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Research Article

A Class of p-Valent Close-to-Convex Functions Defined Using Gegenbauer Polynomials

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Abstract: A new class of p-valent close-to-convex functions is introduced in this paper, which is defined using Gegenbauer Polynomials within the open unit disk $\mathbb{D}$. This investigation sheds light on the properties and behaviors of these p-valent close-to-convex functions, providing estimations for the modulus of the coefficients a_{p+1} and a_{p+2}, with p being a natural number, for functions falling under this particular class. Additionally, this paper also investigates the classical Fekete-Szegö functional problem for functions f that are part of the aforementioned class.

Keywords: analytic functions, holomorphic functions, univalent functions, p-valent functions, principle of subordination, gegenbauer polynomials, chebyshev polynomials, coefficient estimates, fekete-szegö inequality

MSC: 30C45, 30C50, 33C45, 33C05, 11B39

1. Introduction

Consider the set $\mathscr{H}$, which consists of all functions $f(\zeta)$ that are holomorphic within the open unit disk denoted as $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. A function f is classified as p-valent if, within a specific domain $\mathscr{D} \subset \mathbb{C}$, the equation $f(\zeta) = w$ can yield no more than p distinct roots for any given $w \in \mathbb{C}$. Consequently, there exists a particular value $w_0 \in \mathbb{C}$ for which the equation $f(\zeta) = w_0$ possesses precisely p roots in the domain $\mathscr{D}$. Now, define the class $\mathscr{H}_p$ as the collection of all holomorphic functions f that belong to the set $\mathscr{H}$ and satisfy the aforementioned conditions. This classification highlights the significance of p-valency in the study of analytic functions, particularly in understanding their root structures and the implications of their behavior within the unit disk. The exploration of such functions contributes to a deeper comprehension of complex analysis and its applications. Moreover, any function f belongs to the class $\mathscr{H}_p$ can be written as

$$f(\zeta) = \zeta^p + \sum_{n=p+1}^{\infty} a_n \zeta^n, \quad \text{where} \quad \zeta \in \mathbb{D}. \tag{1}$$

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In this paper, $\mathscr{S}$ represents the set of functions that are univalent in the unit disk $\mathbb{D}$ and belong to the class $\mathscr{H} = \mathscr{H}_1$. Moreover, $\mathscr{S}_p^*$ denotes the class of p-valent starlike functions, and we say $f(\zeta) \in \mathscr{H}_p$ in the class $\mathscr{S}_p^*$ if the following condition holds for all $\zeta \in \mathbb{D}$:

$$\mathscr{R}\rceil\left\{\frac{\zeta f'(\zeta)}{f(\zeta)}\right\} > 0.$$

Moreover, let S_p^c denote the set of p-valent convex functions. We can state that $f(\zeta) \in \mathscr{H}_p$ within the class $\mathscr{S}_p^c$ when the following condition is satisfied for all $\zeta \in \mathbb{D}$:

