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Research Article

# Stored Energy in the Exterior Schwarzschild Space-Time

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**Abstract:** In the present study, we develop a well-behaved null tetrad for the exterior Schwarzschild metric. Additionally, under certain conditions, we were able to extract the tetrad components of the Maxwell field tensors for exterior Schwarzschild. In addition, in the presence of the geometry of the Schwarzschild solution, we found a real behaved energy density stored in electromagnetic.

***Keywords*:** exterior Schwarzschild solution, tetrad formalism, gravito-magnetism, stored energy

**MSC:** 04.25.Nx, 04.80.Cc, 04.50.h, 04.20.Jb

## 1. Introduction

The Newman-Penrose (NR) formalism was originally established by Newman and Penrose as a way to deal with general relativity in terms of spinor notation [1]. The common variables used in GR now have complex versions due to spinor notation. Tetrad formalism, in which tensors are projected onto a whole vector basis at each point in spacetime, is a specific example of NP formalism [2]. Expressions for physical observables that use this vector basis frequently have a simpler structure. A null tetrad is a group of four null vectors in the context of curved spacetime, consisting of two real and one pair of complex-conjugate vectors. The formalism is particularly suited to the treatment of radiation propagation in curved space as the two real members asymptotically point radially inward and outward. On a Lorentzian manifold, this collection of orthonormal vector fields is defined, with one being time-like and the other three being space-like. The world-Lines of observers in space-time are the integral curves constructed by geodesics to which these time-like unit vectors are tangent (tangent bundle). It starts by determining a null tetrad at this location which is then used to find other qualities. Each event $p$ on these world lines has a particular triad associated with it that these observers take with them. The investigation of gravitational relations issues may be resolved with the help of this general relativity procedure. The mathematical notation looks lengthy formulas and laborious calculations make it relatively complicated, but it offers a detailed understanding of the symmetries of spacetime. While general relativity calculations are frequently performed on a coordinate basis, working on a local orthonormal basis when measurements are involved can occasionally be advantageous. The NP-suitability as well as internal freedom for creating exact answers to Einstein field equations and for other studies into the theory of relativity are the primary causes of its success. The theory was formulated by Cohen and Kegeles


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[3] in the notation, covariant language of differentiated forms, as well as powerful theorems allowed for the essential generalization of the Hertzian scheme. To enable the use of the completely explicit NP formalism more easily, they converted these results into the common tensor notation. Gómez and Quiroga [4] provide a survey of the NP formalism and the asymptotic structure of spacetime. The Weyl scalars and spin coefficients for all stationary axisymmetric space times were computed using a tetrad basis, and some elements of this approach were also examined. Wood [5] described how the uncertainty was eliminated when selecting the tetrad for calculating the NP indices. In his proposal, a linear perturbation is a gravitational radiation of the space-time scale that is responsive to the wave equation. The NP formalism based on the null tetrad frame and Kerr metric has both been used to thoroughly study this equation [6]. A Mathematica program was created by Hasmani and Panchal [7] to expedite the examination of tetrad components of the Ricci tensor. The establishment of coordinates and metric tensor components is necessary for such a program. Interestingly, the researcher gains more time by performing calculations more quickly. Even though it was created quite some time ago, NP formalism is still used to address a number of general relativity-related issues today [8]. For spherically symmetric static solutions which allow for stable closed orbits, a null tetrad known as Bertrand Spacetime has been developed.

Ricci tensor, Weyl tensor and Ricci spin coefficients have also been investigated in [9]. The multiple structures of the Kerr-Newman metric near future null infinities according to Janis-Newman were then studied. Gong et al. [10] developed the NP constants for the gravitational and electromagnetic fields of the Kerr-Newman metric were calculated using the near future null infinity Newman-Unti formalism of the Kerr-Newman metric. The symmetry between a Schwarzschild black hole's exterior and inside has been investigated in [11]. The Godel universe's electromagnetic field behaviour has been examined in [12]. Salti and Aydogdu [13] have been valuable in shaping our understanding of energy in Schwarzschild-de Sitter spacetimes.

