Contemporary Mathematics
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Research Article

# Some Parallel Surfaces in Three-Dimensional Minkowski Space as the Discriminant Set of a Certain Family of Functions

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**Received:** 26 July 2024; **Revised:** 27 August 2024; **Accepted:** 5 September 2024

**Abstract:** In this paper we prove that, in three-dimensional Minkowski space, the family of parallel surfaces to a given surface at a certain distance can be obtained as the discriminant set of a certain family of functions. As consequence, we obtain that Minkowski spheres with prescribed radius, having contact of order 1 with a surface, have the centers on the parallel surface at distance equal to the radius. We give an example showing that the parallel surface to a timelike flat surface admits cuspidal edge as singularity.

***Keywords*:** minkowski space, parallel surface, discriminant set

**MSC:** 53A35, 53B30

## 1. Introduction

The global geometric study of sets of a certain type requires results from the theory of singularities of differentiable mappings. These results, very extensive in present, have been suggested by other problems and have found applications in numerous fields. Applications of theory of singularities in sciences such as: fluid theory, optics, elasticity, thermodynamics, laser physics, biology, and ecology) are presented in [1]. Recently, the theory of singularities for surfaces has become a useful tool for radiology and visual computing (see [2] and references therein).

The theory of singularities also influenced the development of the global geometry of surfaces in Euclidean space [3–6], as well as in Minkowski space [7–9]. The main problem was the determination of the nature of singularities on certain classes of surfaces (minimal, respectively constant mean curvature surfaces).

In this paper we deal with the singularities of parallel surfaces and extend some results from [3] to three-dimensional Minkowski space. We prove that the family of parallel surfaces to a given surface can be obtained as the discriminant set of a certain family of functions, the key being suggested by contact theory and particular cases from [10].

The rest of the paper is organized as follows. In Section 2 we briefly review the necessary results on the geometry of surfaces in three-dimensional Minkowski space, and the definitions of discriminant set, front, and frontal. Section 3 is devoted to proving of the main results. In Section 4 we give an example of a timelike flat surface whose parallel surface admits the cuspidal edge as a singularity.


DOI: https://doi.org/10.37256/cm.5420245380

## 2. Preliminaries

Let $\mathbb{R}^3$ be the three-dimensional real vector space.

We recall from [11] some notions from the three-dimensional Minkowski space theory which we will use in the main part of the paper.

**Definition 1** The pair $\left(\mathbb{R}^3, \langle\cdot,\cdot\rangle_1\right)$, denoted $\mathbb{R}^3_1$, where the pseudo - inner product $\langle\cdot,\cdot\rangle_1$ is given by $\langle v, w\rangle_1 = -v_1w_1 + v_2w_2 + v_3w_3$ for any $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3) \in \mathbb{R}^3$, is called three-dimensional Minkowski space.

The notation $\mathbb{R}^3_1$ highlights the signature $(-, +, +)$ of the pseudo-inner product $\langle\cdot,\cdot\rangle_1$.

A useful generalization of the notion of length in Minkowski space is presented in the following, but this is not a norm in the usual sense, since it is not subadditive.

**Definition 2** The pseudo-norm of any arbitrary vector $v \in \mathbb{R}^3_1$ is defined by $\|v\|_1 = \sqrt{|\langle v, v\rangle_1|}$. $v$ is called a unit vector if $\|v\|_1 = 1$.

**Definition 3** A nonzero vector $v \in \mathbb{R}^3_1$ is called a spacelike vector, a lightlike vector, or a timelike vector if $\langle v, v\rangle_1 > 0$, $\langle v, v\rangle_1 = 0$, or $\langle v, v\rangle_1 < 0$, respectively.

**Definition 4** Let $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ be two (spacelike or timelike) vectors in $\mathbb{R}^3_1$. The pseudo-cross product of vectors $v$ şi $w$, in this order, is the only vector, denoted $v \wedge w$, defined by

$$v \wedge w = \begin{vmatrix} -e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \left(-(v_2w_3 - v_3w_2),\ -(v_1w_3 - v_3w_1),\ v_1w_2 - v_2w_1\right)$$

where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3_1$.

