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Research Article

A Wavelet Multi-Scale Takagi-Sugeno Fuzzy Approach for Financial Time Series Modeling

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Abstract: Fuzzy logic has been introduced as a modeler suitable for many situations where the data may be uncertain, and difficult to be described via the existing exact and analytic tools. However, although fuzzy models have succeeded to fit many situations, they fail in many others especially nonlinear, non-stationary, volatile, fuzzy, and fluctuated data such as financial time series. In this context, the need has emerged for more effective models to describe the data while preserving the fuzzy model as a basic descriptor of data fuzziness. In the present work, we develop a hybrid approach combining the Takagi-Sugeno fuzzy model with the wavelet decomposition to investigate financial time series as complex systems. The new approach showed effectively a high performance compared to existing methods via error estimates and Lyapunov theory of stability. The model is applied empirically to the Saudi Arabia Tadawul market traded over the period January 01, 2011, to December 31, 2022, a period characterized by many critical movements and phenomena such as the Arab Spring, Qatar embargo, Yemen war, NEOM project, 2030 KSA vision and the last COVID-19 pandemic, which makes its study of great importance to understand markets situations and also for policymakers, managers and investors.

***Keywords*:** financial time series, modeling, dynamics, wavelets, fuzzy models, Lyapunov stability

MSC: 91B84, 31M10, 65T60, 90C70

1. Introduction

The economic and financial system of any country is based on an essential component known as the financial market. Any country (industrial, developed, developing) owns at least one financial market or has to create it to take its place in the world. Each financial market has its proper measure to be exploited for investigation, analysis, forecasting, management, and so on. Generally, this measure is expressed by a quantitative factor known as the market index, which assures the link to the theory of time series. We call these financial indices the financial time series.

Time series analysis, modeling, and forecasting, especially in economics, finance, and marketing, are, indeed, valuable and exciting research fields providing opportunities for scientists, managers, policymakers, and also consumers to understand the real behavior of these series data, and to answer their important questions about them. Financial and economic time series are characterized by many hidden structures such as dynamics, fluctuations, volatility, heterogeneity, and uncertainty or fuzziness, due to many causes such as the data collection, its origin, its recording, and so on.

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Heterogeneity, for example, is the best property of time series able to reflect the statistical characteristics of the data at different time intervals or localizations. This led researchers to develop suitable tools to describe the data via appropriate models that take into consideration the factors, the characteristics, and the structures related. Nowadays, many powerful tools are known and applied such as wavelets, machine learning, fuzzy logic, Fourier analysis, stochastic calculus, optimization, and so on.

In the present paper, we aim to study the challenges facing time series modelers by providing a new hybrid model combining wavelet decomposition with fuzzy models, especially the Takagi-Sugeno (TS) fuzzy model, initially introduced in [1]. In recent years, the TS fuzzy model has been an effective method to analyze and synthesize complex systems, such as those issued from time series, due to its ability to transform complex systems into several simple ones using suitable membership functions, approximating the complex functions behind the data smoothly with arbitrary precision in appropriate spaces. Understanding these series, and thus developing suitable models will be of great utility in forecasting purposes, understanding the time factor impact, substantive phenomena, empirical generalizations, quantifying short/long-term, and permanent effects, and so on. See for instance [2–7] for backgrounds on fuzzy logic, fuzzy systems, fuzzy models, and applications.

The TS fuzzy model was shown to be effective for nonlinear cases, meanwhile its accuracy needs a high regularity [8], which is not the case for time series such as financial ones. More precisely, a concept of the universal fuzzy controller improving Sugeno model was introduced by defining controllable processes. Such a concept was applied for continuous functions' approximation using a transfer function. In [9], the authors proposed an iterative scheme to improve the TS model by updating the structure and the parameters of the model in each step. The evolved model was shown effective against many complex processes, and competitive against neural networks. In [10], a review of fuzzy regression and several applications in management science was developed. The efficiency of fuzzy logic against data problems was discussed via empirical cases. Fuzzy regression has been widely investigated as the simplest way to involve fuzzy logic in financial or actuarial mathematics. Chang and Ayyub [11] investigated three types of fuzzy regression and concluded that a hybrid model combining least-squares regression and fuzzy regression may be suitable for investigating both randomness and fuzziness types of uncertainty. In [12], a two-stage approach was developed for a fuzzy linear regression model based on least-squares crisp regression and fuzzy error estimation. Tanaka et al. [13] introduced the concept of the fuzzy structure of a system via a fuzzy linear function with fuzzy sets as parameters using the original Zadeh extension principle. Such a concept was exploited to formulate a fuzzy linear regression. The model was applied to specific financial time series. Wang and Tsaur [14] investigated the use of fuzzy regression for limited and imprecise data, and where the variables may interact uncertainly, qualitatively, or in a fuzzy way. Watada et al. [15] applied fuzzy regression to analyze the research and development proposals experts' evaluation. The proposed model combined fuzzy regression with the principal component analysis, and dual scaling. Yen et al. [16] investigated a linear fuzzy regression model possessing symmetric triangular fuzzy number coefficients, and proposed an extension to the case of non-symmetric fuzzy triangular coefficients. The model was shown to be suitable for many concrete cases of applications. Bakdi et al. [17] showed that fuzzy logic combined with genetic algorithms may provide a performant modeler for mobile robots. In a close study, Mehran [18] discussed the ability of the TS fuzzy model in control and automation, based on applications in robotics and combining with artificial intelligence tools. In [19], a TS fuzzy model has been applied based on an $l_2 - L_\infty$ performance analysis and control approach for time series by using an associated Lyapunov function for stability. In [20], a TS fuzzy model was applied for a nonlinear vehicle lateral. Both the controller and the quantizer were investigated via a Lyapunov function and a two-step design strategy. Calderaro et al. [21], and Galdi et al. [22] developed a fuzzy controller for improving wind power production. The model is based on a data-driven design scheme able to generate the fuzzy model ensuring thus a maximum energy extraction. Fantuzzi and Rovatti [23] showed the ability of the TS fuzzy model to approximate functions accurately and analytically.