The stored energy impacts black hole dynamics, gravitational radiation, and energy loss mechanisms, contributing to observable effects like gravitational waves. The NP formalism provides researchers with a framework to model these energy interactions and better understand how they influence black hole behavior and the surrounding spacetime. In the context of the geometry of the Schwarzschild exterior solutions to Einstein's field equations, this paper evaluates and discusses the stored energy in electromagnetic fields, one of the most fundamental scientific problems.

## 2. Brief review of newman penrose formalism

As a starting point, NP first proposed a Riemannian space in four dimensions with signature $-2$. This four-dimensional Riemannian manifold is given with basis vectors in a null tetrad at each point. A pair of real null vectors $l_\sigma,\ n_\sigma$ and a pair of complex null vectors $m_\sigma,\ \bar{m}_\sigma$ consisting of two real, orthonormal vectors $c_\sigma,\ d_\sigma$ the following: $m_\sigma = \frac{1}{\sqrt{2}}(c_\sigma - i d_\sigma)$. The tetrad satisfies the pseudo-orthogonality relations:

$$m_\sigma \bar{m}^\sigma = -l_\sigma n^\sigma = -1, \tag{1}$$

with all other scalar products vanishing. The generic symbol $z_n^\sigma$ for null tetrad $l_\sigma,\ n_\sigma,\ m_\sigma$ and $\bar{m}_\sigma$ is introduced, where the tetrad vectors are listed as $n = 0,\ 1,\ 2,$ and 3.

The contravariant components of the metric tensor are provided, in terms of the tetrad, by [14] as a result of pseudo-orthogonality relations (1).

$$g^{\alpha\beta} = z_m^\alpha z_n^\beta \eta^{mn} = 2\left(l^{(\alpha} n^{\beta)} - m^{(\alpha} \bar{m}^{\beta)}\right). \tag{2}$$

where, $\eta^{mn}$ is the flat space metric used for raising and lowering tetrad indices, given by the representation:

$$\eta_{mn} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \tag{3}$$

The complex Ricci rotation coefficients, which describe the behavior and curvature of spacetime in a tetrad-based framework, are written for the tetrad vector in the following form [14]:

$$\gamma^{mnp} = \nabla_\nu z^m_\mu z^n z^\mu z^{p\nu} \tag{4}$$

Using the null tetrad components, the intrinsic derivatives are:

$$\mathit{\Pi} = l^\alpha \frac{\partial}{\partial x^\alpha},$$

$$\Delta = n^\alpha \frac{\partial}{\partial x^\alpha},$$

$$\delta = m^\alpha \frac{\partial}{\partial x^\alpha},$$

$$\bar{\delta} = \bar{m}^\alpha \frac{\partial}{\partial x^\alpha}. \tag{5}$$

The twelve spin coefficients are an essential component of general relativity. They are derived from the null tetrad components and provide crucial information about the behavior of spacetime. The spin coefficients describe the way in which null geodesics are transported along the null tetrad vectors. They also provide information about the curvature of spacetime and how it affects light rays. The spin coefficients have important applications in astrophysics, particularly in the study of black holes and gravitational waves. They are also used in cosmology to model the evolution of the universe. Despite their importance, the spin coefficients can be challenging to calculate and interpret, requiring advanced mathematical techniques and a deep understanding of general relativity. Nonetheless, they remain a vital tool for physicists seeking to understand the fundamental nature of space and time. The twelve spin coefficients are:

