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Research Article

Fourth-Order Delay Differential Equations: New Monotonic Properties of Solutions

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Abstract: New oscillation criteria for the fourth-order delay differential equation are presented. An essential feature of our results is that oscillation of the studied equation is ensured via some conditions. Furthermore, new criteria are provided to delay differential equations by using the Riccati transform technique, the integrated averaging approach, and the comparison method with second order. Our results essentially improve, extend, and simplify some known ones reported in the literature. The results are illustrated with examples.

***Keywords*:** delay, oscillation, fourth-order DEs

MSC: 34K10, 34K11

1. Introduction

This work focuses on the oscillatory behaviours expressed through a fourth-order delay differential equation.

$$\left(r(v)\left(y'''(v)\right)^{\alpha}\right)' + q(v)y^{\beta}(\sigma(v)) = 0, \tag{1}$$

where $v \geq v_0$. We consider the following presumptions to be hold throughout this paper:

- β, α, are quotient of odd positive integers,
- r, $q \in C[v_0, \infty)$, $r(v) > 0$, $r'(v) \geq 0$, $q(v) > 0$, $\sigma \in C[v_0, \infty)$, $\sigma(v) \leq v$, $\lim_{v\to\infty} \sigma(v) = \infty$, and

$$\int_{v_0}^{\infty} \frac{1}{r^{1/\alpha}(s)} ds = \infty. \tag{2}$$

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In the study of differential equations, oscillation theory is essential. It is used in many disciplines, including physics, engineering, biology, and economics. It helps with the comprehension of wave phenomena and mechanical vibrations in physics, which is important for the stability and design of structures. Oscillation theory is used in engineering applications, particularly in control systems and signal processing, to guarantee system stability and efficient signal manipulation. It aids in the modelling of population dynamics and biological cycles in biology, shedding light on phenomena such as circadian rhythms and predator-prey relationships. It is useful in economics for examining market dynamics and business cycles, forecasting trends, and evaluating the effects of legislation.

Comprehending oscillations is essential for evaluating nonlinear dynamics, which is prevalent in real-world systems, and for forecasting resonant events and stability studies. Oscillation theory makes it possible to comprehend and control complicated system behaviours, which promotes improvements in economic modelling, scientific study, and technology. Because of this, it is a crucial component of differential equation research and application. Due to the importance of nonlinear differential equations in numerous disciplines, research on oscillatory and non-oscillatory solutions to these problems has been conducted [1]. The use of fourth-order differential equations allows for the mathematical modelling of a wide variety of biochemical, physical, and biological phenomena.

Many physical, chemical, and biological processes frequently use fourth-order differential equations in their mathematical models [1, 2]. Applications include issues with soil settlement, elasticity, and structural deformation, among other things. In mechanical and engineering problems, questions about the existence of oscillatory and nonoscillatory solutions are crucial [3]. In particular, the oscillatory behaviour of ordinary differential equations plays a crucial role in these applications. Non-linear ordinary differential equations of fractional and non-fractional order in series forms were thoroughly analyzed, see [4]. The authors provided a novel approach to solve non-linear non-integer differential equations, in which ST and Adomian polynomials are combined in this algorithm.

An extensive analysis of the oscillation requirements for fourth-order linear delay differential equations was given by Jadlovska et al. [5]. The authors contributed to the theoretical knowledge required for real-world applications in a variety of fields by providing insights into the circumstances under which solutions oscillate.

Karpuz [6] introduced the Hille-Nehari nonoscillation/oscillation test and examined the nonoscillation and oscillation characteristics of solutions to the second-order linear dynamic problem.

$$\left(rx^{\Delta}\right)^{\Delta}(t)+p(t)x(t)=0 \quad \text{for } t\in[t_0,\,\infty)_{\mathbb{T}}.$$

Fourth-order nonlinear advanced differential equations, which are used to solve challenging real-world situations, were the subject of the study's investigation of oscillation [7]

$$\left(r(v)\left(y'''(v)\right)^{\alpha}\right)'+p(v)f\left(y'''(v)\right)+q(v)g(y(\sigma(v)))=0.$$

They applied the Riccati transformation technique to produce new oscillation criteria.

Using a Philos-type technique, additional oscillation conditions are formulated for third-order mixed neutral differential equations with distributed deviating arguments. Three Riccati transformation approaches were combined with the integral averaging methodology by Kumar et al. [8]. Furthermore, the oscillating characteristics of solutions to a class of third-order differential equations

$$\left(a(t)\left(\left(r(t)x'(t)\right)'\right)^{\alpha}\right)'+\left(p(t)\left(\left(r(t)x'(t)\right)\right)'\right)^{\alpha}+q(t)f(x(\sigma(t)))=0,$$

were investigated by using the generalized Riccati technique by Wang and Dong [9], where $f(y)/y^{\rho}\geq k>0$ and α and β are quotients of odd positive integers. Atta et al. [10] presented a spectral tau solution to the heat conduction equation in order to convert the issue and its underlying conditions into a suitable system of equations that may be solved successfully by the Gaussian elimination method.