Wavelets are effectively the better candidates nowadays in localizing extreme values and anomalies in the data. In [24], wavelets are involved in a neural network model to ensure an accurate functional approximation, which is applied empirically with the SP500 index. Makarichev et al. [25] developed an atomic wavelet scheme for data processing via an approximation of coefficients. Ben Mabrouk and Zaafrane [26] developed a hybrid dynamic wavelet fuzzy approach for time series issued from finance. The proposed scheme showed that fuzzy models may be improved by wavelets by allowing them to take into consideration the extreme values in the data and the high fluctuations. Ebrahimi et al. [27] combined wavelets, neural networks, and the TS fuzzy model to construct an efficient observer-based controller design able to describe the uncertain disturbance in the TS fuzzy systems. In [28], the authors investigated the capability of fuzzy systems in functional approximation in the case of overlapping Gaussian concepts. A connection with radial basis functions is used to compute accurately a system of coefficients. Notice that wavelets overcome the overlapping Gaussian models by their orthogonality which assures independent components. Meanwhile, the most important task in fuzzy models or approximators is with no debt the stability. Guelton et al. [29] applied the concept of Lyapunov function for the investigation of the stability of a fuzzy system via a polynomial form. See also [30, 31]. Han et al. [32], and Zhang et al. [33] investigated the problem of disturbance for an observers-based polynomial fuzzy controller. Joh et al. [34] studied the stability of a linear TS fuzzy model via a common matrix issued from subsystems under a pairwise commutative assumption.

The main model to be used here is the TS fussy model which uses as usual fuzzy rules, composed of linguistic statements based on the 'if-then' structures, where both the condition(s) and the result are fuzzy (sets, numbers, etc). Therefore, a fuzzy model is essentially a model that relates the input variables issued from some fuzzy controllers and/or models to output variables, which in turn may use fuzzy rules. We know that all the financial time series cannot be represented by linear models due to their natural structure. To approximate them via a linear model, we need in fact a very local linear control on a small range. Such a control will be therefore lost for large ranges. Applying it there may result in instability (uncertainty, high nonlinearity, high volatility, fuzziness, and so on). The TS fuzzy model is aimed to represent the local dynamics adequately as universal approximators. Compared to Mamdani fuzzy rules, TS rules use functions of input variables as the rule consequent. See for instance [18, 35–40]. The TS fuzzy model has advantages in realizing multi-objective, expert, and robust control. Meanwhile, the TS fuzzy model intersects all fuzzy models via a major drawback of being non-analytic. Therefore, the need to develop an extension of the TS model into systems involving analytic variables is of interest. Our idea to overcome these drawbacks is based on the involvement of the wavelet decomposition as a pre-processing step that proceeds the TS fuzzy modeling to permit detecting the fluctuations or the volatile behavior accurately, preserving already the fuzzy nature of the data.

The main mathematical tool to be involved in the improvement of the TS fuzzy model is the wavelet analysis of time series which is characterized by its efficiency in representing the data robustly in front of the model specification. It permits also a reduction of the computation time due to the so-called decomposition and reconstruction algorithms or the wavelet filters. In addition, wavelet analysis permits a time-frequency or time-space localization. The literature on wavelets is nowadays large. Readers may refer to [24, 27, 41–43] for backgrounds on the wavelet toolkit. Additionally, economic/financial data has been shown to possess fractal/multifractal structures such as self-similarity, and scaling laws which were already investigated using wavelet multifractal methods. One of the efficient tools used in wavelet multifractal processing of time series is the so-called multifractal spectrum, which is estimated via wavelets in [44, 45]. Arfaoui and Ben Abdallah [46] applied some types of multifractal models based on wavelets for the estimation of financial signals, provided with eventual confidence intervals and statistical tests for performance measuring. Balalaa and Ben Mabrouk [47] applied a wavelet multifractal technique to estimate social indicators such as the quality of life index over critical periods, such as pandemics and economic/financial crises. The study showed effectively that social indicators possess some multifractal behavior with a nonlinear piecewise linear) multifractal spectrum, which joins somehow the behavior or the shape of the multifractal spectrum of the present case. A wavelet change-point analysis was applied there to localize the instants of piecewise linearity changes. In [48], some extensions of self-similar models have been developed and applied for approximating financial signals based on empirical examples and error estimates. In [49], wavelet multifractal techniques have been applied for the study of the multifractality in the capital asset pricing model for the Qatar state during a critical period. In the same line, Sarraj and Ben Mabrouk [50] introduced a non-uniform wavelet multifractal technique to investigate the capital asset pricing model for countries possessing critical movements and political changes such as the Arab Spring. The study showed that, effectively, these changes may induce or cause a multifractality in the economic data, and the market in general. Saadaoui et al. in [51] developed a wavelet multifractal detrended fluctuation analysis to explore the multifractality of some economic data, provided with a two-stage method involving change-point analysis followed by a data segmentation. More details, tools, techniques, and empirical cases may also be found in [52].

There are several basic characters in financial time series, the uncertainty of data, the non-stationarity, the nonlinearity, missing data, the dynamics, stochasticity, and the absence of a priory model lining the inputs to outputs. The original MAM model is unable to take into consideration all these complex characters simultaneously, as it only approximates the data via a linear model, necessitating thus a linearization of the data. However, such a linearization may not be possible in many cases, such as the financial time series whose structure is, in fact, unknown in advance, and posses moreover high fluctuations and volatile aspects, where the linearization, even possible, leads to a loss of information. The hybrid TS-wavelet model overcomes these circumstances by localizing the bad instants in the data via the wavelet tool, and projecting the time series in the suitable approximation space, preparing thus to tackle the data (filtered or projected) by the TS model. This means that the TS model when applied with the wavelet processed data does not depend on the anomalies of the original version of the data (contrarily to the existing models).

In addition to what has been mentioned above, it is worth noting or recalling that the first major addition to the proposed model is the application of the fractal test for time series since the test proved this hypothesis, which in turn eliminates the possibility of applying classical models such as regression models despite their feasibility and importance in several cases. The presence of the fractal property in time series suggests the application of wavelets as an ideal tool for data modeling.