$$k = \gamma_{131} = \nabla_\nu l_\mu m^\mu l^\nu,$$

$$\rho = \gamma_{134} = \nabla_\nu l_\mu m^\mu \bar{m}^\nu,$$

$$\sigma = \gamma_{133} = \nabla_\nu l_\mu m^\mu m^\nu,$$

$$\tau = \gamma_{132} = \nabla_\nu l_\mu m^\mu n^\nu,$$

$$\nu = -\gamma_{242} = -\nabla_\nu n_\mu \bar{m}^\mu n^\nu,$$

$$\mu = -\gamma_{243} = -\nabla_\nu n_\mu \bar{m}^\mu m^\nu,$$

$$\lambda = -\gamma_{244} = -\nabla_\nu n_\mu \bar{m}^\mu \bar{m}^\nu,$$

$$\pi = -\gamma_{241} = -\nabla_\nu n_\mu \bar{m}^\mu l^\nu,$$

$$\alpha = \frac{1}{2}(\gamma_{124} - \gamma_{344}) = \frac{1}{2}(\nabla_\nu l_\mu n^\mu \bar{m}^\nu - \nabla_\nu m_\mu \bar{m}^\mu \bar{m}^\nu),$$

$$\beta = \frac{1}{2}(\gamma_{123} - \gamma_{343}) = \frac{1}{2}(\nabla_\nu l_\mu n^\mu m^\nu - \nabla_\nu m_\mu \bar{m}^\mu m^\nu),$$

$$\gamma = \frac{1}{2}(\gamma_{122} - \gamma_{342}) = \frac{1}{2}(\nabla_\nu l_\mu n^\mu n^\nu - \nabla_\nu m_\mu \bar{m}^\mu n^\nu),$$

$$\varepsilon = \frac{1}{2}(\gamma_{121} - \gamma_{341}) = \frac{1}{2}(\nabla_\nu l_\mu n^\mu l^\nu - \nabla_\nu m_\mu \bar{m}^\mu l^\nu). \tag{6}$$

The above spin coefficients describe how tetrad vectors change as one moves through spacetime, capturing the influence of curvature on the behavior of test particles or fields. The decoupling equation for the complex scalar potential, which simplifies the analysis of electromagnetic fields in curved spacetime by transforming the field equations into scalar wave equations, is given by [3]:

$$[(\Delta - \bar{\gamma} + \gamma + \bar{\mu})(\Pi + 2\varepsilon + \rho) - (\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau})(\delta + 2\beta + \tau)]\psi = 0, \tag{7}$$

this equation facilitates the separation of the scalar field from the gravitational field, providing valuable insights into the dynamics of both. Additionally, the decoupling equation is significant in theories beyond general relativity, such as string theory and quantum gravity, offering a mathematical framework that aids in understanding the fundamental nature of our universe.

The tetrad components of the Maxwell field tensor in terms of $\psi$ are:

$$\phi_o = [-(\Pi - \varepsilon + \bar{\varepsilon} - \bar{\rho})(\Pi + 2\bar{\varepsilon} + \bar{\rho})]\,\bar{\psi}$$

$$\phi_1 = \left[-(\Pi + \bar{\varepsilon} + \varepsilon)(\bar{\delta} + 2\bar{\beta} + \bar{\tau}) + (\pi + \bar{\tau})(\Pi + 2\bar{\varepsilon} + \bar{\rho})\right]\bar{\psi}$$

$$\phi_2 = \left[-(\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau})(\bar{\delta} + 2\bar{\beta} + \bar{\tau}) + \lambda(\Pi + 2\bar{\varepsilon} + \bar{\rho})\right]\bar{\psi}. \tag{8}$$

The existence of only three tetrad components of the Maxwell field tensor instead of four might raise some confusion, which necessitates clarification. The explanation is quite simple: these three components provide a sufficient and compact representation of the Maxwell field tensor. The complex nature of the potential allows these three components to encapsulate all the necessary information about the electromagnetic field. One component is often related to the electric field, while the other two typically relate to the components of the magnetic field. Therefore, the fourth component $(\phi_4)$ is not required and does not provide additional unique information in this specific formalism.