Bohner et al. [11] presented new oscillation criteria for the second-order half-linear neutral delay differential equation, which essentially improve a number of related ones from the literature

$$\left(r\left(z'\right)^{\alpha}\right)'(v)+q(v)y^{\alpha}(\sigma(v))=0,\ v\geq t_0>0,$$

under the condition

$$\pi\left(t_0\right):=\int_{t_0}^{\infty}r^{-1/\alpha}(s)ds<\infty.$$

They utilised a recently created method for successively changing the monotonicities of nonoscillatory binomial differential equation solutions. This method has proven useful in the analysis of second-order half-linear functional differential equations and higher-order linear differential and difference equations. Oscillation of even-order neutral differential equations presented by Bazighifan [12]. Using the technique of Riccati and comparison with first-order differential equations, new Kamenev-type oscillation criteria are established, and they essentially improve and complement some the well-known results reported in the literature.

Using a spectral collocation algorithm, Abd-Elhameed et al. [13] introduced a novel method for efficiently obtaining numerical solutions of the nonlinear time-fractional generalized Kawahara equation (NTFGKE) and their methodology is validated by means of a set of numerical experiments with comparison evaluations to show its efficacy and reliability.

New adequate conditions were presented for the oscillation of a second-order quasilinear neutral delay differential equation

$$\left(a(v)\left(z'(v)\right)^{\beta}\right)'+q(v)y^{\gamma}(\sigma(v))=0.$$

The oscillation results were contingent upon a single condition and significantly enhanced, supplemented, and streamlined several related ones in the existing literature [14]. Furthermore, Nagehan et al. [15] studied the oscillatory behaviour of solutions to a fourth-order linear delay differential equation

$$\left(r_2\left(r_1y'\right)'\right)''+q_1(v)y\left(\tau_1(v)\right)=0.$$

New oscillation criteria were given for a special type of fourth-order differential equation. A new approach for changing fourth-order semi-canonical nonlinear neutral difference equations into canonical form has been proposed by Ganesan et al. [16]

$$D_4z(n)+q(n)x^{\alpha}(n-\tau)=0,\ n\geq n_0>0.$$

The authors developed some novel oscillation criteria by comparing them to first-order delay difference equations. Ahmed and Abd-Elhameed [17] used a novel numerical method to address three types of high-order singular boundary value problems. They proposed three third-kind modified Chebyshev polynomials (CPs) as basis functions for these equations, transforming them into algebraic systems amenable to numerical solutions. Next, they applied the collocation approach using the operational matrices of derivatives of the third-kind modified CPs.

We studied the oscillatory behaviour of the nonlinear delay differential equation of the fourth order. The fundamental idea of our study relies on three mathematical approaches: the Riccati substitution, the integral averaging technique, and the comparison technique. These techniques enable us to derive novel findings on the oscillatory of equation (1).

The paper is organized as follows. In Section 2, we present some lemmas which play important roles in the proofs of the main results. In Section 3, we shall use the integral averaging technique and comparison method to obtain some sufficient conditions for oscillation of every solution of equation (1). In Section 4, we give some examples in order to illustrate the main result.

2. Some basics

First we need the following definitions:

Definition 1 If $r(v)\left(y^{(n-1)}(v)\right)^{\alpha} \in C^1[v_y, \infty)$ and $y(v)$ satisfies (2) on $[v_y, \infty)$, then $y \in C^3[v_y, \infty)$, $v_y \geq v_0$, is a solution of (1).

Definition 2 If a solution to (1) has arbitrarily large zeros on $[v, \infty)$, it is commonly known as oscillatory; if not, it is called nonoscillatory.

Definition 3 Let $D = \left\{(v, s) \in \mathbb{R}^2 : v \geq s \geq t_0\right\}$ and $D_0 = \left\{(v, s) \in \mathbb{R}^2 : v > s \geq v_0\right\}$. The function class $\Im$ is said to have the kernel function $N_i \in C(D, \mathbb{R})$ if

• (i) $N_i(v, s) = 0$ for $v \geq t_0$, $N_i(v, s) > 0$, $(v, s) \in D_0$;

• (ii) $N_i(v, s)$ has a continuous and nonpositive partial derivative $\partial N_i/\partial s$ on D_0 and there exist functions ϑ, $v \in C^1([t_0, \infty), (0, \infty))$ and $n_i \in C(D_0, \mathbb{R})$ for $i = 1, 2$ that give

$$\frac{\partial}{\partial s} N_1(v, s) + \frac{\delta'(s)}{\delta(s)} N_1(v, s) = n_1(v, s) N_1^{\alpha/(\alpha+1)}(v, s), \tag{3}$$

and

$$\frac{\partial}{\partial s} N_2(v, s) + \frac{\vartheta'(s)}{\vartheta(s)} N_2(v, s) = n_2(v, s) \sqrt{N_2(v, s)}. \tag{4}$$