In the present paper, we are concerned with the Saudi Arabia Tadawul market actively traded over a critical period from January 15 2011 to December 31 2022, which is strongly and directly related to many movements such as the Arab Spring, GCC embargo against Qatar country, COVID-19, Yamen war, and so on. Saudi Arabia has certainly an important geographical position, a great worldwide economy, and the largest petroleum exportation. Each year, it receives more than 2 million visitors to the saint cities from all over the world, which may import pandemics and spread them to the rest of the world. The Kingdom of Saudi Arabia has also implemented the NEOM worldwide project as an essential basis in the 2030-vision of the kingdom, which involves international economic, tourist, biological, and financial activities. Therefore, understanding this market, and highlighting its characteristics, and its movement will be of great importance for investors, managers, policymakers as well as consumers. The sample of data to be applied is essentially extracted from the website www.investing.com.

The present paper will be structured as follows. In section 2, the new hybrid approach based on wavelets and the TS fuzzy model combination will be developed. Section 3 is devoted to empirical results based on the Saudi Arabia Tadawul time series. Section 4 is a conclusion.

2. The multi-scaling wavelet TS fuzzy model

The Mamdani fuzzy model has been combined with wavelets in [26] to yield a dynamic model, which consisted of taking into account the change points of the time series localized by the wavelet decomposition, and approximating piece-wisely the time series on intervals between consecutive extreme values. Ebrahimi et al. [27] developed a hybrid fuzzy approach based on TS fuzzy model and wavelet neural networks for the control of uncertain disturbed systems.

The TS fuzzy model aims to describe the local link between input and output variables, by exploring the local behavior such as dynamics, volatility, and fluctuation via the fuzzy rule "IF-THEN" involving functional variables instead of fuzzy sets [8, 23]. More precisely, for an input vector $X = (x_1, x_2, \ldots, x_N)$, a fuzzy statement A (which may be a set), an output vector $Y = (y_1, y_2, \ldots, y_N)$ ($N \geq 1$ in $\mathbb{N}$), and $F : \mathbb{R}^N \longrightarrow \mathbb{R}^n$ a function (eventually unknown), the TS fuzzy control model takes the form

$$\text{IF } X \text{ is } A \ \text{ THEN } \ Y = F(X). \tag{1}$$

Generally, let $Y_n = (y_k, y_{k+1}, \ldots, y_{n-2}, y_{n-1}, y_n)$ ($1 \leq k \leq n$ fixed) be a part of the output of the system that we search to model at the time interval $[k, n]$, U_n the system input (at the time interval $[l, n]$), and Y_{n+1} the next-time $(n+1)$

output. Let also $M = M_1 \times M_2 \times \cdots \times M_N$ be a fuzzy set, and $F : \mathbb{R}^{n-k} \times \mathbb{R}^{n-l} \longrightarrow \mathbb{R}^n$ be any arbitrary function eventually unknown. The TS fuzzy model states that

$$\text{IF } Y_n \text{ is } A \text{ AND } U_n \text{ is } M \text{ THEN } Y_{n+1} = F(Y_n,\ U_n). \tag{2}$$

The system may be interpreted as a forecasting form, Y_n represents the past of Y, with some autoregressive order k.

Our approach consists of replacing the variables in the system with their wavelet coefficients, as these coefficients constitute the coordinates of the variable in the wavelet basis. Let $(\varphi,\ \psi)$ be a pair of (father, mother) wavelets in $L^2(\mathbb{R})$, and $(\varphi_{j,\ k}, \psi_{j,\ k})_{j,\ k}$ be the corresponding orthonormal basis of $L^2(\mathbb{R})$, where $\psi_{j,\ k}(x) = 2^{-j/2}\psi(2^j x - k)$ and $\psi_{j,\ k}(x) = 2^{-j/2}\psi(2^j x - k)$. A time series $X = X(t)$ is decomposed into a series called its wavelet series as

$$X = \sum_k a_{j,\ k}(X)\varphi_{j,\ k} + \sum_{j \geq J}\sum_k d_{j,\ k}(X)\psi_{j,\ k}, \tag{3}$$

where the coefficients $a_{j,\ k}(X)$ and $d_{j,\ k}(X)$ are, respectively, the approximation and the detail coefficients of the time series X estimated via the discrete convolutions

$$a_{j,\ k}(X) = \sum_n X(n)\varphi_{j,\ k}(n), \text{ and } d_{j,\ k}(X) = \textstyle\sum_n X(n)\psi_{j,\ k}(n). \tag{4}$$

For j fixed, the components

$$A_j = \sum_k a_{j,\ k}(X)\varphi_{j,\ k}, \text{ and } D_j = \sum_k d_{J,\ k}(X)\psi_{j,\ k}, \tag{5}$$

are called, respectively the approximation and the detail component of the time series X at the level j. The time series is then written as

$$X = A_J + \sum_{j \geq J} D_j. \tag{6}$$

The component A_J describes the global shape of the time series X, and it belongs to the space $V_J = spann(\varphi_{J,\ k},\ k)$ called the approximation space at the level J, while the component D_j reflects the details or the fluctuations of the time series X, and belongs to the detail space at the level j defined by $W_j = spann(\psi_{j,\ k},\ k)$. Details about all these concepts may be found in [41–43].

The role of the multifractal analysis is to link the function estimation near its singularities to its Hurst-Hölder regularity via the spectrum of singularities, which is initially defined as the fractal dimension of the set of points possessing the same Hurst-Hölder regularity. The function support is then splitted into subsets according to the Holder regularity values. For each subset, the signal will be monofractal, and the computation of the spectrum will be easy. However, many signals such as economic and financial indices are pointwisely irregular, and their Hurst-Hölder exponent is itself irregular, which involves new difficulties for the evaluation of the spectrum. A reconsideration of the spectrum by Arneodo et al. [44, 45] using wavelets permitted to simplify the task. Instead of evaluating the maxima modules directly from the signal, the wavelet transform is evaluated. The support of the signal is then splitted into subsets $A_{j,\ k}$ composed of all coefficient

$d_{j,\,k}$ with the same estimation $2^{-\alpha j}$ near some points. Arneodo et al. way to explore the fractal/multifractal nature of the time series yields the wavelet multifractal spectrum, which is estimated by the function

$$d(\alpha) = \inf_{q}(\alpha q - \tau(q)), \tag{7}$$

where α is the Hurst-Hölder exponent of the series, and B is the so-called scaling function defined via the wavelet coefficients of the series $X(t)$ as

$$\tau(q) = \liminf_{j\to\infty} \frac{\log \sum_{k} |d_{j,\,k}(X)|^q}{-j\log 2}, \tag{8}$$

where, the $d_{j,\,k}(X)$ is as in (4). If $d(\alpha)$ is linear, we deduce that the data is mono-fractal, otherwise (strictly concave), we say that it is multifractal.