**Definition 5** Let $U$ be an open domain and $S$ a surface parametrized by

$$X:\ U \subset \mathbb{R}^2 \to \mathbb{R}^3_1,\ (u, v) \to X(u, v) = (x(u, v),\ y(u, v),\ z(u, v)). \tag{1}$$

The unit normal vector field $n$ on $S$ is given by

$$n = \frac{X_u \wedge X_v}{\|X_u \wedge X_v\|_1}.$$

$S$ is called a timelike surface, a spacelike surface, or a lightlike surface if the unit normal vector $n$ is spacelike, timelike, or lightlike, respectively, at each point of $S$.

By non-lightlike surface we mean both the spacelike surface and the timelike surface.

Throughout the following, let $S$ be an orientable non-lightlike surface from $\mathbb{R}^3_1$ and $n$ its unit normal vector field.

The Gauss map $\gamma$ is defined as follows:

$$\gamma:\ S \longrightarrow \mathscr{S}^2(-1) := \left\{(x, y, z) \in \mathbb{R}^3_1:\ -x^2 + y^2 + z^2 = -1\right\}$$

if $S$ is spacelike, respectively

$$\gamma: S \longrightarrow \mathscr{S}^2(1) := \left\{(x, y, z) \in \mathbb{R}^3_1:\ -x^2 + y^2 + z^2 = 1\right\}$$

if $S$ is timelike.

With the Gauss maps defined above, the operator $A = -\gamma_{*,p}$ is called the Weingarten operator of immersion $X$ at the point $p$.

In the local parametrization $P(u,\ v) = X(u,\ v)$ of the surface $S$, the Weingarten equations are:

$$\begin{cases} -A(X_u) = n_u = \dfrac{FM-GL}{EG-F^2}X_u + \dfrac{FL-EM}{EG-F^2}X_v \\ -A(X_v) = n_v = \dfrac{FN-GM}{EG-F^2}X_u + \dfrac{FM-EN}{EG-F^2}X_v \end{cases} \tag{2}$$

where $\{E,\ F,\ G\}$ and $\{L,\ M,\ N\}$ are the coefficients of the first, respectively, of the second, fundamental form of surface $S$, defined as

$$E = \langle X_u,\ X_u\rangle_1,\ F = \langle X_u,\ X_v\rangle_1,\ G = \langle X_v,\ X_v\rangle_1,$$

$$L = -\langle n_u,\ X_u\rangle_1,\ M = -\langle n_u,\ X_v\rangle_1 = -\langle n_v,\ X_u\rangle_1,\ N = -\langle n_v,\ X_v\rangle_1.$$

The coefficients of $X_u$ and $X_v$ in (2) determine the matrix of $-A$.

We recall from [12] the following definition.

**Definition 6** Let $S$ be an orientable surface and let $\delta$ be a constant positive real number. The surface $\widetilde{S}$ is parallel to $S$ at distance $\delta$ if the points $\widetilde{P}(u,\ v) \in \widetilde{S}$ are defined by $\widetilde{P}(u,\ v) = P(u,\ v) + \delta \cdot n(u,\ v)$, where $n$ is the unit normal vector field on $S$ in $P$.

Formally, the parallel surface $\tilde{S}$ to a given surface $S$ is given by the formula

$$\widetilde{X} : (u,\ v) \to \widetilde{X}(u,\ v) = X(u,\ v) + \delta \cdot n(u,\ v), \tag{3}$$

where $\delta$ is a real positive constant.

For $\delta$ variable, we obtain a family of parallel surfaces with the given surface.

Let $f:\ \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a differentiable function. For a point $c \in \mathbb{R}^m$, the restriction of $f$ to $\mathbb{R}^n \times \{c\}$ (i.e. $x \to f_c(x)$), denoted $f_c$, is called potential.

**Definition 7** The discriminant set of family $f$, denoted $\mathscr{D}(f)$, is the set of all critical points of all potentials $f_c$ in family $f$, i.e., the subset $M \subset \mathbb{R}^n \times \mathbb{R}^m$ characterized by

$$\begin{cases} \dfrac{\partial f_c}{\partial x_1}(x_1,\ \ldots,\ x_n) = 0 \\ \dfrac{\partial f_c}{\partial x_2}(x_1,\ \ldots,\ x_n) = 0 \\ \qquad\vdots \\ \dfrac{\partial f_c}{\partial x_n}(x_1,\ \ldots,\ x_n) = 0 \end{cases},$$

where $(x_1,\ \ldots,\ x_n) \in \mathbb{R}^n$.