It is important to note that the study of the asymptotic behaviour of the positive solutions to (1) only considers two cases:

Case 1 : $y^{(j)}(v) > 0$ for $j = 0, 1, 2, 3$,

Case 2 : $y^{(j)}(v) > 0$ for $j = 0, 1, 3$ and $y''(v) < 0$,

for $v \geq v_1$, where $v_1 \geq v_0$ is sufficiently large. We need the following notations:

$$Q_1(v) = \delta(v) q(v) A_1^{\beta-\alpha} \left(\frac{\sigma(v)}{v}\right)^{3\beta},$$

$$\Phi(v) = \vartheta(v) A_2^{\beta/\alpha-1}(v) \int_v^{\infty} \left(\frac{1}{r(u)} \int_u^{\infty} q(s) \frac{\sigma^{\beta}(s)}{s^{\beta}} ds\right)^{1/\alpha} du,$$

and

$$\Theta(v)=\alpha\mu_1\frac{v^2}{2r^{1/\alpha}(v)\delta^{1/\alpha}(v)}.$$

Some relevant lemmas are as follows:

Lemma 1 [18] Let $y\in C^n([v_0,\ \infty),\ (0,\ \infty))$ and $y^{(n-1)}(v)y^{(n)}(v)\leq 0$ for all $v\geq v_1$, then

$$y(v)\geq\frac{\mu}{(n-1)!}v^{n-1}\left|y^{(n-1)}(v)\right|,$$

for $v\geq v_\mu$.

Lemma 2 [19] If the function y satisfies $y^{(i)}(v)>0,\ i=0,\ 1,\ \ldots,\ n$, and $y^{(n+1)}(v)<0$, then

$$\frac{y(v)}{v^n/n!}\geq\frac{y'(v)}{v^{n-1}/(n-1)!}.$$

Lemma 3 [20] Let α be a ratio of two odd numbers, $V>0$ and U are constants. Then

$$Uy-Vy^{(\alpha+1)/\alpha}\leq\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{U^{\alpha+1}}{V^\alpha}.$$

Lemma 4 Assume that y is an eventually positive solution of (1) and

$$\xi'(v)\leq\frac{\delta'(v)}{\delta(v)}\xi(v)-Q_1(v)-\alpha\mu_1\frac{v^2}{2r^{1/\alpha}(v)\delta^{1/\alpha}(v)}\xi^{\frac{\alpha+1}{\alpha}}(v),\tag{5}$$

and

$$\varphi'(v)\leq-\Phi(v)+\frac{\vartheta'(v)}{\vartheta(v)}\varphi(v)-\frac{1}{\vartheta(v)}\varphi^2(v),\tag{6}$$

where

$$\xi(v):=\delta(v)\frac{r(v)\left(y'''(v)\right)^\alpha}{y^\alpha(v)},\tag{7}$$

and

$$\varphi(v):=\vartheta(v)\frac{y'(v)}{y(v)},\ v\geq v_1.\tag{8}$$

Proof. Let y represent a positive solution to (1) on $[v_0, \infty)$. Assume that Case 1 holds. Since $\xi(v) > 0$ for $v \geq v_1$ and from the definition of y we get

$$\left(r(v)\left(y'''(v)\right)^{\alpha}\right)' + q(v)y^{\beta}(\sigma(v)) \leq 0, \tag{9}$$

which with (1) we obtain

$$\xi'(v) \leq \frac{\delta'(v)}{\delta(v)}\xi(v) - \delta(v)q(v)\frac{y^{\beta}(\sigma(v))}{y^{\alpha}(v)} - \alpha\delta(v)\frac{r(v)\left(y'''(v)\right)^{\alpha}}{y^{\alpha+1}(v)}y'(v). \tag{10}$$

From Lemma 2, we have $y(v) \geq \frac{v}{3}y'(v)$, accordingly,

$$\frac{y(\sigma(v))}{y(v)} \geq \frac{\sigma^3(v)}{v^3}. \tag{11}$$

It follows from Lemma 1 that

$$y'(v) \geq \frac{\mu_1}{2}v^2 y'''(v), \tag{12}$$

for all $\mu_1 \in (0, 1)$ and every large v. From (10)-(12), we find

$$\xi'(v) \leq \frac{\delta'(v)}{\delta(v)}\xi(v) - \delta(v)q(v)y^{\beta-\alpha}(v)\left(\frac{\sigma(v)}{v}\right)^{3\beta} - \alpha\mu_1\frac{v^2}{2r^{1/\alpha}(v)\delta^{1/\alpha}(v)}\xi^{\frac{\alpha+1}{\alpha}}(v).$$