To control our model, especially, from the numerical point of view, we need to guarantee some range of control. The advantage of involving wavelets in the model is their ability to fix this error in advance, depending on the level of wavelet decomposition. Given a function F and its wavelet decomposition F_J at a level J, and as for the $d_{j,\,k}$, we put $M_{j,\,k}$ the set of approximation coefficients $a_{j,\,k}$ for which $\|A_j - X\|_2 \sim 2^{-J}$ [53–56]. The hybrid wavelet TS fuzzy model consists of a J-level model using the functional wavelet bases. More precisely, consider the vector $\mathscr{V}_{j,\,k}(f) = (d_{j,\,k}(f))$ and $\mathscr{W}_{j,\,k}(f) = (a_{j,\,k}(f))$. We introduce a wavelet TS fuzzy control model as

$$\text{IF } \mathscr{V}_{j,\,k}(X) \text{ is } A_{j,\,k} \text{ THEN } \mathscr{V}_{j,\,k}(Y) = F(\mathscr{V}_{j,\,k}(X)), \tag{9}$$

and a wavelet TS fuzzy modeling rule as

$$\text{IF } \mathscr{V}_{j,\,k}(Y_n) \text{ is } A_{j,\,k} \text{ AND } \mathscr{V}_{j,\,k}(U_n) \text{ is } M_{j,\,k} \text{THEN } \mathscr{V}_{j,\,k}(Y_{n+1}) = F(\mathscr{V}_{j,\,k}(Y_n),\, \mathscr{W}_{j,\,k}(U_n)). \tag{10}$$

Figure 1 below shows the functioning of the hybrid wavelet TS fuzzy model provided with the heading steps of the associated algorithm 1.

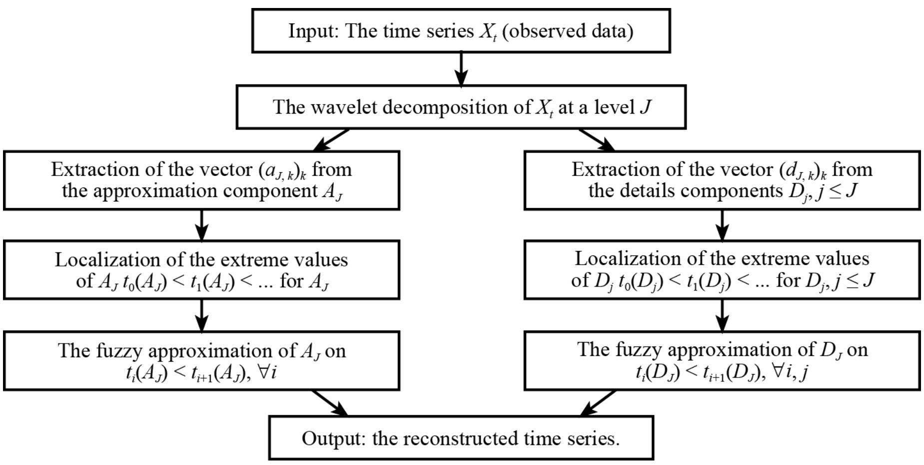

Figure 1. A flowchart illustrating the hybrid wavelet TS fuzzy model

Algorithm 1

Step 1. Load the input time series.

Step 2. Decomposition of time series into the wavelet decomposition $(A_J, D_j, j \leq J)$.

Step 3. Generate the vectors $\mathscr{V}_{j,\,k}(f) = (a_{j,\,k}(f),\ d_{j,\,k}(f))_k$ for each j.

Step 4. Peak up the extreme points $t_i(A_J),\ t_i(D_j)$ of the components $(A_J, D_j, j \leq J)$, respectively.

Step 5. Application of the fuzzy approach (9) and (10) to the components $(A_J, D_j, j \leq J)$ restricted to the intervals $[t_i(A_J),\ t_{i+1}(A_J))$, $[t_i(A_J),\ t_{i+1}(A_J))$, respectively.

Step 6. Regroup the fuzzy models obtained in Step 5 on the whole time interval to get the fuzzy reconstruction of the components $(A_J, D_j, j \leq J)$.

Step 7. Sum up the fuzzy models of A_J with those of the D_j's, $j \leq J$ to get the wavelet fuzzy reconstruction.

To better control the stability of our approach, we propose to apply a wavelet-based Lyapunov stability. Recall that the Lyapunov exponent is shown effective for the recognition of sensitivity to model parameters, and is used to measure the chaos amount due to the model provided with a measure of the convergence/divergence rate of the estimated model solution to the observed data. Given a time series $X = X_k$ and its approximation via some model $\widehat{X} = \widehat{X}_k$, The distance $|X_k - \widehat{X}_k|$ at a time k is estimated by

$$|X_k - \widehat{X}_k| \sim e^{\lambda k}, \text{ where } \lambda = \frac{1}{k} \log \prod_{l=0}^{k-1} |F'(X_l)|, \tag{11}$$

and F is the dynamical system function (to be modeled) defining the time series $X_k = F(X_t,\ 0 \leq t \leq k-1)$. Whenever $\lambda > 0$, there is divergence, and for $\lambda < 0$, there is convergence of the model. Whenever F is regular, the Lyapunov exponent λ is evaluated by

$$\lambda = \lim_{k \to \infty} \frac{1}{k} \sum_{l=0}^{k-1} \log |F'(X_l)|. \tag{12}$$

The wavelet approach for the Lyapunov stability applies a wavelet multi-scale Lyapunov exponent for stability at a scale j as

$$\lambda_j = \lim_{k \to \infty} \frac{1}{k} \sum_{l=0}^{k-1} \log |D_j'(X_l)|. \tag{13}$$

The concept of wavelet Lyapunov exponent was, in fact, investigated by many authors such as [57–60].