The definition of the Maxwell's strength tensor in terms of the Newman-Penrose components are given by

$$F_{\mu\nu} = 2(\phi_1 + \bar{\phi}_1)nl_{[\mu\nu]} + 2\phi_2 lm_{[\mu\nu]} + 2\bar{\phi}_2 l\bar{m}_{[\mu\nu]} + 2\phi_0 \bar{m}n_{[\mu\nu]} + 2\bar{\phi}_0 mn_{[\mu\nu]} + 2(\phi_1 - \bar{\phi}_1)m\bar{m}_{[\mu\nu]}. \tag{9}$$

The Maxwell field tensor $F_{\mu\nu}$ is a second rank antisymmetric tensor in four-dimensional spacetime, possessing six independent components due to its antisymmetric nature.

## 3. Stored energy in the exterior schwarzschild space-time

The exterior Schwarzschild space-time metric is defined by [15]:

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{10}$$

The ccontravariant and metric tensors of equation (10) are:

$$g^{\mu\nu} = \begin{pmatrix} \left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 & 0 \\ 1 & -\left(1 - \frac{2M}{r}\right) & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2 \sin^2\theta} \end{pmatrix}. \tag{11}$$

Substituting equation (11) in equation (2), the metric tensor's contravariant components in terms of the tetrad are given by:

$$2\left(l^0 n^0 - m^0 \bar{m}^0\right) = \left(1 - \frac{2M}{r}\right)^{-1},$$

$$2\left(l^1 n^1 - m^1 \bar{m}^1\right) = -\left(1 - \frac{2M}{r}\right),$$

$$2\left(l^2 n^2 - m^2 \bar{m}^2\right) = -\frac{1}{r^2},$$

$$2\left(l^3n^3 - m^3\bar{m}^3\right) = -\frac{1}{r^2\sin^2\theta},$$

$$\left(l^0n^1 + l^1n^0 - m^0\bar{m}^1 - m^1\bar{m}^0\right) = 0,$$

$$\left(l^0n^2 + l^2n^0 - m^0\bar{m}^2 - m^2\bar{m}^0\right) = 0,$$

$$\left(l^0n^3 + l^3n^0 - m^0\bar{m}^3 - m^3\bar{m}^0\right) = 0,$$

$$\left(l^1n^2 + l^2n^1 - m^1\bar{m}^2 - m^2\bar{m}^1\right) = 0,$$

$$\left(l^1n^3 + l^3n^1 - m^1\bar{m}^3 - m^3\bar{m}^1\right) = 0,$$

$$\left(l^2n^3 + l^3n^2 - m^2\bar{m}^3 - m^3\bar{m}^2\right) = 0. \tag{12}$$

In general relativity, the choice of tetrad can influence the form of the field equations and the corresponding solutions. While the physical content of the solutions should remain consistent, the specific representation of the solutions can vary depending on the tetrad used. This means that different tetrads might lead to different forms of electromagnetic field equations, but these should ultimately describe the same physical phenomena when transformed back to a common coordinate system. One of the solutions of the system represented by (12) is

$$l^\alpha = \left[\frac{1}{\sqrt{2}}\left(1 - \frac{2M}{r}\right)^{\frac{-1}{2}},\ 0,\ 0,\ \frac{-1}{\sqrt{2}r\sin\theta}\right],$$

$$n^\alpha = \left[\frac{1}{\sqrt{2}}\left(1 - \frac{2M}{r}\right)^{\frac{-1}{2}},\ 0,\ 0,\ \frac{1}{\sqrt{2}r\sin\theta}\right],$$

$$m^\alpha = \left[0,\ \frac{i}{\sqrt{2}}\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}},\ \frac{1}{\sqrt{2}r},\ 0\right],$$

$$\bar{m}^\alpha = \left[0,\ \frac{-i}{\sqrt{2}}\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}},\ \frac{1}{\sqrt{2}r},\ 0\right]. \tag{13}$$