Since $y'(v) > 0$, there exist $v_2 \geq v_1$ and $A_1 > 0$ such that

$$y(v) > A_1. \tag{13}$$

Thus, we obtain

$$\xi'(v) \leq \frac{\delta'(v)}{\delta(v)}\xi(v) - \delta(v)q(v)\left(1-p_0\right)^{\beta}A^{\beta-\alpha}\left(\frac{\sigma(v)}{v}\right)^{3\beta} - \alpha\mu_1\frac{v^2}{2r^{1/\alpha}(v)\delta^{1/\alpha}(v)}\xi^{\frac{\alpha+1}{\alpha}}(v),$$

which yields

$$\xi'(v) \leq \frac{\delta'(v)}{\delta(v)}\xi(v) - Q_1(v) - \alpha\mu_1\frac{v^2}{2r^{1/\alpha}(v)\delta^{1/\alpha}(v)}\xi^{\frac{\alpha+1}{\alpha}}(v).$$

Consequently (5) holds.

Now consider that Case 2 holds. Integrating (9) from v to u, we acquire

$$r(u)\left(y'''(u)\right)^{\alpha}-r(v)\left(y'''(v)\right)^{\alpha}\leq-\int_{v}^{u}q(s)y^{\beta}(\sigma(s))ds. \tag{14}$$

From Lemma 2 we get that $y(v)\geq vy'(v)$, and hence

$$y(\sigma(v))\geq\frac{\sigma(v)}{v}y(v). \tag{15}$$

For (14), letting $u\to\infty$ and using (15),

$$r(v)\left(y'''(v)\right)^{\alpha}\geq y^{\beta}(v)\int_{v}^{\infty}q(s)\frac{\sigma^{\beta}(s)}{s^{\beta}}ds.$$

By doing another integration of this inequality, ranging from v to ∞, we obtain

$$y''(v)\leq-y^{\beta/\alpha}(v)\int_{v}^{\infty}\left(\frac{1}{r(u)}\int_{u}^{\infty}q(s)\frac{\sigma^{\beta}(s)}{s^{\beta}}ds\right)^{1/\alpha}du. \tag{16}$$

Since $\varphi(v)>0$ for $v\geq v_1$, as can be shown from its definition and using (13) and (16), we find

$$\begin{aligned}\varphi'(v)=&\frac{\vartheta'(v)}{\vartheta(v)}\varphi(v)+\vartheta(v)\frac{y''(v)}{y(v)}-\vartheta(v)\left(\frac{y'(v)}{y(v)}\right)^{2}\\ \leq&\frac{\vartheta'(v)}{\vartheta(v)}\varphi(v)-\frac{1}{\vartheta(v)}\varphi^{2}(v)-\vartheta(v)y^{\beta/\alpha-1}(v)\int_{v}^{\infty}\left(\frac{1}{r(u)}\int_{u}^{\infty}q(s)\frac{\sigma^{\beta}(s)}{s^{\beta}}ds\right)^{1/\alpha}du.\end{aligned}$$

Since $y'(v)>0$, there exist $v_2\geq v_1$ and $A_2>0$ such that

$$y(v)>A_2. \tag{17}$$

Hence, we obtain

$$\varphi'(v)\leq-\Phi(v)+\frac{\vartheta'(v)}{\vartheta(v)}\varphi(v)-\frac{1}{\vartheta(v)}\varphi^{2}(v).$$

Thus, (6) holds and the proof is complete.

3. Main results

In this section we shall present our results.

Theorem 1 Let (2) holds. If there exist positive functions $\delta,\ \vartheta \in C^1([v_0,\ \infty),\ \mathbb{R})$ such that

$$\limsup_{v\to\infty} \frac{1}{N(v,\ v_1)} \int_{v_1}^{v} N(v,\ s)Q_1(s) - \frac{h_1^{\alpha+1}(v,\ s)N_1^{\alpha}(v,\ s)}{(\alpha+1)^{\alpha+1}} \frac{2^{\alpha} r(s)\delta(s)}{(\mu_1 s^2)^{\alpha}} ds = \infty, \tag{18}$$

and

$$\limsup_{v\to\infty} \frac{1}{N_2(v,\ v_1)} \int_{v_1}^{v} \left(N_2(v,\ s)\Phi(s) - \frac{\vartheta(s)n_2^2(v,\ s)}{4} \right) ds = \infty, \tag{19}$$

for all $\mu_1 \in (0,\ 1)$, then (1) is oscillatory.

Proof. Let y be a non-oscillatory solution on $[v_0,\ \infty)$ and Case 1 holds.