3. A wavelet TS fuzzy model for financial series

In this section, we propose to apply the wavelet TS fuzzy model proposed previously to an economic/financial series deduced from the Saudi Arabia Tadawul market traded over the period January 01 2011, to December 31 2022. Recall that financial data is always subject to uncertainty and fuzziness due to many factors such as data availability, recording, and so on. Indeed, financial indicators such as the Saudi Tadawul are always and already subject to volatility, high fluctuations, stochasticity, self-similarity, scaling law, and fractality. Due to their essential role in the economy, their modeling and understanding are of great interest. The first step to check the multifractality of the data is the evaluation of its wavelet multifractal spectrum. Figure 2 below illustrates the multifractal spectrum of the Tadawul time series.

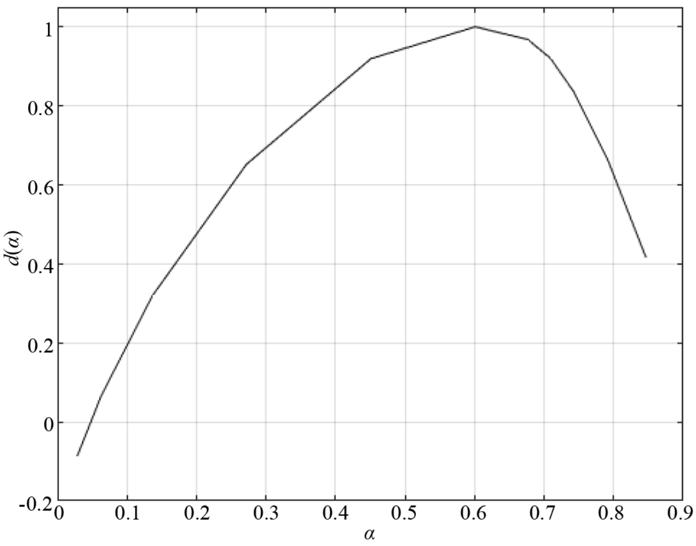

Figure 2. The multifractal spectrum of the Tadawul time series

In the first step, the application of Mamdani model results in two bounds, an upper and lower approximation, which are both linear and which result from a linear optimization (fuzzy regression) program. The original series and its fuzzy estimation are shown in Figure 3.

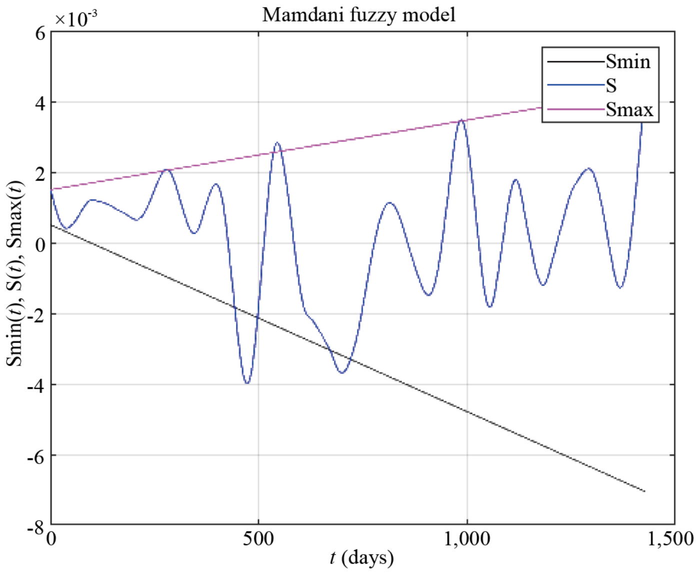

Figure 3. Original series and its Mamdani fuzzy model estimation

Notice that the major problem with this model is that it leads to approximations which become more and more inefficient as the approximations move away linearly from the real solution. Notice also that, as usual, the wavelet-free Mamdani fuzzy model, although it yields a domain of approximation, this domain becomes larger on the whole time interval. This is natural as this model cannot describe the volatile behavior of the data. A major problem that is clearly illustrated is its incapability to localize the extreme points of the data.

To show clearly the volatile aspect of the data and to take into consideration in the model, we conducted a wavelet decomposition at a wavelet level $J = 6$ using the Daubechies wavelet Db12 as the wavelet mother. The result is shown in Figure 4.

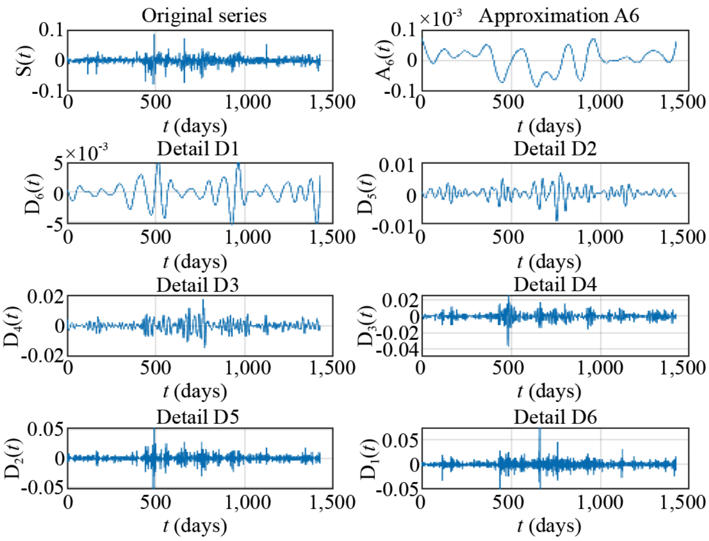

Figure 4. The wavelet decomposition of Tadawul index at level 6

Next, as the volatile behavior of the data is illustrated clearly, and the Mamdani fuzzy model did not permit a good description of the data, we applied the advanced fuzzy logic model due to TS in two forms, a linear and a nonlinear quadratic one. Each version is combined with wavelets to take into consideration the oscillations of the time series. From Figure 4, we notice easily that the hidden fluctuations in the data become more and more clear from the detail components, as we increase the decomposition level. This led us to combine the wavelet decomposition with the TS fuzzy model (in its linear and nonlinear forms) to model both the approximation component A_6 (Figure 5) and the detail component D_6 (Figure 6) using the new hybrid wavelet TS fuzzy model (10). The other components may be estimated similarly. More precisely, the main steps of our hybrid wavelet TS fuzzy model may be resumed as follows,

Stage 1: Decompose the financial time series into wavelet approximation and details components.