$$\mathscr{R}\rceil\left\{1 + \frac{\zeta f''(\zeta)}{f'(\zeta)}\right\} > 0.$$

The fact that if f is analytic in a convex domain $\mathscr{D} \subset \mathbb{C}$ and $\mathscr{R}\rceil\left\{e^{i\theta} f'(\zeta)\right\} > 0$ for some real θ and for all $\zeta \in \mathscr{D}$ is well established (see, for details [1] and [2]). This implies that $f(\zeta)$ is univalent in $\mathscr{D}$. This conclusion is supported by the findings presented in the literature, particularly in the works of Noshiro and Warschawski. In 1935, Ozaki [3] expanded upon the previous findings by demonstrating that if the function $f(\zeta)$, as described in equation (1), is analytic within a convex domain $\mathscr{D} \subset \mathbb{C}$ and $\mathscr{R}\rceil\left\{e^{i\theta} f^{(p)}(\zeta)\right\} > 0$ for a certain real θ and for all $\zeta \in \mathscr{D}$, then $f(\zeta)$ can be at most p-valent in $\mathscr{D}$. This extension of results sheds light on the behavior of analytic functions and provides valuable insights into the properties of functions satisfying the given criteria. Furthermore, it can be proven that if f is a member of the function space $\mathscr{H}_p$ and the real part of $f^{(p)}(\zeta)$ is positive for every ζ in the domain $\mathbb{D}$, then the function $f(\zeta)$ is constrained to being at most p-valent within the unit disk $\mathbb{D}$. Moreover, in 1989, Nunokawa [4] established that if $f \in \mathscr{H}_p$, where $p \geq 2$, and $\arg\left\{f^{(p)}(\zeta)\right\} < \frac{3\pi}{4}$ for all $\zeta \in \mathbb{D}$, then $f(\zeta)$ is at most p-valent in $\mathbb{D}$. For further information and better understanding of p-valent, readers are encouraged to consult the articles [1–8], and the references provided therein.

The functions f and g being analytic in the open unit disk $\mathbb{D}$ implies that f is subordinated by g in $\mathbb{D}$, denoted as $f(z) \prec g(\zeta)$ for all $\zeta \in \mathbb{D}$, if there exists a Schwarz function w satisfying $w(0) = 0$ and $|w(\zeta)| < 1$ for all $\zeta \in \mathbb{D}$, such that $f(\zeta) = g(w(\zeta))$ for all $\zeta \in \mathbb{D}$. This relationship between f and g is a fundamental concept in complex analysis, providing a way to compare the behavior of two analytic functions within the unit disk. Notably, when the function g is univalent over $\mathbb{D}$, the condition $f(\zeta) \prec g(\zeta)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. This equivalence highlights the significance of the subordination principle in understanding the relationship between analytic functions. For further insights and detailed discussions on the Subordination Principle, interested readers are encouraged to explore the monographs [9–12]. These sources provide comprehensive explanations and applications of this principle in the context of complex analysis .

The research conducted in geometric function theory sheds light on the intricate relationships between coefficients and the geometric properties of functions. By examining the bounds placed on the modulus of a function's coefficients, researchers can gain a deeper understanding of how these functions behave and interact within the mathematical framework. This analytical approach not only enhances our comprehension of the underlying principles governing geometric function theory but also paves the way for further exploration and discovery in this dynamic field of study. For example, within the class $\mathscr{S}$, it is established that the modulus of the coefficient a_n is bounded by the value of n. These bounds on the modulus of coefficients provide valuable insights into the geometric characteristics of these functions. Specifically, the restriction on the second coefficients of functions belonging to the class $\mathscr{S}$ offers crucial details regarding the growth and distortion bounds within this class.

The exploration of coefficient-related properties of functions within the bi-univalent class Σ commenced in the 1970s. Notably, Lewin's work, in 1967 [13], marked a significant milestone as he examined the bi-univalent function class and established a bound for the coefficient $|a_2|$. Following this, Netanyahu's research, in 1969 [14], determined that the maximum value of $|a_2|$ is $\frac{4}{3}$ for functions categorized under Σ. Furthermore, Brannan and Clunie, in 1979

[15], demonstrated that for functions in this class, the inequality $|a_2| \leq \sqrt{2}$ holds true. This foundational work has spurred numerous investigations into the coefficient bounds for various subclasses of bi-univalent functions. Despite the extensive research conducted on the coefficient bounds for bi-univalent functions, there remains a significant gap in knowledge regarding the general coefficients $|a_2|$ for cases where $n \geq 4$. The challenge of estimating the coefficients, particularly the general coefficient $|a_n|$, continues to be an unresolved issue in the field. This ongoing inquiry highlights the complexity and richness of the bi-univalent function class, suggesting that further exploration is necessary to fully understand the behavior of these coefficients in higher dimensions.