From Lemma 4, multiplying (5) by $N(v,\ s)$ and integrating from v_1 to v, we obtain

$$\int_{v_1}^{v} N(v,\ s)Q_1(s)ds \le \xi(v_1)N(v,\ v_1) + \int_{v_1}^{v} \left(\frac{\partial}{\partial s}N(v,\ s) + \frac{\delta'(s)}{\delta(s)}N(v,\ s) \right) \xi(s)ds$$

$$- \int_{v_1}^{v} \Theta(s)N(v,\ s)\xi^{\frac{\alpha+1}{\alpha}}(s)ds.$$

From (3), we get

$$\int_{v_1}^{v} N(v,\ s)Q_1(s)ds \le \xi(v_1)N(v,\ v_1) + \int_{v_1}^{v} n_1(v,\ s)N_1^{\alpha/(\alpha+1)}(v,\ s)\xi(s)ds - \int_{v_1}^{v} \Theta(s)N(v,\ s)\xi^{\frac{\alpha+1}{\alpha}}(s)ds. \tag{20}$$

Using Lemma 3 with $V = \Theta(s)N(v,\ s)$, $U = n_1(v,\ s)N_1^{\alpha/(\alpha+1)}(v,\ s)$ and $y = \xi(s)$, we get

$$n_1(v,\ s){N_1}^{\alpha/(\alpha+1)}(v,\ s)\xi(s) - \Theta(s)N(v,\ s)\xi^{\frac{\alpha+1}{\alpha}}(s) \le \frac{n_1^{\alpha+1}(v,\ s)N_1^{\alpha}(v,\ s)}{(\alpha+1)^{\alpha+1}} \frac{2^{\alpha} r(v)\delta(v)}{(\mu_1 v^2)^{\alpha}}.$$

Upon using (20) then yields

$$\frac{1}{N(v,\ v_1)} \int_{v_1}^{v} \left(N(v,\ s)Q_1(s) - \frac{n_1^{\alpha+1}(v,\ s)N_1^{\alpha}(v,\ s)}{(\alpha+1)^{\alpha+1}} \frac{2^{\alpha} r(s)\delta(s)}{(\mu_1 s^2)^{\alpha}} \right) ds \le \xi(v_1),$$

which contradicts (18).

From Lemma 4 and Case 2, we have that (6) holds. Multiplying (6) by $N_2(v,\ s)$ and integrating from v_1 to v, we obtain

$$\int_{v_1}^{v} N_2(v,\ s)\Phi(s)ds \leq \varphi\left(t_1\right) N_2\left(v,\ v_1\right) + \int_{v_1}^{v}\left(\frac{\partial}{\partial s} N_2(v,\ s) + \frac{\vartheta'(s)}{\vartheta(s)} N_2(v,\ s)\right)\varphi(s)ds$$

$$-\int_{v_1}^{v} \frac{1}{\vartheta(s)} N_2(v,\ s)\varphi^2(s)ds.$$

Thus,

$$\int_{v_1}^{v} N_2(v,\ s)\Phi(s)ds \leq \varphi\left(v_1\right) N_2\left(v,\ v_1\right) + \int_{v_1}^{v} n_2(v,\ s)\sqrt{N_2(v,\ s)}\varphi(s)ds - \int_{v_1}^{v} \frac{1}{\vartheta(s)} N_2(v,\ s)\varphi^2(s)ds$$

$$\leq \varphi\left(v_1\right) N_2\left(v,\ v_1\right) + \int_{v_1}^{v} \frac{\vartheta(s) n_2^2(v,\ s)}{4}\ \mathrm{d}s,$$

and so

$$\frac{1}{N_2\left(v,\ v_1\right)} \int_{v_1}^{v}\left(N_2(v,\ s)\Phi(s) - \frac{\vartheta(s) n_2^2(v,\ s)}{4}\right) ds \leq \varphi\left(v_1\right),$$

which clearly contradicts (19). This completes the proof. □

Corollary 2 Suppose (2) holds. If $\delta,\ \vartheta \in C^1\left(\left[v_0,\ \infty\right),\ \mathbb{R}\right)$ are positive functions such that

$$\int_{v_0}^{\infty}\left(Q_1(s) - \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{r(s)\left(\delta'(s)\right)^{\alpha+1}}{\mu_1^{\alpha} s^{2\alpha} \delta^{\alpha}(s)}\right) ds = \infty, \tag{21}$$

and

$$\int_{v_0}^{\infty}\left(\Phi(s) - \frac{\left(\vartheta'(s)\right)^2}{4\vartheta(s)}\right) ds = \infty, \tag{22}$$

for some $\mu_1 \in (0,\ 1)$, then (1) is oscillatory.

If we take $\beta = \alpha$ in equation (1) we obtain some oscillation criteria.