Stage 2: Extract, from stage 1, the vectors of approximation and detail coefficients at the desired level.

Stage 3: Evaluate or estimate the Hölder exponents around the extreme points of the time series to define the fuzzy rules as well as the intervals of the piece-wise estimation.

Stage 4: Apply the fuzzy approach On the successive intervals obtained from the extreme points' localization.

Stage 5: Regroup the fuzzy estimations to get the reconstructed time series.

As mentioned previously, the pure TS fuzzy model was shown to be more accurate for highly regular data, we proceed by a natural way by increasing the order of nonlinearity, illustrated here by the degree of the polynomial approximation, We apply firstly a linear approach followed by another quadratic one to better describe the data. Recall in this context that a highly regular the signal is, a higher degree polynomial approximation is better, such as the Taylor polynomials It is noticeable clearly from Figure 5 and Figure 6 the superiority of the new hybrid model.

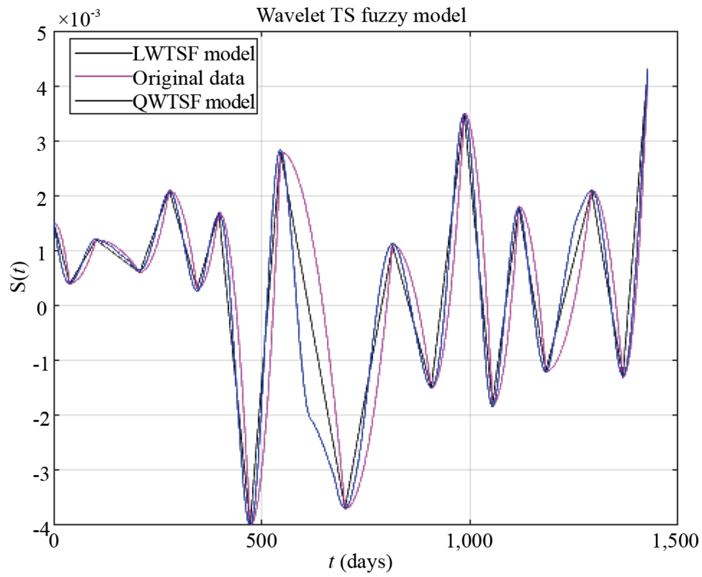

Figure 5. The wavelet TS fuzzy hybrid model for the components A_6

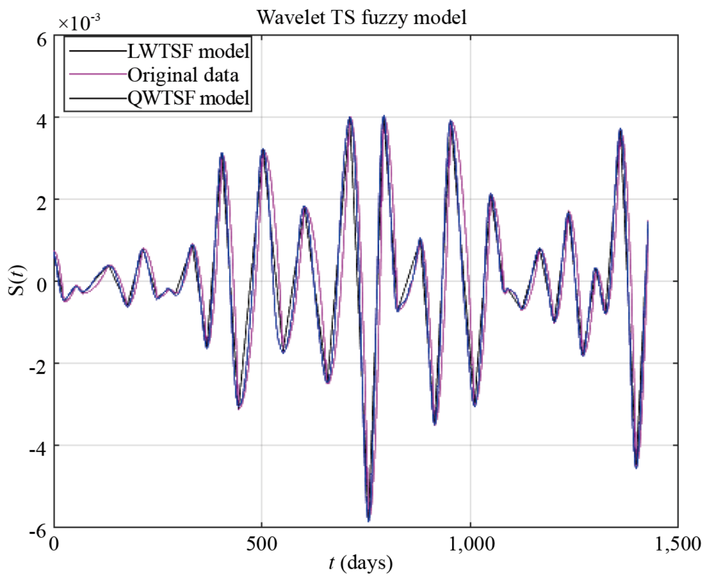

Figure 6. The wavelet TS fuzzy hybrid model for the components D_6

To study the stability of the hybrid model, and to investigate its efficiency, we conducted a Lyapunov exponent stability. Figure 7 illustrates the Lyapunov exponents due to the classical Mamdani fuzzy model. Figure 8 illustrates the Lyapunov exponents due to the linear TS fuzzy model. Figure 9 illustrates the Lyapunov exponents due to the quadratic TS fuzzy model. Figure 10 illustrates the Lyapunov exponents due to the pure wavelet model. Figure 11 illustrates the Lyapunov exponents due to the hybrid Mamdani wavelet fuzzy model. Figure 12 illustrates the Lyapunov exponents due to the hybrid linear TS wavelet fuzzy model. Finally, Figure 13 illustrates the Lyapunov exponents due to the quadratic TS wavelet fuzzy model.