Fekete and Szegö, in 1933 [16], determined the maximum value of $|a_3 - \lambda a_2^2|$ for a univalent function f, with the real parameter $0 \leq \lambda \leq 1$. This result led to the establishment of the Fekete-Szegö problem, which involves maximizing the modulus of the functional $\Psi_\lambda(f) = a_3 - \lambda a_2^2$ for $f \in \mathscr{H}$ with any complex number λ. Numerous researchers have delved into the Fekete-Szegö functional and other coefficient estimates problems. For instance, relevant articles include [16–24], and the references provided therein. These studies have contributed to a deeper understanding of the Fekete-Szegö problem and its implications in the field of geometric function theory.

2. Preliminaries

This section contains essential information that is crucial for the primary findings of this paper. Szynal, in 1994 [25], established and examined a subset $\mathfrak{F}(\mu)$ of the class $\mathscr{H}$, which contains functions of the form

$$f(z) = \int_{-1}^{1} H(z,\ t)\, d\sigma(t). \tag{2}$$

The function defined as $H(z,\ t) = \dfrac{z}{(z^2 - 2tz + 1)^\mu}$ operates under the conditions where $\mu \geq 0$ and $-1 \leq t \leq 1$. In this context, σ represents the probability measure on $[-1,\ 1]$. The Taylor-Maclaurin series expansion of the function $H(z,\ t)$ can be expressed as follows:

$$H(z,\ t) = z + C_1^\mu(t)z^2 + C_2^\mu(t)z^3 + C_3^\mu(t)z^4 + \cdots,$$

where $C_n^\mu(t)$ signifies the Gegenbauer polynomials of order μ.

Addtionally, for any real numbers μ and t, with the stipulation that $\mu \geq 0$ and $-1 \leq t \leq 1$, and $z \in \mathbb{D}$ the generating function for the Gegenbauer polynomials is articulated as:

$$G_\mu(z,\ t) = (z^2 - 2tz + 1)^{-\mu}.$$

This formulation provides a comprehensive framework for understanding the behavior of the Gegenbauer polynomials in relation to the specified parameters. Consequently, for any given value of t, the function $G_\mu(z,\ t)$ is analytic within the unit disk $\mathbb{D}$, and its Taylor-Maclaurin series can be expressed as

$$G_\mu(z,\ t) = \sum_{n=0}^{\infty} C_n^\mu(t)z^n.$$

Furthermore, should $f \in \mathfrak{F}(\mu)$ as denoted in (2), the n^{th} coefficient is expressible as

$$a_n = \int_{-1}^{1} C_{n-1}^{\mu}(t)\, d\sigma(t).$$

Gegenbauer polynomials can also be expressed through a specific recurrence relation, which is articulated as follows:

$$C_n^{\mu}(t) = \frac{2t(n+\mu-1)C_{n-1}^{\mu}(t) - (n+2\mu-2)C_{n-1}^{\mu}(t)}{t}, \tag{3}$$

with the foundational values specified as

$$C_0^{\mu}(t) = 1, \quad C_1^{\mu}(t) = 2\mu t, \quad \text{and } C_2^{\mu}(t) = 2\mu(\mu+1)t^2 - \mu. \tag{4}$$

The Gegenbauer polynomials and their special cases, such as the Legendre polynomials $L_n(t)$ and the Chebyshev polynomials of the second kind $T_n(t)$, are well-known to be orthogonal polynomials. The values of μ for these polynomials are $\mu = 1/2$ and $\mu = 1$ respectively. More precisely, the Legendre polynomials $L_n(t)$ can be expressed as $L_n(t) = C_n^{1/2}(t)$, and the Chebyshev polynomials $T_n(t)$ can be expressed as $T_n(t) = C_n^1(t)$.

Additional information regarding the Gegenbauer polynomials and their specific instances, readers are encouraged to consult the articles referenced as [17–28], as well as the monographs [9, 29, 30], and the related sources. Following this, we proceed to define our category of p-valent close-to-convex functions, denoted as $\mathscr{H}_p(\lambda,\ \alpha,\ \mu)$.