Theorem 3 Let Cases 1 and 2, and (1) hold. For some constant $k_1 \in (0,\ 1)$ suppose that $\delta \in \mathrm{C}^1\left[v_0,\ \infty\right)$ and $\theta \in \mathrm{C}^1\left[v_0,\ \infty\right)$ are positive functions such that

$$\int_{v_0}^{\infty}\left[q(s)\left(\frac{\sigma^2(s)}{s^2}\right)^{\alpha} \delta(s) - \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{r(s)\left(\delta_+'(s)\right)^{\alpha+1}}{\left(k_1 \delta(s) s^2\right)^{\alpha}}\right] ds = \infty, \tag{23}$$

and

$$\int_{v_0}^{\infty}\left[\theta(s)\int_{s}^{\infty}\left[\frac{1}{r(u)}\int_{u}^{\infty}q(\varsigma)\left(\frac{\sigma^2(\varsigma)}{\varsigma^2}\right)^{\alpha}d\varsigma\right]^{\frac{1}{\alpha}}du-\frac{\left(\vartheta'_+(s)\right)^2}{4\vartheta(s)}\right]ds=\infty, \tag{24}$$

then (1) has oscillatory solutions.

Proof. Assume that we have Case 1. We find $y(v)\geq(v/2)y'(v)$, and hence

$$\frac{y(\sigma(v))}{y(v)}\geq\frac{\sigma^2(v)}{v^2}. \tag{25}$$

Using (7) and $\xi(v)>0$ for $v\geq v_1$,

$$\xi'(v)=\delta'(v)\frac{r(v)\left(y'''\right)^{\alpha}(v)}{y^{\alpha}(v)}+\delta(v)\frac{\left(r\left(y'''\right)^{\alpha}\right)'(v)}{y^{\alpha}(v)}-\alpha\delta(v)\frac{y^{\alpha-1}(v)y'(v)r(v)\left(y'''\right)^{\alpha}(v)}{y^{2\alpha}(v)}, \tag{26}$$

for every $\mu_1\in(0,\ 1)$ and all sufficiently large v. Hence, by (25) and (12), we obtain

$$\xi'(v)\leq\delta'(v)\frac{r(v)\left(y'''\right)^{\alpha}(v)}{y^{\alpha}(v)}+\delta(v)\frac{\left(r\left(y'''\right)^{\alpha}\right)'(v)}{y^{\alpha}(v)}-\frac{\alpha\mu_1}{2}v^2\delta(v)\frac{y'''(v)r(v)\left(y'''\right)^{\alpha}(v)}{y^{\alpha+1}(v)}.$$

Therefore, in view of (1), we get

$$\xi'(v)\leq-q(v)\left(\frac{\sigma^2(v)}{v^2}\right)^{\alpha}\delta(v)+\frac{\delta'_+(v)}{\delta(v)}\xi(v)-\frac{\alpha\mu_1}{2}\frac{v^2}{(r(v)\delta(v))^{\frac{1}{\alpha}}}\xi^{\frac{\alpha+1}{\alpha}}(v).$$

We set $A:=\alpha\mu_1v^2/2(r(v)\delta(v))^{1/\alpha},\ B:=\delta'_+(v)/\delta(v)$ and $y:=\xi(v)$. Then using the inequality

$$By-Ay^{\frac{\alpha+1}{\alpha}}\leq\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}},\quad A,\ B>0,$$

results in

$$\frac{\delta'_+(v)}{\delta(v)}\xi(v)-\frac{\alpha\mu_1v^2}{2(r(v)\delta(v))^{\frac{1}{\alpha}}}\xi^{\frac{\alpha+1}{\alpha}}(v)\leq\frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r(v)\left(\delta'_+(v)\right)^{\alpha+1}}{(k\delta(v)v^2)^{\alpha}}.$$

Hence, we obtain

$$\xi'(v)\leq-q(v)\left(\frac{\sigma^2(v)}{v^2}\right)^{u}\delta(v)+\frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r(v)\left(\delta'_+(v)\right)^{\alpha+1}}{(\mu_1\delta(v)v^2)^{\alpha}},$$

which implies that

$$\int_{v_1}^{v}\left[q(s)\left(\frac{\sigma^2(s)}{s^2}\right)^{\alpha}\delta(s)-\frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r(s)\left(\delta_+'(s)\right)^{\alpha+1}}{\left(\mu_1\delta(s)s^2\right)^{\alpha}}\right]ds\le\xi\left(v_1\right),$$

for every $\mu_1\in(0,\ 1)$ and all sufficiently large v, but this contradicts (23).

Now assume that Case 2 holds. Integrating (1) from v to l, we find

$$r(l)\left(y'''\right)^{\alpha}(l)-r(v)\left(y'''\right)^{\alpha}(v)+\int_{v}^{l}q(s)y^{\alpha}(\sigma(s))ds=0.$$

By virtue of $y>0$, $y'>0$ and $y''<0$, we get $y(v)\ge(v/2)y'(v)$. Therefore, (25) holds and hence we obtain

$$r(l)\left(y'''\right)^{\alpha}(l)-r(v)\left(y'''\right)^{\alpha}(v)+\int_{v}^{l}q(s)\left(\frac{\sigma^2(s)}{s^2}\right)^{\alpha}y^{\alpha}(s)ds\le 0.$$