Surfaces with singularities suggested the introduction of more general notions as front and frontal. We recall from [8] some definitions.

**Definition 8** A smooth map $f:\ M^2 \to \mathbb{R}_1^3$, where $M^2$ is a two-dimensional differentiable manifold, is called frontal if there exists a unit vector field $\nu$ on $f(M^2)$, normal on $f_*(TM^2)$.

The field $\nu$ can be seen as an application from $M^2$ to the Minkowski sphere $\mathscr{S}^2$.

**Definition 9** A frontal $f: M^2 \to \mathbb{R}_1^3$ is called wave front if the mapping $(f,\ \nu): M^2 \to \mathbb{R}_1^3 \times \mathscr{S}^2$ is an isotropic immersion.

**Definition 10** A frontal $f: M^2 \to \mathbb{R}_1^3$ is called flat if the Gauss map $\nu:\ M^2 \to \mathscr{S}^2$ is degenerate.

## 3. Main results

**Theorem 1** Let $S$ be a regular non-lightlike surface parametrized by (1) and let $\widetilde{S}$ be the family of parallel surfaces to $S$ at distance $\delta$ given by (3). Then, $\widetilde{S}$ is the discriminant set of the family of functions

$$F: U \times \mathbb{R}^4 \to \mathbb{R},\ (u,\ v,\ x,\ y,\ z,\ \delta) \to -\frac{1}{2}\left(\|(x-x(u,\ v),\ y-y(u,\ v),\ z-z(u,\ v))\|_1^2 + \varepsilon\delta^2\right)$$

with

$$\varepsilon = \begin{cases} +1, & \text{if } S \text{ is spacelike} \\ -1, & \text{if } S \text{ is timelike.} \end{cases} \tag{4}$$

**Proof.** The discriminant set is the set of the solutions of the system

$$\begin{cases} \dfrac{\partial F}{\partial u} = 0 \\ \dfrac{\partial F}{\partial v} = 0 \end{cases},$$

that is, in our case,

$$\begin{cases} -(x-x(u,\ v))x_u + (y-y(u,\ v))y_u + (z-z(u,\ v))z_u = 0 \\ -(x-x(u,\ v))x_v + (y-y(u,\ v))y_v + (z-z(u,\ v))z_v = 0 \end{cases}. \tag{5}$$

The solution of the indeterminate system (5) is:

$$\begin{cases} x - x(u,\ v) = -\lambda \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} \\ y - y(u,\ v) = -\lambda \begin{vmatrix} z_u & x_u \\ z_v & x_v \end{vmatrix} \\ z - z(u,\ v) = -\lambda \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \end{cases}$$

with $\lambda \in \mathbb{R}$, arbitrary.

Choosing $\lambda = \dfrac{\delta}{\|n\|_1}$ we obtain

$$\begin{cases} x = x(u,\ v) + \delta \cdot n_x, \\ y = y(u,\ v) + \delta \cdot n_y, \\ z = z(u,\ v) + \delta \cdot n_z, \end{cases}$$

where $n = (n_x,\ n_y,\ n_z)$, solution that can be written compactly:

$$(x,\ y,\ z) = X(u,\ v) + \delta \cdot n.$$

So, the discriminant set $\mathscr{D}(F)$ of family $F$ is

$$\mathscr{D}(F) = \left\{(x,\ y,\ z,\ \delta) \in \mathbb{R}^4 : (x,\ y,\ z) = X(u,\ v) + \delta n(u,\ v),\ (u,\ v) \in U\right\}.$$

Its intersection with the hyperplane $\delta = \delta_0$ is the parallel surface to $S$ at distance $\delta_0$.

Without loss of generality, it is required that the parameters from the discriminant set to vanish identically the functions of the family. In our case, after replacing we obtain $-\frac{1}{2}(\delta^2(-n_x^2 + n_y^2 + n_z^2) + \varepsilon\delta^2) \equiv 0$, for $\varepsilon$ chosen as in (4).

**Corollary 1** The Minkowski spheres with prescribed radius, which have contact of order 1 with $S$ have the centers on the parallel surface $\widetilde{S}$ at distance equal with that prescribed radius.