Definition 1 A function $f(z)$ belonging to the family $\mathscr{H}_p$ is considered to be part of the class $\mathscr{H}_p(\lambda,\ \alpha,\ \mu)$ if, for every $z \in \mathbb{D}$, it obeys the following subordination condition:

$$\frac{(1-\alpha)}{\lambda p}\left(\frac{zf''(z)}{f'(z)} + 1\right) + \frac{\alpha}{\lambda}\left(\frac{zf''(z)}{f'(z)} - p + 1\right) \prec G_{\mu}(z,\ t),$$

where the parameters λ, α, and μ have specific constraints on their values, namely $\mu \geq 0$, $-1 \leq t \leq 1$, $\alpha \in [0,\ 1]$, and λ is a non-zero complex number.

The subsequent lemma, which is extensively documented in the literature (see for example [22]), is a widely recognized principle that holds significant importance for the work we are presenting.

Lemma 1 Consider the Schwarz function defined as:

$$w(\zeta) = w_1\zeta + w_2\zeta^2 + w_3\zeta^3 + \cdots \text{ where } \zeta \in \mathbb{D}.$$

It follows that $|w_1| \leq 1$ and for any $\gamma \in \mathbb{C}$ the following inequality holds:

$$|w_2 - \gamma w_1^2| \leq 1 + (|\gamma| - 1)|w_1|^2 \leq \max\{1,\ |\gamma|\}.$$

Notably, this result is particularly sharp for the specific cases of the functions $w(\zeta) = \zeta$ and $w(\zeta) = \zeta^2$.

The main objective of this article is to explore a specific category of p-valent functions within the open unit disk $\mathbb{D}$, denoted as $\mathscr{H}_p(\lambda,\ \alpha,\ \mu)$. The article focuses on obtaining estimates for the initial coefficients $|a_{p+1}|$ and $|a_{p+2}|$

for functions belonging to this category. Additionally, the article delves into the analysis of the related Fekete-Szegö functional problem for functions in this particular category.

3. Coefficient bounds of functions in the class $\mathscr{H}_p(\lambda,\ \alpha,\ \mu)$

This section of the paper outlines the bounds for the modulus of the initial coefficients of functions that are part of the class $\mathscr{H}_p(\lambda,\ \alpha,\ \mu)$, as denoted by equation (1).

Theorem 1 If a function f belongs to the class $\mathscr{H}_p(\lambda,\ \alpha,\ \mu)$ and is represented by the equation (1), then

$$|a_{p+1}| \leq \frac{2\mu p^2|t||\lambda|}{(1-\alpha+\alpha p^2)(p+1)}, \tag{5}$$

$$|a_{p+2}| \leq \frac{\mu p|\lambda|}{p+2}\max\{\Theta_1,\ \Theta_2\}, \tag{6}$$

where $\Theta_1 = \dfrac{2(\mu+1)t^2+2|t|+1}{1-\alpha+\alpha p}$ and $\Theta_2 = \dfrac{4\mu p^2 t^2|\lambda|}{(1-\alpha+\alpha p)^2}$.

Proof. Suppose a function f belongs to the class $\mathscr{H}_p(\lambda,\ \alpha,\ \mu)$. According to the Definition 1, we can find a holomorphic function ϕ defined on the open unit disk $\mathbb{D}$ such that

$$\frac{(1-\alpha)}{\lambda p}\left(\frac{zf''(z)}{f'(z)}+1\right)+\frac{\alpha}{\lambda}\left(\frac{zf''(z)}{f'(z)}-p+1\right) = H_\mu(t,\ \phi(z)), \tag{7}$$

where the function ϕ in the form of a power series is expressed as $\phi(z) = d_1 z + d_2 z^2 + d_3 z^3 + \cdots$, satisfying the conditions $\phi(0) = 0$ and $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Additionally, it is a known fact, as referenced in [9], that modulus of the values of the coefficients d_j are bounded by 1 for all $j \in \mathbb{N}$.