The covariant form of this solution is given by

$$l_{\alpha}=\left[\frac{1}{\sqrt{2}}\left(1-\frac{2M}{r}\right)^{\frac{1}{2}},\ 0,\ 0,\ \frac{r\sin\theta}{\sqrt{2}}\right],$$

$$n_{\alpha}=\left[\frac{1}{\sqrt{2}}\left(1-\frac{2M}{r}\right)^{\frac{1}{2}},\ 0,\ 0,\ \frac{-r\sin\theta}{\sqrt{2}}\right],$$

$$m_{\alpha}=\left[0,\ \frac{-i}{\sqrt{2}}\left(1-\frac{2M}{r}\right)^{\frac{-1}{2}},\ \frac{-r}{\sqrt{2}},\ 0\right],$$

$$\bar{m}_{\alpha}=\left[0,\ \frac{i}{\sqrt{2}}\left(1-\frac{2M}{r}\right)^{\frac{-1}{2}},\ \frac{-r}{\sqrt{2}},\ 0\right]. \tag{14}$$

Using egaution (5), the intrinsic derivatives are:

$$\Pi=l^{\alpha}\frac{\partial}{\partial x^{\alpha}}=\frac{1}{\sqrt{2}}\left(1-\frac{2M}{r}\right)^{\frac{-1}{2}}\frac{\partial}{\partial t}-\frac{1}{\sqrt{2}r\sin\theta}\frac{\partial}{\partial\phi},$$

$$\Delta=n^{\alpha}\frac{\partial}{\partial x^{\alpha}}=\frac{1}{\sqrt{2}}\left(1-\frac{2M}{r}\right)^{\frac{-1}{2}}\frac{\partial}{\partial t}+\frac{1}{\sqrt{2}r\sin\theta}\frac{\partial}{\partial\phi},$$

$$\delta=m^{\alpha}\frac{\partial}{\partial x^{\alpha}}=\frac{i}{\sqrt{2}}\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}\frac{\partial}{\partial r}+\frac{1}{\sqrt{2}r}\frac{\partial}{\partial\theta},$$

$$\bar{\delta}=\bar{m}^{\alpha}\frac{\partial}{\partial x^{\alpha}}=\frac{-i}{\sqrt{2}}\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}\frac{\partial}{\partial r}+\frac{1}{\sqrt{2}r}\frac{\partial}{\partial\theta}. \tag{15}$$

Substituting (13) and (14) into equation (6) we get, the zero spin coefficients are:

$$k=\rho=\sigma=\tau=\nu=\mu=\lambda=\pi=\gamma=\varepsilon=0, \tag{16}$$

and the other non-zero spin coefficients are:

$$\alpha=\frac{-mi}{\sqrt{2}r^{2}}\left(1-\frac{2M}{r}\right)^{\frac{-1}{2}},\ \beta=\frac{mi}{\sqrt{2}r^{2}}\left(1-\frac{2M}{r}\right)^{\frac{-1}{2}}. \tag{17}$$

Substituting (15), (16) and (17) into the decoupled equation (7) we get,

$$\left(1-\frac{2M}{r}\right)^{-1}\frac{\partial^2\psi}{\partial t^2}-\left(1-\frac{2M}{r}\right)\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}-\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}-\frac{5M}{r^2}\frac{\partial\psi}{\partial r}$$

$$-\frac{i}{r^2}\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}\frac{\partial\psi}{\partial\theta}+\left[\frac{4Mr-10M^2}{r^3(r-2M)}\right]\psi=0. \tag{18}$$

Equation (18) represent partial second order linear differential equation. The real and imaginary parts of this equation respectively are

$$\left(1-\frac{2M}{r}\right)^{-1}\frac{\partial^2\psi}{\partial t^2}-\left(1-\frac{2M}{r}\right)\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}-\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}-\frac{5M}{r^2}\frac{\partial\psi}{\partial r}+\left[\frac{4Mr-10M^2}{r^3(r-2M)}\right]\psi=0. \tag{19}$$

$$\frac{1}{r^2}\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}\frac{\partial\psi}{\partial\theta}=0. \tag{20}$$