**Proof.** The set $F^{-1}(0)$ can be geometrically seen as a Minkowski sphere (de Sitter space if $\varepsilon = +1$, respectively, hyperbolic plane if $\varepsilon = -1$) of center $(x,\ y,\ z)$, radius $|\delta|$, passing through the point $X(u,\ v)$ of the surface $S$. The conditions required in Theorem 1 for the discriminant set $\mathscr{D}(F)$ shows that this sphere has contact of order 1 with the surface $S$ and the conclusion of the corollary follows.

The surface (3) has, in generally, singular points. In order to determine them, we need to evaluate the vector field $\widetilde{X}_u \wedge \widetilde{X}_v$, where $\widetilde{X}_u = X_u + \delta \cdot n_u$ and $\widetilde{X}_v = X_v + \delta \cdot n_v$. It follows that

$$\widetilde{X}_u \wedge \widetilde{X}_v = X_u \wedge X_v + \delta(n_u \wedge X_v + X_u \wedge n_v) + \delta^2 n_u \wedge n_v. \tag{6}$$

From (2), with $\Delta = F^2 - EG$, we have

$$-\Delta n_u = (FM - GL)X_u + (FL - EM)X_v,\ -\Delta n_v = (FN - GM)X_u + (FM - EN)X_v.$$

After computations, equation (6) becomes

$$\widetilde{X}_u \wedge \widetilde{X}_v = (X_u \wedge X_v)\left(1 + \frac{EN - 2FM + GL}{\Delta}\delta + \frac{LN - M^2}{\Delta}\delta^2\right).$$

Since $S$ is a regular surface, we have $X_u \wedge X_v \neq 0$ on $U$. The second factor reminds of the principal curvatures equation.

The principal curvatures $k_1,\ k_2$ are always real; they can be equal (umbilical points) or both identically zero (planar points). The inverses of these functions of $(u,\ v)$ are called principal radii of curvature. Denoting $\delta = \frac{1}{k}$, it follows that $\widetilde{X}_u \wedge \widetilde{X}_v \neq 0$ in $(u,\ v)$ points in which $(EG-F^2)k^2 + (EN-2FN+GL)k + LN - M^2 = 0$, which represents the principal curvatures equation.

Thus, we have the following:

**Theorem 2** Let the family $\widetilde{X}(u,\ v) = X(u,\ v) + \delta \cdot n(u,\ v)$ of parallel surfaces to $X(u,\ v)$ be.

(i) If $\delta = \frac{1}{k_1(u_0,\ v_0)}$ or $\delta = \frac{1}{k_2(u_0,\ v_0)}$ are the principal radii of curvature in $X(u_0,\ v_0)$, then the point $\widetilde{X}(u_0,\ v_0)$ is singular on the surface $\widetilde{X}$.

(ii) Let $c$ be a curve on $S$ for which $k_1(u,\ v) = k_1^0$ (constant) or $k_2(u,\ v) = k_2^0$ (constant). The imagine of the curve $c$ pe $\widetilde{X}$ defined by $\delta = k_1^0$ or $\delta = k_2^0$ is a curve of singular points on $\widetilde{X}$.

**Proposition 1** Let $\widetilde{S}$ be the parallel surface to $S$ at distance $\delta$ given by (3). Then, $\widetilde{X}(u,\ v)$, viewed as an application $\widetilde{X}:\ U \subset \mathbb{R}^2 \to \mathbb{R}^3_1$, is a flat surface ($K \equiv 0$).

**Proof.** From [12], the unit normal vector field on $\widetilde{X}$ coincides with $n$, the unit normal vector field on $X(u,\ v)$, whence it follows that it is smooth. So, $\widetilde{X}(u,\ v)$, viewed as an application $\widetilde{X}:\ U \subset \mathbb{R}^2 \to \mathbb{R}^3_1$, is a flat surface ($K \equiv 0$), since $\widetilde{X}$ and $X$ have the same Gauss map, which degenerates when its determinant ($= K$) is zero.

Singularities of flat fronts in Euclidean space have been studied in [4] and the general result is that they are cuspidal edge or swallowtail. So, we can say that if a surface is flat, then the parallel surface to it admits as singular points only the cuspidal edge or the swallowtail.

Next we give an example showing that the parallel surface to a timelike flat surface admits a cuspidal edge singularity.

**Example 1** Let the surface $S$ be parametrized by

$$X(u,\ v) = \left(u,\ v^2 + \frac{6v}{\sqrt{9v^2+4}},\ v^3 - \frac{4}{\sqrt{9v^2+4}}\right). \tag{7}$$

The unit normal vector field on $S$ is

$$n = \left(0,\ -\frac{3v}{\sqrt{9v^2+4}},\ \frac{2}{\sqrt{9v^2+4}}\right).$$

Since $\langle n,\ n\rangle_1 > 0$, it follows that $S$ is a timelike surface.

