied.

Example 6 Consider the Kleene algebra $C^{\circ\circ}$ in Figure 8. From Table 2, we show that $C^{\circ\circ}$ is a perfect extension of ${C_1}^{\circ\circ}$. Also, Table 3 shows that $F(C)$ is a perfect extension of $F(C_1)$, respectively.

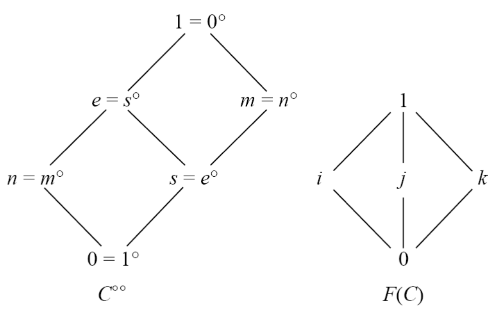


Figure 8. $C^{\circ\circ}$ and $F(C)$

Also, we describe ${C_1}^{\circ\circ}$ and $F(C_1)$ as follows:

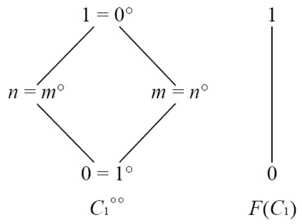


Figure 9. $C^{\circ\circ}$ and $F(C_1)$

Now, we introduce Con$(C^{\circ\circ})$ and Con$({C_1}^{\circ\circ})$ in Table 2.

Table 2. Con$(C^{\circ\circ})$ and Con$({C_1}^{\circ\circ})$

$Con(C^{\circ\circ})$	$Con({C_1}^{\circ\circ})$
$\bigtriangledown_{C^{\circ\circ}} = C^{\circ\circ} \times C^{\circ\circ}$	$\bigtriangledown_{{C_1}^{\circ\circ}} = {C_1}^{\circ\circ} \times {C_1}^{\circ\circ}$
$\bigtriangleup_{C^{\circ\circ}} = \{(i, i) : i \in C^{\circ\circ}\}$	$\bigtriangleup_{{C_1}^{\circ\circ}} = \{(i, i) : i \in {C_1}^{\circ\circ}\}$
$\theta_{C^{\circ\circ}} = \{\{0, m, s\}, \{n, e, 1\}\}$	$\theta_{C_1^{\circ\circ}} = \{\{0, m\}, \{n, 1\}\}$
$\psi_{C^{\circ\circ}} = \{\{0, n\}, \{e, s\}, \{m, 1\}\}$	$\psi_{L_1^{\circ\circ}} = \{\{0, n\}, \{m, 1\}\}$

and again we deduce Con$(F(C))$ and Con$(F(C_1))$ in Table 3.

Table 3. Con$(F(C))$ and Con$(F(C_1))$

$Con(F(C))$	$Con(F(C_1))$
$\bigtriangledown_{F(C)} = F(C) \times F(C)$	$\bigtriangledown_{F(C_1)} = F(C_1) \times F(C_1)$
$\bigtriangleup_{F(C)} = \{(i, i) : i \in F(C)\}$	$\bigtriangleup_{F(C_1)} = \{(i, i) : i \in F(C_1)\}$

Table 4 contains the restrictions.

Table 4. Restrictions

$\mid$	${C_1}^{\circ\circ}$
$\bigtriangledown_{C^{\circ\circ}}$	$\bigtriangledown_{{C_1}^{\circ\circ}}$
$\bigtriangleup_{C^{\circ\circ}}$	$\bigtriangleup_{{C_1}^{\circ\circ}}$
$\theta_{C^{\circ\circ}}$	$\theta_{C_1^{\circ\circ}}$
$\psi_{C^{\circ\circ}}$	$\psi_{C_1^{\circ\circ}}$

We observe from Table 4 that
$\bigtriangledown_{C^{\circ\circ}}$ is a unique extension of $\bigtriangledown_{C_1^{\circ\circ}}$,
$\bigtriangleup_{C^{\circ\circ}}$ is a unique extension of $\bigtriangleup_{C_1^{\circ\circ}}$,
$\theta_{C^{\circ\circ}}$ is a unique extension of $\theta_{C_1^{\circ\circ}}$,
$\psi_{C^{\circ\circ}}$ is a unique extension of $\psi_{C_1^{\circ\circ}}$.