Next, we will search for a solution to the system of equations (19) and (20) under specific conditions. Initially, we consider a stationary solution that is symmetric with respect to $\theta$ and $\phi$ i.e., $\psi$ depends solely on the coordinate $r$. In this scenario, the system simplifies

$$\left(1-\frac{2M}{r}\right)\frac{d^2\psi}{dr^2}+\frac{5M}{r^2}\frac{d\psi}{dr}-\left[\frac{4Mr-10M^2}{r^3(r-2M)}\right]\psi=0. \tag{21}$$

Equation (21) represent linear second order differential equation. The solution of the equation are given by,

$$\psi=\left(1-\frac{2M}{r}\right)^{-1}\left[c_1+c_2\left(r\sqrt{1-\frac{2M}{r}}+2M\ln\left(\sqrt{r}+\sqrt{r-2M}\right)\right)\right]. \tag{22}$$

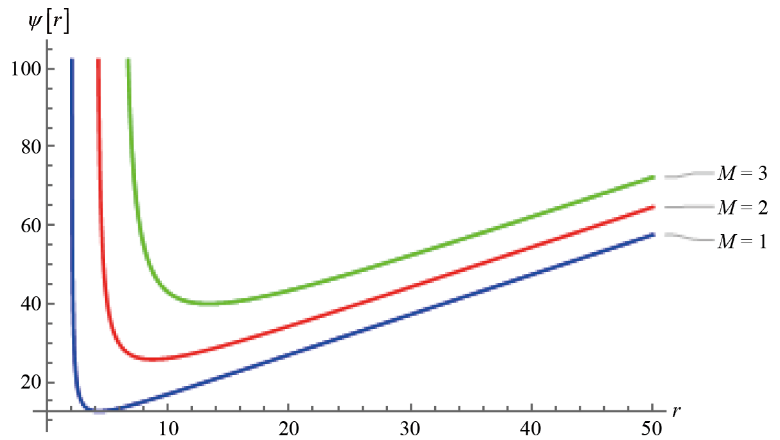


**Figure 1.** Scalar field $\psi(r)$ due to (22) when $c_1=1$, $c_2=1$ at various values of $M$

Equation (22) represents the solution of the complex scalar potential's decoupled equation (7), and it gives us the scalar potential function $\psi$ as a function of the Schwarzschild radius $r$ and mass $M$. The behavior of the scalar potential function can be explained by the fact that the Schwarzschild mass $M$ creates a gravitational field that affects the surrounding space-time. Understanding these complex interactions between mass and space-time is crucial for our understanding of the universe and its many mysteries. To try to understand this relationship, this function is plotted in Figure 1 for different values of mass $M$. It is clear from figure that the change in mass has a slight effect. It can be seen that the three curves are similar in behaviour, but it is noted that the values of the scalar potential function increase with the increase in the Schwarzschild mass $M$. It is clear from the drawing that the value of $\psi$ is very large near the center of the hard body, and then the value of the scalar potential function decreases very quickly with the increase of $r$ until it reaches its minimum value. The value of $\psi$ begins to increase again from its minimum value and increases steadily with the increase of the variable $r$.