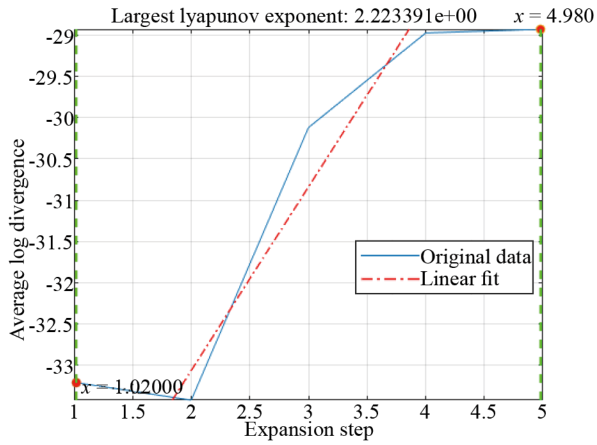

Figure 7. Lyapunov exponent for MAM fuzzy models estimation

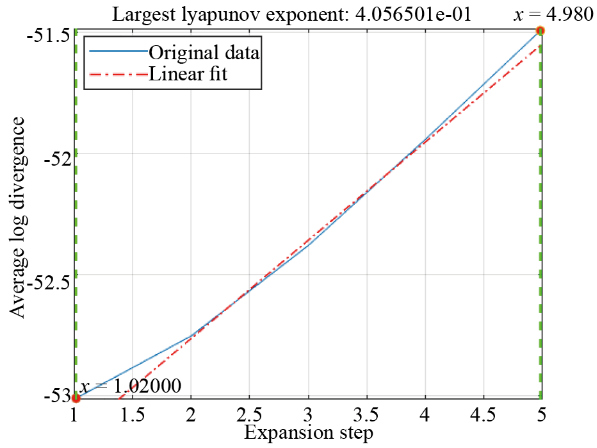

Figure 8. Lyapunov exponent for linear TS fuzzy model estimation

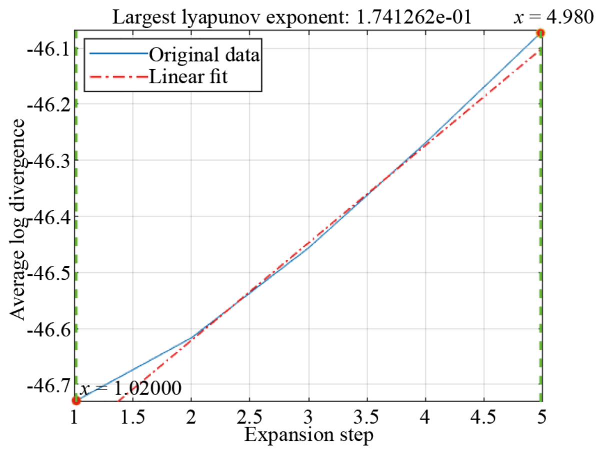

Figure 9. Lyapunov exponent for quadratic TS fuzzy model estimation

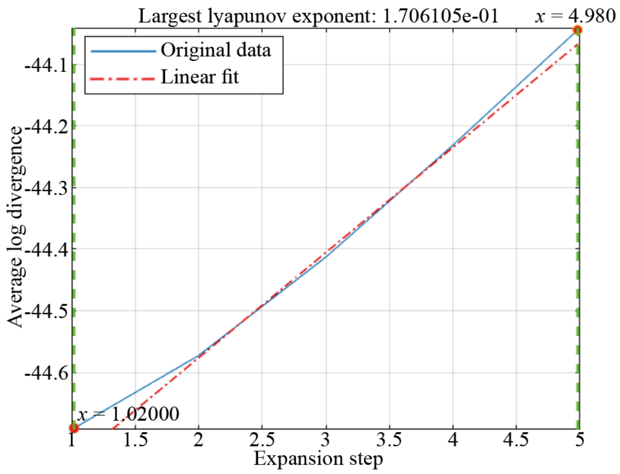

Figure 10. Lyapunov exponent for the pure wavelet estimation

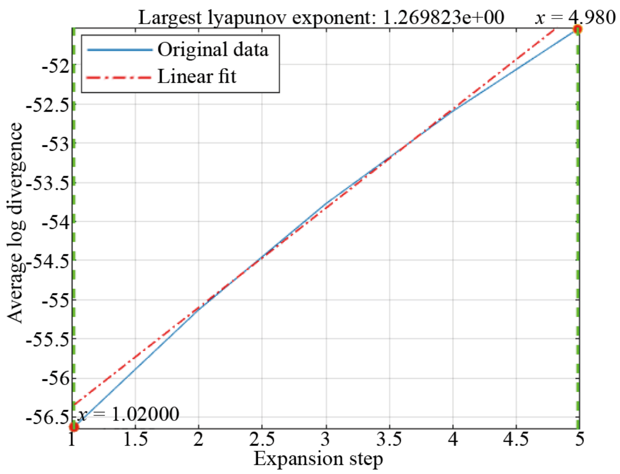

Figure 11. Lyapunov exponent for Wavelet Mamdani fuzzy model estimation

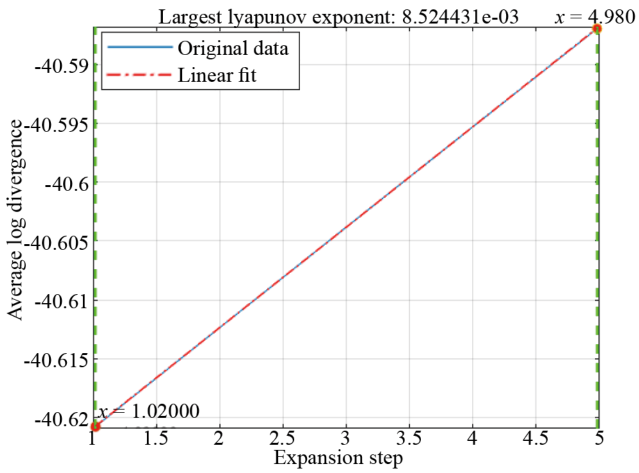

Figure 12. Lyapunov exponent for Wavelet linear TS fuzzy model estimation

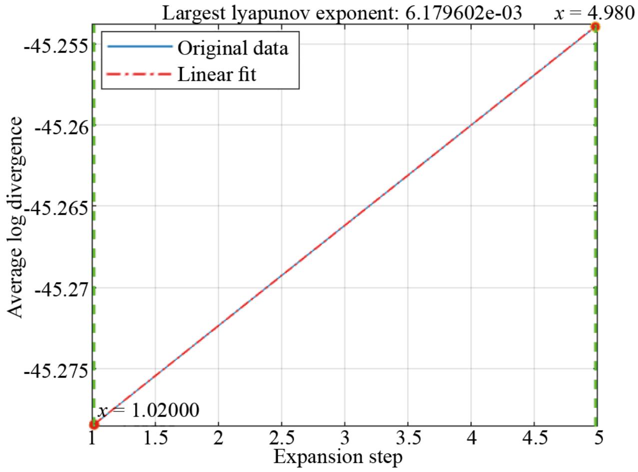

Figure 13. Lyapunov exponent for Wavelet quadratic TS fuzzy model estimation

By examining the Lyapunov exponent figures, we notice easily the instability, and a non-fit of the Mamdani fuzzy model (Figure 7), with its largest Lyapunov exponent being positive. Next, the application of the wavelet-free TS fuzzy model (in a polynomial form) yielded an improvement in the evaluation of the Lyapunov exponent, and consequently in the stability of the model. Figure 8 was due to a 1-degree polynomial (linear) variant and led to a reduction in the Lyapunov exponent, with a better fit to the original data. These facts are more improved with the 2-degree polynomial (quadratic) variant illustrated by Figure 9, where effectively, the largest Lyapunov exponent is more reduced indicating more stability. From Figure 10 to Figure 13, we notice more stability for all models by the involvement of the wavelets, which explains a good correction of fuzzy models by this involvement. Indeed, Figures 10 to 13 show a decreasing order of the largest Lyapunov exponent. Moreover, we notice a reduced divergence range being also well controlled. Yet the maximum exponent is positive which means that a chaotic aspect can take place for this data. The fitting becomes more and more effective until it reaches an optimal form with the application of the last hybrid wavelet TS fuzzy model.