Therefore every congruence on $C_1^{\circ\circ}$ has exactly one extension to $C^{\circ\circ}$. Thus $C^{\circ\circ}$ is a perfect extension of $C_1^{\circ\circ}$.
and

Table 5. $F(C_1)$

$\mid$	$F(C_1)$
$\bigtriangledown_{F(C)}$	$\bigtriangledown_{F(C_1)}$
$\bigtriangleup_{F(C)}$	$\bigtriangleup_{F(C_1)}$

We observe from Table 5 that
$\bigtriangledown_{F(C)}$ is a unique extension of $\bigtriangledown_{F(C_1)}$,
$\bigtriangleup_{F(C)}$ is a unique extension of $\bigtriangleup_{F(C_1)}$,
Therefore every congruence on $F(C_1)$ has exactly one extension to $F(C)$. Thus $F(C)$ is a perfect extension of $F(C_1)$.

In the following example, we give a $\underline{PGK}_2$-algebra C which has not perfect extensions of C_1 as well as $F(C)$ has not also perfect extension of $F(C_1)$.

Example 7 Figure 10 represents a $\underline{PGK}_2$-algebra L with $F(C) = [d)$.

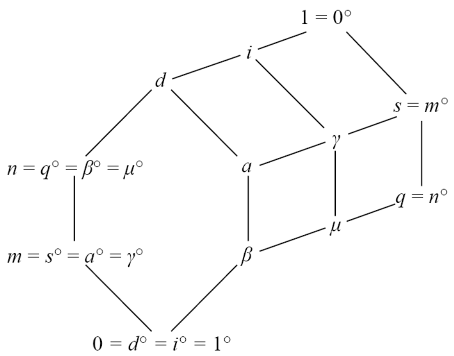


Figure 10. C is a $\underline{PGK}_2$-algebra with $F(C) = [d)$

The set of all congruences on C are:
$\theta_1 = \bigtriangledown_C = C \times C$,
$\theta_2 = \bigtriangleup_C = \{(n, n) : n \in C\}$,
$\theta_3 = \{\{0\}, \{n\}, \{m\}, \{\alpha, \gamma, s\}, \{\beta, \mu, q\}, \{1, i, d\}\}$,
$\theta_4 = \{\{0\}, \{n\}, \{m\}, \{\alpha, \gamma\}, \{\beta, \mu\}, \{s\}, \{q\}, \{d, i\}, \{1\}\}$,
$\theta_5 = \{\{0\}, \{n\}, \{m\}, \{\alpha\}, \{\beta\}, \{\gamma, s\}, \{\mu, q\}, \{d\}, \{i, 1\}\}$,
$\theta_6 = \{\{0\}, \{m, n\}, \{\alpha, \beta, \gamma, \mu, s, q\}, \{d, i, 1\}\}$,
$\theta_7 = \{\{0\}, \{m, n\}, \{\alpha, \beta\}, \{\gamma, \mu, s, q\}, \{d\}, \{i, 1\}\}$,
$\theta_8 = \{\{0\}, \{m, n\}, \{s, q\}, \{\alpha, \beta, \mu, \gamma\}, \{d, i\}, \{1\}\}$,
$\theta_9 = \{\{0\}, \{n, m\}, \{\gamma, \mu\}, \{\alpha, \beta\}, \{s, q\}, \{d\}, \{i\}, \{1\}\}$,
$\theta_{10} = \{\{0\}, \{s, \alpha, \gamma\}, \{\beta, \mu, q\}, \{m\}, \{n\}, \{1, d, i\}\}$,
$\theta_{11} = \{\{0, n, m\}, \{1, i, d, \alpha, \beta, \gamma, \mu, s, q\}\}$,
$\theta_{12} = \{\{1, d, i, m, n\}, \{0, \alpha, \beta, \gamma, \mu, s, q\}\}$,
$\theta_{13} = \{\{0, m, n\}, \{d, \alpha, \beta\}, \{i, \gamma, \mu\}, \{s, q, 1\}\}$,
$\theta_{14} = \{\{0, n, m\}, \{d, i, \alpha, \beta, \gamma, \mu\}, \{q, s, 1\}\}$,
$\theta_{15} = \{\{0, n, m\}, \{d, \alpha, \beta\}, \{i, \gamma, \mu, s, q, 1\}\}$,

Consider the d-subalgebra C_1 of a $\underline{PGK_2}$-algebra C as in Figure 11.

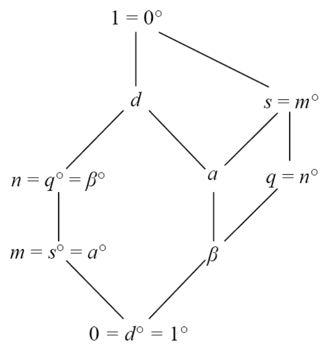
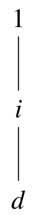