Now, by equating the coefficients on both sides of equation (7), a set of equations can be derived as follows:

$$\left(\frac{(1-\alpha)(p+1)}{\lambda p^2}+\frac{\alpha(p+1)}{\lambda}\right)a_{p+1} = C_1^\mu(t)d_1, \tag{8}$$

and

$$\frac{1-\alpha+\alpha p}{\lambda p}\left(2(p+2)a_{p+2}-\frac{(p+1)^2}{p}a_{p+1}^2\right) = C_1^\mu(t)d_2 + C_2^\mu(t)d_1^2. \tag{9}$$

Hence, using equation (8), we get

$$a_{p+1} = \frac{\lambda p^2 C_1^\mu(t)d_1}{(1-\alpha)(p+1)+\alpha p^2(p+1)}. \tag{10}$$

Therefore, considering the initial values presented in equation (4) alongside the constraint $|d_1| < 1$, we arrive at the conclusion that:

$$|a_{p+1}| \leq \frac{2\mu p^2|t||\lambda|}{(1-\alpha)(p+1)+\alpha p^2(p+1)},$$

which confirms the sought-after estimate for $|a_{p+1}|$.

In the next step, we seek to determine the coefficient estimate for $|a_{p+2}|$. By applying equation (9), we can derive the following equation

$$2p(p+2)a_{p+2} = \frac{\lambda p^2[C_1^{\mu}(t)d_2 + C_2^{\mu}(t)d_1^2]}{1-\alpha+\alpha p} + (p+1)^2 a_{p+1}^2. \tag{11}$$

By utilizing the equation (10), the final equation is transformed into

$$2p(p+2)a_{p+2} = \frac{\lambda p^2[C_1^{\mu}(t)d_2 + C_2^{\mu}(t)d_1^2]}{1-\alpha+\alpha p} + \frac{\lambda^2 p^4 [C_1^{\mu}(t)]^2 d_1^2}{(1-\alpha+\alpha p^2)^2}.$$

Consequently, by employing the initial values (4) and considering the condition $|d_j| \leq 1$ for all $j \in \mathbb{N}$, we obtain

$$|a_{p+2}| \leq \frac{|\lambda| p\mu}{2(p+2)} \left\{ \frac{2(\mu+1)t^2 + 2|t| + 1}{1-\alpha+\alpha p} + \frac{4|\lambda| p^2 \mu t^2}{(1-\alpha+\alpha p^2)^2} \right\}.$$

This leads to the required estimation of $|a_{p+2}|$. Consequently, the proof of Theorem 1 is now concluded. □

The following corollary emerges as a direct outcome of Theorem 1 with the assumption that $\mu = 1$. These initial estimations of coefficients are intricately linked to the Chebyshev polynomials of the second kind. The methodology employed in proving this corollary closely resembles that of the preceding theorem, hence, we have chosen to exclude the detailed proof for brevity.

Corollary 1 If a function $f \in \mathscr{H}_p$ is expressed as (1) and is a member of the class $\mathscr{H}_p(\lambda, \alpha, 1)$, then it can be concluded that

$$|a_{p+1}| \leq \frac{2p^2|t||\lambda|}{(1-\alpha+\alpha p^2)(p+1)},$$

and

$$|a_{p+2}| \leq \frac{p|\lambda|}{p+2} \max\left\{ \frac{4t^2 + 2|t| + 1}{1-\alpha+\alpha p}, \frac{4p^2 t^2 |\lambda|}{(1-\alpha+\alpha p)^2} \right\}.$$

4. Fekete-Szegö inequality of the function class $\mathscr{H}_p(\lambda, \alpha, \mu)$

In this section, we examine the classical Fekete-Szegö functional applied to functions that are memebers of our specified class $\mathscr{H}_p(\lambda, \alpha, \mu)$.