Substituting equation (22) in equation (8) we get the tetrad components of the Maxwell field tensor as follow:

$$\phi_o = 0,\ \phi_1 = 0,$$

$$\phi_2 = \frac{M^2}{r^2(r-2M)^2}\left(-\frac{10M}{r} - 3\left(c_1 + c_2\sqrt{r}\sqrt{r-2M} - 3\right) + \frac{r}{M}\left(2c_1 + c_2\sqrt{r}\sqrt{r-2M} - 2\right)\right)$$

$$+ c_2\frac{2M^2(2r-3M)}{r^2(r-2M)^2}\ln\left(\sqrt{r} + \sqrt{r-2M}\right). \tag{23}$$

Substituting equation (14) and (23) in equation (9) we get

$$E_x = -F_{01} = 0,\ E_z = -F_{03} = 0,$$

$$E_y = -F_{02} = \frac{2M^2}{r^3\left(1-\frac{2M}{r}\right)^{\frac{3}{2}}}\left(-\frac{10M}{r} - 3\left(c_1 + c_2\sqrt{r}\sqrt{r-2M} - 3\right) + \frac{r}{M}\left(2c_1 + c_2\sqrt{r}\sqrt{r-2M} - 2\right)\right)$$

$$+ c_2\frac{4M^2(2r-3M)}{r^3\left(1-\frac{2M}{r}\right)^{\frac{3}{2}}}\ln\left(\sqrt{r} + \sqrt{r-2M}\right)$$

$$H_x = F_{23} = \frac{2M^2\sin\theta}{(r-2M)^2}\left(-\frac{10M}{r} - 3\left(c_1 + c_2\sqrt{r}\sqrt{r-2M} - 3\right) + \frac{r}{M}\left(2c_1 + c_2\sqrt{r}\sqrt{r-2M} - 2\right)\right)$$

$$+ c_2\frac{4M^2(2r-3M)\sin\theta}{(r-2M)^2}\ln\left(\sqrt{r} + \sqrt{r-2M}\right)$$

$$H_y = -F_{13} = 0,\ H_z = F_{12} = 0. \tag{24}$$

Then the energy density

$$c^2 S^{00} = \frac{E^2 + H^2}{2} = -\frac{4M^2 \aleph^2 (2M - r - r^3 \sin^2\theta)}{r^5 (r - 2M)^4} \tag{25}$$

where

$$\aleph = 10M^2 + 3Mr\left(-3 + c_1 + c_2 r\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}\right)$$

$$-r^2\left(-2 + 2c_1 + c_2 r\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} + 2M(3M - 2r) r c_2 \ln[\sqrt{r} + \sqrt{r - 2M}]\right) \tag{26}$$

And the linear momentum density

$$cS^{0i} = \frac{(\vec{E} \times \vec{H})^i}{c} = \left(0,\ 0,\ \frac{-4M^2 \sin\theta}{r^7\left(1 - \frac{2M}{r}\right)^{\frac{7}{2}}} \aleph^2\right). \tag{27}$$

where the quantities $(\vec{E} \times \vec{H})^i$ represent the Poyinting vector components and $c$ ia the speed of light.