Figure 11. C_1 is a d-subalgebra of C

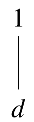
The set of all congruences on C_1 are:
$\psi_1 = \bigtriangledown_{C_1} = C_1 \times C_1$,
$\psi_2 = \bigtriangleup_{C_1} = \{(i,\ i) : i \in C_1\}$,
$\psi_3 = \{\{1,\ d\},\ \{m,\ n\},\ \{\alpha,\ \beta,\ s,\ q\},\ \{0\}\}$,
$\psi_4 = \{\{1,\ d\},\ \{n\},\ \{m\},\ \{\alpha,\ s\},\ \{\beta,\ q\},\ \{0\}\}$,
$\psi_5 = \{\{1\},\ \{d\},\ \{n,\ m\},\ \{\alpha,\ \beta\},\ \{s,\ q\},\ \{0\}\}$,
$\psi_6 = \{\{1,\ d,\ \alpha,\ \beta,\ s,\ q\},\ \{0,\ n,\ m\}\}$,
$\psi_7 = \{\{1,\ d,\ m,\ n\},\ \{0,\ \alpha,\ \beta,\ s,\ q\}$,
$\psi_8 = \{\{1,\ s,\ q\},\ \{d,\ \alpha,\ \beta\},\ \{0,\ n,\ m\}\}$.
It is clear that

$$F(C) = \{d,\ i,\ 1\},\ F(C_1) = \{1,\ d\}.$$



$F(C)$

The set of all lattice congruences on $F(C)$ are:
$\phi_1 = \bigtriangledown_{F(C)}$,
$\phi_2 = \bigtriangleup_{F(C)}$,
$\phi_3 = \{\{d,\ i\},\ \{1\}\}$,
$\phi_4 = \{\{d\},\ \{i,\ 1\}\}$.



$F(C_1)$

The set of all lattice congruences on $F(C_1)$ are:

$\omega_1 = \triangle_{F(C_1)}$,

$\omega_2 = \triangledown_{F(C_1)}$.

To clarify that C has not perfect extensions of C_1 as well as $F(C)$ has not also perfect extension of $F(C_1)$. We consider $\theta_8,\ \theta_9,\ \theta_{10} \in \mathrm{Con}(C)$ and $\psi_5,\ \psi_8 \in \mathrm{Con}(C_1)$ as follows:

$$\theta_8 = \{\{0\},\ \{n,\ m\},\ \{s,\ q\},\ \{\alpha,\ \beta,\ \mu,\ \gamma\},\ \{d,\ i\},\ \{1\}\},$$

$$\theta_9 = \{\{0\},\ \{n,\ m\},\ \{\gamma,\ \mu\},\ \{\alpha,\ \beta\},\ \{s,\ q\},\ \{d\},\ \{i\},\ \{1\}\},$$

and

$$\psi_5 = \{\{0\},\ \{n,\ m\},\ \{s,\ q\},\ \{\alpha,\ \beta\},\ \{1\},\ \{d\}\},$$

We observe that the restrictions $\theta_8 \mid C_1 = \psi_5$ and $\theta_9 \mid C_1 = \psi_5$. Then we say that C has not perfect extension of C_1 as a congruence on C_1 has more than one extension to C. On the other hand, we again consider $\phi_3,\ \phi_4 \in \mathrm{Con}(F(C))$ and $\omega_1,\ \omega_2 \in \mathrm{Con}(F(C_1))$ as follows:

$$\phi_3 = \{\{d,\ i\},\ \{1\}\},$$

$$\phi_4 = \{\{d\},\ \{i,\ 1\}\}.$$

and

$$\omega_1 = \triangle_{F(C_1)},$$

$$\omega_2 = \triangledown_{F(C_1)}.$$

We observe that the restrictions $\phi_3 \mid F(C_1) = \omega_1$ and $\phi_4 \mid F(C_1) = \omega_1$. Then we say that $F(C)$ has not perfect extension of $F(C_1)$ as a congruence on $F(C_1)$ has more than one extension to $F(C)$.

5. Conclusions

In this paper, we focused on the construction $\underline{PGK}_2$-algebras via certain $\underline{PGK}_2$-triples and studied their related properties. We determined S-algebras, principal S-algebras, modular S-algebras and studied their main properties. Also, S-triples, principal S-triples and principal Stone triples are introduced and explained. Principal S-algebras and principal Stone algebras were constructed via principal S-triples and principal Stone triples, respectively. In addition, we determined and described the d-S-subalgebras and the largest d-Stone subalgebra of a modular $\underline{PGK}_2$-algebra. Finally, we characterized perfect extensions of PGK_2-algebras in terms of congruence pairs. The present paper motivates many points for future work. For example, many aspects of $\underline{PGK}_2$-algebras as ideals, and filters of $\underline{PGK}_2$-algebras can be investigated using the findings of the paper. Also, permutability of $\underline{PGK}_2$-algebras can be considered via the triple technique that we modified.

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Conflict of interest

The authors declare no competing interest.

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