Since $y'>0$, this yields

$$r(l)\left(y'''\right)^{\alpha}(l)-r(v)\left(y'''\right)^{\alpha}(v)+y^{\alpha}(v)\int_{v}^{l}q(s)\left(\frac{\sigma^2(s)}{s^2}\right)^{\alpha}ds\le 0.$$

Letting $l\to\infty$, we arrive at the following inequalities

$$-r(v)\left(y'''\right)^{\alpha}(v)+y^{\alpha}(v)\int_{v}^{\infty}q(s)\left(\frac{\sigma^2(s)}{s^2}\right)^{s}\le 0,$$

thus, we see that

$$-y'''(v)+y(v)\left[\frac{1}{r(v)}\int_{v}^{\infty}q(s)\left(\frac{\sigma^2(s)}{s^2}\right)^{\alpha}ds\right]^{\frac{1}{\alpha}}\le 0.$$

Integrating again from v to ∞ gives

$$y''(v)+y(v)\int_{v}^{\infty}\left[\frac{1}{r(u)}\int_{u}^{\infty}q(s)\left(\frac{\sigma^2(s)}{s^2}\right)^{\alpha}ds\right]^{\frac{1}{\alpha}}du\le 0. \tag{27}$$

From (8), $\varphi(v)>0$ for $v\ge v_1$ and

$$\varphi'(v) = \vartheta'(v)\frac{y'(v)}{y(v)} + \vartheta(v)\frac{y''(v)y(v) - (y')^2(v)}{y^2(v)}$$

$$= \vartheta(v)\frac{y''(v)}{y(v)} + \frac{\vartheta'(v)}{\vartheta(v)}\varphi(v) - \frac{\varphi^2(v)}{\vartheta(v)}.$$

Hence, by (27) we find

$$\varphi'(v) \leq -\vartheta(v)\int_v^\infty \left[\frac{1}{r(u)}\int_u^\infty q(s)\left(\frac{\sigma^2(s)}{s^2}\right)^\alpha ds\right]^{\frac{1}{\alpha}} du + \frac{\vartheta'(v)}{\vartheta(v)}\varphi(v) - \frac{\varphi^2(v)}{\vartheta(v)}. \tag{28}$$

Thus, we have

$$\varphi'(v) \leq -\vartheta(v)\int_v^\infty \left[\frac{1}{r(u)}\int_u^\infty q(s)\left(\frac{\sigma^2(s)}{s^2}\right)^\alpha ds\right]^{\frac{1}{\alpha}} du + \frac{(\vartheta'(v))^2}{4\vartheta(v)}.$$

This yields

$$\int_{v_1}^v \left[\vartheta(s)\int_s^\infty \left[\frac{1}{r(u)}\int_u^\infty q(\varsigma)\left(\frac{\sigma^2(\varsigma)}{\varsigma^2}\right)^\alpha d\varsigma\right]^{\frac{1}{\alpha}} du - \frac{(\vartheta'(s))^2}{4\vartheta(s)}\right] ds \leq \varphi(v_1),$$

which contradicts (24) and thus Theorem 3 is proven. □

Theorem 4 Let (2) holds and assume that the following equations

$$\left(\frac{r(v)}{v^{2\alpha}}\left(y'(v)\right)^\alpha\right)' + q(v)\left(\frac{\mu_1\sigma^2(v)}{2v^2}\right)^\alpha y^\alpha(v) = 0, \tag{29}$$

and

$$y''(v) + y(v)\int_t^\infty \left[\frac{1}{r(u)}\int_u^\infty q(s)\left(\frac{\sigma^2(s)}{s^2}\right)^\alpha ds\right]^{\frac{1}{\alpha}} du = 0, \tag{30}$$

are oscillatory, therefore every solution of equation (1) exhibits oscillatory behaviour.

Proof. Following the steps in Theorem 3, we have (26) and (27). If we set $\delta(v) = 1$ in (26), then we obtain

$$\xi'(v) + \frac{\alpha\mu_1 v^2}{2(r(v))^{\frac{1}{\alpha}}}\xi^{\frac{\alpha+1}{\alpha}}(v) + q(v)\left(\frac{\sigma^2(v)}{v^2}\right)^\alpha \leq 0,$$

$\forall \mu_1 \in (0,\ 1)$. Thus, we can see that equation (29) is nonoscillatory for every constant $\mu_1 \in (0,\ 1)$, which is a contradiction. If we now set $\vartheta(v) = 1$ in (28), then we get

$$\varphi'(v) + \varphi^2(v) + \int_t^\infty \left[\frac{1}{r(u)} \int_u^\infty q(s) \left(\frac{\sigma^2(s)}{s^2} \right)^\alpha ds \right]^{\frac{1}{\alpha}} du \leq 0.$$

Hence equation (30) is nonoscillatory, which contradicts the previous assumption. The proof of Theorem 4 is now complete. □

It is generally accepted (see [16]) that if

$$\int_{v_0}^\infty \frac{1}{a(v)} = \infty,$$

and

$$\liminf_{v\to\infty} \left(\int_{v_0}^v \frac{1}{a(s)} ds \right) \int_v^\infty q(s) ds > \frac{1}{4}.$$

Equation (28) with $\alpha = 1$ is oscillatory. From the previous results that we have concluded and Theorem 4, we can easily obtain the Hille and Nehari type oscillation criteria for (1), in the next theorem.