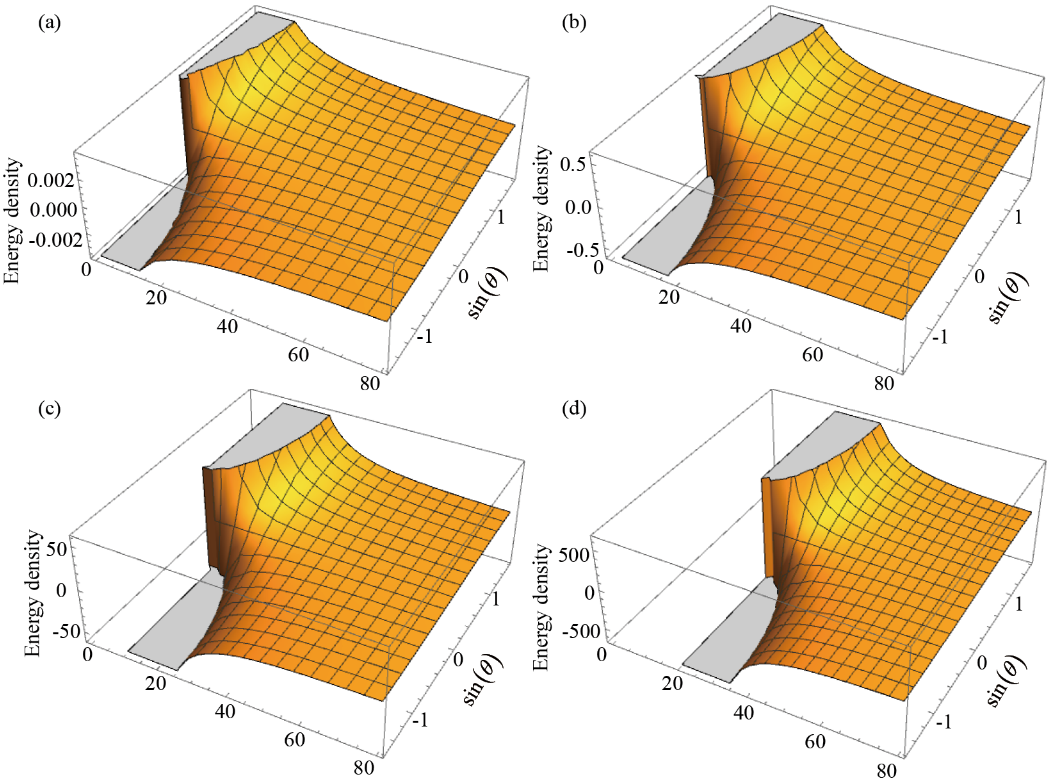


Figure 2. Energy density due to (25) for various values of the black hole's mass (a) $c_1 = 1$, $c_2 = 1$ and $M = 0.1$ (b) $c_1 = 1$, $c_2 = 1$ and $M = 1$ (c) $c_1 = 1$, $c_2 = 1$ and $M = 5$ (d) $c_1 = 1$, $c_2 = 1$ and $M = 10$

The energy density for Schwarzschild spacetime is a crucial concept in general relativity. It refers to the amount of energy per unit volume present in the spacetime surrounding spherically symmetric black hole. This energy density is a direct result of the curvature of spacetime caused by the massive object at its center. Figure 2, obtained from plotting the result obtained for equation (25) give us the energy density of the exterior Schwarzschild spacetime as a function of the variables $r$ and $\theta$ for various values of the black hole's mass. We notice in all figures for the small values of $r$ near the center of the black hole, the amplitude of the energy density curves is very large, and this amplitude begins to diminish little by little with the increase in the values of the radius $r$. This means that the energy density is highest near the event horizon, where gravitational forces are strongest and space is most severely curved. As one moves further away from the black hole, the energy density decreases rapidly until it becomes negligible at large distances. All figures clearly show a region where the energy is undefined for small values of $r$, specifically where $r$ is less than $2M$. This is because this region was not included in the black hole solution being considered. Understanding the energy density of Schwarzschild spacetime is essential for studying the behavior of matter and radiation near black holes, as well as for developing accurate models of astrophysical phenomena.

## 4. Conclusions

In the setting of curved space-time, a null tetrad is a set of four null vectors made up of two real and one pair of complex-conjugate vectors. Because the two real members asymptotically point radially inward and outward, the formalism is particularly well suited to the consideration of radiation propagation in curved space. For the exterior Schwarzschild metric, we create a well-behaved null tetrad in the current work. We were able to obtain the tetrad components of the Maxwell field tensors for exterior Schwarzschild under certain conditions. Moreover, we obtained a real-valued energy density stored in the electromagnetic field within the context of the Schwarzschild geometry.

All of the graphical representations show that the energy density curves have very large amplitudes for small values of r close to the black hole's centre, and that as r increases, this amplitude gradually decreases. As a result, the event horizon, which is also the location of the greatest gravitational forces and the most extreme curvature of space, has the largest energy density. The energy density progressively drops until it is insignificant as one gets farther away from the black hole.

## Conflict of interest

The authors declare no competing financial interest.

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