Finally, to reinforce the previous results, we provide an error estimate

$$Er = \left(\sum_t |X(t) - X_{app}(t)|^2 \right)^{1/2}, \tag{14}$$

where X_{app} is the approximation of the series X due to the model applied. Table 1 illustrates the results for the wavelet decomposition level $J = 6$ conducted via the Daubechies wavelet Db12 using the error (14). We notice easily the superiority of the wavelet fuzzy models, especially the last hybrid model.

Table 1. Error estimates due to different models

The model	Er
Mamdani fuzzy model	2.4123
Linear TS fuzzy model	0.4301
Quadratic TS fuzzy model	0.0524
Wavelet model	1.2301×10^{-4}
Mamdani wavelet fuzzy model	0.0153×10^{-5}
Linear TS wavelet fuzzy model	0.0121×10^{-5}
Quadratic TS wavelet fuzzy model	0.0022×10^{-6}

In the literature, the fuzzy systems aim generally to transform the problem tackled to local (or global, such as MAM) linear approximating versions using linearization via fuzzy rules. The present hybrid wavelet fuzzy model is a mixture of many non-necessary linear subsystems evolving independently, contrary to the existing cases where the subsystems are operating simultaneously. The wavelet decomposition leads effectively to independent components due to the orthogonality of the wavelet bases. In addition, our hybrid model does not necessitate the linearization of the time series at the extreme points.

Financial time series are also characterized by high volatilities, high dynamics, and other properties, which may be well analyzed and synthesized using time/frequency or space/frequency wavelet tools. Wavelet approximation overcomes the linearization method as it does not neglect disturbers, and thus, keeps full information about the approximated data. Recall that the nonlinear structure which is behind the dynamics in data may be the richer part of the system. Linearizing with MAM, regression or other tools may affect the number of equilibrium points which may thus be different in the linearized version than the original one. Another implication is related to the stability, which is converted in the majority of existing works to a dynamical system of the form $X_k = F(X_{k-1})$, and which in turn needs a control of the function F. Many required conditions to guarantee this control may not be satisfied by the data, and thus the use of iterative schemes, for example, to predict or estimate the unknown function or parameters will be inadequate.

Looking at Table 1, we notice that the involvement of the wavelets in the fuzzy modeling approaches decreased the error estimation to $2^{-2J} \sim 10^{-4}$, which aligns with the estimation $\|X - X_J\| \sim 2^{-J}$ already guaranteed via the wavelet theory. We also notice the impact of the polynomial estimation or the nonlinearity level (linear = polynomial of degree 1, quadratic = polynomial of degree 2) which improves approximation. These facts confirm that the data is never linear and that keeping this character in the model is more adequate. Polynomial fuzzy approaches have been already investigated for issues of control, and Lyapunov functions such as [29, 32, 33].

To resume, one main idea in the present paper, especially from the experimental point of view, may be resumed in the exclusion of the extreme points of the data from being modeled around, which as we know, may result in redundancies. These points are considered instead as initial or boundary data. The application of the pure fuzzy models induced, as shown in Table 1, a higher error, whereas the hybrid version gives a better description and a smaller error. On the other hand, by examining the control of the method through the Lyapunov exponent, we see that the hybrid approximators reflect greater stability and adequacy, as shown in Figures 7-13.

In the present work, the Saudi Tadawul financial series is considered to illustrate the performance and efficiency of the hybrid TS-wavelet model compared to the closest models. Recall that financial series possess several strange aspects, characteristics, and structures requiring advanced models for approximation and control. These include extreme values, lack of data, uncertainty, and essentially nonlinearity. The application of the hybrid TS-wavelet model has shown superiority over other models in terms of graphical fitting illustrations as well as error estimation. The hybrid model has proven successful in localizing extreme values and overcoming the nonlinear nature of the data. Moreover, compared to the classical fuzzy MAM model, the functional relationship between input and output which is always unknown, is nevertheless well estimated through the hybrid TS-wavelet model. We can also clearly note the efficient control in terms of the Lyapunov stability exponent which clearly shows the superiority of the hybrid TS-wavelet model.

4. Conclusion

In the present paper, the concept of wavelet decomposition is combined with the Takagi-Sugeno fuzzy model to develop a hybrid model for time series modeling and control. The new model consists of involving the vector of wavelet coefficients of the time series as input and outputs the ‘best’ eventual functional model capable in the data description. The hybrid model is applied to financial data due to the Saudi Arabia Tadawul market traded over a special period (2011-2022) characterized by many critical, and severe movements from the economic, financial, and political points of view, leading to many difficulties arising in the analysis, and strange behaviors such as extreme values, dynamics, fluctuations, uncertainty, and so on. The idea consists of a dynamic approach, which applies the TS fuzzy model on the vector of wavelet coefficients on a specific set relative to the estimation at consecutive extreme points. The theoretical results are provided with numerical simulations and eventual comparison with existing models via tests based on linear and polynomial TS approximators. The present model is shown to be performant, accurate, and able to extract the hidden properties and behavior of the data, opening thus several future directions such as the combination of fuzzy logic with wavelets in other fields where the nonstationary, chaotic, noised, and strange behaviors are hidden in the data, such as time series issued from climate, pollution, wind, and so on.

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Conflict of interest

The author declares here that no conflicts of interest in this paper.

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