Theorem 2 If a function f is a member of the class $\mathscr{H}_p(\lambda, \alpha, \mu)$ and is represented by equation (1), then for a real number ζ and a positive number t such that the following inequality holds

$$|a_3-\zeta a_2^2| \leq \begin{cases} \dfrac{p\mu t|\lambda||A|}{(p+2)(1-\alpha+\alpha p)}, & \text{if } \zeta \in [\zeta_1,\ \zeta_2] \\ \\ \dfrac{p\mu t|\lambda|}{(p+2)(1-\alpha+\alpha p)}, & \text{if } \zeta \notin [\zeta_1,\ \zeta_2], \end{cases} \tag{12}$$

where

$$\zeta_1 = \frac{(p+1)^2}{2p(p+2)} + \frac{(2(\mu+1)t^2-2t-1)(p+1)^2(1-\alpha+\alpha p^2)^2}{8t^2\mu\lambda p^3(p+2)(1-\alpha+\alpha p)},$$

$$\zeta_2 = \frac{(p+1)^2}{2p(p+2)} + \frac{(2(\mu+1)t^2+2t-1)(p+1)^2(1-\alpha+\alpha p^2)^2}{8t^2\mu\lambda p^3(p+2)(1-\alpha+\alpha p)},$$

and

$$A = \frac{2(\mu+1)t^2-1}{2t} + \frac{2t\mu\lambda p^2(1-\alpha+\alpha p)\left((p+1)^2-2\zeta p(p+2)\right)}{(p+1)^2(1-\alpha+\alpha p^2)^2}.$$

Proof. For every real number ζ, the utilization of equations (11) and (10) leads to the following outcome

$$\begin{aligned} a_{p+2}-\zeta a_{p+1}^2 &= \frac{\lambda p[C_1^\mu(t)d_2+C_2^\mu(t)d_1^2]}{2(p+2)(1-\alpha+\alpha p)} + \left(\frac{(p+1)^2}{2p(p+2)}-\zeta\right)a_{p+1}^2 \\ &= \frac{\lambda p[C_1^\mu(t)d_2+C_2^\mu(t)d_1^2]}{2(p+2)(1-\alpha+\alpha p)} + \frac{((p+1)^2-2\zeta p(p+2))\lambda^2 p^3[C_1^\mu(t)]^2d_1^2}{2(p+2)(p+1)^2(1-\alpha+\alpha p^2)^2} \\ &= \frac{\lambda pt\mu}{(p+2)(1-\alpha+\alpha p)}\left\{d_2+Ad_1^2\right\}. \end{aligned}$$

Therefore, with the assistance of Lemma 1, we are able to achieve the following inequality

$$|a_{p+2}-\zeta a_{p+1}^2| \leq \frac{p\mu|\lambda||t|}{(p+2)(1-\alpha+\alpha p)}\max\{1,\ |A|\}.$$

For $t>0$, if the following inequality holds, then we can proceed to solve for the real number ζ

$$\left|\frac{C_2^\mu(t)}{C_1^\mu(t)} + \frac{((p+1)^2-2\zeta p(p+2))\lambda p^2(1-\alpha+\alpha p)[C_1^\mu(t)]}{(p+1)^2(1-\alpha+\alpha p^2)^2}\right| \leq 1.$$

Now, by solving for ζ we obtain

$$-1-\frac{C_2^{\mu}(t)}{C_1^{\mu}(t)} \leq ((p+1)^2-2\zeta p(p+2))\lambda p^2(1-\alpha+\alpha p)[C_1^{\mu}(t)] \leq 1-\frac{C_2^{\mu}(t)}{C_1^{\mu}(t)}.$$