Theorem 5 Let (2) hold. Assume that

$$\int_{v_0}^\infty \frac{v^2}{r(v)} = \infty,$$

and

$$\liminf_{v\to\infty} \left(\int_{v_0}^v \frac{s^2}{r(s)} ds \right) \int_v^\infty q(s) \frac{\sigma^2(s)}{s^2} ds > \frac{1}{2k_1}, \tag{31}$$

for some constant $k_1 \in (0,\ 1)$, and

$$\liminf_{v\to\infty} \int_v^\infty \int_\eta^\infty \frac{1}{r(u)} \int_u^\infty q(s) \frac{\sigma^2(s)}{s^2} ds\, du\, d\eta > \frac{1}{4}. \tag{32}$$

Then every solution of (1) is oscillatory.

Proof. The proof is clear. □

4. Examples

In this section we shall illustrate our main results via some examples.

Example 1 Consider the fourth-order delay differential equation

$$\left(vy'''(v)\right)' + \frac{q_0}{2v^3}y^3(v) = 0,\ v \geq 1, \tag{33}$$

where $\eta \in (0,\ 1)$ and $q_0 > 0$. We note that $\alpha = 1, \beta = 3, r(v) = v, \sigma(v) = v$ and $q(v) = q_0/2v^3$. Hence, if we set $\delta(s) := v^2$ and $\vartheta(v) := v, A_1 = \frac{1}{2}, A_2 = 1$.

Using Corollary 2, we have

$$Q_1(v) = v^2 q_0/2v^3 \frac{1}{4}\left(\frac{v}{v}\right)^9 = q_0/8v,$$

$$\Phi(v) = v\int_v^\infty \left(q_0/4u^3\right) du = \frac{q_0}{8v},$$

$$\int_{v_0}^\infty \left(Q_1(s) - \frac{2^\alpha}{(\alpha+1)^{\alpha+1}}\frac{r(s)\left(\delta'(s)\right)^{\alpha+1}}{\mu_1^\alpha s^{2\alpha}\delta^\alpha(s)}\right) ds = \int_{v_0}^\infty \left(q_0/8s - \frac{2}{\mu_1 s}\right) ds = \infty,\ \text{ if }\ q_0 > \frac{16}{\mu_1}, \tag{34}$$

$$\int_{v_0}^\infty \left(\boldsymbol{\Phi}(s) - \frac{\left(\boldsymbol{\vartheta}'(s)\right)^2}{4\boldsymbol{\vartheta}(s)}\right) ds = \int_{v_0}^\infty \left(\frac{q_0}{8s} - \frac{1}{4s}\right) ds = \infty,\ \text{ if }\ q_0 > 2.$$

Equation (33) is oscillatory if (34) holds.

Example 2 Consider the fourth-order delay differential equation

$$\left(y'''(v)\right)' + \frac{q_0}{v^4}y(3v) = \mathbf{0},\ v \geq \mathbf{1}, \tag{35}$$

where $q_0 > 0$. Let $r(v) = 1$, and $\sigma(v) = 3v$. If we set $l = 1$, then condition (31) becomes

$$\liminf_{v\to\infty}\left(\int_{v_0}^v \frac{s^2}{r(s)}ds\right)\int_v^\infty \frac{q_0}{s^4}ds = \liminf_{v\to\infty}\left(\frac{v^3}{3}\right)\int_v^\infty \frac{q_0}{s^4}ds = \frac{q_0}{9} > \frac{1}{4},$$

and condition (32) now reads as

$$\liminf_{v\to\infty}\int_v^\infty\left(\int_\eta^\infty\left(\frac{\ell}{r(v)}\int_u^\infty q(s)\frac{\sigma^\gamma(s)}{s^\gamma}ds\right)du\right)d\eta = \liminf_{v\to\infty}\int_v^\infty\left(\int_\eta^\infty\left(\int_u^\infty \frac{3q_0}{s^4}ds\right)du\right)d\eta$$

$$= q_0 > \frac{1}{4}.$$

Therefore, from Theorem 5, all solutions of (35) are oscillatory if $q_0 > 2.25$.

5. Conclusion

We have examined the oscillatory behaviour of (1) in this article. Conditional oscillation criterion for (1) when $\beta \neq \alpha$ and $\beta = \alpha$ were obtained. New oscillatory properties have been found through the use of comparison tools, the Riccati transformation, and the integral averaging methodology. We will attempt to establish the oscillation criterion for higher-order differential equations in our future research. Some examples have been presented to highlight the significance of the main results.

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Conflict of interest

All authors have declared they do not have any competing interests.

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