The coefficients of the first and of the second fundamental form are:

$$E = -1,\ F = 0,\ \text{and } G = \left(v\sqrt{9v^2+4} + \frac{12}{9v^2+4}\right)^2,$$

respectively,

$$L = 0,\ M = 0,\ \text{and } N = \frac{6v\left(9v^2+4\right)\sqrt{9v^2+4} + 72}{\left(9v^2+4\right)^2}.$$

Thus,

$$K = \frac{LN - M^2}{EG - F^2} \equiv 0,$$

so $S$ is flat.

The parallel surface to $S$ at distance $\delta = 2$ is parametrized by

$$\widetilde{X}(u, v) = (u, v^2, v^3), \tag{8}$$

which represents the equation of a surface that admits the cuspidal edge as singularity.

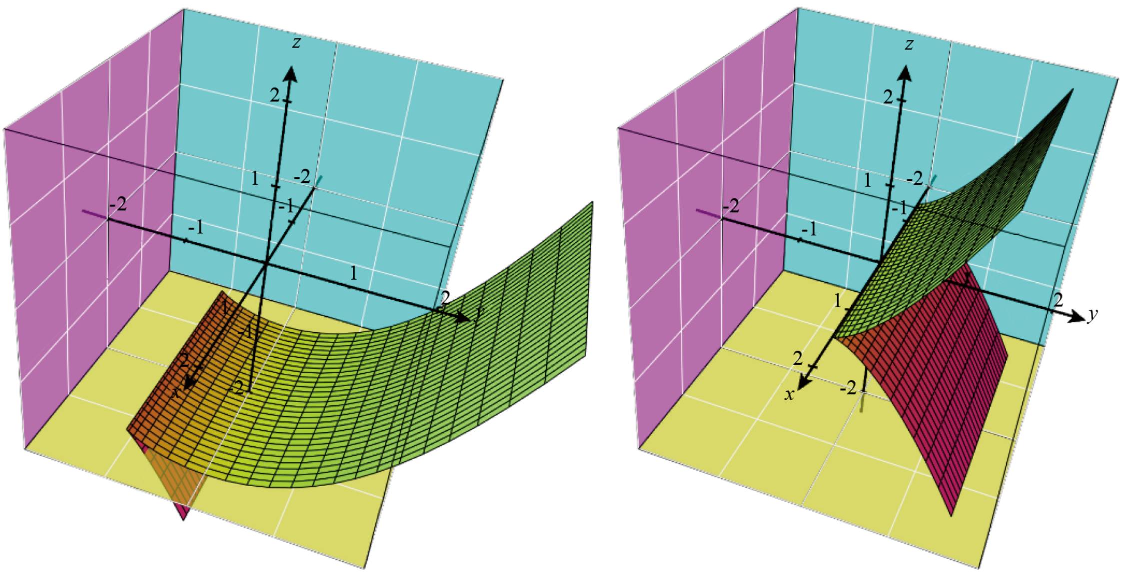


**Figure 1.** The flat timelike surface (7) (left) and its parallel surface (8) (right)

## 4. Conclusion

In this paper we proved that the family of parallel surfaces of a given surface can be obtained as the discriminant set of a certain family of functions. As a consequence, we obtained that Minkowski spheres with prescribed radius, which have contact of order 1 with a surface, have the centers on the parallel surface at distance equal with the radius. Furthermore, we showed that the parallel surface to a flat timelike surface admits cuspidal edge as singularity.

As a further research direction, we intend to study, in three-dimensional Minkowski space, the lightlike situation of the problem treated in the paper, and also, different non-flat surfaces, which in the Euclidean space, also admit other singularities, according to [3].

## Acknowledgement


The author is grateful to the anonymous reviewers for their careful reading of the manuscript, valuable suggestions, and insightful comments, which significantly improved the content of the paper, and to the editors of the Journal

Contemporary Mathematics for their support. The author wishes to express her gratitude to Professor Mihai Anastasiei from “Al. I. Cuza” University of Iaşi, for all the support given during and after the doctoral stage.

## Conflict of interest

The author declares no competing financial interest.

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