Hence, simple calculations give us the following inequality

$$\zeta_1 \leq \zeta \leq \zeta_2,$$

where

$$\zeta_1 = \frac{(p+1)^2}{2p(p+2)} + \frac{[C_2^{\mu}(t)-C_1^{\mu}(t)](p+1)^2(1-\alpha+\alpha p^2)^2}{2[C_1^{\mu}(t)]^2\lambda p^3(p+2)(1-\alpha+\alpha p)},$$

and

$$\zeta_2 = \frac{(p+1)^2}{2p(p+2)} + \frac{[C_2^{\mu}(t)+C_1^{\mu}(t)](p+1)^2(1-\alpha+\alpha p^2)^2}{2[C_1^{\mu}(t)]^2\lambda p^3(p+2)(1-\alpha+\alpha p)}.$$

Finally, utilizing the initial values referenced in equation (4), we obtain the anticipated outcome as indicated in the inequality (12). This marks the conclusion of the proof for the Theorem. □

The following corollary is a natural outcome of the previously stated Theorem 2. When μ is assigned a value of 1, the resulting Fekete-Szegö inequality becomes associated with the Chebyshev polynomials of the second kind. The approach used to establish this corollary closely mirrors that of the previous theorem, therefore, we have opted not to include the detailed proof for the sake of conciseness.

Corollary 2 If a function $f \in \mathscr{A}_p$ is given by (1) belong to the class $\mathscr{A}_p(\lambda,\ \alpha,\ 1)$, then for some $\zeta \in \mathbb{R}$ and $t > 0$,

$$|a_3-\zeta a_2^2| \leq \begin{cases} \dfrac{pt|\lambda||B|}{(p+2)(1-\alpha+\alpha p)}, & \text{if } \zeta \in [\tau_1,\ \tau_2] \\ \\ \dfrac{pt|\lambda|}{(p+2)(1-\alpha+\alpha p)}, & \text{if } \zeta \notin [\tau_1,\ \tau_2], \end{cases}$$

where

$$\tau_1 = \frac{(p+1)^2}{2p(p+2)} + \frac{(4t^2-2t-1)(p+1)^2(1-\alpha+\alpha p^2)^2}{8t^2\lambda p^3(p+2)(1-\alpha+\alpha p)},$$

$$\tau_2 = \frac{(p+1)^2}{2p(p+2)} + \frac{(4t^2+2t-1)(p+1)^2(1-\alpha+\alpha p^2)^2}{8t^2\lambda p^3(p+2)(1-\alpha+\alpha p)},$$

and

$$B = \frac{4t^2-1}{2t} + \frac{2t\lambda p^2(1-\alpha+\alpha p)\left((p+1)^2-2\zeta p(p+2)\right)}{(p+1)^2(1-\alpha+\alpha p^2)^2}.$$

Remark 1 This research paper is a generalization of some results published recently. For example:

• Taking $\alpha = 0$, $\lambda = 1$ and $p = 1$, we have the class $B_C(\mu)$ that is investigated in [31]. We also have the class $\mathscr{G}_\Sigma(x,\ \mu,\ 1)$ that is investigated in [32].

• Taking $\alpha = 0$, $\lambda = 1$, $p = 1$ and $\mu = 1$, we have the class $L(0,\ t)$ that is investigated in [33]. We also have the class $K(0,\ t)$ that is investigated in [34].

5. Conclusion

This research paper explored a novel family of p-valent close-to-convex functions that are associated with Gegenbauer polynomials. The author has established estimates for the initial coefficients and addressed the Fekete-Szegö functional problem for functions within this class. The findings of this study are expected to yield various results for subclasses defined through Horadam polynomials and their specific variations, such as Fibonacci polynomials, Lucas polynomials, Pell polynomials, and both the first and second kinds of Chebyshev polynomials. Furthermore, the insights provided in this paper are anticipated to motivate researchers to broaden these concepts to encompass harmonic functions and symmetric q-calculus.

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Conflict of interest

The authors declare no competing financial interest.

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